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FOURTH ORDER VACUUM POLARIZATION CONTRIBUTION TO THE SIXTH ORDER ELECTRON MAGNETIC MOMENT

J.A. Mignaco *)
Institut de Physique, Université de Genève

and

E. Remiddi
CERN -- Geneva

ABSTRACT

The contribution of fourth order vacuum polarization to sixth order radiative correction to the electron magnetic moment is analytically evaluated and found to be $0.055(\alpha/\pi)^3$.

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1. INTRODUCTION

We report here a calculation of the contribution to the 6th order radiative corrections to the magnetic moment of the electron g_e coming from 4th order vacuum polarization. The relevant Feynman graphs are shown on Fig. 1. The result is

$$\left(\frac{g_e - 2}{2}\right)^{(vi)}_{vac.\,pol.} = .055 \left(\frac{\alpha}{\pi}\right)^3$$

For convenience, we recall that the 4th order electron magnetic moment coming from the 2nd order vacuum polarization is

$$\left(\frac{g_e - 2}{2}\right)^{(iv)}_{vac.\,pol.} = .0157 \left(\frac{\alpha}{\pi}\right)^2$$

whereas the total contribution is [1),2)]

$$\left(\frac{g_e - 2}{2}\right)^{(iv)} = -.328 \left(\frac{\alpha}{\pi}\right)^2$$

On the other hand, an estimate of the whole 6th order radiative corrections made [3)] by neglecting, among other, the contributions we consider here, gives

$$\left(\frac{g_e - 2}{2}\right)^{(vi)}_{estimated} = .13 \left(\frac{\alpha}{\pi}\right)^3$$

The present calculation is based on three tools :

i) the dispersive approach to perturbation theory [4)];

ii) the theory of generalized dilogarithmic functions [5)];

iii) the SCHOONSCHIP program written by Veltman [6)] to perform algebraic manipulations.

2. METHOD

Consider in general a graph of the form shown in Fig. 2, with

$$\Delta = p_1 - p_2$$
$$t = -\Delta^2$$
$$p_1^2 = p_2^2 = -1$$

(we take the electron mass equal to 1).

Let us define $p \equiv p_2 - q$; the vacuum polarization insertion can be written as

$$\Pi_{\sigma\tau}(p) = i\left(p_\sigma p_\tau - p^2 \delta_{\sigma\tau}\right) G(-p^2) \tag{1}$$

where $G(p^2)$ satisfies the subtracted dispersion relation :

$$G(-p^2) = -p^2 \frac{1}{\pi} \int_4^\infty \frac{\operatorname{Im} G(t_1)}{t_1(t_1 + p^2)} \, dt_1 \tag{2}$$

Introducing the usual electric and magnetic Dirac form factors $F_1(t)$ and $F_2(t)$, for the vertex of Fig. 2 we have

$$-e\left[F_1(t)\gamma_\mu - \frac{i}{4} F_2(t)\left(\gamma_\mu \not\Delta - \not\Delta \gamma_\mu\right)\right] =$$

$$= -e\left(\frac{\alpha}{\pi}\right)\frac{i}{(2\pi)^2}\, i\,\bar{u}_2 \int d^4q\, \frac{\gamma_\rho(-i\not q + 1)\gamma_\mu(-i\not\Delta - i\not q + 1)\gamma_\nu}{(q^2+1)\left[(\Delta+q)^2 + 1\right]} \frac{1}{p^2} \Pi_{\nu\rho}(p) \frac{1}{p^2} \tag{3}$$

The form factors are then given by the dispersion relations in t

$$F_1(t) = t\, \frac{1}{\pi} \int_4^\infty \frac{\operatorname{Im} F_1(t')}{t'(t'-t)} \, dt'$$

$$F_2(t) = \frac{1}{\pi} \int_4^\infty \frac{\operatorname{Im} F_2(t')}{t'-t} \, dt' \tag{4}$$

and the contribution to the magnetic moment of the electron is

$$\frac{g_e - 2}{2} \equiv F_2(0) = \frac{1}{\pi}\int_4^\infty \frac{\operatorname{Im} F_2(t')}{t'}\, dt' \tag{5}$$

