Fractal regularity results on optimal irrigation patterns

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Abstract

In this paper the problem of the regularity of the minima of the branched transport problem is addressed. We show that, under suitable conditions on the irrigated measure, the minima present a fractal regularity, that is on a given branch of length l the number of branches bifurcating from it whose length is comparable with ε can be estimated both from above and below by l/ε.

Keywords: optimal transport problem, branched transport problem, irrigation problem, landscape function, fractal regularity

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Contents

1 Introduction

1.1 Optimal transport problems

Optimal transport in the Monge-Kantorovich viewpoint. Optimal transport problems were first considered by Monge in 1781. In optimal transport problems the datum is a couple (μ^+, μ^-) of probability measures (respectively named *initial* and *final* measure). The problem is then to minimize

$$M(t) := \int_{\mathbf{R}^N} c(x, t(x)) \mathrm{d}\mu^+(x)$$

among *transport maps*, i.e. maps $t : \mathbf{R}^N \to \mathbf{R}^N$ such that, for every Borel set B, $\mu^-(B) = \mu^+(t^{-1}(B))$. The function $c : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ is a positive lower semi-continuous function, usually the p-th power of the Euclidean distance.

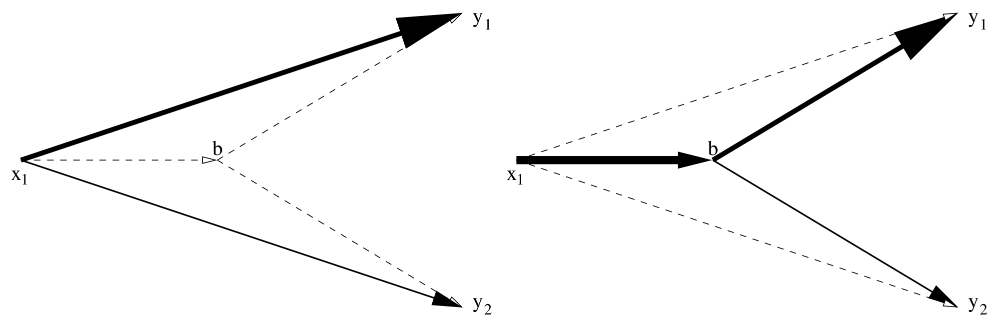

Figure 1: V-shaped versus Y-shaped transport.

In 1948 Kantorovich proposed a relaxed version of Monge's problem. Instead of transport maps, *transport plans* are considered, i.e. probability measures $\pi \in \mathcal{P}(\mathbf{R}^N \times \mathbf{R}^N)$ such that $\pi(A \times \mathbf{R}^N) = \mu^+(A), \pi(\mathbf{R}^N \times B) = \mu^-(B)$. The problem is then to minimize

$$K(\pi) = \int_{\mathbf{R}^N \times \mathbf{R}^N} c(x, y) \mathrm{d}\pi(x, y).$$

Note that, if t is a transport map, the transport plan defined as

$$\pi_t(C) := \mu^+(\{x \in \mathbf{R}^N \ : \ (x, t(x)) \in C\})$$

satisfies $K(\pi_t) = M(t)$. Because of this, Kantorovich's Problem extends Monge's one (it is actually its relaxation w.r.t. the weak convergence of measures, see [1]).

Branched transport problems. Branched transport problems were introduced in order to model many artificial and natural systems (like roads, pipelines, bronchial, and cardiovascular ones) which can naturally be viewed as transport problem, but the Monge-Kantorovich setting is not suitable to describe them, since the minima of the functionals M or K do not show a ramified structure.

For example, if we consider the transport problem of a Dirac mass onto the convex combination of two Dirac masses, the solution of the Monge-Kantorovich will be the one on the left of Figure 1: the initial mass is split and brought on the support of the final measure on a straight line. On the other hand, one would like a functional whose minima were those on the right of Figure 1, where the mass is not split from the beginning, since in branched transport it is cheaper to move it together as much as possible.

In order to describe such systems, Maddalena, Morel, and Solimini in [11] proposed a model based on the Lagrangian formulation of the fluid flow in a system of pipes. In Maddalena, Morel, and Solimini's approach curves with a common initial point $S \in \mathbf{R}^N$ are considered. Such curves represent the trajectory of fluid particles or veins in the cardiovascular system. In this model the initial measure is then a Dirac mass in S, while the final one is obtained counting how many fibres stop in a given volume.

The attempts to model such situations are several. Let us recall some of them to the reader. The first one is Xia's model (see [18]). This model considers a functional which is the relaxation of an appropriate cost defined on weighted directed graphs. Bernot, Caselles, and Morel's Traffic Plans (see [3]) are instead another Lagrangian approach to the problem, while in [5] Brancolini, Buttazzo, and Santambrogio propose a functional defined on curves in the Wasserstein spaces, which penalizes curves which do not take value in the set of discrete measures. Their model is not equivalent to Maddalena, Morel, and Solimini's one, but can be modified to be equivalent as pointed out in [8].

In this paper we will consider the general framework introduced by Maddalena and Solimini in [13] and [14]. We briefly describe it here referring to the cited papers for the details.

Definition 1.1 (Irrigation pattern)**.** Let $I = [a, b] \subseteq \mathbf{R}$ and $(\Omega, \mathcal{B}(\Omega), \mu_\Omega)$ be a probability space (the *reference space*). By *irrigation pattern* we will mean a measurable function $\chi : \Omega \times I \to \mathbf{R}^N$ such that for μ_Ω-a.e. p the function $\chi_p := \chi(p, \cdot) \in \mathrm{AC}(I)$ for almost all p. The pattern $\tilde{\chi}$ will be *equivalent* to χ if the images of μ_Ω through the maps $p \mapsto \chi_p, p \mapsto \tilde{\chi}_p$ are the same. Every $p \in \mu_\Omega$ will be called *particle* and the function χ_p will represent the particle trajectory (which we will refer to as *fibre*). With little abuse of language we sometimes identify the particle p with the fibre χ_p.

Notation. As far as Definition 1.1 is concerned, we fix the notation as follows for the whole paper.

- We will always denote by a (respectively, b) the minimum (respectively, maximum) of I.
- Recall that if Ω is a complete separable metric space and μ_Ω has no atoms (hence Ω is uncountable), then $(\Omega, \mathcal{B}(\Omega), \mu_\Omega)$ is isomorphic (i.e., there exists a one-to-one map preserving the measure) to the standard space $([0, 1], \mathcal{B}([0, 1]), \mathcal{L}^1_{|[0,1]})$ (see, for example, [15, Proposition 12 or Theorem 16 in Section 5 of Chapter 15] or [17, Chapter 1]). We will then always assume that we are in the hypothesis of that result.

Definition 1.2 (Irrigating and irrigated measure)**.** The *irrigating (or initial) measure* is the image of μ_Ω via the map $p \mapsto i^+_\chi(p) := \chi(p, a)$. The irrigating measure will be denoted by μ^+_χ.

The *irrigated (or final) measure* is the image of μ_Ω via the map $p \mapsto i^-_\chi(p) := \chi(p, b)$. This measure will be denoted by μ^-_χ.

Definition 1.3 (Masses)**.** Given an irrigation pattern χ, for every $(p, t) \in \Omega \times I$ we consider the sets

$$\begin{aligned}
[p]^0_t &:= \{q \in \Omega \ : \ \chi(q, s) = \chi(p, s), \ \ \forall s \in [a, t]\}, \\
[p]^1_t &:= \{q \in \Omega \ : \ \chi(q, t) = \chi(p, t)\}, \\
[p]^2_t &:= \{q \in \Omega \ : \ \chi(p, t) \in \chi_q(I)\}.
\end{aligned}$$

For every $i \in \{0, 1\}$ and every $t \in I$, $\{[p]^i_t \ : \ p \in \Omega\}$ is a partition of Ω. The *masses* m^i_χ are given by:

$$m^i_\chi(p, t) := \mu_\Omega([p]^i_t). \tag{1.1}$$

Definition 1.4 (Cost densities)**.** Given an irrigation pattern χ, for $i \in \{0, 1, 2\}$ we consider the following *cost densities*:

$$s^i_{\alpha,\chi}(p, t) := [m^i_\chi(p, t)]^{\alpha-1}.$$

Definition 1.5 (Cost functionals)**.** The *cost functionals* we are interested in will be:

$$J^i_\alpha(\chi) := \int_{\Omega \times I} s^i_{\alpha,\chi}(p, t)|\dot{\chi}(p, t)|\mathrm{d}p\mathrm{d}t.$$

In this paper we will study the regularity of the minima of the *irrigation problem* now stated.

Problem (Irrigation problem)**.** *Let* $\mu^+, \mu^- \in \mathcal{P}(\mathbf{R}^N)$ *be given. The* irrigation problem *is then*

$$\min\{J^i_\alpha(\chi) \ : \ \mu^+_\chi = \mu^+, \mu^-_\chi = \mu^-\}. \tag{1.2}$$

An irrigation pattern χ_{opt} *such that*

$$J^i_\alpha(\chi_{opt}) = \min\{J^i_\alpha(\chi) \ : \ \mu^+_\chi = \mu^+, \mu^-_\chi = \mu^-\},$$

will be called optimal pattern *for the functional* J^i_α.

For $i = 0, 1$ the functional is *synchronous*, i.e. if the trajectories of two particles given by an optimal pattern are the same, then they will move together. For $i = 2$, the functional is *asynchronous*, since each particle can move independently on its trajectory, i.e. for every $p \in \Omega$ the function χ_p

can be re-parametrized (independently) without changing the value of the functional.

We refer to [14] or [6] for a proof of the next theorem, which is the fundamental tool to present the unified theory of the irrigation functionals.

Theorem 1.6 (Synchronization Theorem)**.** *The following statements hold:*

- $J^2_\alpha \leq J^1_\alpha \leq J^0_\alpha$*;*
- $\inf J^0_\alpha = \inf J^1_\alpha = \inf J^2_\alpha$*;*
- J^0_α, J^1_α *share the same minima, if the initial mass is a Dirac mass; so,* χ *is optimal for* J^0_α *if and only if it is optimal for* J^1_α*;*
- *every optimal pattern for* J^1_α *is optimal for* J^2_α*;*
- *every optimal pattern for* J^2_α *can be re-parametrized fibre by fibre to be a minimum for* J^1_α*, i.e. every optimal pattern for* J^2_α *can be synchronized.*

Remark 1.7. Notice that by Theorem 1.6 if a result involving quantities invariant under time scaling fibre by fibre (as, for instance, the landscape function introduced in Definition 1.9) holds for optimal patterns for J^0_α must also hold for minima of J^2_α.

Notation. We finally introduce some notation which will be frequently used in the following.

- Let $\mu^+, \mu^- \in \mathcal{P}(\mathbf{R}^N)$ be given. When we will say that χ is *optimal*, we will always mean that χ solves the irrigation problem with $i = 1$ (i.e. χ is a *minimum*, with given the irrigating and irrigated measures, for J^1_α). Note that in such case χ is also a minimum of J^0_α or a synchronized minimum of J^2_α.
- We will denote by $d_\alpha(\mu^+_\chi, \mu^-_\chi)$ the minimum value in (1.2), which is the same for all the functionals considered ($i = 0, 1, 2$) as proved in [14] (see Theorem 1.6).
- In spite of the fact that the irrigation problem can be stated for a generic pair $\mu^+, \mu^- \in \mathcal{P}(\mathbf{R}^N)$, the irrigating measure μ^+_χ will always be the Dirac mass δ_S (where $S \in \mathbf{R}^N$ is given) in this paper. This is due to the fact that the *landscape function* (one of the main tools used here, see Definition 1.9) can be defined only in this setting. Then, since the irrigating measure is supposed to be a Dirac mass, the final boundary datum μ^- cannot be confused to the initial boundary datum μ^+ and we usually simply write μ instead of μ^-. For the same reason we will simply write μ_χ instead of μ^-_χ as far as the irrigated measure is concerned.

1.2 The fractal regularity for the optimal branched patterns

The branched transport functional differs from the usual functionals of the Calculus of Variations. In the latter case, the minima are forced to have the "finite" gradient and be regular functions because of the convexity in the gradient of the unknown.

In branched transport problems the regularity issue is a completely different problem from the usual one and presents two terms which cause opposite behaviours.

On one side, the term in $|\dot{\chi}|$ (convex) can be treated with the usual regularity techniques. In order to minimize a convex term, the variation of the density tends to be uniform (by Jensen inequality), giving the regularity of the particle trajectories of optimal patterns. In [12] Morel and Santambrogio consider the regularity of the derivative of the particles trajectories for an optimal traffic plan, showing that it is locally of bounded variation if the initial measure is a Dirac mass and the final one is the Lebesgue one on a sufficiently regular set. We will call this "classical regularity".

On the other side, the "concave" behaviour in the mass forces the minima to be concentrated on 1-dimensional sets (by an opposite Jensen inequality), to create a branched structure and, moreover, cannot be attacked with the common regularity tools. We will call this "fractal regularity".

Let us remind that both problems need some regularity on the final measure, which is required both to have the existence of the minimizers and, for example, the regularity of the landscape function. In [4] and [10] conditions on the Ahlfors dimension of the final measure in order to assure the existence of minimizers for the branched transport problem are found.

The main result of this paper is Theorem 6.17. We prove that, for a suitable *universal* constant W (we will call it *scale window*), the number of branches with length between ε and $W\varepsilon$ bifurcating from a given branch of given length l on the support of the irrigated measure, called *wanted branches* (see Figure 2), is bounded from above (easy estimate) and from below (difficult one) by positive multiples of l/ε. The two estimates are obtained via mass balance arguments.

The easy estimate is simply obtained comparing the mass irrigated by the branches we are interested in with the mass of the tubular neighbourhood $U_{W\varepsilon}$ of radius $W\varepsilon$ of the branch.

The difficult estimate is that from below. In fact, the mass of $U_{W\varepsilon}$ can be irrigated by many types of branches: long, far away, and short ones. *Long branches* (see Figure 3) are branches that start in the given part and irrigate $U_{W\varepsilon}$ but are not all contained $U_{W\varepsilon}$. *Far away branches* (see Figure

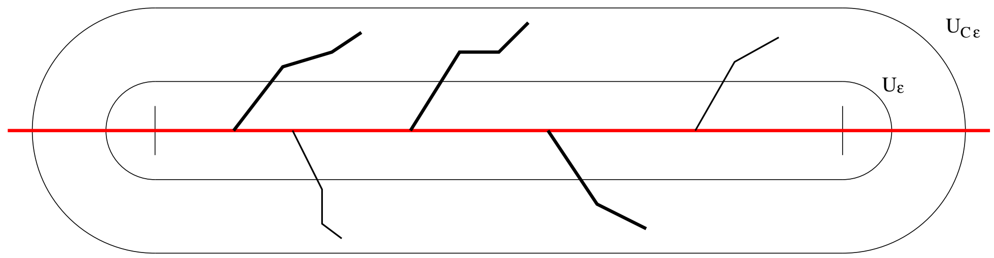

Figure 2: Wanted branches.

