PERGAMON

Chaos, Solitons and Fractals 11 (2000) 289–296

CHAOS
SOLITONS & FRACTALS

www.elsevier.nl/locate/chaos

Fractal market hypothesis and two power-laws

Aleksander Weron [a,*], Rafał Weron [b,1]

[a] *Hugo Steinhaus Center for Stochastic Methods, Technical University of Wrocław, 50-370 Wrocław, Poland*
[b] *Institute of Mathematics, Technical University of Wrocław, 50-370 Wrocław, Poland*

Abstract

A fractal approach is used to analyze financial time series by applying different degrees of time resolutions. This leads to the heterogenous market hypothesis (HMH), where different market participants analyze past events and news with different time horizons. A new general model for asset returns is studied in the framework of the fractal market hypothesis (FMH). It concerns capital market systems in which the conditionally exponential dependence (CED) property can be attached to each investor on the market.

1. Introduction

For some years now, the set of available data from financial markets has grown rapidly. In the 1960s and 1970s, most of the empirical studies were based on yearly, quarterly, or monthly data. During the 1980s, the study of weekly and daily data led to discovery of new properties such as autoregressive hetereroskedasticity. The studies of intraday data in the 1990s revealed a new wealth of complexity (see [2,13,14]).

It is well known starting from [3,8], see also [1,11,14] that market returns are not normally distributed, but this information has been downplayed or rationalized away over the years to maintain the crucial assumption of the traditional capital market theory (CMT). A variety of alternatives to the normal law can be found in literature and it is undeniable that as long as the distribution that is implied by these models is more leptokurtic than the Gaussian law, it will provide a better fit, see Fig. 1.

Mandelbrot introduced fractal model to describe a certain class of objects exhibiting a complex behavior. He first applied it to financial data in [8]. The fractal view starts from a basic principle: analyzing an object on different scales, with different degrees of resolution, and comparing and interrelating the results. For financial time series, this means using different time yardsticks from hourly through daily to monthly and yearly, within the same study. This is far from the conventional time series analysis, which focusses on regularly spaced observations with fixed time intervals.

In a search for satisfactory descriptive models of economic data, large numbers of distributions have been tried and many new distributions have been discovered. Entire classes of distributional types have been constructed and these often serve to direct the search process for a suitable choice, see [11] and references therein where a variety of alternative distributions for asset returns is analyzed. In any particular case it is always possible to find a distribution that fits the data well, provided one works within a suitably broad and flexible class of candidates, see Fig. 1. Some alternatives to the normal distribution, like the stable Pareto distribution [5,10] were often rejected even though they fit the data without modification.

[*] Corresponding author.
E-mail address: weron@1dhpux.immt.pwr.wroc.pl (A. Weron)
[1] Research was partially supported by the Foundation for Polish Science (FNP) Scholarship.