From Eqs. (1) and (3), we derive easily

$$\operatorname{Im} F_2(t) = \left(\frac{\alpha}{\pi}\right) 4 \int d^4q\, \theta(-q_0)\,\delta(q^2+1)\,\theta(\Delta_0+q_0)\,\delta((\Delta+q)^2+1)\cdot$$

$$\cdot\left[3\frac{p^4}{(t-4)^2} - 2\frac{p^2}{t-4}\right]\frac{G(-p^2)}{p^2}$$

which gives

$$F_2(0) = \left(\frac{\alpha}{\pi}\right)\int_4^\infty \frac{dt}{t}\sqrt{1-\frac{4}{t}}\int_{-1}^{1} dz \left[3\frac{p^2}{(t-4)^2} - 2\frac{1}{t-4}\right] G(-p^2) \tag{6}$$

We introduce the new variables x, θ through

$$t = \frac{(1+x)^2}{x} \quad , \qquad p^2 = \frac{(1-\theta)^2}{\theta}$$

so that Eq. (6) reads

$$F_2(0) = 2\left(\frac{\alpha}{\pi}\right)\int_0^1 \frac{dx}{(1+x)^2}\int_x^1 \frac{d\theta}{\theta}\,\frac{1+\theta}{1-\theta}\left[3\frac{(1-\theta)^2}{\theta}\frac{x^2}{(1-x)^4} - 2\frac{x}{(1-x)^2}\right] G(-p^2)\, p^2$$

By exchanging the order of integration

$$\int_0^1 dx \int_x^1 d\theta = \int_0^1 d\theta \int_0^\theta dx$$

$$F_2(0) = -\left(\frac{\alpha}{\pi}\right)\int_0^1 d\theta\, (\theta-1)^2\,\frac{G(-p^2)}{p^2}$$

We use now Eq. (2), by performing the change of variable $t_1 = (1+x_1)^2/x_1$

$$F_2(0) = \left(\frac{\alpha}{\pi}\right) \int_0^1 d\theta\,(\theta-1)^2 \frac{1}{\pi} \int_0^1 dx_1 \frac{1-x_1}{1+x_1} \frac{\theta}{(\theta+x_1)(1+\theta x_1)} \; \mathrm{Im}\, G(x_1)$$

By exchanging the order of integration, we obtain

$$\begin{aligned} F_2(0) = \left(\frac{\alpha}{\pi}\right) \frac{1}{\pi} \int_0^1 \frac{dx_1}{(1+x_1)^2} & \left\{ \frac{3}{2} x_1 + x_1^2 - \frac{1}{x_1^2} - \frac{3}{2}\frac{1}{x_1} + \right. \\ & + x_1 (1+x_1)^2 \log x_1 + \\ & \left. + \left[-x_1(1+x_1)^2 + \frac{1}{x_1}\left(1+\frac{1}{x_1}\right)^2 \right] \log(1+x_1) \right\} \mathrm{Im}\, G(x_1) \end{aligned} \tag{7}$$

This is our basic expression.

3. RESULTS

As an example, the second order vacuum polarization discontinuity is

$$\mathrm{Im}\, G^{(II)}(x) = \left(\frac{\alpha}{\pi}\right) \pi \frac{1}{3} \frac{1-x}{1+x} \left[1 + \frac{2x}{(1+x)^2} \right]$$

which, inserted in Eq. (7), gives [1),2)]

$$\left(\frac{g_e - 2}{2}\right)^{(IV)}_{\text{vac. pol.}} = \left(\frac{\alpha}{\pi}\right)^2 \left[\frac{119}{36} - 2\,\zeta(2) \right] = .0157 \left(\frac{\alpha}{\pi}\right)^2$$

