Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2010, Article ID 157264, 26 pages
doi:10.1155/2010/157264

Review Article

Fractal Time Series—A Tutorial Review

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Received 23 September 2009; Accepted 29 October 2009

Academic Editor: Massimo Scalia

Fractal time series substantially differs from conventional one in its statistic properties. For instance, it may have a heavy-tailed probability distribution function (PDF), a slowly decayed autocorrelation function (ACF), and a power spectrum function (PSD) of $1/f$ type. It may have the statistical dependence, either long-range dependence (LRD) or short-range dependence (SRD), and global or local self-similarity. This article will give a tutorial review about those concepts. Note that a conventional time series can be regarded as the solution to a differential equation of integer order with the excitation of white noise in mathematics. In engineering, such as mechanical engineering or electronics engineering, engineers may usually consider it as the output or response of a differential system or filter of integer order under the excitation of white noise. In this paper, a fractal time series is taken as the solution to a differential equation of fractional order or a response of a fractional system or a fractional filter driven with a white noise in the domain of stochastic processes.

1. Introduction

Denote by $\mathbb{R}^n$ the n-dimensional Euclidean space for $n \in \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the set of positive integers. Then, things belonging to $\mathbb{R}^n$ for $n = 1, 2, 3$ are visible, such as a curve for $n = 1$, a picture for $n = 2$, and a three-dimensional object for $n = 3$.

Denote an element belonging to $\mathbb{R}^n$ by $f(x_1, \ldots, x_n)$ and $x_n \in \mathbb{R}$. Denote a regularly orthogonal coordinate system in $\mathbb{R}^n$ by $\{e_1, e_2, \ldots, e_n\}$. Then, the inner product (e_l, e_m) is given by

$$(e_l, e_m) = \begin{cases} 1, & l = m, \\ 0, & l \neq m. \end{cases} \tag{1.1}$$

Then,

$$f = \sum_{l=1}^{n} (f, e_l) e_l. \tag{1.2}$$

In the domain of the Hilbert space, $n \to \infty$ is allowed (Griffel [1], Liu [2]). Unfortunately, due to the limitation of the eyes of human being, a high-dimensional image of f, for example, $n > 4$, is invisible unless some of its elements are fixed. One can only see an image f for $n > 4$ partly. For example, if we fix the values of x_n for $n \geq 3$, $f(x_1, x_2, x_3, \ldots, x_n)$ is visible. Luckily, human being has nimbus such that people are able to think about high-dimensional objects in $\mathbb{R}^n$ even in the case of $n \to \infty$.

Note that the nature is rich and colorful (Mandelbrot [3], Korvin [4], Peters [5], Bassingthwaighte et al. [6]). Spaces of integer dimension are not enough. As a matter of factor, there exist spaces with fractional dimension, such as $\mathbb{R}^{n+d}$, where $0 < d < 1$ is a fraction. Therefore, even in the low-dimensional case of $n = 1, 2, 3$, those in $\mathbb{R}^{n+d}$ are not completely visible.

We now turn to time series. Intuitively, we say that $x(t)$ is a conventional series if $x(t) \in \mathbb{R}^1 \triangleq \mathbb{R}$. On the other side, $x(t)$ is said to be a fractal time series if it belongs to $\mathbb{R}^{1+d}$ for $0 < d < 1$. A curve of $x \in \mathbb{R}^{1+d}$ we usually see, such as a series of stock market price, is only its integer part belonging to $\mathbb{R}$. However, it is the fractional part of $x(t)$ that makes it substantially differ from a conventional series in the aspects of PDF, ACF, and PSD, unless d is infinitesimal.

The theory of conventional series is relatively mature; see, for example, Fuller [7], Box et al. [8], Mitra and Kaiser [9], Bendat and Piersol [10], but the research regarding fractal time series is quite academic. However, its applications to various fields of sciences and technologies, ranging from physics to computer communications, are increasing, for instance, coastlines, turbulence, geophysical record, economics and finance, computer memories (see, e.g., Mandelbrot [11]), network traffic, precision measurements (Beran [12], Li and Borgnat [13]), electronics engineering, chemical engineering, image compression; see, for example, Levy-Vehel et al. [14], physiology; see, for example, Bassingthwaighte et al. [6], just naming a few. The goal of this paper is to provide a short tutorial with respect to fractal time series.

The remaining article is organized as follows. In Section 2, the concept of fractal time series from the point of view of systems of fractional order will be addressed. The basic properties of fractal time series are explained in Section 3. Some models of fractal time series are discussed in Section 4. Conclusions are given in Section 5.

2. Fractal Time Series: A View from Fractional Systems

A time series can be taken as a solution to a differential equation. In terms of engineering, it is often called signal while a differential equation is usually termed system, or filter. Therefore, without confusions, equation, system, or filter is taken as synonyms in what follows.

