Excerpt from

FRACTAL SPACE-TIME AND MICROPHYSICS
Towards a Theory of Scale Relativity

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Chapter 5

THE FRACTAL STRUCTURE OF QUANTUM SPACE-TIME

5.6. Non-Differentiable Space and Stochastic Quantum Mechanics.

This section is devoted to one of the essential points to be clarified in our approach. We have demonstrated in the previous sections that the basic features of the quantum behaviour can be understood in the frame of a fractal non-differentiable approach. But if we want these ideas to be developed one day into a full theory, one must demonstrate that quantum mechanics itself would indeed be recovered as an approximation or a limit of such a theory. By quantum mechanics, we mean here not only its basic laws like the de Broglie and Heisenberg relations, but the full theory itself: the concepts of *complex* probability amplitude and wave function, the

operator description, and finally the Schrödinger equation itself (in the nonrelativistic case).

The question that is asked here is a fundamental one. Underlying is the problem of the completeness of quantum mechanics and the understanding of the origin of the inescapable probabilistic description in microphysics. Let us recall again Einstein's far reaching position on this question (often misunderstood in our opinion).

Einstein's main criticism on the quantum theory was that, in this theory, *the probability* (more precisely the probability amplitude concept, completed by Born's statistical interpretation) *is set as a founding concept*, rather than *deduced from more fundamental principles*. In other words, contrary to what is often pretended, Einstein was perfectly ready to accept undeterminism as a limitation of our capability to make predictions,[24] but not undeterminism of the *fundamental* physical laws. Following Einstein's prescriptions for a real understanding of microphysics, the future theory must be able to *describe the processes affecting individual systems*, and it is *from such a description that the probabilistic description should naturally emerge*.[25] The confusion came from the fact that for most physicists, it seemed logical to admit that perfectly determined elementary physical laws are unable to yield undeterministic predictions. We shall try to demonstrate in the following that the concept of a fractal non-differentiable space-time actually leads to such a situation, in which one may get both an elementary description in terms of individual space-time events and non-predictability of particle trajectories.

In this respect, we remark that the various attempts that have been made to interpret quantum mechanics while keeping classical concepts are unsatisfactory, since none of them has proved to be able to satisfy to the hereabove “Einstein prescriptions”. Three of these attempts are particularly interesting in the present context and have received considerable attention in recent years: the *quantum potential* approach[26,27] of de Broglie and Bohm, *stochastic quantum mechanics*,[29,31] mainly developed by Nelson, and *geometric quantum mechanics*,[32,34] as proposed by Santamato.

In the quantum potential approach, the trajectories are deterministic and the statistical behaviour is artificially introduced into the formalism by

assuming that the initial conditions are at random; the physical origin of the "quantum force" that produces quantum effects remains mysterious.

In geometric quantum mechanics, the statistical description is obtained by the same postulate of random initial conditions; then "the theory does not describe the motion of an individual particle; rather it describes the statistical behaviour of an ensemble of identical particles".[32] Though not fulfilling Einstein's prescriptions, this theory is however very interesting, since the quantum behaviour is found to be a consequence of the underlying Weyl geometry. As demonstrated by Castro,[34] the Bohm quantum potential equation is recovered (rather than set) from a least-action principle acting on the Weyl gauge potential. Such an approach may be partly related to ours, since Weyl's geometry is closely related to conformal transformations, which include dilatations (i.e., scale transformations).

Stochastic quantum mechanics is a less "classical" approach than the two previously cited. Indeed the trajectories in the quantum potential approach and in geometric quantum mechanics remain deterministic: as such these theories are some kind of "hidden parameter" theories and so should be disproved by Bell's theorem. In stochastic quantum mechanics one assumes that an underlying Brownian motion, of unknown origin, is at work on every particle. This Brownian force induces a Wiener-like process which is at the origin of the quantum behaviour. In this theory the trajectories are *continuous, non-differentiable and non-deterministic*, as prescribed by Feynman's analysis of quantum trajectories. The Schrödinger equation can be recovered as a transcription of Newton's equation and of the Fokker-Planck equations for the diffusion process. We shall see in the following how our own fractal approach may be connected to Nelson's: the main link is apparent in the now well-known fact that Brownian motion is of fractal dimension 2,[35] while this is exactly the dimension which has been computed for quantum particle trajectories (Chapter 4). Let us only remark here that this theory does not satisfy Einstein's prescription either. The basic tool is a stochastic process, so that, as in standard quantum mechanics, the statistical description is set as a fundamental unexplained principle. Nothing is said, either on individual phenomena, or on the physical origin of the underlying stochastic process. Subsequent attempts have tried to ascribe quantum fluctuations to some subquantum medium, second-quantized into

hypothetical particles.[36,37] But the physical nature of such particles remains unclear: there is presently no place for them in the standard model of elementary particles. (See also Rosen[36] for a critical point of view on stochastic quantum mechanics by one of its originators).

Before jumping to the fractal approach, let us say a few additional words on Feynman's path integral reformulation of quantum mechanics.[8] As was perhaps realized at that time by Wheeler (see Ref. 39) this interpretation of quantum mechanics is far more “realistic” (in Einstein's meaning) than the Copenhagen one. Recall that one demonstrates that the probability amplitude for a particle to go from a point a to another point b is given by the sum over all possible trajectories:

$$K(a,b) = \int_a^b exp\{\frac{i}{\hbar} S(a,b)\}\ \mathcal{D}x(t) \quad ,$$

where $S(a,b)$ is the classical action for the path considered and $\mathcal{D}x$ is a differential element coined for this special integral. The integration should be performed over all possible continuous paths connecting a and b, however distant or complicated may they be. Actually the paths which are too distant from the classical trajectory are very unprobable, due to destructive interferences between the $exp(iS/\hbar)$ terms. One finds, for a free particle of de Broglie wavelength λ travelling between two points a and b separated by a distance ℓ, that the most probable paths are included in an ellipsoïd of revolution with thickness at half-way of order $\sqrt{(\lambda\ \ell)}$. Indeed the paths which deviate from the a-b axis by more than y, with $y^2=(\lambda\ell/2)[1-(2x/\ell)^2]$, (where x is the coordinate on this axis, the origin being at the middle of the a-b segment) become increasingly destroyed by destructive interferences.