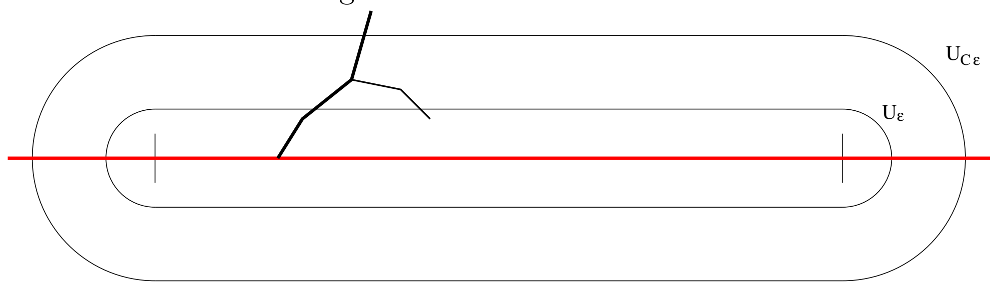

Figure 3: Branches we do not want: long ones.

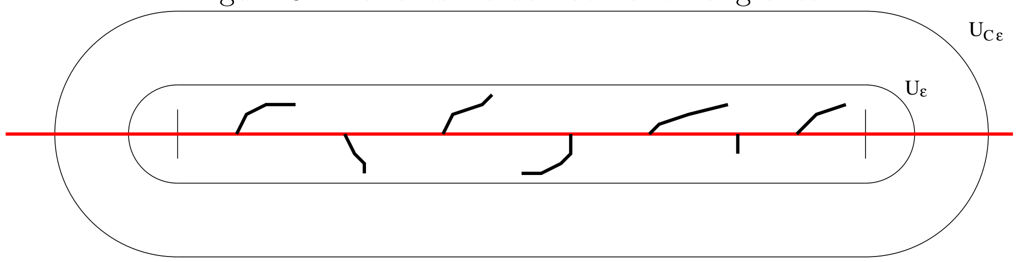

Figure 4: Branches we do not want: short ones.

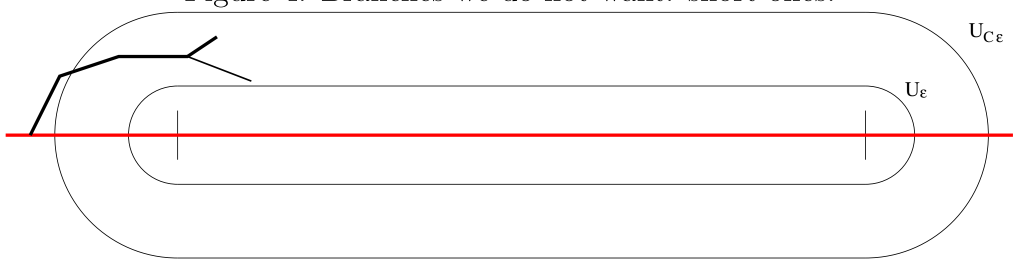

Figure 5: Branches we do not want: faraway ones.

5) are branches that irrigate $U_{W\varepsilon} \setminus U_\varepsilon$, but the bifurcation is not on the given part, but before or after it. *Short branches* (see Figure 4) are branches that bifurcate in the given part and their residual length is smaller than ε. All the mass (or a very large one) part of the mass of $U_{W\varepsilon}$ could be brought by such branches, so that the number of wanted branches bifurcating from the given part would be too few and the estimate from below would not hold.

This estimate is achieved showing that, for a suitable choice of W, long, far away, and short branches can irrigate a fraction of the mass of $U_{W\varepsilon}$ smaller than a given $0 < \lambda < 1$. The constant W is called *scale window*, since it gives the width of a tubular neighbourhood such that the number of wanted sub-branches bifurcating is of order l/ε.

In Section 2 we give the formal definition of branch. For us a branches will be will be a flow line which maximizes the residual distance.

In Section 3 we study gain formulas for the linear mass by-pass of the branched transport functional and derive from it some useful results (among them, the equivalence of the branch distance and the Euclidean one).

In Section 4 we give some useful estimates for the measure of the tubular neighbourhood of a curve when the irrigated measure is Ahlfors regular (from above or below).

In Section 5 we will prove second order gain formulas for double and single by-pass and some technical results which are crucial to rule out far away branches.

In Section 6 we finally prove the fractal regularity result following the argument line depicted here.

1.3 The landscape function

In view of Theorem 1.6, in this section and in the following ones we will consider only the functional J^0_α, so we will then drop the superscript and if we write J_α instead of J^0_α.

Consider the J_α cost in the extended setting. By Fubini's Theorem, it is the integral on Ω of

$$p \mapsto c_\alpha(p) := \int_I s_\alpha(p,t)|\dot\chi_p(t)|\mathrm{d}t. \tag{1.3}$$

The *particle cost* $c_\alpha(p)$ is finite for μ_Ω-a.e. $p \in \Omega$ whenever $J_\alpha(\chi) < +\infty$.

Before going on, we introduce the definition of *domain* of a pattern.

Definition 1.8 (Domain of a pattern)**.** Let χ be a pattern. The *domain of the pattern* χ denoted by D_χ is the set defined by

$$D_\chi = \{x \ : \ \exists A \subseteq \Omega, \mu_\Omega(A) > 0, \exists t \in I \text{ s.t. } \forall p \in A, \chi(p,t) = x\}.$$

We now define the *landscape function*. The landscape function was first introduced in [16] in an equivalent way for optimal patterns. We remark then that in the following we are *not* supposing that χ is an optimal pattern, but only a finite cost one, i.e. $J_\alpha(\chi) < +\infty$. We will always implicitly assume the pattern has a finite cost, whenever the landscape function is considered. The definitions and the proofs of the results recalled here can be found in [7].

Definition 1.9 (Landscape function)**.** For μ_Ω-a.e. p and all $t \in I$, we define the function $Z_\chi : \Omega \times I \to \mathbf{R}^N$ as

$$Z_\chi(p,t) := \int_a^t s_\alpha(p,s)|\dot\chi_p(s)|\mathrm{d}s.$$

A lower semi-continuous function $\varphi : \mathbf{R}^N \to \mathbf{R}$ is *admissible* for χ if

$$\varphi(\chi(p,t)) \le Z_\chi(p,t)$$

holds for μ_Ω-a.e. p and for all $t \in I$. The *landscape function* $\overline{Z}_\chi$ of the pattern χ is then defined by:

$$\overline{Z}_\chi := \sup\{\varphi \ : \ \varphi \text{ admissible for } \chi\}.$$

If there is no misunderstanding, we will simply write $\overline{Z}$ instead of $\overline{Z}_\chi$.

Remark 1.10. Some remarks:

1. $\overline{Z}_\chi$ is lower semi-continuous;
2. $\overline{Z}_\chi$ is the maximal l.s.c. extension of its restriction to D_χ;
3. $\overline{Z}(\chi(p,t)) \le Z(p,t)$ for a.e. $p \in \Omega$ and for all $t \in I$;
4. If $\chi : \Omega \times I \to X$ is an *optimal* pattern, for a.e. $p \in \Omega$ and all $t \in I$ we have (see [7, Theorem 2.8])

$$\overline{Z}(\chi(p,t)) = Z(p,t).$$

We recall the following definition from [9].

Definition 1.11 (Simple patterns)**.** We say that a pattern χ is *simple* if all the fibres which share a common point coincide as functions of the time parameter. In other words, if $\chi(p,t) = \chi(p',t')$, then $t = t'$ and $\chi(p,s) = \chi(p',s)$ for all $s \in [0,t]$. See [9, Definition 6.1].

Remark 1.12. Any optimal pattern χ is simple (see [9]).

If χ is a simple pattern, the function $Z(p,t)$ does not actually depend on (p,t), meaning that if $x = \chi(p,t)$ then Z depends actually on x (and not on the particular couple (p,t) which realizes x). Thanks to this fact, if χ is a simple pattern and $x = \chi(p,t)$, we will write $Z(x)$ instead of $Z(\chi(p,t))$.

Finally, recall that, if the pattern χ is optimal, the function Z is lower semi-continuous, hence $Z = \overline{Z}$ on the domain of χ (and we usually will write Z for $\overline{Z}$).

Note that the same holds for the mass and we will write $m(x)$ instead of $m(\chi(p,t))$ whenever the pattern χ is simple and $x = \chi(p,t)$.

The two main results on the landscape function (see [7, Theorem 6.2 and Corollary 7.3]) are the following ones. Before stating them, we recall the following definitions.

Definition 1.13 (Lower Ahlfors regular measure)**.** A measure μ is *Ahlfors regular from below* in dimension h, if there exists $c_A > 0$ such that

$$\mu(B(x,r)) \geq c_A r^h,$$

for all $r \in [0,1]$ and for all $x \in \operatorname{spt}\mu$.

Definition 1.14 (Upper Ahlfors regular measure)**.** A measure μ is *Ahlfors regular from above* in dimension h, if there exists $C_A > 0$ such that

$$\mu(B(x,r)) \leq C_A r^h.$$

Theorem 1.15 (Hölder continuity of the landscape function)**.** *Let Z be the landscape function associated to the optimal pattern χ. Suppose that the irrigated measure μ_χ is Ahlfors regular from below in dimension h. Let $\beta := 1 + h(\alpha - 1)$. Then, Z is Hölder continuous on $\overline{D}_\chi$, i.e. there exists $c > 0$ such that for all $x, y \in \overline{D}_\chi$*

$$|Z(x) - Z(y)| \leq c|x-y|^\beta.$$

Theorem 1.16 (Upper bound for the Hölder exponent)**.** *Let χ be an optimal pattern. Suppose that the landscape function is Hölder continuous of exponent $\beta \leq \alpha$ and that the irrigated measure μ_χ is Ahlfors regular from above in dimension h'. Then, the following inequality must hold:*

$$h' \leq h = \frac{1-\beta}{1-\alpha},$$

i.e. $\beta \leq 1 + h'(\alpha - 1)$.

2 Branches

2.1 Towards the definition of branch through a point in the domain of the pattern

For the reader's convenience, we recall some notions introduced and studied in [7].

Definition 2.1 (First transit)**.** Given a fibre $p \in \Omega$ and a point x define $t_p(x)$ as

$$t_p(x) := \inf\{t \in I \ : \ \chi(p,t) = x\}.$$

Coherently, $t_p(x) = \sup I$, if the fibre χ_p does not pass through x.

Definition 2.2 (Residual length from a point on a fibre)**.** The *residual length on the fibre χ_p from the point x* is given by

$$l_p(x) := \int_{t_p(x)}^{b} |\dot{\chi}_p(s)| \mathrm{d}s.$$

Definition 2.3 (Residual length from a point)**.** The *residual length from the point x* is the function l defined on the domain of an optimal pattern χ which associates to every x the supremum of the distance along the fibre χ_p from x to the terminal point of the fibre and is given by

$$l(x) := \operatorname{ess\,sup} l_p(x).$$

The essential supremum is taken on $p \in \Omega$ and is equal to the supremum taken among finite cost particles.

From [7, Point 3 in Theorem 4.3 and Theorem 6.2] we obtain a lower estimate on the mass in a point x involving the residual length of the fibre at x. This is the content of the next proposition.

Proposition 2.4. *Suppose that the pattern χ is optimal and the irrigated measure is Ahlfors regular from below in dimension h. Then,*

$$m(x) \geq C_{H,g}^{1/(\alpha-1)} l(x)^h. \tag{2.1}$$

$C_{H,g}$ is the Hölder constant of the landscape function (see Appendix A).

We also have an estimate from above in the case the irrigated measure is Ahlfors regular from above.

Proposition 2.5. *Suppose that the irrigated measure is Ahlfors regular from above in dimension h. Then,*

$$m(x) \leq C_A l(x)^h.$$

Proof. The region irrigated from x must be included in $B_{l(x)}(x)$. The mass irrigated from x must be then less than

$$m(x) \leq \mu(B_{l(x)}(x)) \leq C_A l(x)^h. \qquad \square$$

Let us recall a definition. We refer to [9] and [14].

Definition 2.6 (Flow ordering)**.** Consider a pattern χ. Let $x, y \in \mathbf{R}^N$. We say that x precedes y in the *flow order* if there exists $A \subseteq \Omega$, with $\mu_\Omega(A) > 0$, and $t_x, t_y \in I$ such that $\chi_p(t_x) = x, \chi_p(t_y) = y$ for all $p \in A$ and $t_x \leq t_y$. In this case we write $x \preceq y$. Note that $\preceq$ is a partial ordering.

Corollary 2.7. *Suppose that χ is an optimal pattern and the irrigated measure is Ahlfors regular. Suppose also that $x \preceq y$, and $l(y) \leq l(x) \leq kl(y)$, for some $k > 0$. Then we have:*

$$m(y) \leq m(x) \leq \frac{C}{c} k^h m(y).$$

Here c, C are the constants appearing in Proposition 2.4 and Proposition 2.5 respectively.

Proof. By Proposition 2.4 and 2.5 we have

$$m(x) \leq Cl(x)^h, \quad cl(y)^h \leq m(y).$$

Then,

$$m(x) \leq Cl(x)^h = \frac{C}{c}\frac{l(x)^h}{l(y)^h} cl(y)^h \leq \frac{C}{c} k^h m(y). \qquad \square$$

We now introduce the *branch distance*. Let us recall a definition. We refer again to [9] and [14].

Definition 2.8 (Flow curve)**.** Let χ be a simple pattern. Let $x, y \in \mathbf{R}^N$. If $x \preceq y$ and A, t_x, t_y are as in Definition 2.6, then a curve $\Gamma : J \to \mathbf{R}^N$ (J interval $\subseteq I$) such that for μ_Ω-almost every $p \in A$ coincides in the interval $[t_x, t_y]$ with the curve $\chi_p : [t_x, t_y] \to \mathbf{R}^N$ is called *flow curve*. We will sometimes identify the curve Γ with its image when no ambiguity arises.

We will say that the flow curve is *represented* by the particle p if $p \in A$. Notice that, if χ is a simple pattern, the flow curve between x and y is unique.

If we set $\alpha = \inf J, \beta = \sup J$, then Γ is a flow curve if and only if for every $0 < \varepsilon < \beta - \alpha$, there exists a finite cost fibre p such that $\Gamma_{|[\alpha,\beta-\varepsilon]}$ coincides with $\chi_{p|[\alpha,\beta-\varepsilon]}$.

Given a point x, a *maximal flow curve* starting from x is a flow curve between x and y, where y is a point such that $l(y) = 0$.

