RAIRO
INFORMATIQUE THÉORIQUE

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RAIRO – Informatique théorique, tome 15, nº 2 (1981), p. 161-173.
<http://www.numdam.org/item?id=ITA_1981__15_2_161_0>

R.A.I.R.O. Informatique théorique/Theoretical Informatics
(vol. 15, n° 2, 1981, p. 161 à 173)

FPOL SYSTEMS GENERATING COUNTING LANGUAGES (*)

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Communicated by W. Brauer

Abstract. — *Counting languages are the languages of the form* $\{a_1^n a_2^n \ldots a_t^n \mid t \geqq 2, n \geqq 1\}$ *where* $a_1, \ldots, a_t$ *are letters no two consecutive of which are identical. They possess a "clean structure" in the sense that if an arbitrary word from such a language is cut in t subwords of equal length then no two consecutive subwords contain an occurrence of the same letter. It is shown that whenever an FPOL system G is such that its language contains a "dense enough" subset of a counting language then the whole language of G cannot have such a clean structure.*

Résumé. — *Les langages « comptants » sont les langages de la forme* $\{a_1^n a_2^n \ldots a_t^n \mid t \geqq 2, n \geqq 1\}$, *où* $a_1, \ldots, a_t$ *sont des lettres, et deux lettres consécutives étant différentes. Ils possèdent une « bonne structure », en ce sens que si un mot quelconque d'un tel langage est divisé en t facteurs de même longueur, alors deux facteurs consécutifs ne contiennent pas d'occurrence d'une même lettre. On montre que, si un system FPOL G est tel que son langage contient un sous-ensemble d'un langage comptant qui est « assez dense » alors le langage de G complet ne peut pas avoir cette « bonne structure ».*

1. INTRODUCTION

One of the important research areas within formal language theory is the investigation of the combinatorial structure of a single language within a given language family. Here one aims at a result of the form "if K is a language from a given language family X, then if K contains a string α satisfying a property W_1 then K also contains a set of strings A satisfying a property W_2" *or in more general form* "*if* K contains a subset K_1 satisfying a property W_1 then K also contains a subset K_2 satisfying a property W_2". A classical example of this kind is the celebrated "pumping lemma" for context free languages: it says that if a context free language K contains a "long enough" word α then it also contains

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an infinite subset A related in a very specific way to α. Results of this form shed some light on the generating abilities (restrictions) of grammars defining a given class of languages. They are simply "trade-off" results: if some "structure" is present in a language then also another structure must be present in the same language.

In this paper we establish a result a result in this direction for the class of languages generated by OL systems without erasing productions and with finite axiom sets (called FPOL systems). One of the most popular types of languages (serving as examples of strict inclusions of some classes of languages in others) in formal language theory are t-counting languages (which form a subclass of the so called bounded languages). Those are languages of the form $\{ a_1^n a_2^n \ldots a_t^n \mid n \geqq 1 \}$ where $t \geqq 2$ and $a_1, \ldots, a_t$ are letters no two consecutive of which are identical. They possess a "clean structure" in the sense that if an arbitrary word from such a language is cut into t subwords of equal length then no two consecutive subwords share an occurrence of a common letter. We demonstrate that if an FPOL system G is such that its language contains a "dense enough" subset of a counting language, then the whole language cannot have such a clean structure (or even a structure "approximating" it). Thus again a result in this line: if certain words are in the language from the given class, then other words must also be in the same language.

Certainly there are very few results like this for the class of FPOL languages and we believe that this result together with its proof sheds some new light on the structure of derivations in FPOL systems. Since t-counting languages are obviously EPOL languages, our main result points out a special role (that of a "garbage collector") that the mechanism of nonterminals plays in defining languages of L systems.

Perhaps it is also worthwhile to mention that results like this are especially valuable in the theory of L forms where one is really interested in the structure of "all sentential forms" that a given system can generate. In particular our result is used in [3].

II. PRELIMINARIES

We assume the reader to be familiar with rudiments of formal language theory and in particular with the rudiments of the theory of L systems (*see*, e. g., [2]). We use a rather standard terminology and perhaps only the following notation requires an explanation.

(1) N, N^+ and $N(t)$ denote the set of nonnegative integers, positive integers and positive integers larger than t, respectively.

(2) For a finite set Z, $\# Z$ denotes its cardinality.