2.1. Realization Resulted from a Filter of Integer Order

A stationary time series can be regarded as the output $y(t)$ of a filter under the excitation of white noise $w(t)$. Denote by $g(t)$ the impulse function of a linear filter. Then,

$$y(t) = \int_0^t g(t - \tau) w(\tau) d\tau. \tag{2.1}$$

On the other side, a nonstationary random function can be taken as the output of a filter under the excitation of nonstationary white noise. In general, filters with different $g(t)$'s may yield different series under the excitation of $w(t)$. Hence, conventionally, one considers $w(t)$ as the headspring or root of random series; see, for example, Press et al. [15]. In this paper, we only consider stationary series.

A stochastic filter can be written by

$$\sum_{i=0}^{p} a_i \frac{d^{p-i}y(t)}{dt^{p-i}} = \sum_{i=0}^{q} b_i \frac{d^{q-i}w(t)}{dt^{q-i}}. \tag{2.2}$$

Denote the Fourier transforms of $y(t)$, $g(t)$, and $w(t)$ by $Y(\omega)$, $G(j\omega)$, and $W(\omega)$, respectively, where $j = \sqrt{-1}$ and ω is angular frequency. Then, according to the theorem of convolution, one has

$$Y(\omega) = G(j\omega)W(\omega). \tag{2.3}$$

Denote the PSDs of $y(t)$ and $w(t)$ by $S_{yy}(\omega)$ and $S_{ww}(\omega)$, respectively. Then, when one notices that $S_{ww}(\omega)$ =1 if $w(t)$ is the normalized white noise [9, 10], one has

$$S_{yy}(\omega) = \left|G(j\omega)\right|^2. \tag{2.4}$$

Denote the Laplace transform of $g(t)$ by $G(s)$, where s is a complex variable. Then (Lam [16]),

$$G(s) = \frac{1 + \sum_{i=1}^{q} b_i s^i}{1 + \sum_{i=1}^{p} a_i s^i}. \tag{2.5}$$

If the system is stable, all poles of $G(s)$ are located on the left of s plan. For a stable filter, therefore, one has (Papoulis [17])

$$G(j\omega) = \mathrm{F}\left[g(t)\right] = G(s)|_{s=j\omega}, \tag{2.6}$$

where F stands for the operator of the Fourier transform. A basic property of a linear stable system of integer order is stated as follows.

Note 1. Taking into account $b_0 = 1$ and (2.6), one sees that $|G(j\omega)|^2$ of a stable system of integer order is convergent for $\omega = 0$ and so is $S_{yy}(\omega)$.

In the discrete case, the system function is expressed by the z transform of $g(n)$. That is,

$$G(z) = \mathrm{Z}\left[g(n)\right] = \sum_{n=0}^{\infty} g(n) z^{-n} = \frac{1 + \sum_{i=1}^{q} b_i z^{-i}}{1 + \sum_{i=1}^{p} a_i z^{-i}}, \tag{2.7}$$

where Z represents the operator of z transform. There are two categories of digital filters (Harger [18], Van de Vegte [19], Li [20]). One is in the category of infinite impulse response

(IIR) filters, which correspond to the case of $a_i \neq 0$. The other is in the category of finite impulse response filters (FIRs), which imply $a_i = 0$ [9, 16], (Harger [18], Van de Vegte [19]). In the FIR case, one has

$$G(z) = \sum_{n=0}^{q} g(n) z^{-n} = 1 + \sum_{i=1}^{q} b_i z^{-i}. \tag{2.8}$$

Thus, an FIR filter is always stable with a linear phase.

Note 2. A realization $y(t)$ resulted from an FIR filter of integer order under the excitation of $w(t)$ is linear. It belongs to $\mathbb{R}$.

2.2. Realization Resulted from a Filter of Fractional Order

Let $v > 0$ and $f(t)$ be a piecewise continuous on $(0, \infty)$ and integrable on any finite subinterval of $[0, \infty)$. For $t > 0$, denote by ${}_0D_t^{-v}$ the Riemann-Liouville integral operator of order v [21, page 45]. It is given by

$${}_0D_t^{-v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t-u)^{v-1} f(u) du, \tag{2.9}$$

where Γ is the Gamma function. For simplicity, we write ${}_0D_t^{-v}$ by D^{-v} below.