Notation (Integral on a flow curve). Suppose χ is a simple pattern. If $x \preceq y$ and $\Gamma : [\alpha, \beta] \to \mathbf{R}^N$ is the unique flow curve from x to y, given a function $g : \Gamma([\alpha, \beta]) \to \mathbf{R}$, we define

$$\int_x^y g := \int_\alpha^\beta g(\Gamma(w))|\dot{\Gamma}(w)|\mathrm{d}w = \int_{\Gamma([\alpha,\beta])} g(w)\mathrm{d}\mathcal{H}^1(w).$$

Definition 2.9 (Branch distance)**.** If χ is a simple pattern, we know that, given $x, y \in \operatorname{spt} \chi$ such that $x \preceq y$ or $x \succeq y$, there is one flow curve between x and y. Let p be a particle representing the flow curve between x, y. Then, the *branch distance* between x and y is defined by:

$$l(x, y) := \int_{\min\{t_p(x),t_p(y)\}}^{\max\{t_p(x),t_p(y)\}} |\dot{\chi}_p(s)| \,\mathrm{d}s = \int_{\min\{x,y\}}^{\max\{x,y\}} g(w)\mathrm{d}\mathcal{H}^1(w),$$

where $t_p(x), t_p(y)$ (see Definition 2.1) satisfy $\chi(p, t_p(x)) = x$ and $\chi(p, t_p(y)) = y$ and $g(w) \equiv 1$.

2.2 Branch through a point in the domain of a pattern

Proposition 2.10. *Suppose χ is optimal. If $x \preceq y$, then*

$$l(x, y) \leq l(x) - l(y). \tag{2.2}$$

Proof. Given $\varepsilon > 0$, let p be a finite cost particle (i.e., $c_\alpha(p) < +\infty$) such that the flow curve between x and y is given by χ_p and $l(y) \leq l_p(y) + \varepsilon$. Since x is on such fibre, we also have that $l(x) \geq l_p(x)$. Then,

$$l(x, y) = l_p(x) - l_p(y) \leq l(x) - l(y) + \varepsilon.$$

Since ε can be chosen arbitrarily we get (2.2). □

We now introduce one of the main concepts of the paper.

Definition 2.11 (Branch)**.** Suppose χ is optimal. A flow curve Γ is a *branch* if for every couple of points x, y in the image of Γ with $x \preceq y$ we have:

$$l(x, y) = l(x) - l(y),$$

that is, equality holds in (2.2).

Given a point x in the domain of the pattern χ, a *maximal branch* starting from x is a maximal flow curve starting in x, which is also a branch. Equivalently, a maximal branch starting in x is a flow curve starting in x whose length is $l(x)$.

Remark 2.12. Since inequality (2.2) is true for a flow curve Γ, it is sufficient to prove the equality choosing as x the minimum element and y as the maximum in the image of the Γ (w.r.t. $\preceq$) if they exist.

Remark 2.13. Γ is a branch if and only if

$$\operatorname{ess\,sup} l_p(x) + \operatorname{ess\,sup} l_p(y) = \operatorname{ess\,sup}\{l_p(x) + l_p(y)\}. \tag{2.3}$$

In fact, if Γ is a branch, let $x \preceq y$ and choose p such that $y \in \chi_p(I)$. We have:

$$l_p(x) + l_p(y) = l_p(x, y) + 2l_p(y).$$

Taking the supremum, we have from equality in (2.2):

$$\begin{aligned}&\operatorname{ess\,sup}\{l_p(x) + l_p(y)\} = \\ &\qquad = \operatorname{ess\,sup}\{l_p(x, y) + 2l_p(y)\} = l(x, y) + 2l(y) = (l(x) - l(y)) + 2l(y).\end{aligned}$$

Vice versa, if (2.3) holds, we have:

$$l(x) + l(y) = \sup\{l_p(x) + l_p(y)\} = \sup\{l_p(x, y) + 2l_p(y)\} = l(x, y) + 2l(y),$$

showing then the equality holds in (2.2) and, therefore, that Γ is a branch.

Remark 2.14. Suppose that Γ_1, Γ_2 are branches such that the maximum element of the image of Γ_1 coincides with the minimum of the image of Γ_2. From Definition 2.11 follows that the union of the images of Γ_1 and Γ_2 is a branch.

Remark 2.15. A maximal flow curve whose minimal element is x is a branch if and only if its length is $l(x)$. In fact, by the maximality of Γ follows that $\inf_{y\in\Gamma} l(y) = 0$. Since we have equality in (2.2), it follows that

$$\sup_{y\in\Gamma} l(x, y) = l(x) - \inf_{y\in\Gamma} l(y) = l(x).$$

Proposition 2.16. *Suppose that χ is an optimal pattern and the irrigated measure is Ahlfors regular from below. The function l is continuous on a branch starting from a given point x. Vice versa, if l is continuous on a flow curve, then it must be a branch.*

Proof. The first part of the proposition is straightforward, since by (2.2) l is actually a 1-Lipschitz function.

Suppose now that l is continuous on a flow line Γ, but Γ is not a branch. Then, inequality (2.2) holds strictly for some $x_1, y \in \Gamma$ with $x_1 \preceq y$. Since l is continuous, we can suppose that $l(y) > 0$ (in fact, if $l(y) = 0$, by continuity of l, we could choose a point $y' \preceq y$ such that inequality (2.2) would still hold strictly). By Definition 2.3 there exists a finite cost fibre p_1 such that

$$l(x_1, y) < l_{p_1}(x_1) - l(y). \tag{2.4}$$

Moreover, we have $x_1 \in \chi_{p_1}$ and for all $x \in \chi_{p_1}$ with $x_1 \preceq x \preceq y$:

$$l(x_1, x) = l_{p_1}(x_1, x) = l_{p_1}(x_1) - l_{p_1}(x). \tag{2.5}$$

From $l(x_1, x) + l(x, y) = l(x_1, y)$, (2.4), and (2.5) we get:

$$l(x, y) < l_{p_1}(x) - l(y). \tag{2.6}$$

So $y \notin \chi_{p_1}(I)$, since inequality (2.6) does not hold for $x = y$. This implies that Γ and $\chi_{p_1}(\cdot)$ bifurcate in a point $x_1' \prec y$. By the continuity of l, inequality (2.6), which still holds for $x = x_1'$, is still true for some x_2 such that $x_1' \prec x_2 \prec y$, i.e. inequality (2.4) holds if x_1 is replaced by x_2.

The same argument can be repeated in the interval $[x_2, y]$. We find p_2 and x_2' analogously to p_1 and x_1'. Repeating it n times we find p_n and x_n' analogously to p_1 and x_1'. Note that by (2.1) in every step the flow line bifurcating in x_n' loses a mass greater than $cl(y)^h$ which is a contradiction. □

The next proposition shows that given a point x there always exists a branch starting from x, if x is in the domain of χ.

Lemma 2.17 (Existence of a branch starting from a point)**.** *Suppose that the irrigated measure μ_χ is Ahlfors regular from below (in dimension h) and χ is an optimal pattern. Then, for every x in the domain of χ and $l_0 < l(x)$, there exists a branch starting from x whose length is l_0.*

Proof. By Definition 2.3 there must be a sequence of particles p_n such that $l_{p_n}(x) \to l(x)$ (note that $l(x) < +\infty$ by inequality (2.1)), $x \preceq x_n$,

$$x_n \in \chi_{p_n}(I), \quad l(x, x_n) = l_0.$$

Let

$$P_n = \{p \ : \ x_n \in \chi_p(I)\}.$$

Since the irrigated measure is Ahlfors regular from below, by Proposition 2.4 we have

$$|P_n| \geq c(l_{p_n}(x) - l_0)^h.$$

Let $P := \limsup_{n\to+\infty} P_n$. We have $|P| \geq c(l(x)-l_0)^h$. Let $q_0 \in P$ with $c(q_0) < +\infty$; for infinitely many n, $q_0 \in P_n$. Since χ is a minimum of the synchronous functional J^1_α, $\chi_{q_0}(t) \equiv \chi_{p_n}(t)$ if $q_0 \in P_n$ and $t \in [0,t_n]$ where $\chi(p_n,t_n) = \chi(q_0,t_n) = x_n$. Then, the curve χ_{q_0} coincides with χ_{p_n} for infinitely many n.

This implies that for infinitely many n that $x_n = \overline{x}$ since the fibres p_n, q_0 coincide up to x_n and $l(x,x_n) = l_0$ does not depend on n. We can then write:

$$l_{p_n}(\overline{x}) = l_{p_n}(x) - l_0.$$

Taking the supremum we have

$$l(\overline{x}) \geq l(x) - l_0 = l(x) - l(x,\overline{x}).$$

By Proposition 2.10 and Remark 2.12 the thesis follows. □

Corollary 2.18 (Existence of a maximal branch starting from a point)**.** *Suppose that the irrigated measure μ_χ is Ahlfors regular from below and χ is an optimal pattern. Then, for every x in the domain of the pattern, there exists a maximal branch starting from x.*

Proof. The maximal branch is built iteratively. First, set $l_0 = l(x)/2$ and consider the branch from x whose length is l_0. Let the final point be x_1. Now consider the branch from x_1 whose length is $l_1 = l_0/2 = l(x)/4$. This construction can be continued iteratively and the maximal branch is obtain glueing together such curves. □

3 Linear by-pass gain formulas and main consequences

The results in this section rely on the following estimates. In order to give clearer proofs note that it is easy to prove by concavity of the power function with exponent $0 < \alpha < 1$ that

$$(x+x_0)^\alpha - x_0^\alpha \leq \alpha x_0^{\alpha-1} x, \quad \forall x_0 > 0, -x_0 \leq x. \tag{3.1}$$

Moreover, it is easy to see that when $x \leq 0$ we have:

$$(x+x_0)^\alpha - x_0^\alpha \leq \alpha x_0^{\alpha-1} x - c_\alpha x_0^{\alpha-2} x^2, \quad \forall x_0 > 0, -x_0 \leq x \leq 0. \tag{3.2}$$

where $c_\alpha = \frac{1}{2}\alpha(1-\alpha)$. Formula (3.2) follows from the fact that by Taylor expansion:

$$(x+x_0)^\alpha = x_0^\alpha + \alpha x_0^{\alpha-1} x + c_\alpha x_0^{\alpha-2} x^2 + \frac{c_\alpha(\alpha-2)}{3}(\xi + x_0)^{\alpha-3} x^3,$$

with a suitable $x \leq \xi \leq 0$ so that the last term is negative.

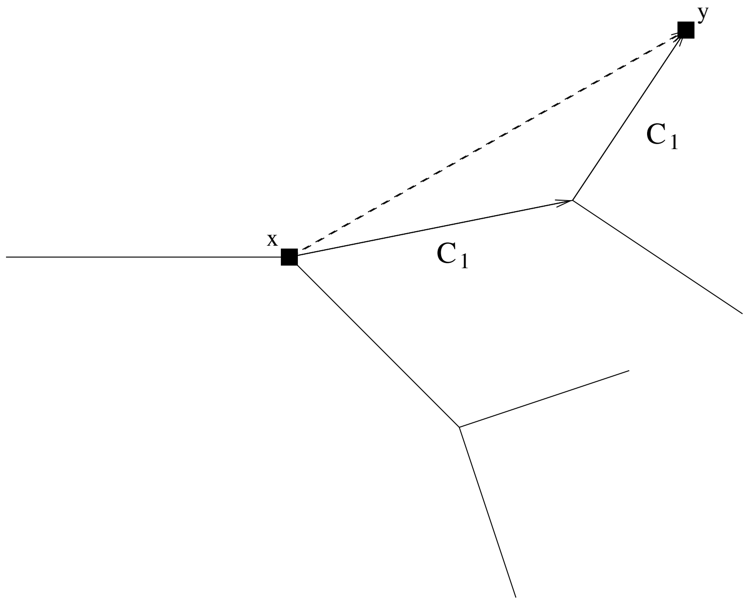

Figure 6: Linear mass by-pass (Definition 3.1).

3.1 Linear by-pass gain formulas

In this section we derive first order gain formulas, that is gain formulas which are consequence of inequality (3.1). We will refer to Figure 6.

Suppose that χ is a simple pattern. Let x and y be points on the same flow line with $x \preceq y$. These two fibres coincide up to a certain bifurcation point. Let C_1 be the flow curve between x and y. First we remove a mass $m \leq m_\chi(x)$ from C_1, and restore it moving a mass given by m through a straight line from x to y. The particles involved in this change of path form a set given by $M \subset \Omega$, such that $\mu_\Omega(M) = m$. A new pattern defined in this way will be named a *linear mass by-pass* along a fibre of the pattern χ and denoted by $\overline{\chi}_{x,y,M}$ (see Figure 6).

Definition 3.1 (Linear mass by-pass)**.** Suppose that χ is a simple pattern. Let $x \preceq y$. Let $x = \chi(p_1, t_1)$ and $y = \chi(p_1, t_2)$ with $t_1 < t_2$. Let $M \subseteq [p_1]_{t_2}$ such that $\mu_\Omega(M) = m$. Define:

$$\overline{\chi}_{x,y,M}(p,t) = \begin{cases} x + \frac{t-t_1}{t_2-t_1}(y-x) & \text{if } p \in M, t_1 \leq t \leq t_2, \\ \chi(p,t) & \text{otherwise} \end{cases}$$

We call this new pattern a *linear mass by-pass* of χ. If there is not ambiguity on x, y, M we will simply write $\overline{\chi}$. When $M = [p_1]_{t_2}$ (the most frequent case) we shall write $\overline{\chi}_{x,y}$.

Remark 3.2. Note that the linear mass by-pass of the previous definition does not change the irrigated measure, i.e. $\mu_\chi = \mu_{\overline{\chi}}$.

Theorem 3.3 (First order gain formula for linear mass by-pass)**.** *Suppose that χ is a simple pattern. Then, the pattern $\overline{\chi}$ satisfies*

$$J_\alpha(\overline{\chi}) - J_\alpha(\chi) \leq m^\alpha |x-y| - \alpha m(Z(y) - Z(x)).$$

Proof. We have:

$$\begin{aligned} J_\alpha(\overline{\chi}) - J_\alpha(\chi) &= \\ &= m^\alpha|x-y| + \int_x^y (m_\chi(w) - m)^\alpha \mathrm{d}\mathcal{H}^1(w) - \int_x^y m_\chi(w)^\alpha \mathrm{d}\mathcal{H}^1(w). \end{aligned}$$

Inequality (3.1) gives:

$$(-m+u)^\alpha - u^\alpha \leq -\alpha m u^{\alpha-1}.$$

Then,

$$J_\alpha(\overline{\chi}) - J_\alpha(\chi) \leq m^\alpha |x-y| - \alpha m(Z(y) - Z(x)).$$ □

Corollary 3.4. *If χ is optimal, then*

$$Z(y) - Z(x) \leq \frac{1}{\alpha} m(y)^{\alpha-1}|x-y|. \tag{3.3}$$

Corollary 3.5. *Suppose that χ is an optimal pattern and $x \preceq y$. Then*

$$l(x,y) \leq \frac{|x-y|}{\alpha}\left[\frac{m(y)}{m(x)}\right]^{\alpha-1}. \tag{3.4}$$

Proof. Indeed, we have that

$$Z(y) - Z(x) = \int_x^y m(w)^{\alpha-1}\mathrm{d}\mathcal{H}^1(w) \geq m(x)^{\alpha-1} l(x,y).$$

The last inequality and inequality (3.3) together end the proof. □

3.2 Comparison of branch distance with the Euclidean one

In this section we shall see how the first order by-pass formula implies the equivalence between the branch distance and the Euclidean one when the pair of points x, y belong to the same branch.