0960-0779/00/$ - see front matter
PII: S0960-0779(98)00295-1

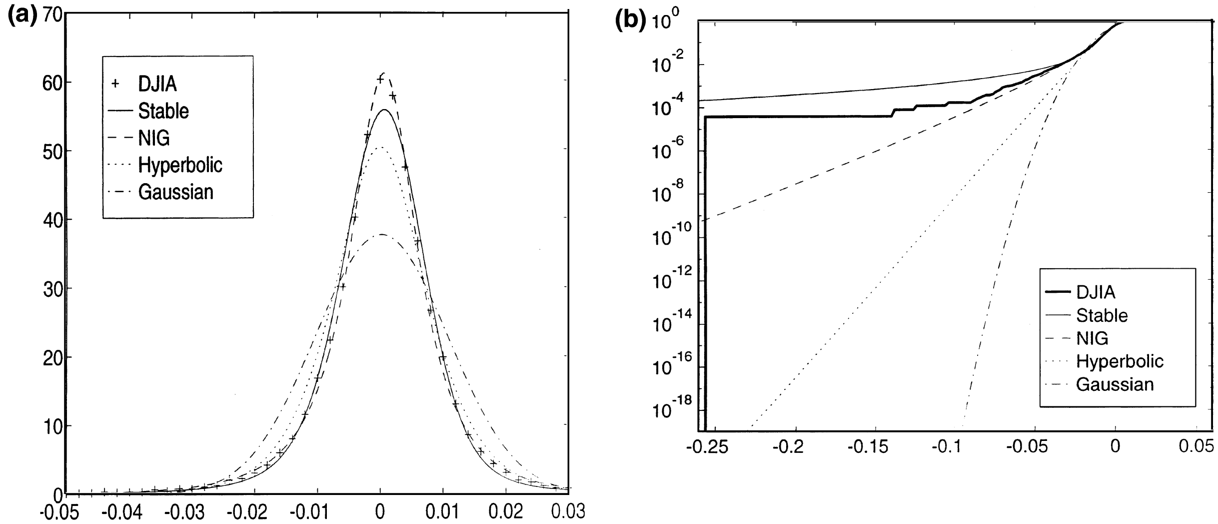

Fig. 1. Histogram of the empirical density (kernel estimator) of the Dow Jones Industrial Average (DJIA) daily returns and four estimating densities: stable, Normal Inverse Gaussian (NIG), hyperbolic and Gaussian (left panel). Left tails of the empirical distribution function and the four countinuous distributions (right panel).

Why? – Standard statistical analysis cannot be applied using those distributions. Besides that, it is one thing to fit given data well through the choice of a "good distribution", but it is an entirely different matter to explain return's data through the use of a statistical model that predicts the data's main characteristics. To deal with such a problem, this paper employs a new conditionally exponential dependence (CED) model, introduced recently in [17] to describe global distributional structure of asset returns.

The efficient market hypothesis (EMH) [4] including arbitrage pricing theory (APT) of Ross [18] and capital asset pricing model (CAPM) originally developed by Sharp [19], Linter [7] and Mossin [12], was very successful in making the mathematical environment easier, but unfortunately is not justified by the real data. Instead, there is a need to seek for a market hypothesis that fits the observed data better and takes into account why markets exist to begin with. In the EMH place, fractal market hypothesis (FMH) has been recently proposed by Peters [15,16], and heterogeneous market hypothesis (HMH) for foreign exchange markets by Müller et al. [13]. Based on current developments of chaos theory and using the fractal objects whose disparate parts are self-similar, these hypotheses provide a new framework for a more precise modeling of turbulence, discontinuity, and non-periodicity that truly characterize today's financial markets. They seem to be robust tools for understanding the conflicting market randomness and determinism we experience every trading and investing day.

The goal of this paper is to demonstrate how the basic ideas of the FMH lead to a rigorous mathematical model, which can be used to explain some empirical facts. For this purpose, we adopt here a recent idea of [6,20] to model asymptotic behavior of general complex systems with local conditionally exponential decay property. Our model leads uniquely to two power-laws describing asset returns. It also explains how the two contrary states: *local randomness* and *global determinism* coexist leading, in a natural way, to the universally observed non-Gaussian distributions of returns. This approach makes a significant step toward explaining features of the statistical mechanism of data generation and, moreover, it predicts the data's main characteristics. In this framework the class of possible distributions, well representing the observations is uniquely determined by the general return equation.

2. Fractal market hypothesis

The FMH emphasizes the impact of information and investment horizons on the behavior of investors. In traditional finance theory, information is treated as a generic item. The investor is also generic. Basically, an investor is any one who wants to buy, sell, or hold a security because of the available information. The investor is also considered rational price-taker, i.e. someone who always wants to maximize return and knows how to value current information. This generic approach, where information and investors are general cases, implies that all types of information impact all investors equally. That is where it fails.

