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Fractal Dimensions of Chaotic Flows of Autonomous Dissipative Systems

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The fractal dimensions of chaotic flows are shown to be given by $D = m^0 + m^+\{1 + |\lambda^+/\lambda^-|\}$, where m^0 and m^+ are the numbers of zero and positive Lyapunov characteristic exponents λ_α, and $\lambda^\pm$ are the mean values of positive and negative λ_α, respectively.

The fractal dimensions of irregular curves and surfaces have turned out to give a useful physical concept, for example, providing a useful exponent for characterizing the energy cascade and the vortex stretching in fully-developed turbulence.[1)~3)] In this note, we shall show by a heuristic argument that the fractal dimensions also give a useful exponent for characterizing chaotic flows and strange attractors of dissipative dynamical systems. For example, the strange attractor of the Lorenz model[4),5)] will turn out to have the Hausdorff dimension $D \simeq 2.06$ at $r=40$, $\sigma=16$, $b=4$.

Let $x(t)$ be an orbit of an autonomous dissipative system which starts from x_0 at time $t=0$ in an m-dimensional phase space, and assume that, as $t \to \infty$, $x(t)$ and nearby orbits are all eventually trapped in an attractor which is *bounded* in the phase space. Let us take one such attractor A and its basin, and consider the time evolution of a small cell $V(t)$ which is initially a small cube $V_0 = l_0{}^m$ in the neighborhood of x_0. Then the cell $V(t)$ is eventually trapped in A. If the flow is mixing on A so that $V(t)$ *covers* A asymptotically, then the Hausdorff dimension of A may be given by the fractal dimension of $V(t)$ in the limit $t \to \infty$.

As long as the cell volume $V(t)$ is infinitesimal it is expected to contract exponentially, and for a large time T we may write as[6),7)]

$$V(T)/V_0 = \exp[T\lambda^{(m)}(x_0)]. \qquad (1)$$

Indeed this is exactly satisfied by the Lorenz model[4)] and the CTA oscillator model[8)] with $\lambda^{(m)}$ being a constant negative rate.

In order to construct a cascade model for the chaotic flow, let τ be a short but non-zero time interval and divide T into a sequence of short time intervals

$$t_i = i\tau\ , \qquad (i=1, 2, \cdots n) \qquad (2)$$

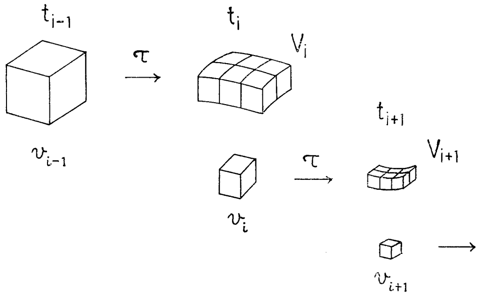

Fig. 1. Cascade model for chaotic flows, where the local self-similarity is assumed:
$V(t_i)/V(t_{i-1}) \fallingdotseq V_i/v_{i-1} = N_i r_i{}^m$,
$V_i = N_i v_i\ . \quad (i=1, 2, \cdots \infty)$

with $n = T/\tau$. As shown in Fig. 1 schematically, let us divide the cell $V(t_i)$ into small cubes with an identical volume $v_i = l_i{}^m$ in the following way. Let us suppose that a small cube $v_{i-1} = l_{i-1}^m$ in the neighborhood of $x(t_{i-1})$ evolves into a small region with a volume

$$V_i(\tau) = \prod_{\alpha=1}^{m} l_{\alpha i}(\tau) \qquad (3)$$

in the short time τ. Let m^- be the number of contracting directions with $l_{\alpha i}(\tau)<l_{i-1}$, and introduce their geometric average

$$l_i \equiv [\prod_{\alpha\varepsilon(-)}^{m^-} l_{\alpha i}(\tau)]^{1/m^-}. \qquad (4)$$

This defines $v_i = l_i^m$ successively from $i=1$ to $i=n$. In the following, we assume that m^- is constant over the flow $V(t)$ except a region of negligible measure. The cascade model shown in Fig. 1 is constructed by assuming that the contraction ratio of the cell volume $V(t)$ for τ does not depend on the shape and volume of the cell itself so that we may put

$$V(t_i)/V(t_{i-1}) \doteqdot V_i(\tau)/v_{i-1}. \qquad (5)$$

This is exactly satisfied by the Lorenz model[4] and the CTA oscillator model,[8] and will be called the *local similarity*.

The local properties of the flow are characterized by the local expansion rates at $x(t_i)$,

$$k_{\alpha i}(\tau) \equiv \tau^{-1} \log[l_{\alpha i}(\tau)/l_{i-1}]. \qquad (6)$$

The global properties of the flow are characterized by the one-dimensional Lyapunov characteristic numbers $\lambda_\alpha(x_0)$. Extending (5), one may assume the local similarity about the deformation of the cell during τ. Then one gets[6),7)]

$$\lambda_\alpha \doteqdot \langle k_{\alpha i}\rangle \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} k_{\alpha i}(\tau). \qquad (7)$$

The angular brackets represent the long-time average along the orbit $x(t_i)$. In terms of the local rates (6) we have

$$V_i(\tau)/v_{i-1} = \exp[\tau k_i^{(m)}(\tau)] \qquad (8)$$

with $k_i^{(m)}(\tau) \equiv \sum_{\alpha=1}^{m} k_{\alpha i}(\tau)$. Hence (1) and (5) lead for large T to

$$\lambda^{(m)} \doteqdot \langle k_i^{(m)}\rangle = m^+\langle k_i^+\rangle + m^-\langle k_i^-\rangle, \qquad (9)$$

where $m = m^+ + m^0 + m^-$,

$$k_i^{\pm}(\tau) \equiv (1/m^{\pm}) \sum_{\alpha\varepsilon(\pm)}^{m^\pm} k_{\alpha i}(\tau) \qquad (10)$$

with $m^\pm$ and m^0 being the numbers of positive, negative and zero $\langle k_{\alpha i}\rangle$, respectively.