Here, and in what follows, we refer to Section 4 for the definition of the mathematical symbols used. In order to settle a basis for comparison, we prefer to present the partial results for the 6th order magnetic moment corresponding to the various contributions to the 4th order vacuum polarization discontinuity (see Fig. 1). For graph 1 in Fig. 1, we have

$$
\begin{aligned}
\operatorname{Im} G^{(IV)}_{(1)}(x) = 2\left(\frac{\alpha}{\pi}\right)^2 \frac{\pi}{27}\Big\{ & 5 - \frac{48}{(1+x)^5} + \frac{120}{(1+x)^4} - \frac{52}{(1+x)^3} - \\
& - \frac{42}{(1+x)^2} + \frac{12}{(1+x)} + 3\log x\Big[-1 - \frac{16}{(1+x)^6} + \\
& + \frac{48}{(1+x)^5} - \frac{36}{(1+x)^4} - \frac{8}{(1+x)^3} + \frac{12}{(1+x)^2}\Big]\Big\}
\end{aligned}
\tag{8}
$$

In graph 2, we separate a two-particle contribution to the vacuum polarization discontinuity

$$
\begin{aligned}
\operatorname{Im} G^{(IV)}_{(2a)}(x) = 2\left(\frac{\alpha}{\pi}\right)^2 \frac{\pi}{12}\Big\{ & \Big[4 + \frac{16}{(1+x)^3} - \frac{24}{(1+x)^2}\Big](1+\log\lambda) \\
& + \Big[1 - \frac{4}{(1+x)^4} + \frac{8}{(1+x)^3} - \frac{4}{(1+x)^2}\Big]\Big[8\zeta(2) + 4\,\mathrm{Li}_2(x) - \\
& - \log^2 x + 4\log x\log(1-x) - 8\log x\log\lambda\Big] \\
& + \Big[-3 + \frac{16}{(1+x)^4} - \frac{32}{(1+x)^3} + \frac{8}{(1+x)^2} + \frac{8}{1+x}\Big]\log x\Big\}
\end{aligned}
\tag{9}
$$

and a three-particle contribution

$$
\begin{aligned}
\operatorname{Im} G^{(IV)}_{(2b)}(x) = 2\left(\frac{\alpha}{\pi}\right)^2 \frac{\pi}{3}\Big\{ & -1 - \frac{2}{(1+x)^3} + \frac{3}{(1+x)^2} + \frac{1}{1+x} + \\
& + \Big[1 - \frac{2}{(1+x)^4} + \frac{4}{(1+x)^3} - \frac{1}{(1+x)^2} - \frac{1}{1+x}\Big]\log x \\
& + \Big[1 - \frac{3}{(1+x)^4} + \frac{6}{(1+x)^3} - \frac{3}{(1+x)^2}\Big](3\,\mathrm{Li}_2(x) + \log\lambda) \\
& + \Big[1 - \frac{4}{(1+x)^4} + \frac{8}{(1+x)^3} - \frac{4}{(1+x)^2}\Big]\Big(2\zeta(2) + 2\,\mathrm{Li}_2(-x) + \\
& + \frac{1}{4}\log^2 x + \log x\log(1+x) + \log x\log(1-x)\Big)\Big\}
\end{aligned}
\tag{10}
$$

Finally, graphs 3 and 4 give

$$\mathrm{Im}\, G^{(iv)}_{(3)}(x) = 2\left(\frac{\alpha}{\pi}\right)^2 \pi \left\{ -\frac{1}{8} - \frac{13}{6}\frac{1}{(1+x)^3} + \frac{7\times 17}{36}\frac{1}{(1+x)^2} - \right.$$

$$-\frac{5}{6}\frac{1}{1+x}$$

$$+ \left[-\frac{13}{12} + \frac{1}{2}\frac{1}{(1+x)^4} - \frac{3}{(1+x)^3} + \frac{7}{2}\frac{1}{(1+x)^2} \right] \log x \tag{11}$$

$$\left. + \frac{1}{3}\left[1 + \frac{4}{(1+x)^3} - \frac{6}{(1+x)^2} \right] \left[\log(1+x) + 2\log(1-x) - \log\lambda \right] \right\}$$