The following five basic assumptions were proposed by Peters [16] for his FMH:

- FMH 1. The market is made up of many individuals with a large number of different investment horizons.
- FMH 2. Information has a different impact on different investment horizons.
- FMH 3. The stability of the market is largely a matter of liquidity (balancing of supply and demand). Liquidity is available when the market is composed of many investors with many different investment horizons.
- FMH 4. Prices reflect a combination of short-term technical trading and long-term fundamental valuation.
- FMH 5. If a security has no tie to the economic cycle, then there will be no long-term trend. Trading, liquidity, and short-term information will dominate.

The purpose of the FMH is to give a model of investor behavior and market price movements that fits our observations. When markets are considered stable, the EMH and CAPM seem to work fine. However, during panics and stampedes, those models break down, like singularities in physics. This is not unexpected, because the EMH, APT, and the CAPM are equilibrium models. They cannot handle the transition to turbulence. Unlike the EMH, the FMH says that information is valued accordingly to the investment horizon of the investor. The key is that under the FMH the market is stable when it has no characteristic timescale or investment horizon. Instability occurs when the market loses its fractal structure and assumes a fairly uniform investment horizon, see [16].

3. Heterogeneous market hypothesis

A statistical study of financial time series from the fractal point of view is based on the analysis of time intervals Δt of different sizes. An elementary example is the scaling law reported in [14], which relates the mean of the absolute logarithmic price change in foreign exchange markets

$$\overline{|\Delta x|} = c(\Delta t)^D,$$

where the bar over $|\Delta x|$ denotes the mean over a long sample, c is an empirical constant and D is the empirical drift exponent. A different empirical scaling behavior in the dynamics of Standard & Poor's 500 index has been presented by Mantegna and Stanley [9]. In spite of its elementary nature, a scaling law study is immediately able to reject the Gaussian hypothesis and reveal an important property of financial time series. For the Gaussian case the above formula is true with a drift exponent of 0.5, while the empirical values of drift exponents D for USD-DEM exchange rate are clustered around a significantly higher value of 0.59. For DEM-NLG rate a drift exponent of only 0.24 was measured, see [13]. These and other recently found properties of empirical time series lead these authors to the HMH as opposed to the assumption of a homogeneous market where all participants interpret news and react to news in the same way. The HMH is characterized by the following interpretations of the empirical findings:

- HMH 1. Different actors in the heterogeneous market have different time horizons and dealing frequencies. The market is heterogeneous with a fractal structure of the participants' time horizons as it consists of short-term, medium-term and long-term components.
- HMH 2. Different actors are likely to settle for different prices and decide to execute their transactions in different market situations. In other words, they create volatility.
- HMH 3. The market is also heterogeneous in the geographic location of the participants.

The market participants of the HMH also differ in the other aspects beyond the time horizons and the geographic locations: they can have different degrees of risk aversion, institutional constraints, and transaction costs. Further empirical evidence in favor of the HMH is given in [13].

4. CED model and two power-laws

Below, we propose a new statistical mechanism that explains the observed market local randomness and global determinism. We hope that this approach clarifies also the ideas of the FMH or the HMH and provides a rigorous mathematical framework for further analysis of financial complex processes. The distributional form of returns on financial assets has important implications for theoretical and empirical analyses in economics and finance. For example, asset, portfolio and option pricing theories are typically based on distributional assumptions. In empirical tests, statistical inference concerning the efficient market hypothesis, the excess volatility question or option pricing models may be sensitive to the distributional assumptions for the returns of the underlying assets.

The market is made up of participants, from tick traders to long-term investors. Each has a different investment horizon that can be ordered in time. When all investors with different horizons are trading simultaneously the market is stable. The stability of the market relies, however, not only on a random diversification of the investment horizons of the participants but also on the fact that the different horizons value the importance of the information flow differently. Hence, both the information flow and the investment horizons should have their own contribution to the observed global market features. In general, the locally random markets have a global statistical structure that is non-random.