(3) If α is a word over Σ then *alph* α denotes the set of all letters from Σ that occur in α, $pref_k(\alpha)$ denotes the prefix of α of the length k and $suf_k(\alpha)$ denotes the suffix of α of the length k. $|\alpha|$ denotes the length of α and $\#_a \alpha$ denotes the number of occurrences of the letter a in α.

(4) If K is a language then:

$$alph\ K = \bigcup_{\alpha \in K} alph\ \alpha, \qquad \mathrm{ALPH}(K) = \{ alph\ \alpha \mid \alpha \in K \}$$

and

$$less_q K = \# \{ |\alpha| \mid \alpha \in K \text{ and } |\alpha| \leqq q \}.$$

(5) In our notation we often identify a singleton set with its element.

To establish the basic notation for this paper we recall now the definition of an FPOL system.

DEFINITION: (1) An FPOL *system* is a construct $G = (\Sigma, P, A)$ where Σ is a finite nonempty alphabet, P is a finite set of *productions*, each of the form $a \to \alpha$ with $a \in \Sigma$, $\alpha \in \Sigma^+$ satisfying the condition:

$(\forall a)_\Sigma (\exists \alpha)_{\Sigma^+} [a \to \alpha$ is in $P]$.

A is a finite nonempty set (of *axioms*), $A \subseteq \Sigma^+$.

(2) Given words $x, y \in \Sigma^+$ we say that *x directly derives y in G* if $x = a_1 \ldots a_t$ and $y = \alpha_1 \ldots \alpha_t$ where $a_1 \to \alpha_1, \ldots, a_t \to \alpha_t$ are productions from P. We write then

$x \underset{G}{\Rightarrow} y$.

(3) For a positive integer m we say that *x derives y in m steps* if there exist $x_1, \ldots, x_m$ such that:

$$x_0 \underset{G}{\Rightarrow} x_1, \quad x_1 \underset{G}{\Rightarrow} x_2, \quad \ldots, \quad x_{m-1} \underset{G}{\Rightarrow} x_m \quad \text{and} \quad x_m = y.$$

We denote it by $x \overset{m}{\underset{G}{\Rightarrow}} y$. If $x = y$ or there exists an m such that $x \overset{m}{\underset{G}{\Rightarrow}} y$ then we say that *x derives y in G* and denote it by $x \overset{*}{\underset{G}{\Rightarrow}} y$.

(4) The *language of* G, denoted as $L(G)$, is defined by:

$$L(G) = \{ \alpha \in \Sigma^+ \mid (\exists w)_A [w \overset{*}{\Rightarrow} \alpha] \}. \quad \Box$$

DEFINITION: Let $G=(\Sigma, P, A)$ be an FPOL system.

(1) Let $\alpha \in \Sigma^+$. Then $G_\alpha=(\Sigma, P, \alpha)$.

(2) Let $n \in N^+$. Then

$$L^n(G)=\{\alpha \in L(G) : (\exists w)_A [w \overset{n}{\underset{G}{\Rightarrow}} a]\} \qquad \text{and} \qquad L^n(G, \alpha)=L^n(G_\alpha).$$

(3) *inf* $G \subseteqq \Sigma$ where $a \in$ *inf* G if and only if $\{\alpha \in \mathrm{L}(\mathrm{G}) : a \in alph\ \alpha\}$ is infinite; elements of *inf* G are called *infinite letters* (*in* G).

(4) *fin* $G=\Sigma \setminus$ *inf* G; elements of *fin* G are called *finite letters* (*in* G).

(5) *mult* $G \subseteqq$ *inf* G where $a \in$ *mult* G if and only if:

$$(\forall n)_{N+} (\exists \alpha)_{L(G)} [\#_a \alpha > n];$$

elements of *mult* G are called *multiple letters* (*in* G).

(6) *copy* $G=\{m \in N^+ \mid (\exists \alpha)_{\Sigma+} [\alpha^m \in L(G)]\}$.

(7) The *growth relation of* G, denoted as f_G, is a function from N^+ into finite subsets of N^+ defined by $f_G(n)=\{|\alpha| \mid \alpha \in L(n, G)\}$.