Equations (8) - (11) agree with the calculation of Källén and Sabry [7], provided one makes the replacements :

$$x = \frac{1-\delta}{1+\delta} \quad , \quad Li_2(\pm x) = -\Phi\left(\mp \frac{1-\delta}{1+\delta}\right) - \frac{1}{2}\zeta(2)$$

By inserting the above expressions into Eq. (7), we obtain

$$F^{(vi)}_{2\,(1)}(0) = \left(\frac{\alpha}{\pi}\right)^3 \left[-\frac{23\times 41}{4\times 81} - \frac{8}{45}\zeta(2) + \frac{5}{3}\zeta(3) \right] = .0\,?\,2558 \left(\frac{\alpha}{\pi}\right)^3 \tag{12}$$

$$F^{(vi)}_{2\,(2a)}(0) = \left(\frac{\alpha}{\pi}\right)^3 \left[-\frac{7\times 11}{8\times 9} + \frac{5?}{16}\zeta(2) + \frac{11}{12}\zeta(3) - \right.$$

$$- \frac{16}{15}\zeta^2(2) + \frac{16}{3}a_4 - \frac{19}{3}\zeta(2)\log 2 + \frac{8}{3}\zeta(2)\log^2 2$$

$$\left. + \frac{2}{9}\log^4 2 + \log\lambda\left(-\frac{61}{36} + \frac{8}{3}\zeta(3) - \frac{8}{9}\zeta(2) \right) \right] \tag{13}$$

$$= (.087011 + .048877 \log\lambda)\left(\frac{\alpha}{\pi}\right)^3$$

$$F_{2_{(2b)}}^{(VI)}(0) = \left(\frac{\alpha}{\pi}\right)^3 \Big[\frac{1607}{16\times 27} + \frac{11\times 109}{16\times 27}\zeta(2) + \frac{65}{36}\zeta(3) + \frac{16}{3}a_4 +$$

$$+ \frac{2}{15}\zeta^2(2) - \frac{25}{3}\zeta(2)\log 2 - \frac{16}{3}\zeta(2)\log^2 2 +$$

$$+ \frac{2}{9}\log^4 2 + \log\lambda\left(-\frac{59}{12} - \frac{8}{3}\zeta(3) + \frac{44}{9}\right)\Big]$$

$$= (-.088816 - .080251\log\lambda)\left(\frac{\alpha}{\pi}\right)^3 \qquad (14)$$

$$F_{2_{(3)}}^{(VI)}(0) = \left(\frac{\alpha}{\pi}\right)^3 \Big[\frac{7\times 13\times 17}{16\times 27} - 9\zeta(2) - 2\zeta(3) +$$

$$+ 12\zeta(2)\log 2$$

$$+ \log\lambda\left(\frac{119}{18} - 4\zeta(2)\right)\Big]$$

$$= (.054675 + .031374\log\lambda)\left(\frac{\alpha}{\pi}\right)^3 \qquad (15)$$

Collecting all the results

$$\left(\frac{g_e - 2}{2}\right)^{(VI)}_{\text{vac.pol.}} \equiv F_{2_{(1)}}^{(VI)}(0) + F_{2_{(2a)}}^{(VI)}(0) + F_{2_{(2b)}}^{(VI)}(0) + F_{2_{(3)}}^{(VI)}(0) =$$

$$= \left(\frac{\alpha}{\pi}\right)^3 \left\{ \frac{269}{81} - \frac{2\times 31\times 7}{5\times 27}\zeta(2) + \frac{61}{18}\zeta(3) - \frac{14}{15}\zeta^2(2) + \right.$$

$$\left. + \frac{32}{3}a_4 - \frac{8}{3}\zeta(2)\log 2 - \frac{8}{3}\zeta(2)\log^2 2 + \frac{4}{9}\log^4 2 \right\} = .00554\left(\frac{\alpha}{\pi}\right)^3$$