Let $v_p, v_{p-1}, \ldots, v_0$ and $u_q, u_{q-1}, \ldots, u_0$ be two strictly decreasing sequences of nonnegative numbers. Then, for the constants a_i and b_i, we have

$$\sum_{i=0}^{p} a_{p-i} D^{v_i} y(t) = \sum_{i=0}^{q} b_{q-i} D^{u_i} w(t), \tag{2.10}$$

which is a stochastically fractional differential equation with constant coefficients of order v_p. It corresponds to a stochastically fractional filter of order v_p. The transfer function of this filter expressed by using the Laplace transform is given by (Ortigueira [22])

$$G(s) = \frac{1 + \sum_{i=1}^{q} b_{q-i} s^{-u_i}}{1 + \sum_{i=1}^{p} a_{p-i} s^{-v_i}}. \tag{2.11}$$

In the discrete case, it is expressed in z domain by (Ortigueira [23, 24], Chen and Moore [25], Vinagre et al. [26])

$$G(z) = \frac{1 + \sum_{i=1}^{q} b_{q-i} z^{-u_i}}{1 + \sum_{i=1}^{p} a_{p-i} z^{-v_i}}. \tag{2.12}$$

Denote the inverse Laplace transform and the inverse z transform by L^{-1} and Z^{-1}, respectively. Then, the impulse responses of the filter expressed by (2.10) in the continuous and discrete cases are given by

$$\begin{aligned} g(t) &= \mathrm{L}^{-1}[G(s)], \\ g(n) &= \mathrm{Z}^{-1}[G(z)], \end{aligned} \tag{2.13}$$

respectively.

Without loss of the generality to explain the concept of fractal time series, we reduce (2.10) to the following expression:

$$\sum_{i=0}^{p} a_{p-i} D^{v_i} y(t) = w(t). \tag{2.14}$$

Consequently, (2.11) and (2.12) are reduced to

$$\begin{aligned} G(s) &= 1 + \sum_{i=1}^{q} b_{q-i} s^{-u_i}, \\ G(z) &= 1 + \sum_{i=1}^{q} b_{q-i} z^{-u_i}. \end{aligned} \tag{2.15}$$

Recall that the realization resulted from such a class of filters can be expressed in the continuous case by

$$y(t) = w(t) * g(t), \tag{2.16}$$

where $*$ implies the operation of convolution, or in the discrete case by

$$y(n) = w(n) * g(n). \tag{2.17}$$

Hence, we have the following notes.

Note 3. A realization $y(t)$ resulted from a stochastically fractional differential equation may be unbelonging to $\mathbb{R}$.

Note 4. For a stochastically fractional differential equation, Note 1 may be untrue.

We shall further explain Note 4 in the next section. As an example to interpret the point in Note 3, we consider a widely used fractal time series called the fractional Brownian motion (fBm) introduced by Mandelbrot and van Ness [27].

Replacing v with $H + 0.5$ in (2.9) for $0 < H < 1$, where H is the Hurst parameter, fBm defined by using the Riemann-Liouville integral operator is given by

$${}_0D_t^{-(H+1/2)} B'(t) = \frac{1}{\Gamma(H+1/2)} \int_0^t (t-u)^{H-1/2} dB(u) \triangleq B_H(t), \tag{2.18}$$

where $B(t)$, $t \in (0, \infty)$, is the Wiener Brownian motion; see, for example, Hida [28] for Brownian motion. The differential of $B(t)$ is in the sense of generalized function over the Schwartz space of test functions; see, for example, Gelfand and Vilenkin [29] for generalized functions. Taking into account the definition of the convolution used by Mikusinski [30], we have the impulse response of a fractional filter given by

$$\frac{(-t)^{H-1/2}}{\Gamma(H+1/2)}. \tag{2.19}$$

Consequently, fBm denoted by $B_H(t)$ can be taken as an output of the filter (2.19) under the excitation $dB(t)/dt$ (Li and Chi [31]). That is,

$$B_H(t) = \frac{dB(t)}{dt} * \frac{(-t)^{H-1/2}}{\Gamma(H+1/2)}. \tag{2.20}$$

Therefore, Note 5 comes.

Note 5. FBm is a special case as a realization of a fractional filter driven with $dB(t)/dt$.

Other articles discussing fBm from the point of view of systems or filters of fractional order can be seen in Ortigueira [32], Ortigueira and Batista [33, 34], and Podlubny [35]. In the end of this section, I use another equation to interpret the concept of fractal time series. The fractional oscillator or fractional Ornstein-Uhlenbeck process is the solution of the fractional Langevin equation given by

$$({}_aD_t + A)^{\alpha} y(t) = w(t), \quad \alpha > 0, \tag{2.21}$$

where A is a positive constant, and $w(t)$ is the white noise (Lim et al. [36, 37]). Obviously, the fractal time series $y(t)$ in (2.21) is a realization resulted from a fractional filter under the excitation $w(t)$. More about this will be discussed in Section 4.

3. Basic Properties of Fractal Time Series

Fractal time series has its particular properties in comparison with the conventional one. Its power law in general is closely related to the concept of memory. A particular point, which has to be paid attention to, is that there may usually not exist mean and/or variance in such a series. This may be a main reason why measures of fractal dimension and the Hurst parameter play a role in the field of fractal time series.