It is intuitively clear that in such a volume, the number of non-differentiable paths greatly exceeds those which are differentiable. Indeed, as recalled in Sec. 4.1, Feynman finds that the mean quadratic velocity scales as $\langle v^2\rangle \approx \delta t^{-1}$, corresponding to fractal dimension 2 (see Secs. 3.8 and 4.1).

Hence in Feynman's approach, one can use the concept of particle trajectory, while this was forbidden in the Copenhagen interpretation of quantum mechanics. This is not a return to determinism, since a Feynman path is only one possibility among an infinity: the various possible paths are

precisely characterized as being equiprobable (they have equal probability, but *not* equal probability amplitudes). In Bohr's intepretation of quantum mechanics, following his principle of complementarity, only the wave nature of particles is actualized in the absence of measurement, and no information finer than that given by the wave function has physical sense. On the contrary in Feynman's perspective, one may explicitly consider individual possible trajectories, then analyse them and describe their structure. This does not result in trivial statements, since even though the number of possible trajectories is infinite, they share common properties, among which their non-differentiability and $D=2$ fractal character.

However if one wants to reach “Einstein's realism,” one must go one step further. One should not only consider individual trajectories but must also look at the structure of all the individual points which constitute them. This means looking at the very structure of space-time, and more specifically, asking oneself what are the elementary properties of space-time which drive the particles into such complicated paths. We thus fall back to our original proposal: trajectories are fractal because space-time itself is fractal and non-differentiable and because they follow its geodesical lines.

Let us now make an attempt at explicitely looking at one point of a fractal space-time. Useful intuitive models of what happens may be found in Sec. 3.6 about fractal surfaces, and in the generalization to fractal spaces in Sec. 3.10. Actually every point of a fractal space happens to be singular, and any trajectory passing through such a singular point is expected to be broken, in agreement with its non-differentiability. Hence whatever detailed model of singular space is used, the minimal prescription in the description of what happens at a given point P of the fractal space implies two main features:

(1) Even if one assumes the incoming trajectory to be well-defined, the outcoming trajectory will be defined only in a probabilistic way. This is a strongly chaotic situation, in which *infinitesimal* differences in the initial conditions lead to completely different trajectories (see Fig. 5.7). For example, in a simple conic model, the local properties of each point of a singular space can be described by the value of the ratio of the circonference of a circle over its diameter, π_X, different from the Euclidean value π. In the 2-dimensional case (singular surface), two outcoming

trajectories correspond to each incoming trajectory. The point *P* behaves as if it was attractive when $\pi_X < \pi$ ("spherical") and repulsive when $\pi_X > \pi$ ("hyperbolic"). We conjecture that a generalization to 3-space will yield outcoming trajectories making up a full cone, the opening angle of which is related to the value of π_X at point *P*.

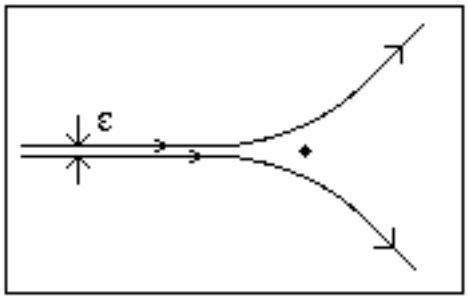

Figure 5.7. Two trajectories initially infinitely close may diverge in a non-differentiable space.

Now, reversing the arrow of time, one finds that to a given outcoming trajectory, there also correspond several incoming trajectories. The general situation at point *P* is finally that several trajectories with broken slopes are possible (an infinity in 3-space).

(2) Once one of the possible incoming and outcoming trajectories is chosen, the trajectory at point *P* is characterized by *two velocities* instead of one classically, a velocity $\boldsymbol{\nu}_-$ before the point and a velocity $\boldsymbol{\nu}_+$ after the point (see Fig. 5.8). So the two velocity vectors $\boldsymbol{\nu}_+$ and $\boldsymbol{\nu}_-$ define a plane which passes through the point *P* and which has no classical counterpart. We shall come back later to the physical meaning of this plane. From these two vectors, one may now define in the same plane

$$\boldsymbol{u} = \frac{\boldsymbol{\nu}_+ - \boldsymbol{\nu}_-}{2} \quad , \qquad \boldsymbol{\nu} = \frac{\boldsymbol{\nu}_+ + \boldsymbol{\nu}_-}{2} \quad .$$

It is straighforward that the classical differentiable case is recovered for $\boldsymbol{\nu}_+ = \boldsymbol{\nu}_-$, so that $\boldsymbol{\nu}$ is the generalization of the classical velocity ($\boldsymbol{\nu} = \boldsymbol{\nu}_+ = \boldsymbol{\nu}_-$), while $\boldsymbol{u}$ is a new quantity which vanishes in the classical (differentiable) approximation.

What is the meaning of this new velocity $\boldsymbol{u}$? Being built from a *difference* of velocities, one may expect it to be related to an acceleration.

However the acceleration at point P is actually given by $\boldsymbol{\Gamma} = (\boldsymbol{v}_+ - \boldsymbol{v}_-)/2\delta t$, with $\delta t \to 0$, so that it is undefined from the view point of standard methods: $\boldsymbol{\Gamma} = \infty$. So let us attack the problem from the fractal point of view. In Sec. 3.8 we have introduced the concept of fractal functions $f(x,\varepsilon)$.

