FRACTAL HEAT CONDUCTION PROBLEM SOLVED BY LOCAL FRACTIONAL VARIATION ITERATION METHOD

by

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This paper points out a novel local fractional variational iteration method for processing the local fractional heat conduction equation arising in fractal heat transfer.

Key words: *local fractional variational iteration method, local fractional heat conduction equation, fractal heat transfer*

Introduction

Various transport phenomena in nano-scale [1, 2, 3] cannot be described by smooth continuum approach and need the fractal nature of the objects to be taken into account, for example in nano-scale porous materials[4] and they are termed Cantor materials. In case of fractal objects the fractal Fourier law should be used [5, 6, 7] in contrast to the continuous case when both the classical and the fractional versions are valid [8-18]. When the transport is performed in fractal objects the local temperature depends on the fractal dimensions and examples for that exist in well -known media such as polar bear hair [17, 20] and wool [21]. In these cases the fractional calculus assuming smooth functions [9-15] due to the continuum concept and the memory effects is not applicable. The problem invokes application of local fractional models and relevant solution approaches providing adequate physical results. The present paper shows how the local version of the variational iteration method [22] can be applied to local fractional heat conduction equation relevant to a fractal heat transfer.

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Local fractional heat conduction equation

Local fractional heat conduction equation with no heat generation in fractal media reads [5-7]

$$K^{2\alpha}\frac{\partial^{2\alpha}T(x,t)}{\partial x^{2\alpha}}-\rho_\alpha c_\alpha\frac{\partial^{\alpha}T(x,t)}{\partial t^{\alpha}}=0 \quad \text{at}\, t>\tau_0 \,\text{and in}\, x\in[0,l], \tag{1a}$$

where $\Delta^\alpha(T(x)-T(x_0))\cong\Gamma(1+\alpha)\Delta(T(x)-T(x_0))$.

In eq. (1a) , the transport coefficient $K^{2\alpha}$ is the fractal thermal conductivity related to fractal dimensions of materials [5-7].

$$\frac{\partial^{2\alpha}T(x,t)}{\partial x^{2\alpha}}-a^\alpha\frac{\partial^{\alpha}T(x,t)}{\partial t^{\alpha}}=0 \tag{1b}$$

The fractal heat diffusivity of the medium is defined as $a^\alpha=\rho_\alpha c_\alpha/K^{2\alpha}$.

The local fractional derivative of $T(x)$ of order α at $x=x_0$ is given by [5-7, 22-24]

$$D_x^{(\alpha)}T(x_0)=\frac{d^\alpha T(x)}{dx^\alpha}\Big|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^\alpha(T(x)-T(x_0))}{(x-x_0)^\alpha}, \tag{1c}$$

where $\Delta^\alpha(T(x)-T(x_0))\cong\Gamma(1+\alpha)\Delta(T(x)-T(x_0))$.

The local fractional integral of $T(x)$ of order α in the interval $[a,b]$ is defined by [5-7, 22-24]

$${}_aI_b^{(\alpha)}T(x)=\frac{1}{\Gamma(1+\alpha)}\int_a^b T(t)(dt)^\alpha=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1}T(t_j)(\Delta t_j)^\alpha, \tag{1d}$$

In eq. (1d) $\Delta t_j=t_{j+1}-t_j$, $\Delta t=\max\{\Delta t_1,\Delta t_2,\Delta t_j,...\}$ and $[t_j,t_{j+1}]$, $j=0,...,N-1$, $t_0=a,t_N=b$, is a partition of the interval $[a,b]$. In order to facilitate the presentation of the solution approach developed we consider the case of the non-dimension which yields

$$\frac{\partial^{2\alpha}T(x,t)}{\partial x^{2\alpha}}-\frac{\partial^{\alpha}T(x,t)}{\partial t^{\alpha}}=0 \tag{2a}$$

with a fractal boundary condition

$$\frac{\partial^{\alpha}T(0,t)}{\partial x^{\alpha}}=E_\alpha(t^\alpha),\ T(0,t)=0. \tag{2b}$$

Local Fractional Variation Iteration Method: Solution

The nonlinear local fractional equation (2a) reads [22] as a sum of linear L_α and non-linear N_α

local fractional operators, $L_\alpha T + N_\alpha T = 0$ which allows the following correction functional to be constructed [22]. We can construct a correction functional as follows [22]

$$T_{n+1}(t) = T_n(t) + {}_{t_0}I_t^{(\alpha)}\left\{\xi^\alpha\left[L_\alpha T_n(s) + N_\alpha \tilde{T}_n(s)\right]\right\}, \tag{3}$$

In eq. (3) $\tilde{T}_n$ is a restricted local fractional variation, while ξ^α is a fractal Lagrange multiplier. The determination of ξ^α require stationary conditions of the functional, that is $\delta^\alpha \tilde{T}_n = 0$ [5, 22].