3.1. Power Law in Fractal Time Series

Denote the ACF of $x(t)$ by $r_{xx}(\tau)$, where $r_{xx}(\tau) = \mathrm{E}[x(t)x(t+\tau)]$. Then, $x(t)$ is called SRD if r_{xx} is integrable (Beran [12]), that is,

$$\int_0^{\infty} r_{xx}(\tau) d\tau < \infty. \tag{3.1}$$

On the other side, $x(t)$ is LRD if r_{xx} is nonintegrable, that is,

$$\int_0^\infty r_{xx}(\tau)d\tau = \infty. \tag{3.2}$$

A typical form of such an ACF for r_{xx} being nonintegrable has the following asymptotic expression:

$$r_{xx}(\tau) \sim c|\tau|^{-\beta} \quad (\tau \longrightarrow \infty), \tag{3.3}$$

where $c > 0$ is a constant and $0< \beta <1$. The above expression implies a power law in the ACF of LRD fractal series.

Denote the PSD of $x(t)$ by $S_{xx}(\omega)$. Then,

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} r_{xx}(t)e^{-j\omega t}dt. \tag{3.4}$$

In the LRD case, the above $S_{xx}(\omega)$ does not exist as an ordinary function but it can be regarded as a function in the domain of generalized functions. Since

$$\mathrm{F}\left(|\tau|^{-\beta}\right) = 2\sin\left(\frac{\pi\beta}{2}\right)\Gamma(1-\beta)|\omega|^{\beta-1} \tag{3.5}$$

see, for example, [29] and Li and Lim [38, 39], the PSD of LRD series has the property of power law. It is usually called $1/f$ noise or $1/f^{\alpha}$ $(\alpha > 0)$ noise (Mandelbrot [40]). Thus, comes Note 6.

Note 6. The PSD of an LRD fractal series is divergent for $\omega = 0$. This is a basic property of LRD fractal time series, which substantially differs from that as described in Note 1.

Denote the PDF of $x(t)$ by $p(x)$. Then, the ACF of $x(t)$ can be expressed by

$$r_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)p(x)dx. \tag{3.6}$$

Considering that r_{xx} is nonintegrable in the LRD case, one sees that a heavy-tailed PDF is an obvious consequence of LRD series; see, for example, Li [41, 42], Abry et al. [43].

Denote μ_x the mean of $x(t)$. Then,

$$\mu_x = \int_{-\infty}^{\infty} xp(x)dx. \tag{3.7}$$

The variance of $x(t)$ is given by

$$\mathrm{Var}(x) = \int_{-\infty}^{\infty} (x-\mu_x)^2 p(x)dx. \tag{3.8}$$

One thing remarkable in LRD fractal time series is that the tail of $p(x)$ may be so heavy that the above integral either (3.7) or (3.8) may not exist. To explain this, we recall a series obeying the Pareto distribution that is a commonly used heavy-tailed distribution. Denote $p_{\text{Pareto}}(x)$ the PDF of the Pareto distribution. Then,

$$p_{\text{Pareto}}(x) = \frac{ab}{x^{a+1}}, \tag{3.9}$$

where $x \geq a$. The mean and variance of $x(t)$ that follows $p_{\text{Pareto}}(x)$ are respectively given by

$$\begin{aligned} \mu_{\text{Pareto}} &= \frac{ab}{a-1}, \\ \text{Var}\,(x)_{\text{Pareto}} &= \frac{ab^2}{(a-1)^2(a-2)}. \end{aligned} \tag{3.10}$$

It can be easily seen that μ_{Pareto} and $\text{Var}(x)_{\text{Pareto}}$ do not exist for $a = 1$. That fractal time series with LRD may not have its mean and or variance is one of its particular points [6].

Note that μ_x implies a global property of $x(t)$ while $\text{Var}(x)$ represents a local property of $x(t)$. For an LRD $x(t)$, unfortunately, in general, the concepts of mean and variance are inappropriate to describe the global property and the local one of $x(t)$. We need other measures to characterize the global property and the local one of LRD $x(t)$. Fractal dimension and the Hurst parameter are utilized for this purpose.

3.2. Fractal Dimension and the Hurst Parameter

In fractal time series, one, respectively, uses the fractal dimension and the Hurst parameter of $x(t)$ to describe its local property and the global one ([3], Li and Lim [39, 44]). In fact, if r_{xx} is sufficiently smooth on $(0, \infty)$ and if

$$r_{xx}(0) - r_{xx}(\tau) \sim c_1|\tau|^{\alpha} \quad \text{for } |\tau| \longrightarrow 0, \tag{3.11}$$

where c_1 is a constant and α is the fractal index of $x(t)$, the fractal dimension of $x(t)$ is expressed by

$$D = 2 - \frac{\alpha}{2}; \tag{3.12}$$

see, for example, Kent and Wood [45], Hall and Roy [46], and Adler [47].

On the other side, expressing β in (3.3) by the Hurst parameter $0.5 < H < 1$ yields

$$\beta = 2 - 2H. \tag{3.13}$$

Therefore,

$$r_{xx}(\tau) \sim c|\tau|^{2H-2} \quad (\tau \longrightarrow \infty). \tag{3.14}$$

Different from those in conventional series, we, respectively, use D and H to characterize the local property and the global one of LRD $x(t)$ rather than mean and variance (Gneiting and Schlather [48], Lim and Li [49]).