Theorem 3.6 (Equivalence of Euclidean and branch distance)**.** *Suppose that χ is an optimal pattern and the irrigated measure is Ahlfors regular. Then, there exists a constant C_{EB} (only depending on $C_A, h, \alpha, C_{H,g}, C_{H,e}$) such that for all $x, y \in D_\chi$ on the same branch we have $l(x,y) \leq C_{EB}|x-y|$.*

Proof. Without loss of generality we suppose $x \preceq y$.

Suppose first that $l(x) \leq 2l(y)$. Thanks to the Ahlfors regularity of the irrigated measure, Proposition 2.4 and Proposition 2.5 imply

$$\frac{m(y)}{m(x)} \geq \frac{C_{H,g}^{1/(\alpha-1)} l(y)^h}{C_A l(x)^h} \geq \frac{C_{H,g}^{1/(\alpha-1)} 2^{-h} l(x)^h}{C_A l(x)^h} = \frac{C_{H,g}^{1/(\alpha-1)} 2^{-h}}{C_A}.$$

By equation (3.4), we have $l(x,y) \leq C_{EB}|x-y|$ with

$$C_{EB} = \frac{C_{H,g} 2^{h(1-\alpha)}}{\alpha C_A^{\alpha-1}}.$$

Suppose instead that $l(x) \geq 2l(y)$. We have:

$$Z(y) - Z(x) = \int_x^y m(w)^{\alpha-1} \mathrm{d}\mathcal{H}^1(w) \geq m(x)^{\alpha-1} l(x,y). \tag{3.5}$$

Since the measure is Ahlfors regular from above, $m(x) \leq C_A l(x)^h$, that is $m(x)^{\alpha-1} \geq C_A^{\alpha-1} l(x)^{\beta-1}$. Since $l(x) \geq 2l(y)$, we have $2l(x,y) = 2[l(x) - l(y)] \geq l(x)$, so that

$$m(x)^{\alpha-1} \geq C_A^{\alpha-1} l(x)^{\beta-1} \geq C_A^{\alpha-1} 2^{\beta-1} l(x,y)^{\beta-1}.$$

Then, by inequality (3.5) we have:

$$Z(y) - Z(x) \geq C_A^{\alpha-1} 2^{\beta-1} l(x,y)^\beta.$$

Since μ_χ is Ahlfors regular from below, by Theorem 1.15, Z is Hölder continuous with exponent β and from the last estimate we get:

$$C_A^{\alpha-1} 2^{\beta-1} l(x,y)^\beta \leq C_{H,e}|x-y|^\beta,$$

that is

$$l(x,y) \leq C_{EB}|x-y|,$$

with

$$C_{EB} = C_A^{(1-\alpha)/\beta} C_{H,e}^{1/\beta} 2^{(1-\beta)/\beta}.$$

□

Remark 3.7. Since the branch distance is greater than the Euclidean one, we have $C_{EB} \geq 1$.

4 Estimates for the measure of the tubular neighbourhood of a curve

In this section we provide two estimates for the measure of the tubular neighbourhood of a branch. The first estimate can be given for a generic connected set (so, we do not ask that it is a branch), while the second can be given only for branches since it requires the equivalence of the Euclidean distance and the branch one.

4.1 Estimate from above for the measure of the tubular neighbourhood of a connected set

In this section we prove an estimate from above on the measure of the tubular neighbourhood of a connected set. The *tubular neighbourhood* of a set C will be

$$U_\varepsilon(C) := \{x \in \mathbf{R}^N \ : \ \operatorname{dist}(x, C) < \varepsilon\}.$$

Lemma 4.1 (Small ε). *Let μ be a Ahlfors regular measure from above in dimension h. Let C be a connected set of finite length. Then, there exists a constant $K_{(4.1)}$ depending only on C_A and h such that, if $\varepsilon < (\operatorname{diam} C)/2$,*

$$\mu(U_\varepsilon(C)) \le K_{(4.1)} \mathcal{H}^1(C) \varepsilon^{h-1}. \tag{4.1}$$

Precisely, we can choose $K_{(4.1)} = C_A 3^h$.

Proof. Let n be the cardinality of elements of a family of disjoint balls of radius ε centred on C. Let $x_i, 1 \le i \le n$ be the centres of such balls.

The cardinality n can be bounded from above as follows. By [2, Lemma 4.4.5], $\mathcal{H}^1(C \cap B_\varepsilon(x_i)) \ge \varepsilon$ if $\varepsilon < (\operatorname{diam} C)/2$, hence

$$n\varepsilon \le \sum_{i=1}^n \mathcal{H}^1(C \cap B_\varepsilon(x_i)) = \mathcal{H}^1 \left(C \cap \bigcup_{i=1}^n B_\varepsilon(x_i) \right) \le \mathcal{H}^1(C).$$

Hence, $n \le \mathcal{H}^1(C)\varepsilon^{-1}$.

Let us now consider a maximal family (i.e., one which maximizes the cardinality which we will be again denoted by n). It can be easily proved that

$$C \subseteq \bigcup_{i=1}^n B_{2\varepsilon}(x_i);$$

otherwise it would exist a point $\overline{x} \in C$ such that

$$B_\varepsilon(\overline{x}) \cap \bigcup_{i=1}^n B_\varepsilon(x_i) = \emptyset,$$

against the maximality of n. It follows then that

$$U_\varepsilon(C) \subseteq \bigcup_{i=1}^{n} B_{3\varepsilon}(x_i).$$

Then, we have:

$$\mu(U_\varepsilon(C)) \leq \sum_{i=1}^{n} \mu(B_{3\varepsilon}(x_i)) \leq nC_A(3\varepsilon)^h \leq 3^h C_A \mathcal{H}^1(C)\varepsilon^{h-1}.$$

Setting $K_{(4.1)} = C_A 3^h$ completes the proof. □

Lemma 4.2 (Large ε)**.** *Let μ be an Ahlfors regular measure from above in dimension h. Let C be a connected set of finite length. Then, there exists a constant $K_{(4.2)}$ depending only on C_A and h such that if $\varepsilon \geq (\operatorname{diam} C)/2$*

$$\mu(U_\varepsilon(C)) \leq K_{(4.2)}\varepsilon^h. \tag{4.2}$$

Precisely, we can choose $K_{(4.2)} = C_A 2^h$.

Proof. Take $x_1 \in C$. We have:

$$U_\varepsilon(C) \subseteq B_{2\varepsilon}(x_1).$$

Then, we have:

$$\mu(U_\varepsilon(C)) \leq \mu(B_{2\varepsilon}(x_1)) \leq C_A(2\varepsilon)^h \leq C_A 2^h \varepsilon^h,$$

since $\varepsilon \geq (\operatorname{diam} C)/2$. Setting $K_{(4.2)} = C_A 2^h$ completes the proof. □

As a consequence of Lemma 4.1 and Lemma 4.2 we have the following corollary.

Corollary 4.3. *Let μ be an Ahlfors regular measure from above in dimension h. Let C be a connected set of finite length. Then, there exists a constant $\widetilde{K}$ depending only on C_A and h such that*

$$\mu(U_\varepsilon(C)) \leq \widetilde{K}(\mathcal{H}^1(C)\varepsilon^{h-1} + \varepsilon^h). \tag{4.3}$$

Precisely, we can choose $\widetilde{K} = C_A 3^h$.

Corollary 4.4. *Let μ be an Ahlfors regular measure from above in dimension h. Let C be a connected set of finite length and set $l = \mathcal{H}^1(C)$. Then, given $m > 0$, there exists a constant $\widehat{K}$ depending only on C_A, h, m such that, if $\varepsilon \leq ml$, we have*

$$\mu(U_\varepsilon(C)) \leq \widehat{K}\mathcal{H}^1(C)\varepsilon^{h-1}.$$

Precisely, we can choose $\widehat{K} = \widetilde{K}(1 + m)$.

Proof. Suppose $\varepsilon \leq ml$. Then, $\varepsilon^h = \varepsilon^{h-1}\varepsilon \leq \varepsilon^{h-1}ml$. Plugging the last estimate in (4.3), we achieve the proof. □

4.2 Estimate from below on the measure of tubular neighbourhood of a branch

In this section we will prove an estimate from below on the measure of the tubular neighbourhood of the image of a Lipschitz curve. We will consider a curve $\gamma \in \mathrm{AC}([a,b],\mathbf{R}^N)$ (see Definition 2.11). We will denote by Γ the image of the γ, i.e. $\gamma([a,b])$, and by μ an Ahlfors regular measure from below.

Lemma 4.5. *Let χ be an optimal pattern and let μ_χ be an Ahlfors regular measure from below. Let $\gamma \in \mathrm{AC}([a,b],\mathbf{R}^N)$ and set $\Gamma = \gamma([a,b])$. Suppose that γ is a branch and that $\Gamma \subseteq \operatorname{spt}\mu$. Then, there exists a constant $K_{(4.4)}$ depending only on c_A and h such that, if $\varepsilon \leq \mathcal{H}^1(\Gamma)$,*

$$\mu_\chi(U_\varepsilon(\Gamma)) \geq K_{(4.4)}\mathcal{H}^1(\Gamma)\varepsilon^{h-1}. \tag{4.4}$$

Precisely, we can choose $K_{(4.4)} = c_A 2^{-h} C_{EB}^{-h}$.

Proof. Let $l = \mathcal{H}^1(\Gamma)$. Choose the point t_k such that $a = t_0, b = t_n$ and $t_k \leq t_{k+1}$, $l(t_k, t_{k-1}) = \varepsilon$ for $0 \leq k \leq n-1$ and $l(t_{n-1}, t_n) \leq \varepsilon$. The number $n+1$ of such points is $[l/\varepsilon]+1$, so that is greater than l/ε. Since the curve γ is a branch,

$$l(\gamma(t_k),\gamma(t_{k+1})) \leq C_{EB}|\gamma(t_k) - \gamma(t_{k+1})|. \tag{4.5}$$

The estimate now follows from considering the balls of radius $\varepsilon/2C_{EB}$ centred at the point $\gamma(t_k)$. By inequality (4.5) such balls are disjoint and

$$\bigcup_k B_{\varepsilon/2C_{EB}}(\gamma(t_k)) \subseteq U_\varepsilon(\Gamma).$$

Then, we have:

$$\mu_\chi(U_\varepsilon(\Gamma)) \geq \mu_\chi\left(\bigcup_{k=0}^n B_{\varepsilon/2C_{EB}}(\gamma(t_k))\right) = \\ = \sum_{k=0}^n \mu_\chi\left(B_{\varepsilon/2C_{EB}}(\gamma(t_k))\right) \geq c_A\left(\frac{\varepsilon}{2C_{EB}}\right)^h \frac{l}{\varepsilon}.$$

Choosing $K_{(4.4)} = c_A 2^{-h} C_{EB}^{-h}$, the statement is proved. □

4.3 Estimate from above of the number of short branches

Lemma 4.6 (Number of branches)**.** *Let χ be an optimal pattern. Suppose that the irrigated measure is Ahlfors regular in dimension h. Consider a*

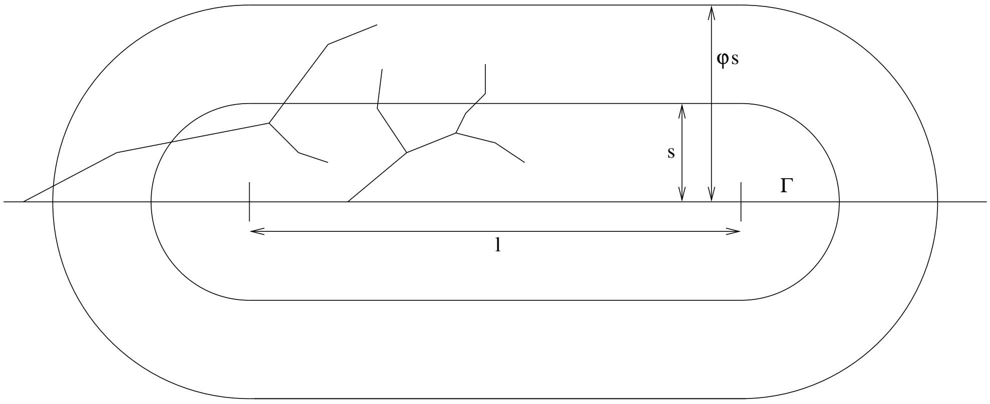

Figure 7: Number of branches (Lemma 4.6).

branch Γ *of length* l. *Suppose that* s *is given. Let also* m *be any number such that* $\varphi s \leq ml$. *Let* $N(s, \varphi)$ *be the number of sub-branches of residual length between* s *and* φs *contained in the tubular neighbourhood* $U_{\varphi s}(\Gamma)$ *(see Figure 7). We then have:*

$$N(s, \varphi) \leq \frac{\widehat{K}\varphi^{h-1}}{C_{H,g}^{1/(\alpha-1)}} \frac{l}{s}. \tag{4.6}$$

Proof. The mass carried by $U_{\varphi s}(\Gamma)$ is estimated from above by the upper Ahlfors regularity thanks to Corollary 4.4:

$$\mu_\chi(U_{\varphi s}(\Gamma)) \leq \widehat{K} l(\varphi s)^{h-1}.$$

Since the irrigated measure is lower Ahlfors regular, by Proposition 2.4 the mass carried by a branch of residual length at least s is at least $C_{H,g}^{1/(\alpha-1)} s^h$. The total mass carried by such branches must then be at most the mass of the tubular neighbourhood $U_{\varphi s}(\Gamma)$. We must then have:

$$N(s, \varphi) C_{H,g}^{1/(\alpha-1)} s^h \leq \widehat{K}\varphi^{h-1} l s^{h-1},$$

that is equation (4.6). □

5 Second order gain formulas

The gain formulas developed in this section depend not only on inequality (3.1), but also on inequality (3.2).

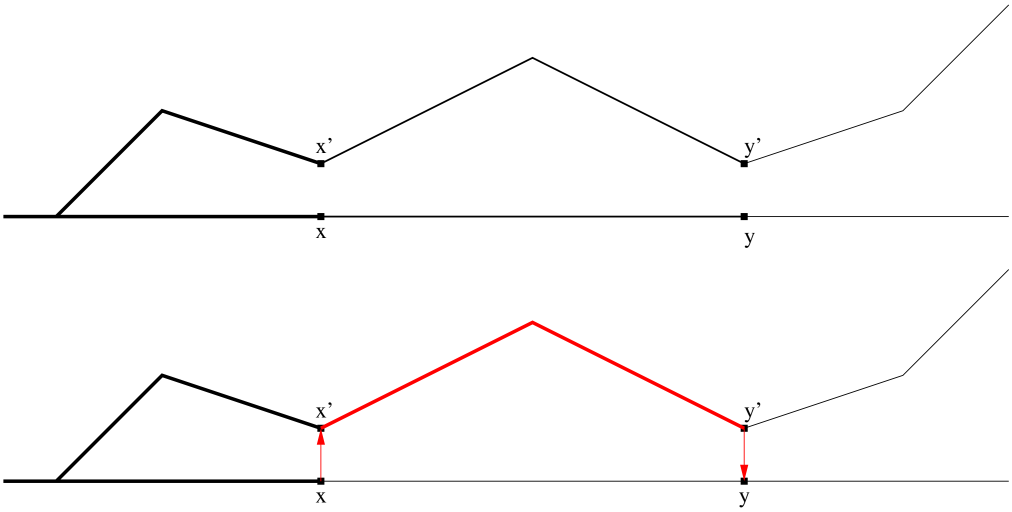

Figure 8: Double by-pass (Definition 5.1).