Following [17] we will assume that the model is a discrete time economy with a finite number of trading dates from time 0 to time T and its uncertainty has a global impact on the market index daily returns on the interval $[0, T]$. (As a proxy of the market index one typically uses a stock index, for example S&P or NASDAQ.) In the family of all world investors let us identify those N who are acting on a given market described by a chosen index. Call them $I_{1N}, I_{2N}, \ldots, I_{NN}$. Let R_{iN} be the positive (or the absolute value of negative) part of the ith investor's return. The economy is populated by a finite, but a large number N of investors on the market. Each ith investor is related to a cluster of agents acting simultaneously on common complement markets. The influence of this cluster of agents is of the type of short-range (inter-cluster) interactions and is reflected by a random risk-aversion factor A_i. The long-range type of interactions are imposed on the ith investor by the inter-cluster relationship manifested by the random risk factors B^i_j for all $j \neq i$. They reflect how fast the information flows to the ith investor.

CED 1. For the ith investor the following CED property holds:

$$\phi_{iN}(r\,|\,a,b) = P\big(R_{iN} \geqslant r\,|\,A_i = a,\ b_N^{-1}\max(B^i_1, \ldots, B^i_{i-1}, B^i_{i+1}, \ldots, B^i_N) = b\big) = \exp(-[a\min(r,b)]^c), \tag{1}$$

where r, a, b are non-negative constants, b_N is a suitable, positive normalizing constant and $c \geqslant 1$. The range of the exponent c is justified by the reversion tendency of the market.

The dependence in the CED model measured by the conditional return excess decays similarly as in EGARCH models in an exponential way, see Eq. (1) , but reflects both short as well as long-range effects. This new probabilistic idea concerns systems in which the behavior of each individual entity strongly depends on its short- and long-range random interactions.

CED 2. Investors have different investment horizons (“short-range interaction”) affected by different information sets (“long-range interaction”). The investment horizon of the investor is reflected by the random variable A_i, while $\{B^i_j,\ j = 1, 2, \ldots, N,\ j \neq i\}$ reflect the information flow to this investor.

The probability that the return R_{iN} will be not less than r is conditioned by the value a taken by the random variable A_i and by the value b taken by the maximum of the set of random variables $\{B^i_j,\ j = 1, 2, \ldots,\ N, j \neq i\}$. Therefore Eq. (1) can be rewritten as follows:

$$\phi_{iN}(r\,|\,a,b) \equiv \begin{cases} 1 & \text{for } r = 0, \\ \exp(-(ar)^c) & \text{for } r < b, \\ \exp(-(ab)^c) & \text{for } r \geqslant b, \end{cases} \tag{2}$$

i.e. the conditional return excess $\phi_{iN}(r\,|\,a,b)$ decays exponentially with a decay rate a and exponent c as r tends to the value b. Then it takes a constant value $\ll 1$. The basic statistical assumption is that

CED 3. The random variables $A_1, A_2, \ldots$ and $B^i_1, B^i_2, \ldots$ form independent and convergent (with respect to addition and maximum, respectively [6]) sequences of non-negative, independent, identically distributed (iid) random variables. The variables $R_{1N}, \ldots, R_{NN}$ are also non-negative, iid for each N.

Let us stress, however, that the dependence on external conditions is expressed by the above relationship, Eq. (1), of each R_{iN} with A_i and $\max(B^i_1, \dots, B^i_{i-1}, B^i_{i+1}, \dots, B^i_N)$. Assumption 3 can be partially justified by the following argument. Institutional trading is a major factor in the determination of security prices. If professional investment managers have similar beliefs, then the iid distributions assumption may hold as a first approximation. Professional managers are likely to have similar beliefs because they have access to similar information sources. This uniformity of information over time would tend to generate similar beliefs. It is important to point out that the assumption of iid of the returns R_{iN} is not as restrictive as it may appear. For example, Mitttnik and Rachev [11] showed that the assumption of iid random asset-price changes can be used to describe large classes of well-known financial models.