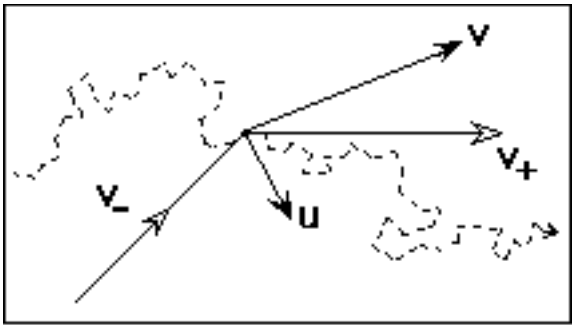

Figure 5.8. The backward and forward velocity on a trajectory whose slope is broken at each of its points, and their combination $\boldsymbol{v}$, which generalizes the classical velocity, and $\boldsymbol{u}$, which vanishes in the differentiable case.

Start with the position vector of a particle. In the fractal approach this is a finite fractal function $\boldsymbol{x}(t,\varepsilon)$, where, depending on the experiment performed, ε is either a spatial resolution ($\varepsilon=\delta x$) or a temporal one ($\varepsilon=\delta t$). Then the three components of the velocity of the particle are divergent fractal functions:

$$\frac{d\boldsymbol{x}}{dt}(t,\varepsilon) = \boldsymbol{w}(t,\varepsilon)\left(\frac{\tau}{\varepsilon}\right)^{\beta} \quad ,$$

where we have assumed the resolution to be a temporal one, i.e., $\varepsilon=\delta t$, so that standard quantum mechanics yields $\beta=1/2$. The finite fractal vector $\boldsymbol{w}(t,\varepsilon)$ is another representation for the hereabove $\boldsymbol{v}_+$ and $\boldsymbol{v}_-$ velocities. The fractal acceleration will itself be a fractal function, now independent of the value of β, from the theorem of Sec. 3.8:

$$\boldsymbol{\Gamma}(t,\varepsilon) = \frac{d\boldsymbol{w}}{dt}(t,\varepsilon) = \boldsymbol{\gamma}(t,\varepsilon)\left(\frac{\tau}{\varepsilon}\right) \quad . \tag{5.6.1}$$

We may now compute the value of the hereabove new velocity $\boldsymbol{u}$ to any resolution ε and eventually take the limit $\varepsilon \to 0$. One may write

$$\boldsymbol{u} = \frac{\boldsymbol{w}(t+\varepsilon,\varepsilon) - \boldsymbol{w}(t-\varepsilon,\varepsilon)}{2} \quad .$$

But we may also compute $\boldsymbol{\Gamma}$ as

$$\boldsymbol{\Gamma}(t,\varepsilon) = \frac{\boldsymbol{w}(t+\varepsilon,\varepsilon) - \boldsymbol{w}(t-\varepsilon,\varepsilon)}{2\ \varepsilon} \quad ,$$

so that we get $\boldsymbol{u} = \varepsilon\ \boldsymbol{\Gamma}(t,\varepsilon)$, and from Eq. (5.6.1) we finally obtain the remarkable result

$$\boldsymbol{u} = \boldsymbol{\gamma}(t,\varepsilon)\ \tau \quad .$$

This formula states that *the new "velocity"* $\boldsymbol{u}$ *is nothing but the finite part of the fractal acceleration* (to the constant multiplicative factor τ, which is fixed by the *classical* state of motion of the particle). The infinite acceleration is naturally renormalized thanks to the fractal structures, and this renormalized acceleration is identified with $\boldsymbol{u}/\tau$. We shall see in the following that this quantity has a profound physical sense (the probability of presence is deduced from its average): this demonstrates that the introduction of the finite part of fractal functions was indeed endowed with a physical meaning. Consider in this respect some of the difficulties encountered by the various models which have been proposed for Brownian motion (see also hereafter). In the Einstein-Smoluchowski and Wiener methods, there is a well-defined position but the velocity is undefined. In the Ornstein-Uhlenbeck method, one assumes a defined velocity, but then the acceleration is undefined, being always infinite. Our method of separation of finite and infinite parts of fractal functions sheds some light on the meaning of these models: the Ornstein-Uhlenbeck case corresponds to taking the finite part $\boldsymbol{w}$ of the fractal velocity, while the Wiener process starts with the position variable $\boldsymbol{x}$, which is itself a finite fractal function. Finally we have shown that the acceleration itself may be defined after our fractal renormalization is performed.

Let us proceed further. We have seen that the non-differentiable fractal structure of space implies emergence of a fundamental and elementary probabilistic behaviour of trajectories at any point of this space. (We shall not explicitely consider in this section the space-time case: let us simply say that the generalization of the concept of non-differentiable and

probabilistic trajectories to space-time is related to the probabilistic creation and annihilation of particle-antiparticle pairs: see a preliminary account of the relativistic case in Sec. 5.8).

This probabilistic behaviour is now a consequence of a more fundamental postulate, the postulate that microphysics space-time is fractal, which is itself a consequence of the principle of relativity of motion, in its extended version ("the laws of nature should apply to systems of coordinates whatever their state of motion, even non differentiable"); or equivalently of the principle of general covariance, itself extended ("the equations of physics should be covariant under general continuous coordinate transformations, not only differentiable ones"). In this respect we hope that, if a future well-developed theory is to be built upon these principles, it will satisfy "Einstein's prescriptions" for a realistic theory of microphysics: one may have, at least in principle, a *completely determined continuous space-time*, the geodesics of which are *undeterministic* because of its *non-differentiability*.

The elementary probability introduced at the infinitesimal level *implies a statistical treatment* of physical laws. We shall see that the setting of such a statistical approach for a fractal motion *of fractal dimension* 2, as imposed by Heisenberg's relations, is fully equivalent to stochastic quantum mechanics. Since the Schrödinger equation is obtained in stochastic quantum mechanics as a consequence of a Wiener process, we may now be certain that a fractal theory of microphysics is expected to include quantum mechanics as an approximation.