(7.1) If there exists a polynomial Φ such that:

$$(\forall n)_N + (\forall m)_{f_{G(n)}} [m < \Phi(n)],$$

then we say that f_G is of *polynomial type;* otherwise f_G is *exponential.*

(7.2) If there exists a constant C such that:

$$(\forall n)_{N^+} (\exists m)_{f_{G(n)}} [m < C],$$

then we can say that f_G is *limited.*

(7.3) If $(\forall n)_{N^+} [\# f_G(n)=1]$, then we can say that f_G is *deterministic.* □

III. AUXILIARY RESULTS

In this section we investigate certain aspects of derivations in FPOL systems in general and in the so called t-balanced FPOL systems in particular.

DEFINITION: Let Σ be a finite alphabet.

(1) Let $\alpha \in \Sigma^+$ and let t be a positive integer $t \geqq 2$. A *t-disjoint decomposition of* α is a vector $(\alpha_1, \ldots, \alpha_t)$ such that $\alpha_1, \ldots, \alpha_t \in \Sigma^+$, $\alpha_1 \ldots \alpha_t=\alpha$ and, for every i in $\{1, \ldots, t-1\}$, *alph* $\alpha_i \cap$ *alph* $\alpha_{i+1}=\emptyset$.

(2) Let $K \subseteqq \Sigma^+$ and let t be a positive integer, $t>2$. We say that K is *t-balanced* if there exist positive rational numbers $c_1, \ldots, c_t$ with $\sum_{i=1}^{t} c_i=1$ and a positive

integer d such that for every α in K there exists a t-disjoint decomposition $(\alpha_1, \ldots, \alpha_t)$ of α such that, for every $i \in \{1, \ldots, t\}$, $c_i . |\alpha| - d \leqq |\alpha_i| \leqq c_i . |\alpha| + d$. In such a case we also say that K *is* (v, d)-*balanced* and that $(\alpha_1, \ldots, \alpha_t)$ is a (v, d)-*balanced* decomposition of α, where $v = (c_1, \ldots, c_t)$.

(3) An FPOL system G is t-*balanced* if $L(G)$ is t-balanced. □

The following three lemmas describe the basic property of growth relations of t-balanced FPOL systems.

LEMMA 1; *If* $G = (\Sigma, P, A)$ *is a t-balanced FPOL system with* $t \geqq 3$, *then there exists a positive integer* k_0 *such that, for every a in* Σ *and for every positive integer* n, $\# f_{G_a}(n) < k_0$.

Proof: Clearly it suffices to show that for every a in Σ there exists a positive integer k_a such that, for every positive integer n, $\# f_{G_a}(n) < k_a$.

Let $v = (c_1, \ldots, c_t)$ and d be such that $L(G)$ is (v, d)-balanced. Let $c_{\min} = \min\{c_1, \ldots, c_t\}$. If $a \in \Sigma$ then either $a \in inf\ G$ or $a \in fin\ G$. We will consider these cases separately.

(i) Let $a \in inf\ G$.

In this case we will prove the result by contradiction. Thus let us assume that: there does not exist a positive integer k_a such that, for every positive integer n,

$$\# f_{G_a}(n) < k_a. \qquad (\star)$$

Then we proceed as follows.

(i.1) There exist a positive integer n_0, a positive integer r larger than $\#\Sigma$ and words $w_1, \ldots, w_r$ in $L^{n_0}(G)$ such that, for every i in $\{1, \ldots, t\}$ and for every j in $\{1, \ldots, r-1\}$, $c_i |w_{j+1}| > c_i |w_j| + 2d$.

This is proved as follows.

Clearly it suffices to show (i.1) with c_i replaced by $c_{\min}$.

Let us take an arbitrary n and let $f_{G_a}(n) = \{x_1, \ldots, x_s\}$ where elements $x_1, \ldots, x_s$ are arranged in the increasing order. Let $x_{i_1}, \ldots, x_{i_r}$ be the longest subsequence of $x_1, \ldots, x_s$ defined as follows: $x_{i_1} = x_1$, and for $1 \leqq j \leqq r-1$, i_{j+1} is the smallest index with the property that:

$$x_{i_{j+1}} - x_{i_j} > \frac{2d}{c_{\min}}.$$

If $r \leqq \#\Sigma$ then $s \leqq \#\Sigma (2d/c_{\min})$. Since n was arbitrary, if we set k_a equal to the smallest positive integer larger than $(\#\Sigma (2d/c_{\min})) + 1$ then we get that, for every positive integer n, $\# f_{G_a}(n) < k_a$, which contradicts $(\star)$.