The cut-off in the return excess Eq. (2) given by the value $r = b$ determines the probability (indeed very small) that the return of ith investor can reach any value greater than b. The value of this probability is the smallest one, since a cut-off by any other value $b_1 < b$ yields a greater probability than $\exp[-(ab)^c]$. This is a manifestation of the unlimited returns. Thus the market contains some arbitrage opportunities. Note that Eq. (1) precisely defines the meaning of random variables related to it. It does not hold for sets of any arbitrarily chosen variables. If R_{iN} has to denote a return, then $A_i = a$ has the sense of an individual risk aversion factor and

$$b_N^{-1} \max(B^i_1, \dots, B^i_{i-1}, B^i_{i+1}, \dots, B^i_N) = b,$$

the sense of a submarket maximal risk factor given by

$$\phi_{iN}(r \,|\, b) = \int_0^\infty \phi_{iN}(r \,|\, a, b)\, \mathrm{d}F_A(a),$$

where F_A is the common distribution function (but unknown) of the sequence of random variables $\{A_i\}$.

Let the global behavior of the asset market be given by

$$\phi(r) = P\Big(\lim_{N\to\infty} r_N \min(R_{1N}, \dots, R_{NN}) \geqslant r\Big), \tag{3}$$

where r_N is a suitable, positive normalizing constant. Under the above assumptions, the function $\phi(r)$, fulfills the global return equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}r}(r) = -\alpha\lambda(\lambda r)^{\alpha-1}\left(1 - \exp\left(-\frac{(\lambda r)^{-\alpha}}{k}\right)\right)\phi(r), \tag{4}$$

where the parameters $\lambda > 0,\ k > 0,\ \mu > 0$ and $\alpha = \alpha' c (c > 1, 0 < \alpha' \leqslant 1)$ are determined by the limiting procedure in Eq. (3).

By Jurlewicz et al. [6] the probability density $f(r) = -\mathrm{d}\phi(r)/\mathrm{d}r$ has the following two power-laws property:

$$f(r) \propto \begin{cases} (\lambda r)^{\alpha-1} & \text{for } \lambda r \ll 1, \\ (\lambda r)^{-(\alpha/k)-1} & \text{for } \lambda r \gg 1. \end{cases} \tag{5}$$

The above equality (3) defines the return excess $P(\tilde{R} \geqslant r)$ of a system as a whole, where $\tilde{R}$ represents the global return. The derivative $f(r) = -(\mathrm{d}\phi/\mathrm{d}r)(r)$ represents the frequency distribution (probability density) of a global market return. Hence the global return distribution is characterized by the following three parameters: (α, λ, k). Here α is the shape and λ is the scale parameter. At this point let us stress the role of the parameter k. It decides how fast the information flow is spread out in the market; $k \to 0$ denotes the case when the long-range interaction is neglected. So, there are no arbitrage opportunities on the market. If $k \to 0$, Eq. (4) takes the form

$$\frac{\mathrm{d}\phi}{\mathrm{d}r}(r) = -\alpha(\lambda r)^{\alpha-1}\phi(r),$$

with the solution

$$\phi(r) = \exp[-(\lambda r)^\alpha].$$

5. Empirical analysis

5.1. Example 1. Standard & Poor 500 composite index

The data set is the S&P500 index which is a composite index based on the performance of the main 500 shares on the New York Stock Exchange. To illustrate the typical characteristics of the global market returns we consider daily observations during the period from 2 July 1962 to 31 December 1991. We note that this period includes the stock market crash of October 1987. Let $S(t)$ be the daily observation of day t for the S&P 500 index and $R(t)$ be the daily return of day t. Then $S(t)$ and $R(t)$ are related by $R(t) = \log S(t) - \log S(t-1)$. The 7420 daily returns of the S&P 500 are splitted to 3879 positive and 3495 negative returns; there are also 47 zero returns.

Applying the CED model separately to both data sets we are getting the following parameters:

$$\alpha^+ = 0.8656, \quad \lambda^+ = 142.5871, \quad k^+ = 0.4977$$

and

$$\alpha^- = 0.7405, \quad \lambda^- = 125.0000, \quad k^- = 0.3392.$$

for positive and absolute value of negative S&P500 returns, respectively.