Let us now recall how the stochastic approach is able to yield Schrödinger's equation. Though the essential information is already contained in Nelson's work,[31,40] we present here a new derivation (with occasional different notations) in which we attempt at fully using the new "bi-velocity" structure $(\boldsymbol{u},\boldsymbol{v})$: this allows us to better grasp the physical origin of the complex probability amplitude and of the complex operators, and to *demonstrate* two of Nelson's main assumptions rather than postulate them.[1]

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[1] Further works have shown that the departure from the fractal space-time approach and stochastic mechanics was larger than initially thought. The use of a description in terms of stochastic variables is motivated by the need to account for the fractal and nondifferentiable fluctuations of the motion in a fractal space. But, besides this common point with Nelson's theory, the subsequent developments of the

Let us place ourselves at point P at some given instant t. The basic hypothesis in the stochastic approach is that, even if the processes considered are no more differentiable in the usual sense, one may still define[31] a mean forward derivative d_+/dt and a mean backward derivative d_-/dt:

$$\frac{d_+}{dt}\boldsymbol{y}(t) = \lim_{\Delta t\to 0+} <\frac{y(t+\Delta t) - y(t)}{\Delta t}> \quad ,$$

$$\frac{d_-}{dt}\boldsymbol{y}(t) = \lim_{\Delta t\to 0+} <\frac{y(t) - y(t-\Delta t)}{\Delta t}> \quad .$$

Once applied to the position vector $\boldsymbol{x}$, they yield *forward and backward mean velocities*:

$$\frac{d_+}{dt}\boldsymbol{x}(t) = \boldsymbol{b}_+ \quad , \quad \frac{d_-}{dt}\boldsymbol{x}(t) = \boldsymbol{b}_- \quad .$$

In our approach, these velocities are defined as the average at point P and time t of the respective velocities of the outcoming and incoming fractal trajectories; in stochastic quantum mechanics, this corresponds to an average on the quantum state.

It is clear that, in the fractal approach, these slopes, averaged over families of virtual trajectories, are expected to be related to the fractal space-time itself. However the hypothesis that one can define mean forward and backward derivatives may not be fulfilled *in the general case*: for most of its points (those which are given by an infinite number of digits of their curvilinear coordinate, see Sec. 3) even this generalized derivative may be

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two theories fundamentally depart, since stochastic mechanics is founded on diffusion processes, which is not the case of the scale-relativity approach. This is a strong problem for stochastic mechanics, since the backward process introduced by Nelson (and its associated so-called backward Fokker-Planck equation) corresponds to no existing physical diffusion process (see, e.g., H. Grabert et al. 1979, Phys. Rev. A 19, 2440; L. Nottale, 1997, Astron. Astrophys. 327, 867). In the present fractal space approach, there is actually no need, neither to introduce the density from the very beginning, nor to found the theory on Fokker-Planck equations (therefore a large part of pages 146-147 where this is done according to stochastic mechanics could be suppressed). Indeed, the new theory is founded on first principles, namely, on the action principle from which a geodesics / generalized complex equation of dynamics (5.6.9) is written in terms of the covariant derivative (5.6.5). Then the Schrödinger equation (5.6.11) is directly derived from this equation, without making use of Fokker-Planck equations. In this framework, the Born interpretation of the square of the modulus of the wave function as a probability density is subsequently derived from the initial geometric description. For a more recent and complete derivation, see Célérier M.N. & Nottale L., 2004, J. Phys. A37, 931 "Quantum-classical transition in scale relativity" http://wwwusr.obspm.fr/~nottale/ardirac.pdf.

undefined on a fractal space. However the "physical" points are those which correspond to a finite resolution (arbitrarily small but never zero), while we have shown that these points, corresponding to a finite number of digits, may indeed be characterized by a “left derivative” and a “right derivative” for some classes of fractals: those whose generator has a null slope at the origin (see Fig. 3.9).

In the following, we shall disregard these difficulties and consider Nelson's assumption as the strongest simplifying assumption one may make about fractals, i.e., in other words, as the simplest possible breaking of differentiability. We shall see that this yields Schrödinger's equation in its simplest form, without spin or charge. One may then expect new structures to emerge when relaxing this hypothesis (this will not be considered in the present book).

The position vector $\boldsymbol{x}(t)$ of the particle is thus assimilated to a stochastic process which satisfies, respectively after ($dt > 0$) and before ($dt < 0$) the instant t,

$$d\boldsymbol{x}(t) = \boldsymbol{b}_{+}[\boldsymbol{x}(t)]\, dt + d\boldsymbol{\xi}_{+}(t) = \boldsymbol{b}_{-}[\boldsymbol{x}(t)]\, dt + d\boldsymbol{\xi}_{-}(t) \ ,$$

where $\boldsymbol{\xi}(t)$ is a Wiener process.[41] It is in the description of $\boldsymbol{\xi}$ that the D=2 fractal character of trajectories is input. Indeed, that $\boldsymbol{\xi}$ is a Wiener process means that the $d\boldsymbol{\xi}$'s are assumed to be Gaussian with mean 0, mutually independent and such that

$$\langle d\boldsymbol{\xi}_{+i}(t)\, d\boldsymbol{\xi}_{+j}(t)\rangle = 2\,\mathcal{D}\,\delta_{ij}\,dt \ , \qquad (5.6.2a)$$

$$\langle d\boldsymbol{\xi}_{-i}(t)\, d\boldsymbol{\xi}_{-j}(t)\rangle = -2\,\mathcal{D}\,\delta_{ij}\,dt \ , \qquad (5.6.2b)$$

where $\langle\rangle$ denotes averaging, and where $\mathcal{D}$ is the diffusion coefficient.