(i.2) Let $\alpha=\alpha_1 a \alpha_2$ be a word in $L(G)$ that is long enough, meaning that, for every $i\in\{1, \ldots, t\}$, $|\alpha| c_i > 3|w_r| + 5d$ where $w_1, \ldots, w_r$ is a sequence (in the order of increasing length) from (i.1) for some fixed n_0 and r. Let:

$$\beta_1 = \bar{\alpha}_1 w_1 \bar{\alpha}_2 \in L^{n_0}(G, \alpha),$$
$$\vdots$$
$$\beta_r = \bar{\alpha}_1 w_r \bar{\alpha}_2 \in L^{n_0}(G, \alpha),$$

where $\bar{\alpha}_1, \bar{\alpha}_2$ are some fixed words such that:

$$\bar{\alpha}_1 \in L^{n_0}(G, \bar{\alpha}_1) \quad \text{and} \quad \alpha_2 \in L^{n_0}(G, \alpha_2).$$

Let, for each $i\in\{1, \ldots, r\}$, $(\beta_i\ [1], \ldots, \beta_i\ [t])$ be a (v, d)-balanced decomposition of β_i.

Since $|\beta_i| \geqq |\alpha|$ and $t \geqq 3$ the condition on the length of α assures us that either w_i is contained in the word resulting from β_i by cutting off its prefix $(\beta_i\ [1])$ $(pref_{|w_r|+2d}(\beta_i\ [2]))$ or w_i is contained in the word resulting from β_i by cutting off its suffix $(suf_{|w_r|+2d}(\beta_i[t-1]))(\beta_i[t])$. Because these two cases are symmetric we assume the first one.

Since, for each $i\in\{1, \ldots, r-1\}$,

$$|w_{i+1}| - |w_i| > \frac{2d}{c_{\min}}, \qquad |\beta_{i+1}| - |\beta_i| > \frac{2d}{c_{\min}}.$$

Consequently $|\beta_{i+1}[1]| - |\beta_i[1]| > 0$ and so $\beta_{i+1}\ [1]$ results from $\beta_i\ [1]$ by catenating to $\beta_i[1]$ a nonempty prefix of $\beta_i\ [2]$. Also:

$$|\beta_r[1]| - |\beta_1[1]| \leqq (c_1 . (|\bar{\alpha}_1 \bar{\alpha}_2| + |w_r|) + d)$$
$$- (c_1 (|\bar{\alpha}_1 \bar{\alpha}_2| + |w_1|) - d) = c_1 (|w_r| - |w_1|) + 2d \leqq |w_r| + 2d.$$

Thus in constructing consecutively $\beta_2\ [1], \beta_3\ [1], \ldots, \beta_r\ [1]$ we use nonempty subwords of a prefix of $\beta_1\ [2]$ and we never reach the occurrence of w_1 indicated by the equality $\beta_1 = \bar{\alpha}_1 w_1 \bar{\alpha}_2$. However $r > \#\Sigma$ and so at least two nonempty subwords used in the process of constructing $\beta_2\ [1], \beta_3\ [1], \ldots, \beta_r\ [1]$ contain an occurrence of the same letter. This implies that there exists a j in $\{2, \ldots, r-1\}$ such that:

$$alph\ (\beta_j\ [1]) \cap alph\ (\beta_j\ [2]) \neq \emptyset$$

which contradicts the fact that $(\beta_j\ [1], \ldots, \beta_j\ [t])$ is a (v, d)-balanced decomposition of β_j.

Thus we have shown that (★) does not hold.

(ii) Let $a \in fin\,G$.

Let Z be the set of all words α such that $alph\,\alpha \subseteqq inf\,G$ and there exists a word β in $L(G)$ such that $\beta \Rightarrow \alpha$ and $alph\,\beta \cap fin\,G \neq \emptyset$. Note that Z is a finite set and so if we set:

$$s = \max\{|\alpha| \mid \alpha \in Z\},$$

$$r = \#\{\beta \in L(G) \mid alph\,\beta \cap fin\,G \neq \emptyset\} + \#Z,$$

and

$$k = \max\{k_b \mid b \in inf\,G\},$$

then $\# f_{G_a}(n) < 1 + r + k^s$ for every $n \geqq 0$. □

LEMMA 2 : *Let G be a t-balanced FPOL system with* $t \geqq 3$ *and let* $a \in \text{mult}\,G$. *Then* f_{G_a} *is deterministic.*

Proof: Let $G = (\Sigma, P, A)$. Clearly there exists a letter b in Σ which for any m can derive a word β such that $\#_a \beta > m$. So let k_0 be the constant from the statement of lemma 1 and let β be a word such that b derives β (in some e steps) and $\#_a \beta > k_0$.