Zipf plots (double logarithmic plots) of empirical densities (kernel estimators) of the S&P500 daily returns vs rank are shown in Fig. 2 for positive and absolute value of negative returns. Observe that the Zipf plots demonstrate clearly the difference in the behavior of positive and negative returns and also visualize the two power-laws (see Eq. (5)) for small and large returns, respectively.

5.2. Example 2. NASDAQ index

The data set is the NASDAQ index which is a composite index based on the performance of the main shares on the NASDAQ over-the-counter market. To illustrate the typical characteristics of the global market returns we consider daily observations during the period from 14 December 1972 to 31 December 1991. We note that this period includes the stock market crash of October 1987. The 4809 daily returns for the NASDAQ index are splitted to 2735 positive and 2067 negative returns; there are also seven zero returns.

Applying the CED model separately to both data sets we are getting the following parameters:

$$\alpha^+ = 0.9228, \quad \lambda^+ = 166.6667, \quad k^+ = 0.4230$$

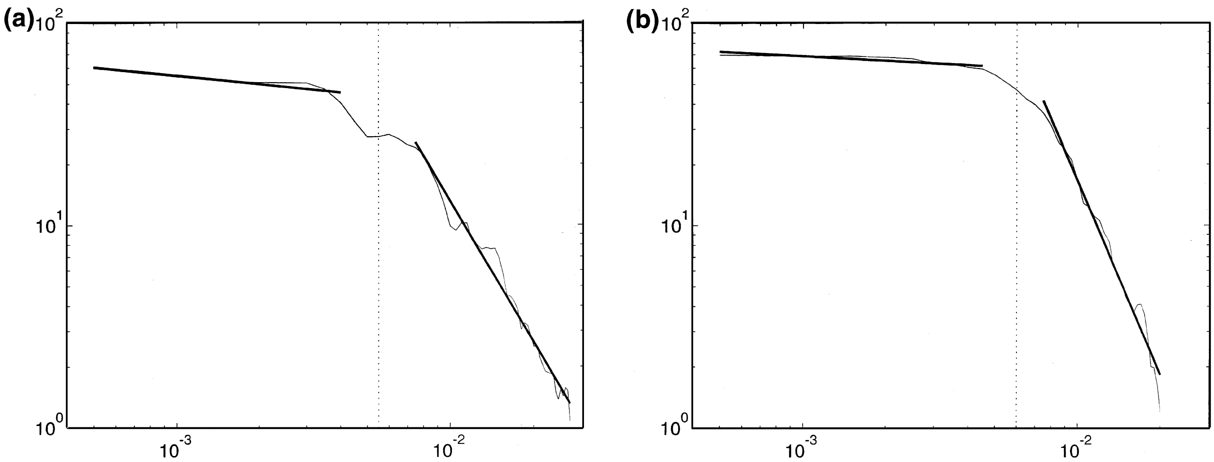

Fig. 2. Zipf plots (double logarithmic plots) of empirical densities (kernel estimators) of the S&P500 daily returns vs rank for positive (left panel) and absolute value of negative returns (right panel). Bold lines represent the two power-laws, see Eq. (5).

and

$$\alpha^- = 0.8596, \quad \lambda^- = 181.8182, \quad k^- = 0.6510$$

for positive and absolute value of negative returns, respectively.

Zipf plots of empirical densities of the NASDAQ daily returns vs rank are shown in Fig. 3 for positive and absolute value of negative returns.

5.3. Example 3. USD/SFR exchange rate

The data set is the USD/SFR exchange rate from the period 20 May 1985 to 12 April 1991. To illustrate the dependence of the empirical result from the time horizons we consider three types of USD/SFR exchange rate returns. Namely, daily, hourly and one minute returns. Zipf plots of empirical densities of the USD/SFR exchange rate Δt-returns vs rank are shown in Fig. 4 for $\Delta t = 1$ day, 1 h and 1 min, respectively. Observe essential differences in the shapes of these densities supporting the hypothesis of a fractal scaling law as a function of the time interval ranging from few minutes up to a day. This demonstrates that, in principle, it is possible to estimate the empirical drift exponent D in the scaling law reported in [13] from the two power-law approach.