5.1 Double by-pass

In view of possible future aims the following formula is going to be established at a higher level of generality than what we need in this paper. In the next section we shall discuss the particular case in which we are really interested. Suppose that y, y' are irrigated by different flow curves as in Figure 8. Suppose that an amount $0 < m \leq m(y)$ of the mass flowing in y is deviated on x' through a new flow curve $\overrightarrow{xx'}$. Then, the mass flows up to y' on the flow curve $\overrightarrow{x'y'}$ and finally restored on y on a further new flow curve $\overrightarrow{y'y}$.

Definition 5.1 (Double by-pass)**.** Suppose that χ is a simple pattern. Suppose that x, y are on the same branch and $x \preceq y$, that x', y' are on another branch and $x' \preceq y'$. Let $m \leq m(y)$. Referring to Figure 8, consider the new pattern defined as follows:

- a mass equal to m is moved on a new branch from x to x';
- on the branch $\overrightarrow{x'y'}$, the mass is given by $m + m(\cdot)$; on the branch $\overrightarrow{xy}$, the mass is given by $-m + m(\cdot)$;
- the irrigated measure is restored through a branch from y' to y.

Any new pattern obtained in this way will be called *double by-pass* and denoted by $\tilde{\chi}_{x,x',y,y',m}$. If it is not ambiguous, we will simple write $\tilde{\chi}$.

Theorem 5.2. *If χ is simple and $\tilde{\chi}$ is a double by-pass (Definition 5.1), we have:*

$$J_\alpha(\tilde{\chi}) - J_\alpha(\chi) \leq \alpha m(Z(y') - Z(x')) - \alpha m(Z(y) - Z(x)) - \\ - c_\alpha m^2 \int_x^y m(w)^{\alpha-2} \mathrm{d}\mathcal{H}^1(w) + m^\alpha(|x - x'| + |y - y'|).$$

Proof. We have:

$$J_\alpha(\tilde{\chi}) - J_\alpha(\chi) = m^\alpha(|y - y'| + |x - x'|) + \\ + \int_x^y [(-m + m(w))^\alpha - m(w)^\alpha] \mathrm{d}\mathcal{H}^1(w) + \\ + \int_{x'}^{y'} [(m + m(w))^\alpha - m(w)^\alpha] \mathrm{d}\mathcal{H}^1(w).$$

By formula (3.1):

$$\int_{x'}^{y'} [(m + m(w))^\alpha - m(w)^\alpha] \mathrm{d}\mathcal{H}^1(w) \leq \\ \leq \int_{x'}^{y'} \alpha m m(w)^{\alpha-1} \mathrm{d}\mathcal{H}^1(w) = \alpha m(Z(y') - Z(x')).$$

By formula (3.2):

$$\int_x^y [(-m + m(w))^\alpha - m(w)^\alpha] \mathrm{d}\mathcal{H}^1(w) \leq \\ \leq \int_x^y \alpha(-m) m(w)^{\alpha-1} - c_\alpha m^2 m(w)^{\alpha-2} \mathrm{d}\mathcal{H}^1(w) \leq \\ \leq -\alpha m(Z(y) - Z(x)) - c_\alpha m^2 \int_x^y m(w)^{\alpha-2} \mathrm{d}\mathcal{H}^1(w).$$

The proof is now completed. □

5.2 Single by-pass

The previous statements are particularly relevant in the case $x = x'$ that we consider in this section.

Definition 5.3 (Single by-pass)**.** We shall call *single by-pass* the pattern $\tilde{\chi}$ introduced in Definition 5.1 when $x = x'$.

Theorem 5.2 can be restated the following form.

Corollary 5.4 (Second order gain formula)**.** *If χ is simple and $\tilde{\chi}$ is a single by-pass (Definition 5.3), we have:*

$$J_\alpha(\tilde{\chi}) - J_\alpha(\chi) \leq \alpha m(Z(y') - Z(y)) + \\ + m^\alpha |y - y'| - c_\alpha m^2 \int_x^y m_\chi(w)^{\alpha-2} \mathrm{d}\mathcal{H}^1(w). \quad (5.1)$$

Corollary 5.5 (Second order gain formula)**.** *If χ is an optimal pattern and $\tilde{\chi}$ is a single by-pass (Definition 5.3), we have:*

$$\Delta Z(y, y') = Z(y') - Z(y) \geq -\frac{1}{\alpha} m^{\alpha-1} |y - y'| + \frac{c_\alpha}{\alpha} m \int_x^y m_\chi(w)^{\alpha-2} \mathrm{d}\mathcal{H}^1(w). \quad (5.2)$$

Remark 5.6. The last term in the right-hand side in inequalities (5.1) and (5.2) can be dropped, leading to first order gain formulas.

A further estimate brings a refinement of Corollary 5.4 and 5.5 we have the following corollary.

Corollary 5.7 (Second order gain formula)**.** *If χ is simple and $\tilde{\chi}$ is a single by-pass (Definition 5.3), we have:*

$$J_\alpha(\tilde{\chi}) - J_\alpha(\chi) \leq \alpha m(Z(y') - Z(y)) + m^\alpha |y - y'| - c_\alpha m^2 \mathcal{H}^1(\overrightarrow{xy}),$$

where $\overrightarrow{xy}$ is the unique flow curve between x and y.

Proof. Just apply Theorem 5.4 recalling that $m(w)^{\alpha-2} \geq 1$. □

Corollary 5.8 (Second order gain formula)**.** *If χ is an optimal pattern and $\tilde{\chi}$ is a single by-pass (Definition 5.3), we have:*

$$\Delta Z(y, y') = Z(y') - Z(y) \geq -\frac{1}{\alpha} m^{\alpha-1} |y - y'| + \frac{c_\alpha}{\alpha} m \mathcal{H}^1(\overrightarrow{xy}),$$

where $\overrightarrow{xy}$ is the unique flow curve between x and y.

5.3 Estimates for ε-cycles and ε-loops

5.3.1 ε-cycles

For the following definitions we refer to Figure 9.

Definition 5.9 (ε-cycle)**.** Let χ be a simple pattern and $y, y' \in D(\chi)$ non comparable for the flow order, i.e. such that $y' \npreceq y$ and $y \npreceq y'$. We say that the couple (y, y') form an *ε-cycle* if there exists $z \in D(\chi)$ such that $z \preceq y, z \npreceq y', l(z) \leq 2l(y)$ and $|y - y'| < \varepsilon l(z, y)$.

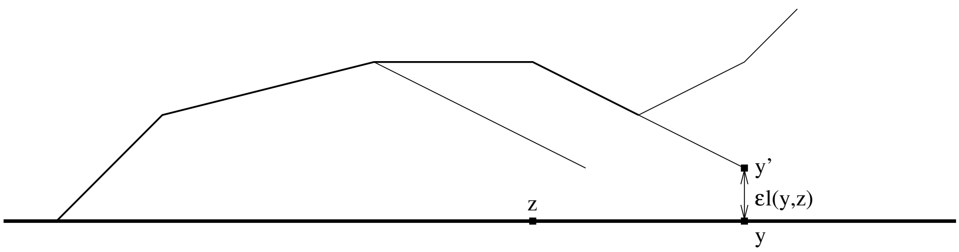

Figure 9: ε-cycle (Definition 5.9).

Definition 5.10 (Double ε-cycle)**.** Let χ be a simple pattern and $y, y' \in D(\chi)$ non comparable for the flow order, i.e. such that $y' \not\preceq y$ and $y \not\preceq y'$. We say that y, y' form a *double ε-cycle* if both (y, y') and (y', y) form an ε-cycle.

Lemma 5.11 (ΔZ lower bound for ε-cycles)**.** *Suppose that χ is optimal and the irrigated measure is Ahlfors regular. Then, there exists $\varepsilon_0(h, c_A, C_A, \alpha) > 0$ such that if $0 < \varepsilon \le \varepsilon_0$ and y, y' form an ε-cycle, we have for some $C'(h, c_A, C_A, \alpha) > 0$*

$$\Delta Z(y, y') = Z(y') - Z(y) \ge C' l(z, y) m(y)^{\alpha-1}. \tag{5.3}$$

In particular, we have

$$\Delta Z(y, y') = Z(y') - Z(y) > 0. \tag{5.4}$$

Proof. Note that from $l(z) \le 2l(y)$ and by inequality (2.2), we have $l(z, y) \le l(y)$. Set $l := l(z, y)$. By Corollary 2.7 we have

$$m(y) \le m(z) \le C l(z)^h \le C(2l(y))^h \le \frac{C}{c} 2^h m(y).$$

Applying Corollary 5.5, we get:

$$\begin{aligned}\Delta Z(y, y') &\ge -\frac{1}{\alpha} m(y)^{\alpha-1} \varepsilon l + \frac{c_\alpha}{\alpha} m(y) \int_z^y m(w)^{\alpha-2} \mathrm{d}\mathcal{H}^1(w) \ge \\ &\ge -\frac{1}{\alpha} m(y)^{\alpha-1} \varepsilon l + \frac{c_\alpha}{\alpha} m(y) m(z)^{\alpha-2} l(z, y) \ge \\ &\ge -\frac{1}{\alpha} m(y)^{\alpha-1} \varepsilon l + \frac{c_\alpha}{\alpha} m(y) [2^h \frac{C}{c} m(y)]^{\alpha-2} l.\end{aligned}$$

From the last formula, it follows that

$$Z(y') - Z(y) \ge \frac{l}{\alpha} m(y)^{\alpha-1} \left(c_\alpha \left(\frac{C2^h}{c} \right)^{\alpha-2} - \varepsilon \right).$$

It is now sufficient to choose

$$\varepsilon_0 = \frac{1}{2} c_\alpha (c^{-1} C 2^h)^{\alpha-2}, \quad C' = \frac{1}{4}(1-\alpha)(c^{-1} C 2^h)^{\alpha-2}$$

to complete the proof. □

In the hypotheses of Lemma 5.11 we directly deduce from inequality (5.4) the following corollary.

Corollary 5.12 (There are no double ε-cycle for small ε)**.** *Suppose that χ is an optimal pattern and the irrigated measure is Ahlfors regular. Then there exists $\varepsilon_0(h, c_A, C_A, \alpha) > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then y, y' do not form a double ε-cycle.*

The next to lemmas are further estimates for ε-cycles that are not required for the main argument of this paper, but we add for completeness in view of possible future applications.

Lemma 5.13 (Mass upper bound for ε-cycles)**.** *Suppose that χ is an optimal pattern and y, y' form an ε-cycle. Then, we have:*

$$m(y')^{1-\alpha} \leq \frac{\varepsilon l}{\alpha[Z(y') - Z(y)]}. \tag{5.5}$$

Proof. Moving an amount of mass $m(y')$ from y' to y and applying Remark 5.6, we get

$$Z(y) - Z(y') \geq -\frac{1}{\alpha} m(y')^{\alpha-1} \varepsilon l.$$

Equation (5.5) then follows. □

Lemma 5.14 (Mass ratio bound for ε-cycles)**.** *In the hypothesis of Lemma 5.11 (or at least if inequality* (5.3) *holds) and Lemma 5.13, we have:*

$$\left(\frac{m(y')}{m(y)}\right)^{1-\alpha} \leq \frac{\varepsilon}{C'}. \tag{5.6}$$

Proof. Inequality (5.3) can be rewritten as

$$\frac{1}{m(y)^{1-\alpha}} \leq \frac{\alpha[Z(y') - Z(y)]}{C' l}.$$

Multiplying term by term the last inequality with (5.5) we obtain (5.6). □

The following two examples and remark will be very useful in the rest of the paper.

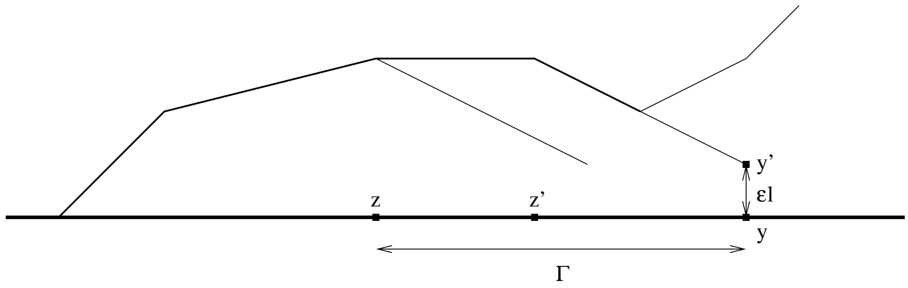

Figure 10: ε-cycle (Example 5.15).

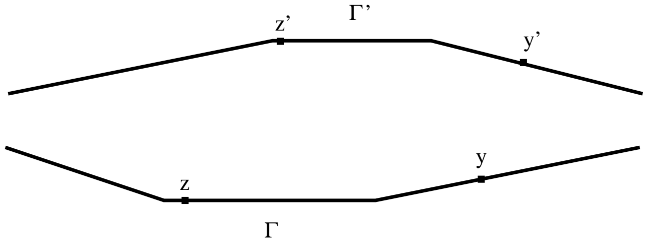

Figure 11: ε-cycle (Example 5.16).

Example 5.15. Consider a branch Γ starting from a point z and two points y, y' such that $y \in \Gamma$, $y' \not\succeq z, z \not\preceq y'$. Suppose also that $|y - y'| < \varepsilon l$. In this case we have an ε-cycle if $l = \min\{l(z, y), l(y)\}$ (see Figure 10).

Example 5.16. Consider two branches Γ, Γ' from the points z, z' respectively such that $z \not\preceq z', z' \not\preceq z$. Let $z \preceq y, z' \preceq y'$. In this case we have a double ε-cycle if $|y - y'| < \varepsilon l$, where $l = \min\{l(z, y), l(y), l(z', y'), l(y')\}$ (see Figure 11).

Remark 5.17. Note that by Corollary 5.12, if we can produce the situation depicted in Example 5.16, we get in contradiction if $0 < \varepsilon < \varepsilon_0$.

5.3.2 ε-loops

In this section we introduce an analogous notion to the ε-cycle which also leads (trivially in this case) to inequality (5.4). In this case, instead of having two flow lines which almost touch, we have a single flow line which almost touch itself producing a loop.

Definition 5.18 (ε-loop)**.** Let χ simple be a pattern. Two points $y, y' \in D_\chi$ form an *ε-loop* if $y \preceq y'$ and $|y - y'| \leq \varepsilon l$, where $l = l(y, y')$. See Figure 12.