Following (3) the local fractional functional becomes [6]

$$T_{n+1}(x) = T_n(x) + {}_0I_x^{(\alpha)}\left\{\xi^\alpha\left[\frac{\partial^{2\alpha}T_n}{\partial x^{2\alpha}} - \frac{\partial^\alpha T_n}{\partial \tau^\alpha}\right]\right\}. \tag{4}$$

and the stationary condition yields

$$\delta^\alpha T_{n+1} = \left[1 - \left(\xi^\alpha\right)^{(\alpha)}\big|_{\tau=x}\right]\delta^\alpha T_n + \xi^\alpha\big|_{\tau=x}\,\delta^\alpha\frac{\partial^\alpha T_n}{\partial x^\alpha} + {}_0I_x^{(\alpha)}\left\{\left(\xi^\alpha\right)^{(2\alpha)}\big|_{\tau=x}\,\delta^\alpha T_n\right\}. \tag{5a}$$

$$1 - \left(\xi^\alpha\right)^{(\alpha)}\big|_{\tau=x} = 0,\quad \xi^\alpha\big|_{\tau=x} = 0,\quad \left(\xi^\alpha\right)^{(2\alpha)}\big|_{\tau=x} = 0. \tag{5b}$$

Then, the Lagrange multiplier is

$$\xi^\alpha = \frac{(\tau - x)^\alpha}{\Gamma(1+\alpha)}. \tag{6}$$

Hence, the successive interaction formula is

$$T_{n+1}(x,t) = T_n(x,t) + {}_0I_x^{(\alpha)}\left\{\frac{(\tau - x)^\alpha}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}T_n(x,\tau)}{\partial x^{2\alpha}} - \frac{\partial^\alpha T_n(x,\tau)}{\partial \tau^\alpha}\right]\right\}. \tag{7}$$

Assuming an initial approximation $T(x,t) = x^\alpha E_\alpha(t^\alpha)/\Gamma(1+\alpha)$, we get

$$u_1(x,t) = u_0(x,t) + {}_0I_t^{(\alpha)}\left\{\frac{(\tau - t)^\alpha}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}T_0(x,\tau)}{\partial x^{2\alpha}} - \frac{\partial^\alpha T_0(x,\tau)}{\partial \tau^\alpha}\right]\right\} = E_\alpha(t^\alpha)\sum_{k=0}^{1}\frac{t^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}, \tag{8a}$$

$$u_2(x,t) = u_1(x,t) + {}_0I_t^{(\alpha)}\left\{\frac{(\tau - t)^\alpha}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}T_1(x,\tau)}{\partial x^{2\alpha}} - \frac{\partial^\alpha T_1(x,\tau)}{\partial \tau^\alpha}\right]\right\} = E_\alpha(t^\alpha)\sum_{k=0}^{2}\frac{t^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}, \tag{8b}$$

$$\vdots$$

Consequently, the local fractional series solution $T = \lim_{n\to\infty} T_n$ is

$$T_n(x,t) = E_\alpha(t^\alpha)\sum_{k=0}^{n}\frac{x^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}. \tag{9a}$$

Then we can derive in a compact form

$$T(x,t) = \lim_{n\to\infty}\left[E_\alpha(t^\alpha)\sum_{k=0}^{n}\frac{x^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}\right] = E_\alpha(t^\alpha)\sinh_\alpha(x^\alpha), \tag{9b}$$

where

$$\sinh_\alpha\left(x^\alpha\right)=\frac{E_\alpha\left(x^\alpha\right)+E_\alpha\left(-x^\alpha\right)}{2}. \tag{9c}$$

As is known, the temperature field can be written in the form

$$\left|E_\alpha\left(t^\alpha\right)-E_\alpha\left(t_0^\alpha\right)\right|\leq E_\alpha\left(t_0^\alpha\right)\left|t-t_0\right|^\alpha<\varepsilon^\alpha \tag{10a}$$

and

$$\left|\sinh_\alpha\left(x^\alpha\right)-\sinh_\alpha\left(x_0^\alpha\right)\right|<\left|\cosh_\alpha\left(x_0^\alpha\right)\right|\left|x-x_0\right|^\alpha<\varepsilon^\alpha. \tag{10b}$$

Hence, the fractal dimensions of both $E_\alpha\left(t^\alpha\right)$ and $\sinh_\alpha\left(x^\alpha\right)$ are equal to α. It is shown that the temperature describes transports processes in fractal media.

Conclusions

This paper presents a local fractional iteration method by an example solving local heat-conduction equation relevant to fractal media. The method is derived on the basis of the local fractional calculus [6, 22, 23, 24]. It differs from the fractional iteration method (VIM) [25-29] based on both fractional and the classical integer calculus [22].The compact solution developed is effective and in describing transports in fractal media.