In passing, we mention that the estimation of H and/or D becomes a branch of fractal time series as can be seen from [11, 12]. Various methods regarding the estimation of fractal parameters are reported; see, for example, Taqqu et al. [50], methods based on ACF regression (Li and Zhao [51] and Li [52]), periodogram regression method (Raymond et al. [53]), generalized linear regression (Beran [54, 55]), scaled and rescaled windowed variance methods ([56–58], Schepers et al. [59], Mielniczuk and Wojdłło [60], Cajueiro and Tabak [61]), dispersional method (Raymond and Bassingthwaighte [62, 63]), maximum likelihood estimation methods (Kendziorski et al. [64], Guerrero and Smith [65]), methods based on wavelet [66–72], fractional Fourier transform (Chen et al. [73]) and detrended method (Govindan [74]).

In the end of this section, we note that self-similarity of a stationary process is a concept closely relating to fractal time series. Fractional Gaussian noise (fGn) is an only stationary increment process with self-similarity (Samorodnitsky and Taqqu [75]). In general, however, a fractal time series may not be globally self-similar. Nevertheless, a series that is not self-similar may be locally self-similar [47].

4. Some Models of Fractal Time Series

Fractal time series can be classified into two classes from a view of statistical dependence. One is LRD and the other is SRD. It can be also classified into Gaussian series or nonGaussian ones. I shall discuss the models of fractal time series of Gaussian type in Sections 4.1–4.4, and 4.6. Series of nonGaussian type will be described in Section 4.5.

4.1. Fractional Brownian Motion (fBm)

FBm is commonly used in modeling nonstationary fractal time series. It is Gaussian (Sinai [76, 77]). The definition of fBm described in (2.18) is called the Riemann-Liouville type since it uses the Riemann-Liouville integral; see, for example, [27], Sithi and Lim [78], Muniandy and Lim [79], and Feyel and de la Pradelle [80]. Its PSD is given by

$$S_{B_H,RL}(t,\omega) = \frac{\pi\omega t}{\omega^{2H+1}}\left[J_H(2\omega t)\mathbb{H}_{H-1}(2\omega t) - J_{H-1}(2\omega t)\mathbb{H}_H(2\omega t)\right], \tag{4.1}$$

where J_H is the Bessel function of order H (G.A. Korn and T.M. Korn [81]), $\mathbb{H}_H$ is the Struve function of order H, and the subscript on the left side implies the type of the Riemann-Liouville integral, see [78] for details. The ACF of the fBm of the Riemann-Liouville type is given by

$$r_{B_H,RL}(t,s) = \frac{t^{H+1/2}s^{H-1/2}}{(H+1/2)\Gamma(H+1/2)^2}\,{}_2F_1\left(\frac{1}{2}-H,1,H+\frac{1}{2},\frac{t}{s}\right), \tag{4.2}$$

where ${}_2F_1$ is the hypergeometric function.

Note that the increment process of the fBm of the Riemann-Liouville type is nonstationary (Lim and Muniandy [82]). Therefore, another definition of fBm based on the Weyl integral [27] is usually used when considering stationary increment process of fBm.

The Weyl integral of order v is given for $v > 0$ by [21]

$$W^{-v} f(t) = \frac{1}{\Gamma(v)} \int_t^{\infty} (u-t)^{v-1} f(u) du. \tag{4.3}$$

Thus, the fBm of the Weyl type is defined by

$$B_H(t) - B_H(0) = \frac{1}{\Gamma(H+1/2)} \left\{ \int_{-\infty}^{0} \left[(t-u)^{H-0.5} - (-u)^{H-0.5} \right] dB(u) + \int_0^t (t-u)^{H-0.5} dB(u) \right\}. \tag{4.4}$$

It has stationary increment. Its PSD is given by (Flandrin [83])

$$S_{B_H,W}(t,\omega) = \frac{1}{|\omega|^{2H+1}} \left(1 - 2^{1-2H} \cos 2\omega t \right), \tag{4.5}$$

Its ACF is expressed by

$$r_{B_H,W}(t,s) = \frac{V_H}{(H+1/2)\Gamma(H+1/2)} \left[|t|^{2H} + |s|^{2H} - |t-s|^{2H} \right], \tag{4.6}$$

where V_H is the strength of the fBm and it is given by

$$V_H = \mathrm{Var}[B_H(1)] = \Gamma(1-2H) \frac{\cos \pi H}{\pi H}. \tag{4.7}$$

The basic properties of fBm are listed below.

Note 7. Either the fBm of the Riemann-Liouville type or the one of the Weyl type is nonstationary as can be seen from (4.1) and (4.5).