The following remark is the analogous of Lemma 5.11, which in this case is straightforward. We also have for the ε-loops analogous estimates to those

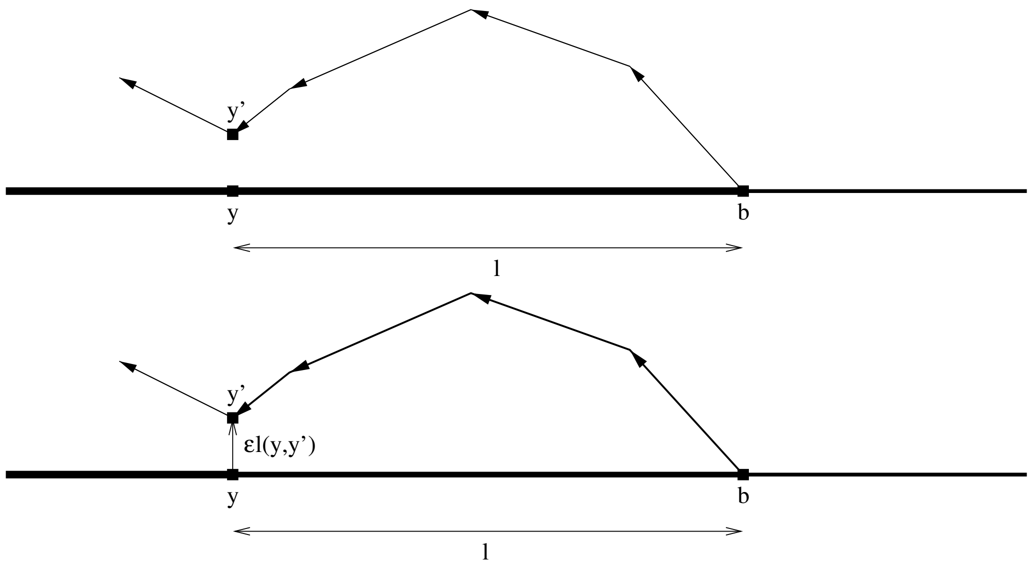

Figure 12: ε-loop (Definition 5.18).

presented for ε-cycles (Lemma 5.13 and Lemma 5.14). We enclose them with the same comments.

Remark 5.19. Since $y \preceq y'$ inequality (5.3) and inequality (5.4) obviously hold.

Lemma 5.20 (Mass upper bound for ε-loops)**.** *Let χ be an optimal pattern irrigating an Ahlfors regular measure. Suppose that y, y' are in ε-loop. Then, we have:*

$$m(y)^{1-\alpha} \leq \frac{l\varepsilon}{\alpha(Z(y') - Z(y))}.$$

Proof. We consider the linear by-pass $\overline{\chi} = \overline{\chi}_{y,y'}$ (introduced in Definition 3.1). Since χ is optimal, by Corollary 3.4, we have:

$$Z(y') - Z(y) \leq \frac{1}{\alpha} l\varepsilon m(y)^{\alpha-1},$$

from which we deduce

$$m(y)^{1-\alpha} \leq \frac{l\varepsilon}{\alpha(Z(y') - Z(y))},$$

proving the lemma. □

Lemma 5.21 (Mass ratio bounds for ε-loops)**.** *Let χ be an optimal pattern. Suppose that y, y' are in ε-loop. Then, we have:*

$$\left(\frac{m(y')}{m(y)}\right)^{1-\alpha} \leq \frac{\varepsilon}{\alpha}.$$

Proof. Just apply formula (3.4) of Corollary 3.5. □

5.3.3 Estimate from above of the number of long branches

Lemma 5.22 (Number of long branches)**.** *Let χ be an optimal pattern and suppose that the irrigated measure is Ahlfors regular. Consider a branch Γ of length l. Then, the number of the branches starting from $U_l(\Gamma)$ whose length is at least l is bounded by a constant only depending on the dimension N, α, h, c_A, C_A.*

Proof. Let $\Lambda(\varepsilon)$ be the least number of points of an ε-net of the unit sphere of $\mathbf{R}^N$. Set $R_1 = \frac{3}{2}l, R_2 = \frac{5}{2}l, R_3 = \frac{7}{2}l$ and denote by B_1, B_2, B_3 the balls of radius R_1, R_2, R_3, respectively, centred in the middle point of Γ. See Figure 13.

We divide the branches we are interested in into two sets: branches that remain in the ball B_3 and the other ones.

The number of the branches of the first set can be estimated using an argument of the same kind of the one used to prove Lemma 4.6. In fact, on one side each branch carries a mass given by cl^h by Proposition 2.4, while on the other the total mass carried cannot exceed the mass of the ball. This mass by the Ahlfors regularity is at most $C_A R_3^h$. Hence the number N_{in} of such branches must satisfy

$$cl^h N_{\text{in}} \leq C_A R^h = C_A \left(\frac{7}{2}l\right)^h,$$

that is $N_{\text{in}} \leq c^{-1} C_A \left(\frac{7}{2}\right)^h$ does not depend on l.

We now consider the other branches. Suppose that the number of such branches exceeds $\Lambda(\frac{1}{5}\varepsilon)$. By a scaling argument one sees that this is the number of a $\frac{1}{2}l\varepsilon$-net on the sphere ∂B_2. Then, there must be at least two of such branches intersecting the boundary of ball B_2 at two y, y' points whose mutual distance is at most $l\varepsilon$. Since the two branches go outside the ball B_3 and start within B_1, y, y' form a double ε-cycle, as in the Example 5.16. We get a contradiction for $\varepsilon < \varepsilon_0$ (see Remark 5.17), so the number of such branches must not exceed $\Lambda(\frac{1}{5}\varepsilon_0)$. □

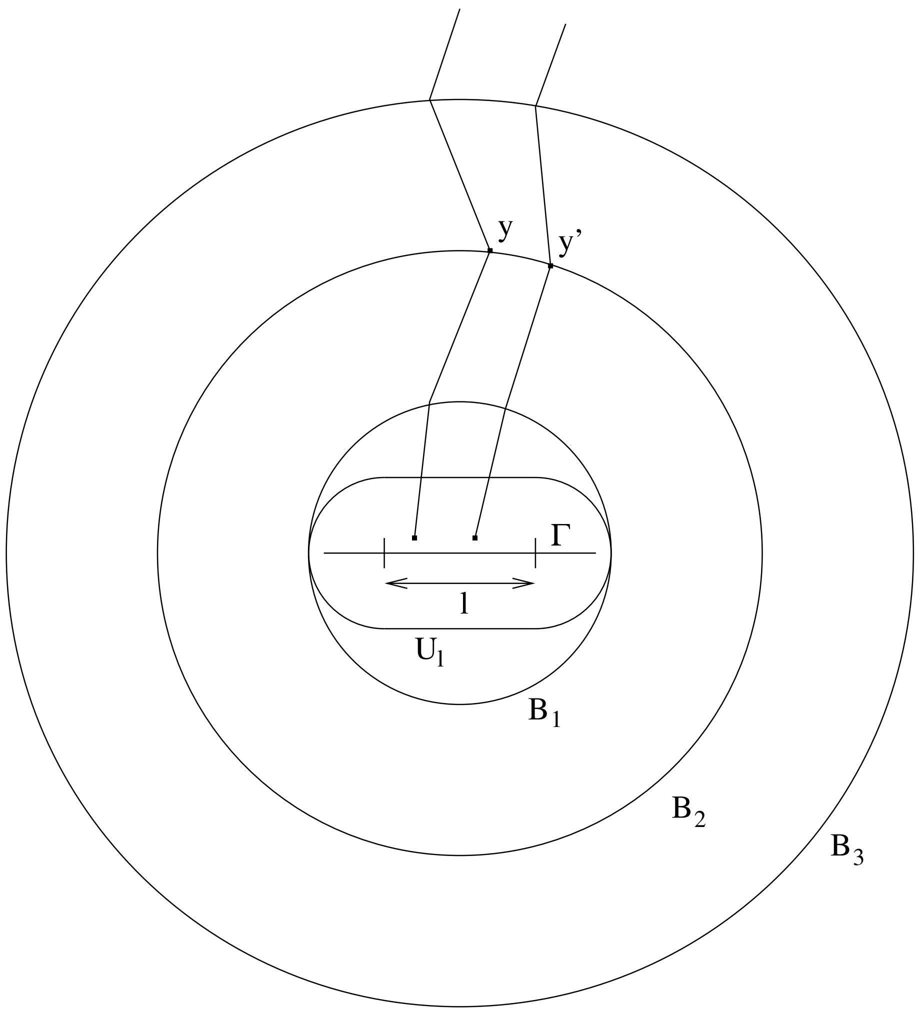

Figure 13: Number of long branches (Lemma 5.22).

6 The regularity result

In this section we will develop the main part of the argument leading to the main result (Theorem 6.17) following the strategy depicted in Section 1.2. At the end of this section we will prove that for a suitable universal constant W the number of branches with length between ε and $W\varepsilon$ bifurcating from a part of a given branch of given length l is bounded from above and from below by some constants times l/ε. The two estimates are obtained via mass balance arguments.

In the following we will suppose that the irrigated measure will be Ahlfors regular in dimension $h > 1$.

6.1 Unwanted branches

6.1.1 Zoom Lemma

In this section we state and prove the *Zoom Lemma* (Lemma 6.3), which is a key tool in the argument leading to the main result (Theorem 6.17). Thanks to this lemma the estimate of Theorem 6.17 are shown, without loss of generality, without considering too long branches, i.e. the estimate is valid even considering a restricted set of branches (this makes the result stronger, of course).

Consider a branch Γ of length l. Given a constant c, one must possibly find branches bifurcating from it of length greater than cl. In the following lemma it is shown that we can select a suitable branch sub-part of length $l' \leq l$ in order to have all the branches bifurcating from it with a length bounded from above by cl'. The scale transition l'/l is bounded from below by a constant depending only on α, N, μ, c.

Definition 6.1 (Good branch)**.** Let Γ be a given branch. We say that Γ is *good* if for all $x \in \Gamma$ and all $y \in D_\chi$ such that if $b = \inf\{z \in \Gamma \ : z \preceq x, z \not\preceq y\}$ and $|x - y| \leq \frac{1}{2}|x - b|$, then $Z(x) - Z(b) \leq \frac{1}{2}(Z(y) - Z(b))$.

Definition 6.2 (Residual length and residual mass after a bifurcation)**.** Let Γ be a flow curve and let b be a bifurcation point on Γ. Let y be a point on the bifurcating line. We refer to Figure 14. The *residual length* after the given bifurcation in b is defined by:

$$l^+(b) := \sup\{l(z) \ : \ b \prec z \preceq y\}.$$

The *residual mass* after the given bifurcation in b is defined by:

$$m^+(b) = \sup\{m(z) \ : \ b \prec z \preceq y\}.$$

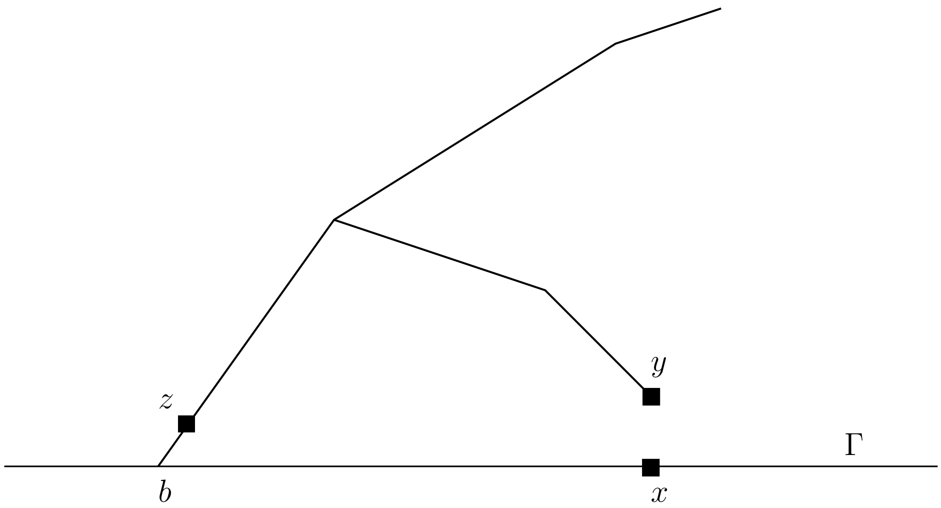

Figure 14: Residual length and residual mass after bifurcation (Definition 6.2).

Lemma 6.3 (Zoom lemma)**.** *Let χ be an optimal pattern and suppose the irrigated measure is Ahlfors regular in dimension $h > 1$. There exists $c_1, c_2 >$ 0 (only depending on α, N, c_A, C_A) such that, if Γ is a branch of length l, then, there exists a good sub-branch (in the sense of Definition 6.1) $\Gamma' \subseteq \Gamma$ whose length is $l' \geq c_1 l$ with no bifurcation of length greater than $c_2 l'$ starting from it.*

Proof. We divide the proof in some steps.

- *First step.* Let $l_1 = l/2$, divide Γ in two parts Γ_1, Γ_2 and let Γ_1 the one closer to the source. We have $l(x) \geq l_1$ for $x \in \Gamma_1$ and, by Proposition 2.4, the $m(x) \geq C_{H,g}^{1/(\alpha-1)} l_1^h = 2^{-h} C_{H,g}^{1/(\alpha-1)} l^h$ for $x \in \Gamma_1$.

- *Second step.* Fix any $c_0 > 0$. By Lemma 5.22 the number of branches bifurcating from Γ_1 of length greater than $c_0 l$ is bounded from above by some constant $n = n(\alpha, N, c_A, C_A, c_0)$.

- *Third step.* Divide Γ_1 in $n + 1$ equal sub-parts of length $l' = l_1/(n + 1)$. Since the number of branches bifurcating from this part of length greater than $c_0 l$ is bounded from above by n, there must be at least one of these sub-parts from which no bifurcation longer than $c_0 l$ starts. Choose such a part as Γ'. So we obtain the desired estimates with $c_1 = \frac{1}{2(n+1)}, c_2 = 2c_0(n + 1)$.

- *Fourth step.* We now prove that Γ' is good when c_0 is suitably chosen. We want to show that, if x, y, b are as in Definition 6.1, then $Z(x) - Z(b) \leq \frac{1}{2}(Z(y) - Z(b))$. We have:

$$Z(x) - Z(b) \leq m(x)^{\alpha-1} l(b,x), \quad m^+(b)^{\alpha-1} l(b,y) \leq Z(y) - Z(b).$$

If we show that $m(x)^{\alpha-1} l(b,x) \leq \frac{1}{2} m^+(b)^{\alpha-1} l(b,y)$, we are done. This condition can be rewritten as:

$$\frac{l(b,x)}{l(b,y)} \leq \frac{1}{2} \left[\frac{m(x)}{m^+(b)}\right]^{1-\alpha}. \tag{6.1}$$

By Proposition 2.4 and Proposition 2.5, since $l(x) \geq l/2$, we have

$$m(x) \geq C_{H,g}^{1/(\alpha-1)} (l/2)^h,$$

while $l^+(b) \leq c_0 l$ gives $m^+(b) \leq C_A l^+(b)^h \leq C_A c_0^h l^h$. Hence

$$\left[\frac{m(x)}{m^+(b)}\right]^{1-\alpha} \geq \left[\frac{C_{H,g}^{1/(\alpha-1)}}{C_A c_0^h}\right]^{1-\alpha}.$$

On the other side,

$$\frac{l(b,x)}{l(b,y)} \leq \frac{C_{EB}|b-x|}{|b-y|} \leq 2C_{EB},$$

since $|b-x| \leq |b-y| + |y-x| \leq |b-y| + \varepsilon \leq \frac{1}{2}|b-x|$. It is clear that choosing $c_0 > 0$ sufficiently small we get

$$2C_{EB} \leq \left[\frac{C_{H,g}^{1/(\alpha-1)}}{C_A c_0^h}\right]^{1-\alpha}.$$

Then inequality (6.1) holds true provided c_0 is chosen sufficiently small.