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References

[1] Hooft. G, A Confrontation with Infinity, *Int. J. Mod. Phys. A.*, 15 (2000), pp.4395

[2] M. Majumder, N. Chopra, R. Andrews, et al. Nanoscale Hydrodynamics - Enhanced Flow in Carbon Nanotubes, *Nature*, 438 (2005), pp.44

[3] Y. Xuan, Wilfried Roetzel, Conceptions for Heat Transfer Correlation of Nanofluids, *Int. J. Heat Mass Transfer*, 43 (2000), pp.3701-3707

[4] Daniel Rayneau-Kirkhope, Y. Mao, Robert Farr, Ultralight Fractal Structures from Hollow Tubes, *Phys. Rev. Letts*, 109 (2012), 204301

[5] X. J. Yang, Heat Transfer in Discontinuous Media, *Adv. Mech. Eng. Appl.*, 1 (3) (2012) pp. 47-53

[6] X. J. Yang, Advanced Local Fractional Calculus and Its Applications, World Science Publisher, New York, USA, 2012.

[7] M. S. Hu, X. J. Yang, J. Fan, Approximation Solution to Local Fractional Volterra Integral Equations Arising in Fractal Heat Transfer, *J. Nano Res.*, accepted, 2012.

[8] T. M. Shih, A literature survey on numerical heat transfer, *Num. Heat Transfer*, 5(4) (1982), pp. 369-420

[9] Meilanov, R., Shabanova, M., Akhmedov, E., A research note on a Solution of Stefan Problem with Fractional Time and Space Derivatives, *Int. Rev. Chem. Eng.*, 3 (6) (2011), pp. 810-813

[10] Siddique, I., Vieru, D., Stokes Flows of a Newtonian Fluid with Fractional Derivatives and Slip at

the Wall, *Int. Rev. Chem. Eng.*, 3(6) (2011), pp. 822- 826.
[12] Y. Z. Povstenko, Thermoelasticity that uses fractional heat conduction equation, *J. Math. Sci.*, 51 (2) (2009), pp. 293-246
[13] J. Hristov, Approximate solutions to fractional sub-diffusion equations: The heat-balance integral method, *Eur. Phys. J. Spec. Topics*, 193 (4) (2011), pp. 229 – 243
[14] J. Hristov, Heat-balance integral to fractional (half-time) heat diffusion sub-model, *Thermal Sci.*, 14 (2) (2010), pp. 291-316
[15] J. Hristov, An exercise with the He's variation iteration method to a fractional bernoulli equation arising in transient conduction with non-linear heat flux at the boundary, *Int. Rev. Chem. Eng.*, 4 (5) (2012), pp.489-497
[16] J. H. He, A new fractal derivation, *Thermal Sci.*, 15 (2011), pp. 145-147
[17] Q. L. Wang, J.H. He, Z.B. Li, Fractional model for heat conduction in polar hairs, *Thermal Sci.*,16 (2) (2012), 339-342.
[18] Molliq, R. Y., Noorani, M. S. M., Hashim, I., Variational Iteration Method for Fractional Heat- and Wave-Like Equations, *Nonlinear Analysis: R.W.A.*, 10 (3) (2009), pp. 1854-1869
[19] Sakar, M. G., Erdogan, F., Yyldirim, A., Variational Iteration Method for the Time-Fractional Fornberg-Whitham Equation, *Comput. Math. Appl.*, 63 (9) (2012), pp. 1382-1388
[20] J. H. He, Q. L. Wang, J. Sun, Can polar bear hairs absorb environmental energy? *Thermal Sci.*, 15 (3) (2011), 911-913
[21] J. Fan, J. F. Liu, J.H. He, Hierarchy of wool fibers and fractal dimensions, *Int. J. Nonlin. Sci. Num.*, 9 (3) (2008), 293-296
[22] X. J. Yang, F. R Zhang, Local fractional variational iteration method and its algorithms, *Adv. Comput. Math. Appl.*, 1(3) (2012), pp.139-145
[23] X. J. Yang, Local Fractional Integral Transforms, *Progress in Nonlinear Science*, 4 (2011), pp. 1-225
[24] X. J. Yang, Local Fractional Functional Analysis and Its Applications, Asian Academic publisher Limited, Hong Kong, China, 2011
[25] J. H. He, Some asymptotic methods for strongly nonlinear equations, *Int. J. Mod. Phys. B*, 20 (10) (2006), 1141-1199
[26]G. C. Wu, Applications of the variational iteration method to fractional diffusion equations: local versus nonlocal Ones, *Int. Rev. Chem. Eng.*, 4 (5) (2012), 505-510
[28] J. H. He, G. C. Wu, F. Austin, The Variational iteration method which should be followed, *Nonlinear Sci. Letts. A*, 1 (1) (2010), pp.1-30
[29] J. H. He, Asymptotic methods for solitary solutions and compactons, *Abstract and Applied Analysis*, 2012, 916793

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