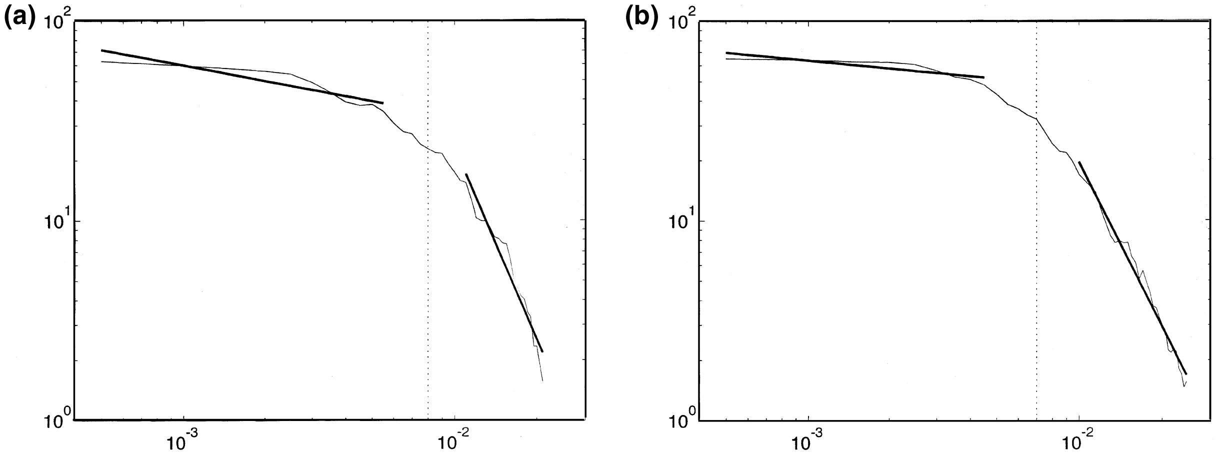

Fig. 3. Zipf plots of empirical densities of the NASDAQ daily returns vs rank for positive (left panel) and absolute value of negative (right panel) returns. Bold lines represent the two power-laws.