Note 8. Both the fBm of the Riemann-Liouville type and the one of the Weyl type are self-similar because they have the property expressed by

$$B_H(at) \equiv a^H B_H(t), \quad a > 0, \tag{4.8}$$

where $\equiv$ denotes equality in the sense of probability distribution.

Note 9. The PSD of fBm is divergent at $\omega = 0$, exhibiting a case of $1/f^{\alpha}$ noise.

Note 10. The process fBm reduces to the standard Brownian motion when $H = 1/2$, as can be seen from (2.18) and (4.4).

Note 11. A consequence of Note 10 is

$$S_{B_{1/2},RL}(t,\omega) = S_{B_{1/2},W}(t,\omega) = \frac{1}{\omega^2}(1-\cos 2\omega t), \tag{4.9}$$

which is the PSD of the standard Brownian motion [78].

Note 12. The fractal dimension of fBm is given by

$$D_{\text{fBm}} = 2 - H_{\text{fBm}}. \tag{4.10}$$

4.2. Generalized Fractional Brownian Motion with Holder Function

Recall that the fractal dimension of a sample path represents its self-similarity. For fBm, however, D_{fBm} is linearly related to H_{fBm} (4.10). On the other hand, (4.8) holds for all time scales. Hence, (4.8) represents a global self-similarity of fBm. This is a monofractal character, which may be too restrictive for many practical applications. Lim and Muniandy [82] replaced the Hurst parameter H in (4.4) by a continuously deterministic function $H(t)$ to obtain a form of the generalized fBm. The function $H(t)$ satisfies $H : [0,\infty) \to (0,1)$. Denote the generalized fBm by $X(t)$, instead of $B_H(t)$, so as to distinguish it from the standard one. Then,

$$X(t) = \frac{1}{\Gamma(H(t)+1/2)}\left\{\int_{-\infty}^{0}\left[(t-u)^{H(t)-0.5} - (-u)^{H(t)-0.5}\right]dB(u) + \int_{0}^{t}(t-u)^{H(t)-0.5}dB(u)\right\}. \tag{4.11}$$

By using $H(t)$, one has a tool to characterize local properties of fBm. The following ACF holds for $\tau \to 0$:

$$\mathrm{E}[X(t)X(t+\tau)] = \frac{V_{H(t)}}{(H(t)+1/2)\Gamma(H(t)+1/2)}\left[|t|^{2H(t)} + |t+\tau|^{2H(t)} - |\tau|^{2H(t)}\right]. \tag{4.12}$$

The self-similarity expressed below is in the local sense as $H(t)$ is time varying

$$X(at) \equiv a^{H(t)}X(t), \quad a > 0. \tag{4.13}$$

Assume that $H(t)$ is a β-Holder function. Then, $0 < \inf[H(t)] \le \sup[H(t)] < \min(1,\beta)$. Therefore, one has the following local Hausdorff dimension of $x(t)$ for $[a,b] \subset \mathbb{R}^+$:

$$\dim\{X(t),\ t \in [a,b]\} = 2 - \min\{H(t),\ t \in [a,b]\}. \tag{4.14}$$

The above expression also exhibits the local self-similarity of $X(t)$.

Based on the local growth of the increment process, one may write a sequence expressed by

$$S_k(j) = \frac{m}{N-1}\sum_{j=0}^{j+k}|X(i+1) - X(i)|, \quad 1 < k < N, \tag{4.15}$$

where m is the largest integer not exceeding N/k. Then, $H(t)$ at point $t = j/(N-1)$ is given by

$$H(t) = -\frac{\log\left(\sqrt{\pi/2}S_k(j)\right)}{\log(N-1)}; \tag{4.16}$$

see Peltier and Levy-Vehel [84, 85] for the details. Li et al. [86] demonstrate an application of this type of fBm to network traffic modeling, and Muniandy et al. [87] in financial engineering.

4.3. Fractional Gaussian Noise (fGn)

The continuous fGn is the derivative of the smoothed fBm that is in the domain of generalized functions. Its ACF denoted by $C_H(\tau;\varepsilon)$ is given by

$$C_H(\tau;\varepsilon) = \frac{V_H\varepsilon^{2H-2}}{2}\left[\left(\frac{|\tau|}{\varepsilon}+1\right)^{2H} + \left|\frac{|\tau|}{\varepsilon}-1\right|^{2H} - 2\left|\frac{\tau}{\varepsilon}\right|^{2H}\right], \quad \tau \in \mathbb{R}, \tag{4.17}$$

where $H \in (0,1)$ is the Hurst parameter and $\varepsilon > 0$ is used by smoothing fBm so that the smoothed fBm is differentiable [27].

FGn includes three classes of time series. When $H \in (0.5,1)$, $C_H(\tau;\varepsilon)$ is positive and finite for all τ. It is nonintegrable and the corresponding series is LRD. For $H \in (0,0.5)$, the integral of $C_H(\tau;\varepsilon)$ is zero and $C_H(0;\varepsilon)$ diverges when $\varepsilon \to 0$. In addition, $C_H(\tau;\varepsilon)$ changes its sign and becomes negative for some τ proportional to ε in this parameter domain [27, page 434]. FGn reduces to the white noise when $H = 0.5$.