The local contraction ratio for τ is given by

$$r_i(\tau) \equiv l_i/l_{i-1} = \exp[\tau k_i^-(\tau)], \qquad (11)$$

where (4) and (6) have been used. The global contraction ratio for large T is given by

$$r(T) \equiv l_n/l_0 = \prod_{i=1}^{n} r_i(\tau) \qquad (12)$$

$$= \exp[T\langle k_i^-\rangle]. \qquad (13)$$

The cell $V(T)$ at time $t=T$ which was a small cube $V_0 = l_0^m$ at $t=0$ now consists of a large number of much smaller cubes with volume $v_n = l_n^m = [r(T)]^m V_0$. Therefore the fractal dimension D of the cell $V(T)$ may be given by

$$N(T) \equiv V(T)/r^m V_0 = r^{-D}, \qquad (14)$$

which leads to

$$D = m - T\lambda^{(m)}/\log[r(T)]. \qquad (15)$$

Inserting (9) and (13) into (15) leads to

$$D \doteqdot m^0 + m^+\{1 + |\langle k_i^+\rangle/\langle k_i^-\rangle|\}. \qquad (16)$$

This gives the fractal dimension of the subspace which the cell $V(T)$ covers as $T\to\infty$. Therefore, if the flow is mixing, then this gives the Hausdorff dimension of the attractor in which the cell is eventually trapped. Thus the global properties can be expressed in terms of the long-time averages of the local rates $k_{\alpha i}(\tau)$ which are calculable from dynamical equations directly.

The local offspring number N_i is given by

$$V_i(\tau) = N_i(\tau) v_i = [N_i r_i^m] v_{i-1}. \qquad (17)$$

Then the local similarity (5) leads to

$$V(t_n) \doteqdot N_n r_n^m V(t_{n-1}) = V_0 \prod_{i=1}^{n} N_i r_i^m. \qquad (18)$$

Therefore (14) and (12) lead to

$$N(T) \doteqdot \prod_{i=1}^{n} N_i(\tau). \tag{19}$$

This is inserted into (14) to give

$$D \doteqdot \langle \log N_i \rangle / \langle \log[1/r_i] \rangle. \tag{20}$$

The global self-similarity holds if and only if the fluctuations of $N_i(\tau)$ and $r_i(\tau)$ along the orbit $x(t_i)$ are negligible. Then (20) reduces to

$$D \simeq \log N_i / \log[1/r_i]. \tag{21}$$

This agrees with Mandelbrot's definition for self-similar processes. Very often, however, the global self-similarity does not hold. Then we have to use (20) or (16).

Let us apply the formula (16) to some attractors. For the fixed points with $m^+ = m^0 = 0$, we have $D=0$. For the quasi-periodic orbits with $m^+=0$, this leads to $D=m^0$. For the measure-preserving systems where $V(T)=V_0$, $m^+\langle k_i^+\rangle = m^-|\langle k_i^-\rangle|$, we have $D=m$. According to Shimada and Nagashima's computer investigation,[7] the Lorenz model has a bounded strange attractor with $m^\pm = m^0 = 1$ and $\langle k_i^+\rangle \simeq 1.37$, $\langle k_i^-\rangle \simeq -22.37$ at $r=40$, $\sigma=16$, $b=4$. Then we have $D\simeq 2.06$. This gives the Hausdorff dimension of the Lorenz attractor since the flow must be mixing.[7] The Lorenz attractor is a doubly-connected infinitely-many sheeted surface whose three-dimensional volume is zero.[4] The 0.06 in $D\simeq 2.06$ represents the thickness of this sheeted surface in the fractal space, ensuring that the nonperiodic orbit $x(t)$ does not cross in the wandering motion between two unstable fixed points.

The formula (16) is valid for strange attractors A of autonomous chaotic systems. The fundamental assumption used is that the cell $V(t)$ which was a small m-dimensional cube V_0 at $t=0$ is eventually trapped in A and covers A as $t\to\infty$. In addition, we have assumed the local similarity and the constancy of the number of contracting directions of a small cube almost everywhere over the flow. A generalization of (16) to non-autonomous systems would be interesting.

The statistics of fully-developed turbulence is specified by the intermittency exponent $\mu = 3-D$, where D is the fractal dimension of the vortex stretching in the ordinary space ($m=3$).[2),3)] The present description of chaotic flows may be applied to the vortex stretching in order to relate μ to the local expansion rates $\langle k_{\alpha i}\rangle$ of the vortex stretching which would be measurable by fluid experiments. Indeed, $D = 1 + |\langle k_i^+\rangle / \langle k_i^-\rangle|$ if the vortices are of ribbon-like structure,[9] while $D = 2 + |\langle k_i^+\rangle / \langle k_i^-\rangle|$ if the vortices are of sheet-like structure.[9]

The β-model for the energy cascade presented by Frisch, Sulem and Nelkin is useful for describing the statistics of turbulence.[2] On the basis of this model, Fujisaka and Mori have proposed a variational principle for determining μ and obtained $\mu \simeq 0.341$, $D\simeq 2.659$ in good agreement with experiments.[3] The present theory enables us to construct a discrete model for the choatic flow in phase space which corresponds to a generalization of the β-model. This and a formulation of the β-model from the viewpoint of the vortex stretching will be discussed in a separate paper.

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