4. INTEGRALS

We used the symbols

$$\zeta(2) \equiv \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6} = 1.64493\ 40668$$

$$\zeta(3) \equiv \sum_{m=1}^{\infty} \frac{1}{m^3} = 1.20205\ 69031$$

$$a_4 \equiv \sum_{m=1}^{\infty} \frac{1}{2^m m^4} = .51747\ 90616$$

According to Lewin's book [5], we define the di- and trilogarithm as :

$$Li_2(x) \equiv -\int_0^x \frac{\log(1-t)}{t}dt$$

$$Li_3(x) \equiv \int_0^x \frac{Li_2(t)}{t}dt$$

By noting that, for real x :

$$Li_2(x+i\epsilon) - Li_2(x-i\epsilon) = 2i\pi\,\theta(x-1)\log x$$

$$Li_3(x+i\epsilon) - Li_3(x-i\epsilon) = i\pi\,\theta(x-1)\log^2 x$$

the following (dispersive) representations follow :

$$\left.\begin{aligned} Li_2(x) &= \int_0^1 \frac{\log t}{t-\frac{1}{x}}dt \\ Li_3(x) &= -\frac{1}{2}\int_0^1 \frac{\log^2 t}{t-\frac{1}{x}}dt \end{aligned}\right\} \qquad (16)$$

Integrals involving the product of at most two logarithms or a dilogarithm only can be explicitly found, for instance, in Lewin's book [5] or in Terent'ev's paper [2].

We limit ourselves to give a list of the integrals occurring in the calculation, which involve three logarithms or a logarithm times a dilogarithm. Many of them can be expressed in terms of $\zeta(2)$, $\zeta(3)$, a_4 and $\log 2$ by simple use of the methods of Nielsen [5]. They are :

$$\int_0^1 \frac{\log^2 x \, \log(1+x)}{1+x}\, dx = -\int_0^1 \frac{\log x \, \log^2(1+x)}{x}\, dx =$$

$$= -\frac{3}{2}\zeta^2(2) + 4a_4 + \frac{7}{2}\zeta(3)\log 2 - \zeta(2)\log^2 2 + \frac{1}{6}\log^4 2$$

$$\int_0^1 \frac{\log x \, \log^2(1+x)}{1+x}\, dx = -\frac{4}{5}\zeta^2(2) + 2a_4 + \frac{7}{4}\zeta(3)\log 2 - \frac{1}{2}\zeta(2)\log^2 2 + \frac{1}{12}\log^4 2$$

$$\int_0^1 \frac{\log^2 x \, \log(1+x)}{x}\, dx = -\frac{1}{3}\int \frac{\log^3 x}{1+x}\, dx = \frac{7}{10}\zeta^2(2)$$

$$\int_0^1 \frac{\log(1+x)\, Li_2(-x)}{x}\, dx = \frac{1}{8}\zeta^2(2)$$

$$\int_0^1 \frac{\log(1+x)\, Li_2(-x)}{1+x}\, dx = \frac{6}{5}\zeta^2(2) - 3a_4 - \frac{21}{8}\zeta(3)\log 2 + \frac{1}{2}\zeta(2)\log^2 2 - \frac{1}{8}\log^4 2$$

$$\int_0^1 \frac{\log x \; Li_2(-x)}{1+x}\, dx = \frac{13}{8}\zeta^2(2) - 4a_4 - \frac{7}{2}\zeta(3)\log 2 + \zeta(2)\log^2 2 - \frac{1}{6}\log^4 2$$