Now we prove the lemma by contradiction as follows. If the lemma is not true then there exist a positive integer n_0 and words α_1, α_2 in $L^{n_0}(G_a)$ such that $|\alpha_1| \neq |\alpha_2|$. But then the number of words of different lengths that β can derive in n_0 steps is larger than k_0 and consequently $\# f_{G_b}(e + n_0) > k_0$, which contradicts lemma 1. □

LEMMA 3: *Let G be an FPOL system such that* f_G *is deterministic and copy G is an infinite set. Then* f_G *is exponential.*

Proof: Let $G = (\Sigma, P, A)$, let $\overline{P}$ be a set of productions containing precisely one production for every $a \in \Sigma$ such that $\overline{P} \subseteqq P$ and let $\omega \in A$. Consider the DOL system $\overline{G} = (\Sigma, \overline{P}, \omega)$. Since f_G is deterministic, $f_G = f_{\overline{G}}$. Note that there are arbitrarily large integers m dividing all numbers $f_{\overline{G}}(n)$ provided that $n \geqq n_m$ for suitably thosen m.

The lemma follows now by the following easy to prove property of DOL growth functions. Assume that a DOL growth function f not identically zero has the following property. For every positive integer m, there are integers $m_0 \geqq m$ and n_0 such that m_0 divides $f(n)$ wherever $n \geqq n_0$. Then f is not of polynomial type. □

After we have established the basic properties of growth relations of t-balanced FPOL systems we move to investigate the structure of t-balanced FPOL systems the languages of which contain counting languages. Those counting languages are defined now.

DEFINITION: Let t be a positive integer, $t \geqq 2$. A language M over Σ is called a *t-counting language* if $M = \{ a_1^n a_2^n \dots a_t^n \mid n \geqq 1 \}$, where for $i \in \{1, \dots, t\}$, $a_i \in \Sigma$ and $a_j \neq a_{j+1}$ for $j \in \{1, \dots, t-1\}$. We also say that a_j and a_{j+1} are *neighbors in* M. □

To prove our main theorem we need the following transformation of an FPOL system.

DEFINITION: Let $G = (\Sigma, P, A)$ be an FPOL system and k a positive integer. The *k-decomposition of* G is a set $\mathcal{G} = \{G_1, \dots, G_k\}$ of FPOL systems (called *components*) such that, for every $i \in \{1, \dots, k\}$, $G_i = (\Sigma, P^k, A_i)$ where $A_1 = A$ and $A_i = \{ \alpha \mid \alpha \in L^{i-1}(G) \}$ for $i \in \{2, \dots, k\}$, and $(a \to \alpha) \in P^k$ if and only if $a \overset{k}{\underset{G}{\Rightarrow}} \alpha$. □

If follows directly from the above definition that $L(G) = \bigcup_{i=1}^{k} L(G_i)$, where $\mathcal{G} = \{G_1, \dots, G_k\}$ is a k-decomposition of G.

A particular kind of decomposition will be useful for our purposes. It is defined as follows. Let $G = (\Sigma, h, A)$ be an FPOL system. We say that G is *well-sliced* if:

(1) for every a in Σ and every k, $l \geqq 1$,

$$\mathrm{ALPH}(L^k(G_a)) = \mathrm{ALPH}(L^l(G_a))$$

and moreover if x is a word such that $|x| \geqq 2$ and $\#$ *alph* $x = 1$ then $x \in L^k(G_a)$ if and only there exists a word y such that $|y| \geqq 2$, *alph* $x =$ *alph* y and $y \in L^l(G_a)$;

(2) for every a in Σ if $\bigcup_{n \geqq 1} L^n(G_a)$ is finite then

$$\bigcup_{n \geqq 1} L^n(G_a) = \{ \alpha \mid a \Rightarrow \alpha \}.$$

The proof of the following result is rather standard (*see*, e. g., [1]) and so it is omitted. (By a *well-sliced decomposition of an* FPOL *system* we understand a decomposition each component of which is well-sliced.)