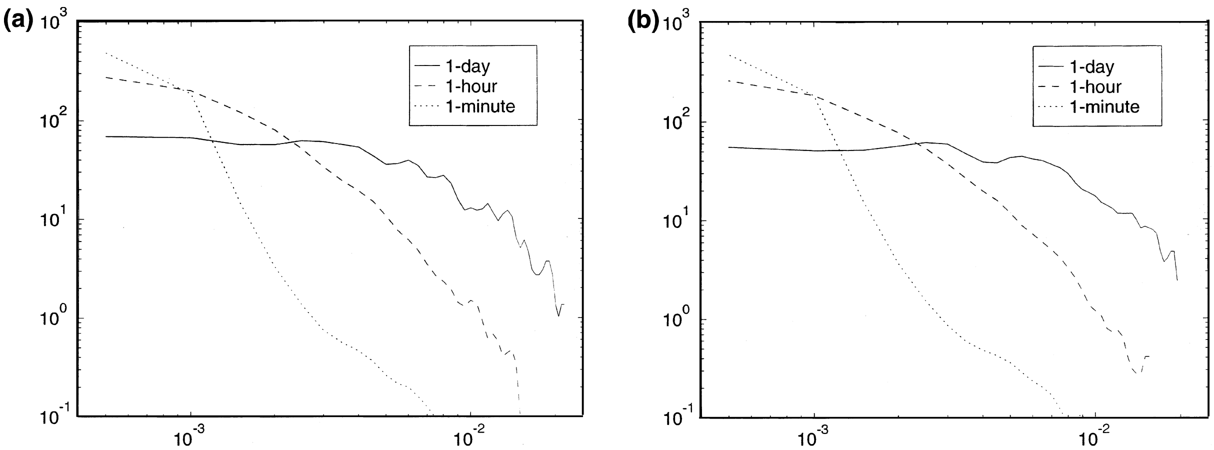

Fig. 4. Zipf plots of empirical densities of the USD/SFR exchange rate returns for three time intervals: 1 day, 1 h and 1 min vs rank. Positive returns are displayed on the left panel and absolute value of negative returns on the right panel.

References

[1] T. Bollerslev, R.Y. Chou, K.F. Kroner, ARCH modeling in finance. A review of the theory and empirical evidence, Journal of Econometrics 52 (1992) 5–59.
[2] M.M. Dacorogna, C.L. Gauvreau, U.A. Müller, R.B. Olsen, O.V. Pictet, Changing time scale for short-term forecasting in financial markets, Journal of Forecasting 15 (1996) 203–227.
[3] E. Fama, The behavior of stock market prices, Journal of Business 38 (1965) 34–105.
[4] R.A. Jarrow, Finance Theory, Prentice-Hall, Englewood Cliffs, NJ, 1988.
[5] A. Janicki, A. Weron, Simulation and Chaotic Behavior of α Stable Stochastic Processes, Marcel Dekker, New York, 1994.
[6] A. Jurlewicz, A. Weron, K. Weron, Asymptotic behavior of stochastic systems with conditionally exponential decay property, Applicationes Mathematicae 23 (1996) 379–394.
[7] J. Lintner, The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets, Review of Economics and Statistics 47 (1965) 13–37.
[8] B.B. Mandelbrot, New methods in statistical economics, Journal of Political Economy 71 (1963) 421–440.
[9] R.N. Mantegna, H.E. Stanley, Scaling behaviour in the dynamics of an economic index, Nature 376 (1995) 46–49.
[10] J.H. McCulloch, Financial applications of stable distributions, in: G.S. Maddala, C.R. Rao (Eds.), Statistical Methods in Finance, Handbook of Statistics XIV, North-Holland, New York, 1996.
[11] S. Mittnik, S.T. Rachev, Modeling asset returns with alternative stable distributions, Econometric Reviews 12 (1993) 261–330; Reply to Comments 12 (1993) 347–389.
[12] J. Mossin, Equilibrium in a capital asset market, Econometrica 34 (1966) 768–783.
[13] U.A. Müller, M.M. Dacorogna, R.D. Davé, O.V. Pictet, R.B. Olsen, J.R. Ward, Fractals and intrinsic time – A challenge to econometricians, Research Report UAM.1993-08-16, Olsen & Associates, Zürich, 1993.
[14] U.A. Müller, M.M. Dacorogna, R.B. Olsen, O.V. Pictet, M. Schwarz, C. Morgenegg, Statistical study of foreign exchange rates, empirical evidence of price change scaling law, and intraday analysis, Journal of Banking & Finance 14 (1990) 1189–1208.
[15] E.E. Peters, Chaos and Order in the Capital Markets, Wiley, New York, 1991.
[16] E.E. Peters, Fractal Market Analysis. Applying Chaos Theory to Investment & Economics, Wiley, New York, 1994.
[17] S.T. Rachev, A. Weron, K. Weron, Conditionally exponential dependence model for asset returns, Applied Mathematics Letters 10 (1997) 5–9.
[18] S. Ross, The arbitrage theory of capital asset pricing, Journal of Economic Theory 13 (1976) 341–360.
[19] W. Sharp, Capital asset price; A theory of market equilibrium under conditions of risk, Journal of Finance 19 (1964) 425–442.
[20] K. Weron, A. Jurlewicz, Two forms of self-similarity as a fundamental feature of the power-law dielectric relaxation, Journal of Physics A 26 (1993) 395–410.