The other integrals can be expressed in an analogous way after some preliminary manipulations. By changing $1+x \to x$, we have

$$\int_0^1 \frac{\log^2(1+x)\log(1-x)}{1+x}\,dx = -\frac{4}{5}\zeta^2(2) + 2a_4 + 2\zeta(3)\log 2 - \zeta(2)\log^2 2 + \frac{1}{3}\log^4 2$$

By using the identities :

$$\log(1+x)\log(1-x) = \frac{1}{2}\left[\log^2(1-x^2) - \log^2(1+x) - \log^2(1-x)\right]$$

$$\log^2(1+x)\log(1-x) = \frac{1}{6}\left[\log^3\left(\frac{1-x}{1+x}\right) + \log^3(1-x^2) - 2\log^3(1-x)\right]$$

we obtain (with obvious change of variables)

$$\int_0^1 \frac{\log x \log(1+x)\log(1-x)}{x}\,dx = -\frac{27}{40}\zeta^2(2) + 2a_4 + \frac{7}{4}\zeta(3)\log 2 - \frac{1}{2}\zeta(2)\log^2 2 + \frac{1}{12}\log^4 2$$

$$\int_0^1 \frac{\log(1+x)\,Li_2(x)}{1+x}\,dx = \frac{1}{2}\zeta(2)\log^2 2 + \frac{1}{2}\int_0^1 \frac{\log^2(1+x)\log(1-x)}{x}\,dx = -\frac{3}{40}\zeta^2(2) + \frac{1}{2}\zeta(2)\log^2 2$$

so that

$$\int_0^1 \frac{\log^2(1+x)\log(1-x)}{x}\,dx = -\frac{3}{20}\zeta^2(2)$$

By using the representation (16), we have :

$$\int_0^1 \frac{\log x\; Li_2(x)}{1+x}\,dx = \int_0^1 dx\,\frac{\log x}{1+x}\int_0^1 d\theta\,\frac{\log\theta}{\theta - \frac{1}{x}}$$

and, exchanging the integrations, it follows :

$$\int_0^1 \frac{\log x\; Li_2(x)}{1+x}\,dx = -\frac{3}{40}\zeta^2(2)$$

The following identity is then verified :

$$\int_0^1 \frac{\log(1+x)\,Li_2(x)}{x}\,dx = \int_0^1 \frac{\log x \log(1+x)\log(1-x)}{x}\,dx - \int_0^1 \frac{\log x\, Li_2(x)}{1+x}\,dx$$

$$= -\frac{3}{5}\zeta^2(2) + 2a_4 + \frac{7}{4}\zeta(3)\log 2 - \frac{1}{2}\zeta(2)\log^2 2 + \frac{1}{12}\log^4 2$$

On the other hand :

$$\int_0^1 \frac{\log(1+x)\,Li_2(x)}{x}\,dx = \zeta(3)\log 2 - \int_0^1 \frac{Li_3(x)}{1+x}\,dx$$

The last integral, with the help of the representation (16) may be led onto :

$$\int_0^1 \frac{Li_3(x)}{1+x}\,dx = \frac{2}{5}\zeta^2(2) - \frac{3}{4}\zeta(3)\log 2 + \frac{1}{2}\int_0^1 \frac{\log^2 x \log(1-x)}{1+x}\,dx$$

$$= \frac{3}{5}\zeta^2(2) - 2a_4 - \frac{3}{4}\zeta(3)\log 2 + \frac{1}{2}\zeta(2)\log^2 2 - \frac{1}{12}\log^4 2$$

By collecting results, we have

$$\int_0^1 \frac{\log^2 x \log(1-x)}{1+x}\,dx = \frac{2}{5}\zeta^2(2) - 4a_4 + \zeta(2)\log^2 2 - \frac{1}{6}\log^4 2$$