LEMMA 4: *For every FPOL system there exists a well-sliced decomposition.* □

We are ready now to prove the main result of this paper.

THEOREM 1: *Let $t \geqq 3$, M be a t-counting language, G be a t-balanced FPOL system and $K = M \cap L(G)$. There exists a constant C such that* $\text{less}_q K \leqq C . \log_2 q$ *for every positive integer q.*

Proof: Let $G = (\Sigma, P, A)$ and $\Delta = alph\ M$. By lemma 4 there exists a well-sliced decomposition of G and since it suffices to prove the theorem for a single component of such a decomposition let us assume that G is well-sliced.

Since the result holds trivially when K is finite, let us assume that K is infinite.

(1) For every letter b in Δ there exists a multiple letter a and a word α in $\{ b \}^*$ such that $a \overset{+}{\Rightarrow} \alpha$. This is obvious.

(2) If $a \in mult\ G$, $b \in \Delta$, $\alpha \in \{ b \}^+$ and $a \overset{+}{\Rightarrow} \alpha$ then:

(i) f_{G_a} is either constant or exponential,
(ii) f_{G_b} is either constant or exponential, and
(iii) f_{G_a} is constant if and only if f_{G_b} is constant.

We prove (2) as follows.

By Lemma 2, f_{G_a} is deterministic and because G is well-sliced, for every positive integer n, $l \in f_{G_a}(n)$ if and only if $b^l \in L^n(G_a)$.

Let $\tau = b^{i_1}, b^{i_2}, \ldots$ be such that $i_j = f_{G_a}(j)$.

If τ contains infinitely many different words then G_a satisfies the assumptions of lemma 3 and so f_{G_a} is exponential.

Otherwise, because G is well-sliced, f_{G_a} is a constant function.

Thus (i) is proved. But a derives strings "through" b and so a and b must have the same type of growth. Consequently (i) implies (ii) and (iii).

(3) Either, for every b in Δ, f_{G_b} is a constant function, or, for every b in Δ, f_{G_b} is exponential.

This is proved as follows.

Let $b \in \Delta$. From (1) and (2) it follows that f_{G_b} is either constant or exponential. Now let a be a neighbor of b (in M). Then if we take a word α from K of the form $\ldots a^n b^n \ldots$ (or symmetrically $\ldots b^n a^n \ldots$) and will derive in G words from it in such a way that each occurrence of b in α will produce the same subtree, then if b is not of the same type as a, we obtain a word β in $L(G)$ that is not t-balanced; a contradiction. Consequently any two neighbors in M must have the same type of growth and (3) holds.

(4) It is not true that f_{G_a} is constant for every a in Δ. We prove it by showing that if f_{G_a} is constant for every a in Δ then the fact that K is infinite leads to a contradiction.

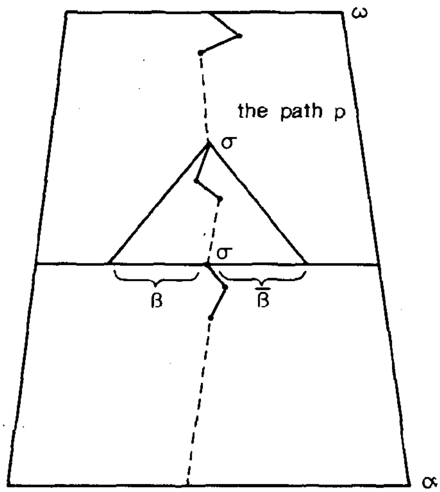
Since K is infinite we can choose α in K which is arbitrarily long, e. g., so long that each derivation graph for α in G is such that on each path in it there exists a label that appears at least twice. In a derivation graph corresponding to a derivation of α from ω in A we choose a path $p=e_0, e_1, \ldots$ as follows:

e_0 is an occurrence in ω such that no other occurrence in ω contributes a longer subword to α,

e_{i+1} is a direct descendant of e_i such that no other direct descendant of e_i contributes a longer subword to α.

Now, on p we choose the first (from e_0) label σ that repeats itself on p. Then we take the first repetition of σ on p (and we let β, $\bar{\beta}$ to be the words such that the contribution of the first σ on p to the level on which the first repetition of σ on p occurs is $\beta\sigma\bar{\beta}$ where the indicated occurrence of σ is the occurrence of σ on p).