The PSD of fGn is given by (Li and Lim [38])

$$S_{\mathrm{fGn}}(\omega) = \sigma^2 \sin(H\pi)\Gamma(2H+1)|\omega|^{1-2H}. \tag{4.18}$$

Denote the discrete fGn by dfGn. Then, the ACF of dfGn is given by

$$r_{\mathrm{dfGn}}(k) = \frac{\sigma^2}{2}\left[(|k|+1)^{2H} + ||k|-1|^{2H} - 2|k|^{2H}\right]. \tag{4.19}$$

Its PSD, see Sinai [77], is given by

$$S_{\mathrm{dfGn}}(\omega) = 2C_f(1-\cos\omega)\sum_{n=-\infty}^{\infty}|2\pi n+\omega|^{-2H-1}, \quad \omega\in[-\pi,\pi], \tag{4.20}$$

where $C_f = \sigma^2(2\pi)^{-1}\sin(\pi H)\Gamma(2H+1)$.

Note that the expression $0.5[(k+1)^{2H} - 2k^{2H} + (k-1)^{2H}]$ is the finite second-order difference of $0.5(k)^{2H}$. Approximating it with the second-order differential of $0.5(k)^{2H}$ yields

$$0.5\left[(k+1)^{2H} - 2k^{2H} + (k-1)^{2H}\right] \approx H(2H-1)(k)^{2H-2}. \tag{4.21}$$

The above approximation is quite accurate for $k > 10$ [11]. Hence, taking into account (3.12) and (3.13), the following immediately appears (Li and Lim [44]):

$$D_{\mathrm{fGn}} = 2 - H_{\mathrm{fGn}}. \tag{4.22}$$

Hence, we have the following notes.

Note 13. The fGn as the increment process of the fBm of the Weyl type is stationary. It is exactly self-similar with the global self-similarity described by (4.22).

Note 14. The PSD of the fGn is divergent at $\omega = 0$.

Again, we remark that the fGn may be too strict for modeling a real series in practice. Hence, generalized versions of fGn are expected. One of the generalization of fGn is to replace H by $H(t)$ in (4.19) ([82]) so that

$$r_{\mathrm{dfGn}}(k;H(t)) = \frac{\sigma^2}{2}\left[(|k|+1)^{2H(t)} + ||k|-1|^{2H(t)} - 2|k|^{2H(t)}\right]. \tag{4.23}$$

Another generalization by Li [88] is given by

$$r_{\mathrm{dfGn}}(k;H,a) = \frac{\sigma^2}{2}\left(\left||k|^a+1\right|^{2H} - 2\left||k|^a\right|^{2H} + \left||k|^a-1\right|^{2H}\right), \quad 0 < a \le 1. \tag{4.24}$$

In (4.23), if $H(t) = \text{const}$, the ACF reduces to that of the standard fGn. On the other side, $r_{\mathrm{dfGn}}(k;H,a)$ in (4.24) becomes the ACF of the standard fGn if $\alpha = 1$.

4.4. Generalized Cauchy (GC) Process

As discussed in Section 2, we use two parameters, namely, D and H, to respectively measure the local behavior and the global one of fractal time series instead of variance and mean. More precisely, the former measures a local property, namely, local irregularity, of a sample path while the latter characterizes a global property, namely, LRD. The parameter $1 < D < 2$ is independent of $0 < H < 1$ in principle as can be seen from [3]. By using a single parameter model, such as fGn and fBm, D and H happen to be linearly related. Hence, a single parameter model fails to separately capture the local irregularity and LRD. To release such relationship, two-parameter model is needed. The GC process is one of such models.

A series $X(t)$ is called the GC process if it is a stationary Gaussian centred process with the ACF given by

$$C_{\mathrm{GC}}(\tau) = \mathrm{E}[X(t+\tau)X(t)] = \left(1+|\tau|^{\alpha}\right)^{-\beta/\alpha}, \tag{4.25}$$

where $0 < \alpha \le 2$ and $\beta > 0$. The ACF $C_{\mathrm{GC}}(\tau)$ is positive-definite for the above ranges of α and β and it is a completely monotone for $0 < \alpha \le 1,\ \beta > 0$. When $\alpha = \beta = 2$, one gets the usual Cauchy process that is modeled by its ACF expressed by

$$C(\tau) = \left(1+|\tau|^{2}\right)^{-1}, \tag{4.26}$$

which has been applied in geostatistics; see, for example, Chiles and Delfiner [89].