The proof is then complete. □

The main use of the Zoom Lemma will consist (approximatively) in knowing that we can replace a given branch with a sub-branch at some scale, which is good branch (see Definition 6.1) and has no branching at large scale. In other words we can assume without restriction to the aim of proving the main result of the paper that any given branch satisfies such properties.

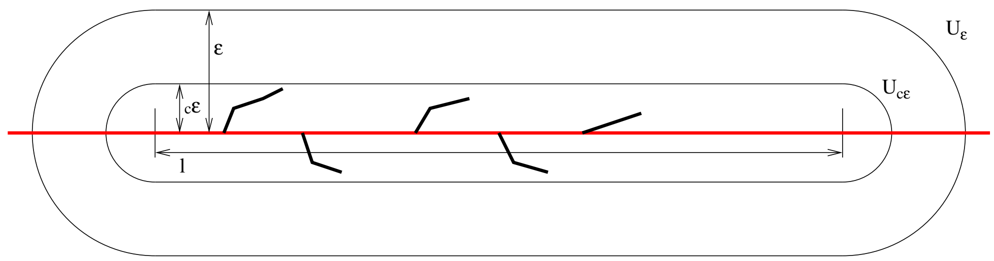

Figure 15: Short branches (Lemma 6.4).

6.1.2 Short branches

In this section we will prove that for a suitable constant c the amount of mass irrigated by the branches whose residual length is less that $c\varepsilon$ can be smaller than a given fraction of the mass carried by the irrigated measure in the tubular neighbourhood of radius ε (see Figure 15).

Lemma 6.4. *Suppose χ is an optimal pattern and the irrigated measure is Ahlfors regular. Consider a branch Γ of length l and $\varepsilon \leq l$. Consider the branches bifurcating from it of residual length less or equal than $c\varepsilon$. Then, we can choose a suitable constant $c < 1$ such that the measure irrigated by such branches is less than a given fraction $0 < \lambda < 1$ of the measure of the tubular neighbourhood $U_\varepsilon(\Gamma)$.*

Proof. The branches of residual length less or equal to $c\varepsilon$ irrigate at the tubular neighbourhood $U_{c\varepsilon}(\Gamma)$. By Lemma 4.1

$$\mu(U_{c\varepsilon}(\Gamma)) \leq K_{(4.1)} l (c\varepsilon)^{h-1}.$$

On the other side, by Lemma 4.5, the mass of tubular neighbourhood U_ε is bounded from below by

$$\mu(U_\varepsilon(\Gamma)) \geq K_{(4.4)} l \varepsilon^{h-1}.$$

Hence,

$$\frac{\mu(U_{c\varepsilon}(\Gamma))}{\mu(U_\varepsilon(\Gamma))} \leq \frac{K_{(4.1)} l (c\varepsilon)^{h-1}}{K_{(4.4)} l \varepsilon^{h-1}} = \frac{K_{(4.1)} c^{h-1}}{K_{(4.4)}}$$

Since $h > 1$, choosing $c < (\lambda K_{(4.4)} K_{(4.1)}^{-1})^{1/(h-1)}$ the statement is proved. □

6.1.3 Far away branches

Counting the branches We now introduce a new kind of points (depending on two parameters γ, ε), that we will call *reference points*.

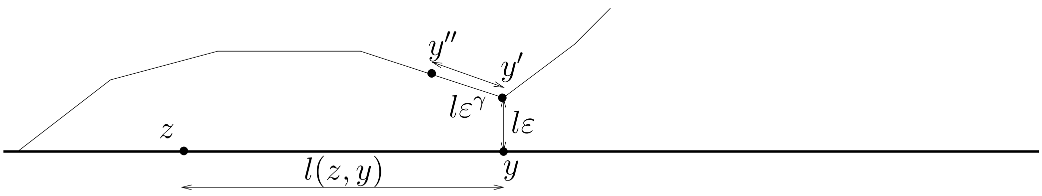

Figure 16: (γ, ε)-reference point (Definition 6.5).

Definition 6.5 ((γ, ε)-reference points)**.** Let χ be a simple pattern. Suppose that, as in Figure 16, the points y, y' form an ε-cycle or an ε-loop. Given $\gamma \in (0, 1)$, we define the (γ, ε)*-reference point* as the point y'' obtained moving towards the source S from y' of a length given by $\varepsilon^\gamma l$ (in the case the point y' is too close to the source, we set $y'' = S$), where $l = l(y, z)$ in the case of an ε-cycle and $l = l(y, y')$ in the case of an ε-loop.

Remark 6.6. Notice that the points y, y'' form an δ-cycle with δ of order of ε^γ. In particular, Lemma 5.11 applies.

The next proposition gives a lower bound for the mass of a reference point.

Proposition 6.7 (Lower bound for the mass of a reference point)**.** *Suppose that y'' is a (γ, ε)-reference point. If $2\varepsilon^\gamma < \varepsilon_0$, where ε_0 is as in Lemma 5.11, we have:*

$$m(y'')^{\frac{1}{h}} \geq C_{H,e}^{\frac{1}{\beta-1}} \varepsilon^{\frac{\beta-\gamma}{\beta-1}} l. \tag{6.2}$$

Proof. Since $\varepsilon < \varepsilon_0$, from (5.4) of Lemma 5.11 we have $Z(y'') - Z(y) > 0$, hence $Z(y') - Z(y'') < Z(y') - Z(y)$. By the Hölder regularity of the landscape function $Z(y') - Z(y) \leq C_{H,e}(\varepsilon l)^\beta$. Since, on the other hand, $Z(y'') - Z(y') \geq m(y'')^{\alpha-1} \varepsilon^\gamma l$, we finally find:

$$m(y'')^{\alpha-1} \varepsilon^\gamma l \leq C_{H,e}(\varepsilon l)^\beta,$$

which gives (6.2). □

Before stating Corollary 6.9 we have to describe a trisection construction and define a set of branches $\mathcal{B}$ whose cardinality will be counted in the corollary.

Definition 6.8 (l-centred part of a branch)**.** A branch Γ is said to be l-centred if there exist two branches Γ', Γ'' such that

- $x' \preceq x$ for all $x' \in \Gamma', x \in \Gamma$;

- $x \preceq x''$ for all $x \in \Gamma, x'' \in \Gamma''$;
- $\max \Gamma' = \min \Gamma$, $\max \Gamma = \min \Gamma''$;
- $l(\Gamma) = l(\Gamma') = l(\Gamma'') = l$;

Consider the tubular neighbourhood $U_{l\varepsilon}(\Gamma)$ of radius $l\varepsilon$ of a l-centred Γ. Consider now the set of points y' in the tubular neighbourhood $U_{l\varepsilon}(\Gamma)$ on the domain of the pattern χ, which are not reached by branches starting from $\Gamma' \cup \Gamma \cup \Gamma''$. Therefore, we consider the set of the points $y' \in D_\chi \cap U_{l\varepsilon}(\Gamma)$ such one of the following conditions hold:

1. $\forall z \in \Gamma', z \npreceq y'$, i.e. the fibre reaching y' leave the branch before Γ';
2. $\forall z \in \Gamma'', z \preceq y'$, i.e. the fibre reaching y' leave the branch after Γ'.

In both cases, since $y' \in U_{l\varepsilon}(\Gamma)$, there exists $y \in \Gamma$ such that $|y - y'| < l\varepsilon$. If the first condition is satisfied, then the pair (y, y') form an ε-cycle as in Example 5.15 (choosing z as the first point of Γ'). If the second one is true, then the pair (y, y') form an ε-loop, since $y \preceq y', l(y, y') \geq l$. Then, in both cases, starting from y' we can consider a (γ, ε)-reference point y''. By Lemma 2.17, from y'' a branch starts. Let $\mathcal{B}$ the set of such branches.

In the next corollary we consider any set $\overline{\mathcal{B}}(\subseteq \mathcal{B})$ of disjoint maximal branches starting from (γ, ε)-reference points w.r.t. a fixed branch Γ.

Corollary 6.9 (Counting the branches)**.** *Suppose the irrigated measure is Ahlfors regular in dimension $h > 1$. Given a part of a l-centred branch Γ, fix $\gamma < \beta$ and consider a set $\overline{\mathcal{B}}$ as above. Then, the cardinality of $\overline{\mathcal{B}}$ can be estimated by a constant $N_b(N, \alpha, h, c_A, C_A, \gamma)$ (hence, not depending on l, ε).*

Proof. Notice that the exponent $\frac{\beta-\gamma}{\beta-1}$ in (6.2) is negative, hence if $\varepsilon \leq 1$, $m(y'')^{\frac{1}{h}} \geq cl$ (where $c = C_{H,e}^{1/(\beta-1)}$). This immediately gives the $l(y'') \geq c^h l$ and applying Lemma 5.22 we obtain the thesis. □

Counting the points Up to now we just estimated the cardinality of some set of branches. What we want to do here is to estimate the cardinality of some sets of reference points.

Lemma 6.10. *Let Γ be a branch, $\gamma' < \gamma$ and $\mathcal{R}$ be a set of (γ, ε)-reference points (w.r.t. the branch Γ) whose reciprocal distance is at least $l\varepsilon^{\gamma'}$. Then, for $\varepsilon < \varepsilon_0$ (only depending on the dimension and the irrigated measure) the cardinality of $\mathcal{R}$ is estimated from above by the same constant of Corollary 6.9. Hence, the estimate does not depend on l, ε.*

Proof. Suppose that the points y_1'', y_2'' are on the same branch and $y_1'', y_2'' \in \mathcal{R}$, $y_1'' \prec y_2''$. Then, there exists $y \in \Gamma$ such that $|y - y_2''| < l\varepsilon^{\gamma'}$. The points y, y_2'' produce a double δ-cycle with δ of order $\varepsilon^{\gamma'-\gamma}$. This is a contradiction if $\delta < \varepsilon_0$, that is $\varepsilon \ll \varepsilon_0^{1/(\gamma'-\gamma)}$.

For every point in $\mathcal{R}$ we consider (by Corollary 2.18) a maximal branch starting from it. These branches are disjoint since χ is simple and the thesis follows applying Corollary 6.9. □

Proposition 6.11. *Let χ be an optimal pattern and suppose the irrigated measure is Ahlfors regular. Given a l-centred branch Γ contained in the support of the irrigated measure, the ratio between the measure of the tubular neighbourhood of Γ of radius ε irrigated by far away branches and the measure of tubular neighbourhood $U_\varepsilon(\Gamma)$ can be taken as small as desired for a suitable choice of ε (depending only on the dimension N and the irrigated measure).*

Proof. Fix $\gamma' < \gamma$ and let $\mathcal{R}_0$ be a maximal set of (γ, ε)-reference points (w.r.t. the branch Γ) whose reciprocal distance is at least $l\varepsilon^{\gamma'}$ (such set exists thanks to Lemma 6.10).

The set $\mathcal{R}_0$ is a $l\varepsilon^{\gamma'}$-net of the set of (γ, ε)-reference points, hence a $2l\varepsilon^{\gamma'}$-net of the set of points in $U_\varepsilon(\Gamma)$ irrigated by far-away branches.

Given $y'' \in \mathcal{R}_0$, the measure $\mu(U_\varepsilon(\Gamma) \cap B_{2l\varepsilon^\gamma}(y''))$ can be estimated from above (thanks to Lemma 4.1) by

$$\mu(U_\varepsilon(\Gamma) \cap B_{2l\varepsilon^\gamma}(y'')) \le K_{(4.1)}[4l\varepsilon^{\gamma'}][\varepsilon^{h-1}] = 4K_{(4.1)}l\varepsilon^{\gamma'+h-1}.$$

Hence, the measure irrigated by far away branches in $U_\varepsilon(\Gamma)$ (denoted by $\mu_{\text{far}}(U_\varepsilon(\Gamma))$) is at most

$$\mu_{\text{far}}(U_\varepsilon(\Gamma)) \le \sum_{y'' \in \mathcal{R}_0} \mu(U_\varepsilon(\Gamma) \cap B_{2l\varepsilon^\gamma}(y'')) \le \#\mathcal{R}_0(4K_{(4.1)}l\varepsilon^{\gamma'+h-1}) \le cl\varepsilon^{\gamma'+h-1},$$

where the constant c depend only on the dimension and the irrigated measure.

Since by Lemma 4.5 we have

$$\mu(U_\varepsilon(\Gamma)) \ge K_{(4.4)}l\varepsilon^{h-1},$$

it follows that

$$\frac{\mu_{\text{far}}(U_\varepsilon(\Gamma))}{\mu(U_\varepsilon(\Gamma))} \le c'\varepsilon^{\gamma'},$$

where the constant c' depend only on the dimension and the irrigated measure. This shows that choosing ε sufficiently small, the fraction can be made as small as desired, proving the statement. □

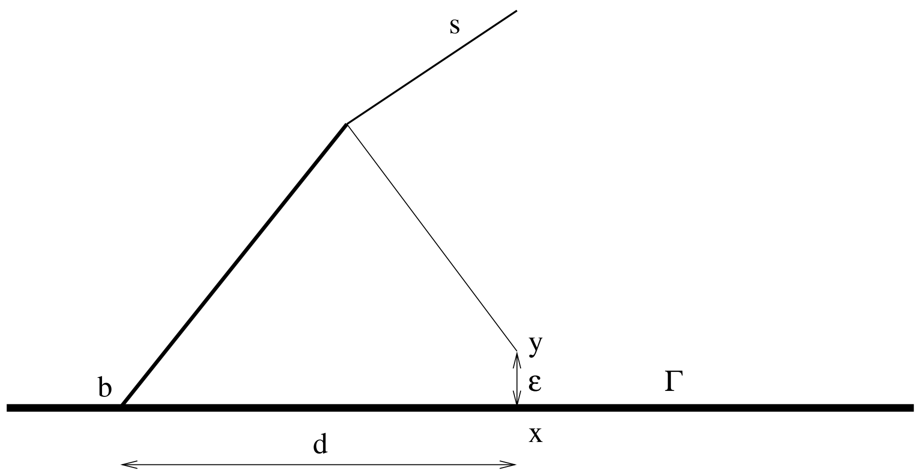

Figure 17: Interpolating–Configuration($l, \varepsilon, l_{\mathrm{res}}$) (Definition 6.12)

6.1.4 Long branches

In this section we consider the case of "long branches". We will count the number of branches coming out from a branch Γ, which (roughly) irrigate a point whose distance from Γ is less or equal than ε.