The situation is illustrated by the following figure:



Now we proceed as follows.

(i) $\beta\bar{\beta} \neq \Lambda$.

We prove it by contradiction. To this aim assume that $\beta\bar{\beta}=\Lambda$.

(i.1) Then every label ρ on p that repeats itself must be such that $\rho \overset{+}{\Rightarrow} \delta\rho\bar{\delta}$ implies $\delta\bar{\delta}=\Lambda$.

This is seen as follows.

Since G is well-sliced, $\sigma \Rightarrow \sigma$, $\sigma \Rightarrow \zeta\rho\bar{\zeta}$ and $\rho \Rightarrow \mu\rho\bar{\mu}$ for some words $\zeta, \bar{\zeta}, \mu, \bar{\mu}$ such that alph $\mu\bar{\mu} = alph\ \delta\bar{\delta}$.

Then:

$$
\begin{array}{ccccccccc}
\sigma & \Rightarrow & \zeta\rho\bar{\zeta} & \Rightarrow & \zeta^{(1)}\mu\rho\bar{\mu}\bar{\zeta}^{(1)} & \Rightarrow & \zeta^{(2)}\mu^{(1)}\mu\rho\bar{\mu}\bar{\mu}^{(1)}\bar{\zeta}^{(2)} & \Rightarrow & \dots \\
\sigma & \Rightarrow & \sigma & \Rightarrow & \zeta\rho\bar{\zeta} & \Rightarrow & \zeta^{(1)}\zeta\rho\bar{\zeta}\bar{\zeta}^{(1)} & \Rightarrow & \dots \\
\sigma & \Rightarrow & \sigma & \Rightarrow & \sigma & \Rightarrow & \zeta\rho\bar{\zeta} & \Rightarrow & \dots \\
\sigma & \Rightarrow & \sigma & \Rightarrow & \sigma & \Rightarrow & \sigma & \Rightarrow & \dots \\
\vdots & & \vdots & & & \vdots & & \vdots &
\end{array}
$$

for some words $\zeta^{(1)}, \bar{\zeta}^{(1)}, \zeta^{(2)}, \bar{\zeta}^{(2)}, \dots, \mu^{(1)}, \bar{\mu}^{(1)}, \mu^{(2)}, \bar{\mu}^{(2)}, \dots$ where all the words $\mu^{(1)}\bar{\mu}^{(1)}, \mu^{(2)}\bar{\mu}^{(2)}, \dots$ are nonempty if $\delta\bar{\delta}$ is nonempty. Consequently if $\delta\bar{\delta} \neq \Lambda$ then there exists a positive integer l, such that $\# f_{G_a}(l) > k_0$, which contradicts lemma 1 (where k_0 is the constant from the statement of lemma 1).

Thus (i.1) holds.

But (i.1) implies that α cannot be longer than *a* fixed *a priori* constant; since α was an arbitrary word in K this contradicts the fact that K is infinite.

Thus indeed $\beta\bar{\beta} \neq \Lambda$ and (i) holds.

(ii) Since G is well-sliced, $\sigma \Rightarrow \gamma\sigma\bar{\gamma}$ for some words $\gamma, \bar{\gamma}$ such that $alph\ \gamma\bar{\gamma} = alph\ \beta\bar{\beta}$ and $\sigma \Rightarrow \pi$ for some $\pi \in \Delta^+$. Since we have assumed that f_{G_a} is constant for every a in Δ, f_{G_π} is constant.

Then:

$$
\begin{array}{l}
\sigma \Rightarrow \pi \Rightarrow \pi^{(1)} \Rightarrow \pi^{(2)} \Rightarrow \pi^{(3)} \Rightarrow \dots \\
\sigma \Rightarrow \gamma\sigma\bar{\gamma} \Rightarrow \gamma^{(1)}\pi\bar{\gamma}^{(1)} \Rightarrow \gamma^{(2)}\pi^{(1)}\bar{\gamma}^{(2)} \Rightarrow \dots \\
\sigma \Rightarrow \gamma\sigma\bar{\gamma} \Rightarrow \gamma^{(1)}\gamma\sigma\bar{\gamma}\bar{\gamma}^{(1)} \Rightarrow \gamma^{(2)}\gamma^{(1)}\pi\bar{\gamma}^{(1)}\bar{\gamma}^{(2)} \Rightarrow \dots \\
\sigma \Rightarrow \gamma\sigma\bar{\gamma} \Rightarrow \gamma^{(1)}\gamma\sigma\bar{\gamma}\bar{\gamma}^{(1)} \Rightarrow \gamma^{(2)}\gamma^{(1)}\gamma\sigma\bar{\gamma}\bar{\gamma}^{(1)}\bar{\gamma}^{(2)} \\
\qquad \Rightarrow \gamma^{(3)}\gamma^{(2)}\gamma^{(1)}\pi\bar{\gamma}^{(1)}\bar{\gamma}^{(2)}\bar{\gamma}^{(3)} \Rightarrow \dots \\
\vdots
\end{array}
$$