Nelson's postulate is that[31] $\mathcal{D} = \hbar/2m$. This value can be easily justified. Indeed the diffusion coefficient is expected to be given by the product L^2T^{-1} of the characteristic length and time period of the system, i.e., in an equivalent way, $2\mathcal{D} = Lv$. In the quantum non-relativistic case, $L = \lambda_{\mathrm{dB}} = \hbar/p = \hbar/mv$, so that we finally obtain $\mathcal{D} = \hbar/2m$. Note however that it has been demonstrated by Davidson[42] that any value of $\mathcal{D}$ may lead to quantum mechanics, and by Shucker[43] that in the limit $\mathcal{D} \to 0$, the theory

becomes equivalent to Bohm's[27] deterministic quantum potential approach. We shall in the following work essentially with the value $\mathcal{D} = \hbar/2m$.

That Eq. (5.6.2) is indeed a consequence of a fractal dimension 2 of trajectories is straighforward: it may be written $<d\boldsymbol{\xi}^2>/dt^2 \approx dt^{-1}$, i.e. precisely Feynman's result $<v^2>^{1/2} \approx \delta t^{-1/2}$. We have demonstrated (Sec. 3.8) that a fractal dimension D leads to the relation $\mathcal{L} \approx <v^2>^{1/2} \approx \delta x^{1-D}$ when the measurement resolution is spatial, and $\mathcal{L} \approx \delta t^{1/D-1}$ when it is temporal. Feynman's power $-1/2$ is thus translated into $D = 2$. In Nelson's approach, this choice for the stochastic basic process is attributed to the hypothesis that quantum particles are subject to a Brownian motion of unknown origin. The connection with fractals is clear: Brownian motion is now known to be of fractal dimension 2.[35] However the fractal conjecture is far more general than the Brownian motion hypothesis: the fractal hypothesis is a general geometrical description which supersedes any of its possible physical cause, the second is a particular choice of a $D = 2$ physical phenomenon which has at present no experimental support, the existence of a diffusing particle being up to now at variance with the standard model of elementary particles.

Let us now examine the implications of this Wiener process for the mean derivatives. Start from any function f of $\boldsymbol{x}$ and t, and expand f in a Taylor series up to order 2. Then take the average and use the properties of the Wiener process ξ (Eq. 5.6.2), one gets[31]

$$d_+f/dt = (\partial/\partial t + \boldsymbol{b}_+ \cdot \boldsymbol{\nabla} + \mathcal{D}\,\Delta)f \quad , \tag{5.6.3a}$$

$$d_-f/dt = (\partial/\partial t + \boldsymbol{b}_- \cdot \boldsymbol{\nabla} - \mathcal{D}\,\Delta)f \quad . \tag{5.6.3b}$$

Let $\rho(\boldsymbol{x},t)$ be the probability density of $\boldsymbol{x}(t)$.[1] It has been demonstrated by Kolmogorov[44] that for any Markov process (the Wiener process is indeed a particular case of a Markov process[41]), the probability density satisfies a forward equation:

$$\partial\rho/\partial t + div(\rho\boldsymbol{b}_+) = \mathcal{D}\,\Delta\rho \quad ,$$

[1] The introduction at this level of the analysis of the density and of Fokker-Planck equations comes under stochastic mechanics, but it is actually irrelevant in the present fractal space approach (see note 1 p. 143).

and a backward equation:

$$\partial\rho/\partial t \; + \; div(\rho \boldsymbol{b}_-) \; = \; -\mathcal{D}\,\Delta\rho \quad .$$

These two equations are often called forward and backward Fokker-Planck equations. We may now define two new *average* velocities:

$$\boldsymbol{V} \; = \; \frac{\boldsymbol{b}_+ + \boldsymbol{b}_-}{2} \qquad ; \qquad \boldsymbol{U} \; = \; \frac{\boldsymbol{b}_+ - \boldsymbol{b}_-}{2}$$

i.e., in the space-time approach, they are the statistical averages of the individual velocities, $\boldsymbol{V} = \langle \boldsymbol{v} \rangle$ and $\boldsymbol{U} = \langle \boldsymbol{u} \rangle$. Adding the two Fokker-Planck equations yields

$$\partial\rho/\partial t \; + div(\rho \boldsymbol{V}) \; = 0$$

which is nothing but the well-known equation of continuity. This confirms $\boldsymbol{V}$ as a generalization of the classical velocity. Subtracting the two Fokker-Planck equations yields

$$div(\rho \boldsymbol{U}) - \mathcal{D}\,\Delta\rho \; = \; 0 \quad ,$$

which may be written

$$div\{\rho \; [\boldsymbol{U} \; - \mathcal{D} \; \boldsymbol{\nabla} ln\rho \;] \} = \; 0 \quad .$$

One may actually demonstrate[31,40] by using the properties of (5.6.3) that the term under the *div* operator is itself null, so that $\boldsymbol{U}$ is a gradient:

$$\boldsymbol{U} \; = \; \mathcal{D} \; \boldsymbol{\nabla} ln\rho \quad .$$

We shall now introduce new notations aimed at fully using the bi-vector new structure $(\boldsymbol{U},\boldsymbol{V})$. From now on, the derivation of the Schrödinger equation which is presented here is original (to the best of our knowledge). We place ourselves in the $(\boldsymbol{U},\boldsymbol{V})$ plane and introduce a new complex velocity

$$\boldsymbol{\mathcal{V}} \; = \; \boldsymbol{V} - \; i \; \boldsymbol{U} \quad .$$

Consider now the backward and forward mean derivatives. In the same way as $\boldsymbol{U}$ and $\boldsymbol{V}$ have been defined, we may set

$$\frac{d_v}{dt} = \frac{1}{2}\,\frac{d_+ + d_-}{dt} \quad , \qquad \frac{d_u}{dt} = \frac{1}{2}\,\frac{d_+ - d_-}{dt} \quad .$$

By combining Eqs. (5.6.3 a) and (5.6.3 b), these mean derivatives write

$$\frac{d_v}{dt} = \frac{\partial}{\partial t} + \boldsymbol{V}\,.\,\boldsymbol{\nabla} \quad , \qquad \frac{d_u}{dt} = \mathcal{D}\,\Delta + \boldsymbol{U}\,.\,\boldsymbol{\nabla} \quad . \tag{5.6.4}$$

From these two new operators, we now define a complex operator

$$\frac{d\!\!\!^{-}}{dt} = \frac{d_v}{dt} - i\,\frac{d_u}{dt} \quad ,$$

which, from (5.6.4), is finally given by

$$\frac{d\!\!\!^{-}}{dt} = \left(\frac{\partial}{\partial t} - i\,\mathcal{D}\,\Delta\right) + \boldsymbol{\mathcal{V}}\,.\,\boldsymbol{\nabla} \quad , \tag{5.6.5}$$

since $\boldsymbol{V} - i\,\boldsymbol{U} = \boldsymbol{\mathcal{V}}$ by definition.