Finally, by making the replacement $x \to (1-x)/(x+1)$, we obtain

$$\int_0^1 \frac{\log x \log(1+x)\log(1-x)}{1+x}\,dx = -\frac{4}{5}\zeta^2(2) + 2a_4 + \frac{21}{8}\zeta(3)\log 2 - \frac{5}{4}\zeta(2)\log^2 2 + \frac{1}{12}\log^4 2$$

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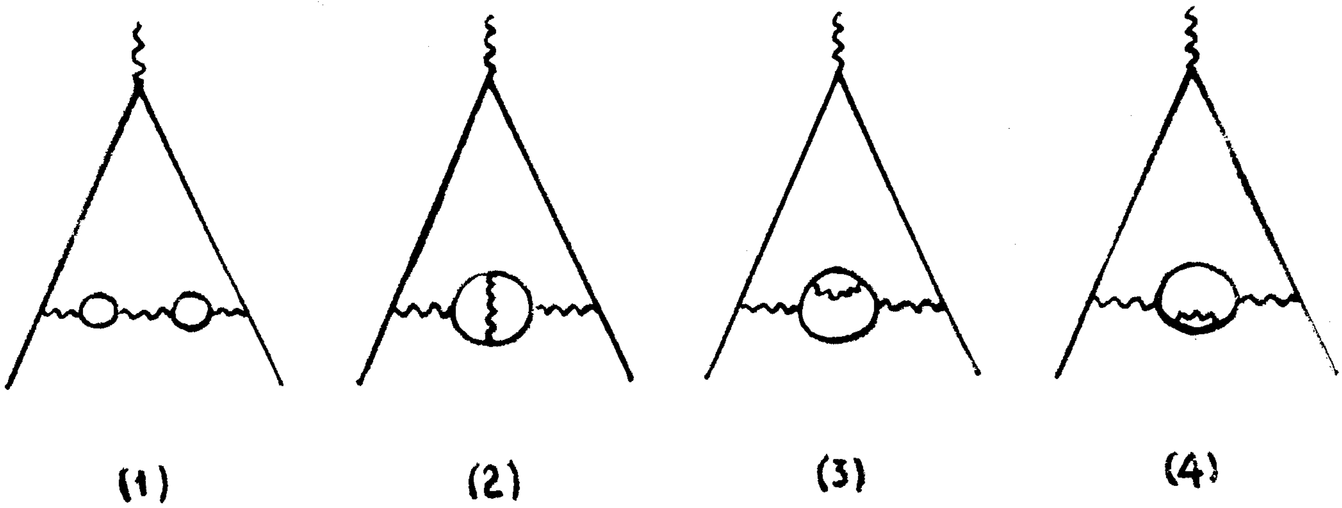

Figure 1

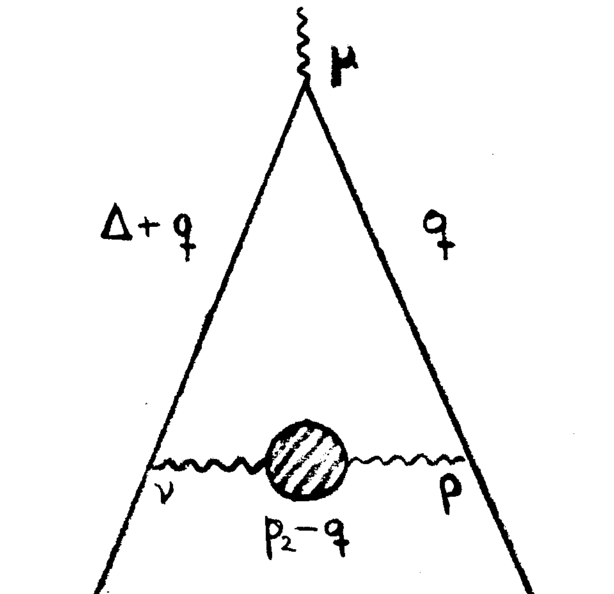

Figure 2