where all $\gamma\bar{\gamma}, \gamma^{(1)}\bar{\gamma}^{(1)}, \dots, \pi, \pi^{(1)}, \dots$ are nonempty words.

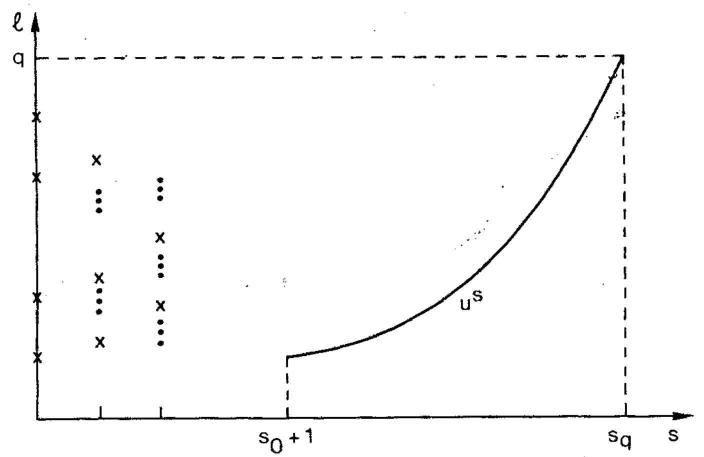
Since f_{G_π} is constant, the above implies that there exists a positive integer l such that $\# f_{G_\sigma}(l) > k_0$ which contradicts lemma 1 (where k_0 is the constant from the statement of lemma 1).

Consequently it cannot be true that f_{G_a} is constant for every a in Δ, and so (4) holds.

(5) f_{G_b} is exponential for every b in Δ. This follows directly from (3) and (4).
(6) There exists a positive integer constant s_0 such that in every derivation without repetitions (in its trace) of a word from k, already after s_0 steps an intermediate word contains an occurrence of a multiple letter a for which there exist b in Δ and α in $\{b\}^+$ such that $a \overset{+}{\Rightarrow} \alpha$. This is obvious.

(7) Now we complete the proof of the theorem as follows: $less_q\, K \leqq U_1 + U_2$, where U_1 is the number of all the words from K of length not larger than q that are obtained by a derivation without a repetition which does not take more than s_0 steps, and U_2 is the number of all the words from K of length not larger than q that are obtained by a derivation without a repetition which takes more than s_0 steps.

The following graphic represents the situation:



where s is the number of steps (in derivations without repetitions) required to derive a word in K and l is the length of a word in K [so that the point (i, j) is on the graphic if in i steps one can derive a word from K of length j].

From (2), (5) and (6) it follows that for $i > s_0$ all the points (i, j) are above the exponential line u^s for some constant $u > 1$. But then lemma 1 implies that there exists a constant h_0 such that (note that $s_q = \log_u q$):

$$less_q\ K \leqq U_1 + U_2 \leqq h_0\, s_0 + h_0 \log_u q.$$

Since $\log_u q = \log_2 q / \log_2 u$,

$$less_q\, K \leqq h_0 . s_0 + h_0 \log_2 q / \log_2 u \leqq C . \log_2 q$$

for a suitable constant C.

Thus the theorem holds. □

As a corollary of the above theorem we get the following result which turns out to be useful in the theory of EOL forms (*see* [3]).

COROLLARY 1: *Let G be an FPOL system such that L (G) contains* $\{a^n b^n c^n \mid n \geqq 1\}$. *Then for no finite language F,* $L(G) \setminus F$ *is 3-balanced.*

Proof: Directly from theorem 1. □

ACKNOWLEDGEMENTS

The authors greatfully acknowledge comments of A. Salomaa, R. Verraedt and W. Brauer concerning this paper.

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