We shall now postulate that the passage from classical (differentiable) mechanics to the new nondifferentiable processes that are considered here can be implemented by a unique prescription: *Replace the standard time derivative d/dt by the new complex operator* $d\!\!\!^{-}/dt$. Let us indicate the main steps by which one may generalize classical mechanics using this prescription.

We assume that any mechanical system can be characterized by a Lagrange function $\mathcal{L}(\boldsymbol{x}_i, \boldsymbol{\mathcal{V}}_i, t)$, from which a *mean* action S is defined:

$$S = \int_{t_1}^{t_2} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\mathcal{V}}, t)\, dt \quad . \tag{5.6.6}$$[1]

[1] In the published version we had taken the mean of the integral. This is not necessary since it is already defined from the mean ("classical", differentiable) parts of the variables.

The action principle[1], applied on this new action with both ends of the above integral fixed, leads to generalized Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \mathcal{V}_i} = \frac{\partial \mathcal{L}}{\partial x_i} \quad , \tag{5.6.7}$$

in agreement with the correspondence ($d/dt \rightarrow d/dt$). Other fundamental results of classical mechanics are also generalized in the same way. In particular, assuming homogeneity of space *in the mean* leads to defining a complex momentum

$$\mathcal{P}_i = \frac{\partial \mathcal{L}}{\partial \mathcal{V}_i} \quad .$$

If one now considers the action as a functional of the upper limit of integration in (5.6.6), the variation of the action from a trajectory to another nearby trajectory, when combined with Eq. (5.6.7), yields a generalization of another well-known result, namely, that the complex momentum is the gradient of the complex action:

$$\boldsymbol{\mathcal{P}} = \boldsymbol{\nabla} S \quad . \tag{5.6.8}$$

We shall now specialize and consider Newtonian mechanics. The Lagrange function of a closed system, $L(\boldsymbol{x},\boldsymbol{v},t)= \frac{1}{2} m\boldsymbol{v}^2 - \mathcal{U}$, is generalized as $\mathcal{L}(\boldsymbol{x},\boldsymbol{\mathcal{V}},t) = \frac{1}{2} m\boldsymbol{\mathcal{V}}^2 - \mathcal{U}$, so that the Euler-Lagrange equation keeps the form of Newton's fundamental equation of dynamics:

$$-\boldsymbol{\nabla}\mathcal{U} = m \frac{d}{dt}\boldsymbol{\mathcal{V}} \quad , \tag{5.6.9}$$

which is now written in terms of complex variables and operator.

Let us separate the real and imaginary parts of the complex acceleration $\boldsymbol{\gamma} = d\boldsymbol{\mathcal{V}}/dt$. We find

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[1] We wrote "least-action principle" in the published version. We have corrected this expression here, since, though it remains a *stationary* action principle, it is no longer a *least*-action principle. Indeed, it is applied here in the complex plane, which is not ordered.

$$d\boldsymbol{\mathcal{V}} = (d_v - i\, d_u)(\boldsymbol{V} - i\, \boldsymbol{U}) = (d_v \boldsymbol{V} - d_u \boldsymbol{U}) - i\,(d_u \boldsymbol{V} + d_v \boldsymbol{U}) \quad .$$

The force $F = -\boldsymbol{\nabla}\mathcal{U}$ being real, the imaginary part of the complex acceleration vanishes. It is given by

$$\frac{d_u}{dt}\boldsymbol{V} + \frac{d_v}{dt}\boldsymbol{U} = \partial \boldsymbol{U}/\partial t + \boldsymbol{U}.\boldsymbol{\nabla}\boldsymbol{V} + \boldsymbol{V}.\boldsymbol{\nabla}\boldsymbol{U} + \mathcal{D}\,\Delta \boldsymbol{V} = 0 \quad ,$$

from which $\partial \boldsymbol{U}/\partial t$ may be obtained. Differentiating the expression $\boldsymbol{U} = \mathcal{D}\,\boldsymbol{\nabla} ln\rho$ and using the equation of continuity yields another expression for $\partial \boldsymbol{U}/\partial t$:

$$\frac{\partial \boldsymbol{U}}{\partial t} = -\mathcal{D}\,\boldsymbol{\nabla}(div\boldsymbol{V}) - \boldsymbol{\nabla}(\boldsymbol{V}.\boldsymbol{U}) \quad .$$

The comparison of these two relations yields $\boldsymbol{\nabla}(div\boldsymbol{V}) = \Delta \boldsymbol{V} - \boldsymbol{U} \wedge curl\boldsymbol{V}$, where the term in $curl\boldsymbol{U}$ vanishes since $\boldsymbol{U}$ is already known to be a gradient. But in the Newtonian case now considered, the complex momentum becomes $\boldsymbol{\mathcal{P}} = m\boldsymbol{\mathcal{V}}$, so that Eq. (5.6.8) implies that $\boldsymbol{\mathcal{V}}$ *is a gradient*. This demonstrates that the "classical" velocity $\boldsymbol{V}$ is a gradient (while this was postulated in Nelson's original paper). We can now introduce a generalization of the classical action S (in dimensionless units) by the relation

$$\boldsymbol{V} = 2\,\mathcal{D}\,\boldsymbol{\nabla} S \quad .$$

Combining this relation with the expression for $\boldsymbol{U}$, we find the complex action to be given by $\mathcal{S} = 2\,m\,\mathcal{D}\,(S - i\, ln\rho^{1/2})$, i.e., its imaginary part is the logarithm of the probability density.