Let Γ be a branch. Let $y \in U_\varepsilon(\Gamma)$ any point in the support of the irrigated measure on a branch of scale $s = l^+(b)$ starting from a point $b \in \Gamma$. Let x be a point of Γ such that $|x - y| < \varepsilon$. The b, x, y, are as in the following definition.

Definition 6.12 (Interpolating–Configuration($l, \varepsilon, l_{\mathrm{res}}$))**.** Let Γ be a branch. Suppose $l, \varepsilon, l_{\mathrm{res}}$ are given. Suppose that x is on the branch through b, $l(x) \leq l(b) \leq 2l(x)$ and y is a point with $d(x, y) \leq \varepsilon$, x is irrigated by a branch starting from b. Suppose also that $l_{\mathrm{res}} = l^+(b)$. Then, we will say that b, x, y are in *Interpolating–Configuration*(l, ε, l_{res}). See Figure 17.

Theorem 6.13 (Interpolation estimate)**.** *Suppose that* b, x, y *are in Interpolating–Configuration*(l, ε, l_{res})*. Suppose that*

- $Z(x) - Z(b) \leq \frac{1}{2}(Z(y) - Z(b))$*;*
- $\varepsilon < Kl(b, x)/2$ *(*$K = C_{EB}^{-1}$*, see Theorem 3.6).*

Then, for some constant $\hat{H}$

$$l(b, x) \leq \hat{H} \varepsilon^\beta l^+(b)^{1-\beta}.$$

Precisely, we can choose $\hat{H} = 4C_H C_A^{1-\alpha} K^{-1}$.

Proof. Note that by hypothesis we have that $Z(y) \geq Z(x)$, since, certainly, $Z(x) - Z(b) \leq \frac{1}{2}(Z(y) - Z(b)) \leq Z(y) - Z(b)$. Now, write $Z(y) - Z(x)$ as $(Z(y) - Z(b)) - (Z(x) - Z(b))$. In this case, we then have:

$$Z(y) - Z(x) \geq \frac{1}{2}(Z(y) - Z(b)) \geq \frac{1}{2} m^+(b)^{\alpha-1} l(b, y).$$

Clearly, we have:

$$\frac{1}{2} K l(b, x) \leq K l(b, x) - \varepsilon \leq |b - x| - \varepsilon \leq |b - x| \leq l(b, y),$$

so that,

$$Z(y) - Z(x) \geq \frac{1}{4} m^+(b)^{\alpha-1} K l(b, x).$$

In the same way as in Proposition 2.5, we can prove that:

$$m^+(b) \leq C_A l^+(b)^h.$$

We then have:

$$C_H \varepsilon^\beta \geq Z(y) - Z(x) \geq \frac{1}{4} C_A^{\alpha-1} l^+(b)^{h(\alpha-1)} K l(b, x).$$

The last inequality shows that $l(b, x) \leq \hat{H} \varepsilon^\beta l^+(b)^{1-\beta}$. □

Remark 6.14. Thanks to the Zoom Lemma (Lemma 6.3) we can suppose that the first hypothesis of Theorem 6.13, that is

$$Z(x) - Z(b) \leq \frac{1}{2}(Z(y) - Z(b)),$$

is always satisfied, eventually restricting to a sub-branch (i.e. a connected subset of a branch). If such hypotheses are not satisfied for all branches, then we will prove the estimate from below of Theorem 6.17 only for a shorter branch at the same scale, but this is, of course, sufficient to have the same estimate for the whole branch.

Theorem 6.15. *Consider a branch Γ of length l. Then, there exists a sufficiently large constant N such that the mass irrigated by all the branches starting from Γ of length $s \geq \varepsilon 2^N$ is less than a given fraction $0 < \lambda < 1$ of the mass of the tubular neighbourhood $U_\varepsilon(\Gamma)$.*

Proof. We have proved in Theorem 6.13 that, if b_i, x_i, y_i are in Interpolating–Configuration$(l, \varepsilon, l_{\text{res}} = l^+(b_i))$,

$$l(b_i, x_i) \leq \hat{H} \varepsilon^\beta l^+(b_i)^{1-\beta}.$$

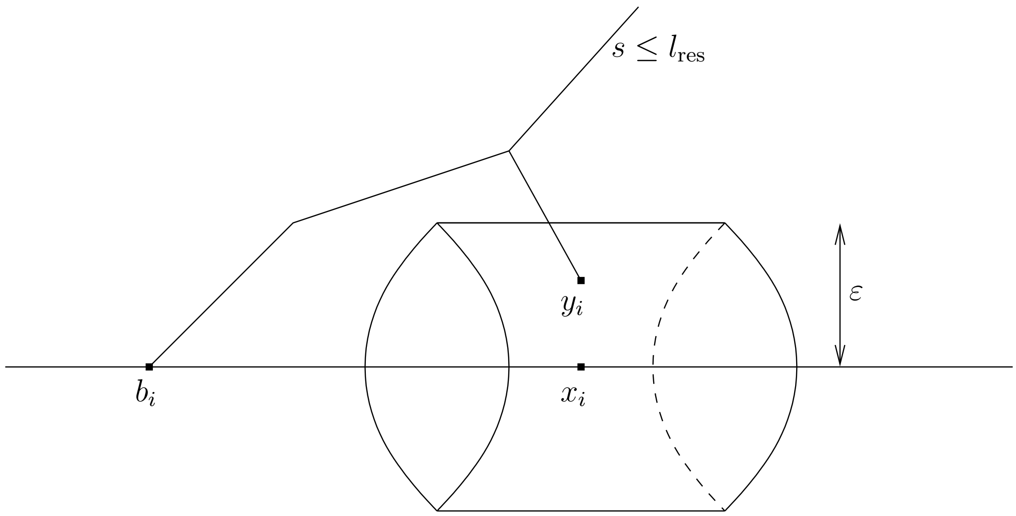

Figure 18: Theorem 6.15.

The configuration is depicted in Figure 18. We know by Lemma 4.6 that the number of branches of residual length l_{res} at least starting from a branch of length l

$$N(l_{\text{res}}) \leq C \frac{l}{l_{\text{res}}},$$

where C is the constant of inequality (4.6).

Each branch can then irrigate a tubular neighbourhood of a part of length $l(b_i, x_i)$ (up to some constant) of the main branch. The total mass of this neighbourhood (since $l(b_i, x_i) \leq \hat{H}\varepsilon^\beta l^+(b_i)^{1-\beta}$) is bounded from above by $l(b_i, x_i)\varepsilon^{h-1} = l^+(b_i)^{1-\beta}\varepsilon^{h\alpha}$. The total mass irrigable from branches of residual length such that $s < l^+(b_i) < 2s$ is bounded by

$$N(s)\varepsilon^{h\alpha} l^+(b_i)^{1-\beta} \leq \frac{l}{s}\varepsilon^{h\alpha}(2s)^{1-\beta} = \varepsilon^{h\alpha}2^{1-\beta} l s^{-\beta}.$$

For $\varepsilon 2^n \leq s \leq \varepsilon 2^{n+1}$ the bound is:

$$N(s)\varepsilon^{h\alpha} l^+(b_i)^{1-\beta} \leq 2^{1-\beta} l 2^{-n\beta}\varepsilon^{h-1}.$$

The total mass is then bounded by the geometric series

$$2^{1-\beta}\sum_n l\varepsilon^{h-1}2^{-n\beta}.$$

Choosing N sufficiently large, the ratio between the the mass irrigated by the branches of length greater that $\varepsilon 2^N$ and the tubular neighbourhood measure is bounded by

$$\frac{\sum_{n\geq N} l\varepsilon^{h-1}2^{-n\beta}}{l\varepsilon^{h-1}},$$

and it can be made as small as desired. □

6.2 The fractal estimate

From Lemma 4.6, we can deduce the following statement.

Lemma 6.16 (Part I: number of branches from above)**.** *Consider a branch Γ of length l. Given two constants $0 < c_1 < c_2$, let $N(\varepsilon, c_1, c_2)$ be the number of branches bifurcating from Γ whose residual length is between $c_1\varepsilon$ and $c_2\varepsilon$. Then,*

$$N(\varepsilon, c_1, c_2) \leq \frac{C_A c_2^{h-1}}{C_{H,g}^{1/(\alpha-1)} c_1^h} \frac{l}{\varepsilon}.$$

The previous lemma is the easy part of the *fractal estimate.* It states that if there are bifurcations in a given range $[c_1\varepsilon, c_2\varepsilon]$, then their number must not exceed l/ε times a suitable constant depending only on the Ahlfors regularity of the measure and α. *A priori* there may be no branches in $[c_1\varepsilon, c_2\varepsilon]$.

We will now state and prove the main theorem of the paper, providing the estimate from below.

Theorem 6.17 (Part II: number of branches from below)**.** *Let χ be an optimal pattern and suppose that the irrigated measure μ_χ is Ahlfors regular in dimension $h \geq 1$. Consider a branch Γ of length l contained the support of μ_χ. Then, there exists a suitable constant W and $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, the number of branches bifurcating from Γ whose residual length is between ε and $W\varepsilon$ is bounded from below by some constant depending only on the Ahlfors regularity of the given measure times l/ε.*

Proof. The proof of the result is obtained merging Lemma 6.4, Theorem 6.15, and Proposition 6.11.

By Lemma 6.4, the measure irrigated by the branches whose length is less than ε ("short branches") is a given fraction λ_1 of the measure of the tubular neighbourhood.

If we choose $W = 2^N$ where N is the integer given by Theorem 6.15, the measure irrigated by "long branches" can be set smaller than a given fraction λ_2 of the measure of the tubular neighbourhood of radius ε.

Finally, if we choose ε_0 as in Proposition 6.11 (using $l/3$ instead of l), the measure irrigated by "far away branches" can be set smaller than a given fraction λ_3 of the measure of the tubular neighbourhood of the middle third.

The measure irrigated by branches whose length is between ε and $W\varepsilon$ is then bounded from below by some constant times $l\varepsilon^{h-1}$. Since each one of them carries a measure given at most by $C_A(W\varepsilon)^h$, by mass balance, we must have:

$$c_0 l\varepsilon^{h-1} \leq \text{(irrigated measure)} \leq c_2 \text{ (number of branches) } \varepsilon^h.$$

Hence the number of branches is then greater than some constant times l/ε. □

A Some estimates on the Hölder constant of the landscape function

In this section we will set:

$$C_{H,e} := \sup_{x\neq y} \frac{|Z(x) - Z(y)|}{|x-y|^\beta}$$

$$C_{H,g} := \sup_{\substack{x\neq y \\ \text{same fibre}}} \frac{|Z(x) - Z(y)|}{l(x,y)^\beta}$$

$C_{H,e}$ and $C_{H,g}$ are the Hölder constants of the landscape function w.r.t. the Euclidean and branch distance respectively. Obviously, $C_{H,g} \leq C_{H,e}$. By [7, Remark 4.6], $C_{H,e} \leq C_{H,g}(1 + 2/\alpha)$.

Proposition A.1. *Suppose that the irrigated measure is Ahlfors regular from below in dimension h. Let c_A be the Ahlfors constant (from below). We then have:*

$$C_{H,g} \leq \frac{4\sqrt{N} c_A^{\alpha-1}}{\alpha 2^{\beta-1}(1-2^{-\beta})^2}. \tag{A.1}$$

Proof. The proof is a by-product of the proof of [7, Lemma 6.1]. Fix a constant $c < C_{H,g}$. By [7, Theorem 4.3], there must be a point x and a terminal point x_0 such that

$$Z(x_0) - Z(x) > cl(x, x_0)^\beta \geq c|x - x_0|^\beta.$$

Going on in the proof of [7, Lemma 6.1] we obtain that

$$2^{\beta-1}(1-2^\beta)c \leq C(c_A, h, N, \alpha),$$

where $C(c_A, h, N, \alpha)$ is the constant of [7, Lemma 5.2]. Substituting to $C(c_A, h, N, \alpha)$ its value, we obtain (A.1). □

Proposition A.2. *Suppose that the irrigated measure in Ahlfors regular from above in dimension h. Let C_A be the Ahlfors constant from above. Then,*

$$C_{H,e} \geq C_A^{\alpha-1}. \tag{A.2}$$

Proof. Consider a point x and the terminal point x_0 of its fibre. As usual, we have:

$$m(x)^{\alpha-1}l(x) \leq z(x_0) - z(x) \leq C_{H,e}|x - x_0|^\beta \leq C_{H,e}l(x)^\beta.$$

We get:

$$C_{H,e} \geq \frac{m(x)^{\alpha-1}}{l(x)^{\beta-1}}.$$

In the case of an Ahlfors regular measure from above, by Proposition 2.5, $m(x) \leq C_A l(x)^h$. Combining the last two formulas, we obtain (A.2). □

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List of symbols

$(\Omega, \mathcal{B}(\Omega), \mu_\Omega)$:	the reference space (Definition 1.1).
$\chi : \Omega \times I \to \mathbf{R}^N$:	a pattern (Definition 1.1).
$p \in \Omega$:	a particle (Definition 1.1).
$\chi_p := \chi(p, \cdot)$:	a fibre (Definition 1.1).
$i_\chi^+(p) := \chi(p, a)$:	see Definition 1.2.
$i_\chi^-(p) := \chi(p, b)$:	see Definition 1.2.
$\mu_\chi^\pm := (i_\chi^\pm)_\# \mu_\Omega$:	irrigating and irrigated measures (Definition 1.2).
$m_\chi^i(p, t)$:	mass carried in $\chi(p, t)$ (see (1.1)).
$m(x)$:	mass carried in x (Remark 1.12).
$J_\alpha^i(\chi)$:	the irrigation functional (Definition 1.5).
$d_\alpha(\mu_\chi^+, \mu_\chi^-)$:	the minimum branched transport cost (see (1.2)).
$c_\alpha(p)$:	fibre cost (see (1.3)).
D_χ:	domain of the pattern χ (Definition 1.8).
$\overline{Z}_\chi$:	the landscape function (Definition 1.9).
Z:	the landscape function (Remark 1.10, 1.12).
simple pattern:	see Definition 1.11.
c_A, C_A, h:	Ahlfors constants and dimension of the measure considered (Definition 1.13, 1.14).
$t_p(x)$:	first transit (Definition 2.1).
$l_p(x), l(x)$:	residual length (Definition 2.2, 2.3).
$C_{H,e}, C_{H,g}$:	landscape function Hölder constant (see Appendix A).
$\preceq$:	flow ordering (Definition 2.6).
flow curve:	see Definition 2.8.
$l(x, y)$:	branch distance (Definition 2.9).
branch:	see Definition 2.11.
$\overline{\chi}$:	linear mass by-pass (Definition 3.1).
C_{EB}:	equivalence constant between Euclidean and branch distance (Theorem 3.6).
$\tilde{\chi}$:	double, single by-pass (Definition 5.1, 5.3).
(double) ε-cycle:	see Definition 5.9, 5.10.
ε-loop:	see Definition 5.18.
good branch:	see Definition 6.1.
$l^+(x), m^+(x)$:	see Definition 6.2.
reference point:	see Definition 6.5.
l-centred branch:	see Definition 6.8.

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