Note that Nelson[31] arbitrarily defines the acceleration as

$$d_N^2\boldsymbol{x}/dt^2 = \frac{1}{2}\,\frac{d_+ d_- + d_- d_+}{dt^2}\,\boldsymbol{x}$$

(it could *a priori* have been any second order combination of d_+ and d_-; however see Nelson[40]). It is easy to show that Nelson's acceleration is nothing but the real part of our complex acceleration $d\boldsymbol{\mathcal{V}}/dt$. It is also noticeable that a stochastic least-action principle was introduced by Guerra

and Morato,[99] based on the *real* Lagrange function $L = \frac{1}{2}m(\boldsymbol{V}^2 - \boldsymbol{U}^2) - \mathcal{U}$, which is nothing but the real part of our complex Lagrange function $\mathcal{L}$.

We shall see that the way to Schrödinger's equation is now remarkably short. We now introduce the complex function $\psi = e^{iS/2m\mathcal{D}}$,[1] that is, in terms of probability density and real part of action:

$$\psi \;=\; \sqrt{\rho}\;\; e^{iS}\;\;,$$

and the complex velocity is now related to this new function by

$$\boldsymbol{\mathcal{V}} \;=\; -2\,i\;\;\mathcal{D}\;\;\boldsymbol{\nabla}\,(ln\psi)\;\;\;. \qquad (5.6.10)$$

Let us stop one moment on this result. In terms of our *complex momentum* $\boldsymbol{\mathcal{P}} = m\boldsymbol{\mathcal{V}}$, it writes (when $\mathcal{D} = \hbar/2m$) $\boldsymbol{\mathcal{P}}\psi = -i\,\hbar\boldsymbol{\nabla}\psi$, i.e., in operator terms

$$\boldsymbol{\mathcal{P}} \;=\; -i\,\hbar\boldsymbol{\nabla}\;\;.$$

Hence one of the most mysterious "recipes" (or postulates) of quantum mechanics, the correspondence rule $\boldsymbol{p} \rightarrow -i\,\hbar\boldsymbol{\nabla}$, finds a natural interpretation once the complex "bi-velocity" is introduced.

Let us now introduce the wave function ψ in the equation of motion (5.6.9), which generalizes Newton's equation to nondifferentiable space. It takes the new form

$$\boldsymbol{\nabla}\mathcal{U} \;=\; 2\,i\;\mathcal{D}\;m\;\frac{d}{dt}\;(\boldsymbol{\nabla}\,ln\psi)\;\;.$$

Being aware that d and $\boldsymbol{\nabla}$ do not commute, we replace d/dt by its expression (5.6.5):

$$\boldsymbol{\nabla}\mathcal{U} = 2i\;\mathcal{D}\;m\;\Big[\;\frac{\partial}{\partial t}\;\;\boldsymbol{\nabla}\,ln\psi - i\;\mathcal{D}\;\Delta(\boldsymbol{\nabla}\,ln\psi) - 2i\;\mathcal{D}\;(\boldsymbol{\nabla}\,ln\psi\;.\boldsymbol{\nabla})(\boldsymbol{\nabla}\,ln\psi)\;\Big]$$

This expression may be simplified thanks to the three following identities, which may be established by straightforward calculation:

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[1] A misprint in the published version has been corrected: the wave function is define as $\exp(iS/2m\mathcal{D})$ instead of $\exp(iS)$.

$$\boldsymbol{\nabla}\,\Delta \;=\; \Delta\,\boldsymbol{\nabla}$$

$$(\boldsymbol{\nabla} f\,.\,\boldsymbol{\nabla})(\boldsymbol{\nabla} f) \;=\; \tfrac{1}{2}\;\boldsymbol{\nabla}(\boldsymbol{\nabla} f)^2$$

$$\frac{\Delta f}{f} \quad = \quad \Delta\, lnf \;+ (\boldsymbol{\nabla}\, lnf)^2$$

This implies

$$\tfrac{1}{2}\,\Delta\,(\boldsymbol{\nabla}\, ln\psi) + (\boldsymbol{\nabla}\, ln\psi\,.\,\boldsymbol{\nabla})(\boldsymbol{\nabla}\, ln\psi) \;=\; \tfrac{1}{2}\;\;\boldsymbol{\nabla}\;\frac{\Delta\psi}{\psi}\quad,$$

and we finally obtain

$$\frac{d}{dt}\;\boldsymbol{v} = -\,\boldsymbol{\nabla}\,\mathcal{U}\,/\,m = \;\; -2\;\;\mathcal{D}\;\boldsymbol{\nabla}\;\;\{\,i\;\;\frac{\partial}{\partial t}\;ln\psi\;+\;\mathcal{D}\;\frac{\Delta\psi}{\psi}\;\;\}\;.$$

Integrating this equation finally yields

$$\mathcal{D}^2\,\Delta\psi\;+\;i\;\mathcal{D}\;\frac{\partial}{\partial t}\,\psi\;\;-\;\;\frac{\mathcal{U}}{2m}\,\psi\;=\;0\quad, \qquad (5.6.11)$$

up to an arbitrary phase factor $\alpha(t)$ which may be set to zero by a suitable choice of the phase S. Replacing $\mathcal{D}$ by $\hbar/2m$, we get Schrödinger's equation

$$\mathcal{U}\;=\;\frac{i\,\hbar}{\psi}\;\frac{\partial}{\partial t}\,\psi\;+\;\frac{\hbar^2}{2m}\;\frac{\Delta\psi}{\psi}\quad.$$

This is, in our opinion, a demonstration that an eventual future theory based on the principle of scale relativity and the fractal space-time conjecture will give back quantum mechanics as an approximation. The very direct route from Newton's equation to Schrödinger's equation suggests to us the following interpretation of quantum mechanics: *quantum mechanics is mechanics in a non-differentiable space*. We shall also suggest in Sec 7.2 that the above formalism may be used in a different situation, namely that it may help solve the problem of structures arising from chaos.

We think that the above formalism solves the problem of the physical origin of the “complex plane” of quantum mechanics. One of the most mysterious feature of this theory was the complex nature of the probability

amplitude: being irreducible to classical laws, it led to the belief that it is impossible to find its origin in space-time and to the interpretation of quantum mechanics in terms of an abstract space. Here we have shown, by extending Nelson's formalism, that in a non-differentiable space-time one may attach to each point of space-time a plane $(\boldsymbol{U},\boldsymbol{V})$ (this plane is a coordinate-dependent "field"), and that the wave function is directly linked to this plane:

$$\boldsymbol{\nabla}\, ln\psi \;=\; \frac{1}{2\mathcal{D}}\;\; (\boldsymbol{U} + i\,\boldsymbol{V}) \;\; .$$

This returns the probability amplitude into 4-dimensional space-time, but into a space-time which is basically and irreducibly non-differentiable.

Paradoxically, if one accepts the above interpretation of quantum mechanics, one of the main difficulties is to understand why this "approximation" is so good. Recent theoretical and experimental work on nonlinear perturbations on the Schrödinger equation[45,46] give a first clue of an answer to this question: in the domain of low energy processes (spectra of atomic transitions and so on), nonlinear corrections are expected to be extremely faint.

In our own approach, the most relevant comparison concerning possible new predictions is in the relations between Newton's and Einstein's theories. The Newtonian theory of gravitation is indeed so precise in its predictions that it was thought for centuries that it was definitive. The relativistic theory, special and general, nevertheless embeds the Newtonian theory, and brings "corrections" to it in well-defined situations: specifically, when a variable or a relevant physical quantity becomes very large. The theory of relativity should be used when describing or using very high velocities ($v \approx c$), large distances (the case of cosmology) [but in special relativity already, even for $v << c$, the time transformation writes $t' = t + vx/c^2$, and there is a non-negligible relativistic correction for $x \gtrsim c^2/v$], high densities (neutron stars) or strong gravitational fields (black holes).

We suggest that the same is true of quantum mechanics. Most current experiments testing quantum mechanics to a very high degree of precision correspond to weak field and/or relatively large length scale. So possible

deviations from quantum mechanical predictions are to be searched in two situations:

**Strong field*: we have suggested[3] that the unexplained anomalous positron and electron lines observed at Darmstadt in heavy ion collisions are precisely a first occurrence of such a breaking of quantum mechanics in strong electromagnetic field.[1] We shall show in Sec. 5.10 that these spectra have indeed fractal properties whose main features are accounted for by fractal derivatives of the kind described in Sec. 3.7.

**Very small scale*, i.e., *very high energies* : we shall address more fully this case in Chapter 6, in which a theory of special scale relativity will be developed. It leads to a new relation between the length-time scale and the mass-energy-momentum scale and to several new predictions, among which that of the value of two fundamental scales in particle physics, namely the scale of "grand unification", and the electroweak symmetry breaking scale.

References

3. Nottale, L., 1989, *Int. J. Mod. Phys*. **A4**, 5047.

24. Einstein, A., in *Albert Einstein, Oeuvres choisies*, Vol. 1 (Seuil-CNRS, Paris, 1990).

25. Pais, A., *'Subtle is the Lord', the Science and Life of Albert Einstein* (Oxford University Press, 1982).

26. de Broglie, L., 1926, *C.R. Acad. Sci. Paris* **183**, 447; **185**, 380.

27. Bohm, D., 1952, *Phys. Rev*. **85**, 166, 180.

28. Dewdney, C., Holland, P.R., Kyprianidis, A., & Vigier, J.P., 1988, *Nature* **336**, 536.

29. Fényes, I., 1952, *Z. Physik* **132**, 81.

30. Kershaw, D., 1964, *Phys. Rev*. **B 136**, 1850.

31. Nelson, E., 1966, *Phys. Rev*. **150**, 1079.

32. Santamato, E., 1984, *Phys. Rev*. **D29**, 216; **D32**, 2615.

33. Santamato, E., 1988, *Phys. Lett.* **A 130**, 199.

34. Castro, C., 1990, *J. Math. Phys*. **31**, 2633.

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[1] This proposal has not been supported by the further developments of the theory.

35. Mandelbrot, B., *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).
36. Bohm, D., & Vigier, J.P., 1954, *Phys. Rev*. **96**, 208.
37. Weizel, W., 1954, *Z. Physik* **134**, 264; **135**, 270; **136**, 582.
38. Rosen, N., 1984, *Found. Phys.* **14**, 579.
39. Wheeler, J.A., 1989, in *Richard Feynman, Physics Today* (special issue, February), p.24.
40. Nelson, E., *Quantum Fluctuations* (Princeton Univ. Press, 1985).
41. Welsh, D.J.A., in *Mathematics Applied to Physics* (Springer,1970), p.465.
42. Davidson, M., 1979, *Physica,* **96A**, 465.
43. Shuker, D., *Letters in Math. Phys*. **4**, 61.
44. Kolmogorov, A.N., 1931, *Math. Annalen* **104**, 415
45. Weinberg, S., 1989, *Phys. Rev. Lett*. **62**, 485.
46. Bollinger, J.J., *et al.*, 1989, *Phys. Rev. Lett*. **63**, 1031.
99. Guerra, F., & Morato, L.M., 1983, *Phys. Rev*. **D27**, 1774.