The function $C_{\mathrm{GC}}(\tau)$ has the asymptotic expressions of (3.11) and (3.14). More precisely, we have

$$\begin{aligned} C_{\mathrm{GC}}(\tau) &\sim |\tau|^{\alpha}, \quad \tau \longrightarrow 0, \\ C_{\mathrm{GC}}(\tau) &\sim |\tau|^{-\beta}, \quad \tau \longrightarrow \infty. \end{aligned} \tag{4.27}$$

According to (3.12) and (3.13), therefore, one has

$$D_{\mathrm{GC}} = 2 - \frac{\alpha}{2}, \tag{4.28}$$

$$H_{\mathrm{GC}} = 1 - \frac{\beta}{2}. \tag{4.29}$$

When considering the multiscale property of a series, one may utilize the time varying D_{GC} and H_{GC} on an interval-by-interval basis. Denote the fractal dimension and the Hurst parameter in the Ith interval by $D_{\mathrm{GC}}(I)$ and $H_{\mathrm{GC}}(I)$, respectively. Then, we have the ACF in the Ith interval given by

$$C_{\mathrm{GC}}(\tau; I) = \left(1+\tau^{\alpha(I)}\right)^{-\beta(I)/\alpha(I)}, \quad \tau \ge 0. \tag{4.30}$$

Consequently, we have

$$\begin{aligned} D_{\mathrm{GC}}(n) &= 2 - \frac{\alpha(n)}{2}, \\ H_{\mathrm{GC}}(n) &= 1 - \frac{\beta(n)}{2}. \end{aligned} \tag{4.31}$$

Denote $\mathrm{Sa}(\omega) = (\sin\omega)/\omega$. Then, the PSD of the GC process is given by (Li and Lim [39])

$$S_{\mathrm{GC}}(\omega) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma[(\beta/\alpha)+k]}{\pi\Gamma(\beta/\alpha)\Gamma(1+k)} I_1(\omega) * \mathrm{Sa}(\omega) + \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma[(\beta/\alpha)+k]}{\pi\Gamma(\beta/\alpha)\Gamma(1+k)} [\pi I_2(\omega) - I_2(\omega) * \mathrm{Sa}(\omega)], \tag{4.32}$$

where

$$I_1(\omega) = -2\sin\left(\frac{\alpha k\pi}{2}\right)\Gamma(\alpha k+1)|\omega|^{-\alpha k-1}, \quad I_2(\omega) = 2\sin\left[\frac{(\beta+\alpha k)\pi}{2}\right]\Gamma[1-(\beta+\alpha k)]|\omega|^{(\beta+\alpha k)-1}. \tag{4.33}$$

In practice, the asymptotic expressions of $S_{\mathrm{GC}}(\omega)$ for small frequency and large one may be useful. The PSD of the GC process for $\omega \to 0$ is given by

$$S_{\mathrm{GC}}(\omega) \sim \frac{1}{\Gamma(\beta)\cos(\beta\pi/2)}|\omega|^{\beta-1}, \quad \omega \longrightarrow 0, \tag{4.34}$$

which is actually the inverse Fourier transform of $C_{\mathrm{GC}}(\tau)$ for $\tau \to \infty$. On the other hand, $S_{\mathrm{GC}}(\omega)$ for $\omega \to \infty$ is given by

$$S_{\mathrm{GC}}(\omega) \sim \frac{\beta\Gamma(1+\alpha)\sin(\alpha\pi/2)}{\pi\alpha}|\omega|^{-(1+\alpha)}, \quad \omega \longrightarrow \infty; \tag{4.35}$$

see [49] for details. As shown in (4.34) and (4.35), one may easily observe the power law that $S_{\mathrm{GC}}(\omega)$ obeys.

Note 15. The GC process is LRD if $0 < \beta < 1$. It is SRD if $1 < \beta$. Its statistical dependence is measured by H (4.29).

Note 16. The GC process has the local self-similarity measured by D_{GC} expressed by (4.28).

Note 17. The GC process is nonMarkovian since $C_{\mathrm{GC}}(t_1, t_2)$ does not satisfy the triangular relation given by

$$C_{\mathrm{GC}}(t_1, t_3) = \frac{C_{\mathrm{GC}}(t_1, t_2)C_{\mathrm{GC}}(t_2, t_3)}{C_{\mathrm{GC}}(t_2, t_2)}, \quad t_1 < t_2 < t_3, \tag{4.36}$$

which is a necessary condition for a Gaussian process to be Markovian (Todorovic [90]). In fact, up to a multiplicative constant, the Ornstein-Uhlenbeck process is the only stationary Gaussian Markov process (Lim and Muniandy [91], Wolpert and Taqqu [92]).

The above discussions exhibit that the GC model can be used to decouple the local behavior and the global one of fractal time series, flexibly better agreement with the real data for both short-term and long-term lags. Li and Lim gave an analysis of the modeling performance of the GC model in Hilbert space [93]. The application of the GC process to network traffic modeling refers to [44], and Li and Zhao [94]. Recently, Lim and Teo [95] extended the GC model to describe the Gaussian fields and Gaussian sheets. Vengadesh et al. [96] applied it to the analysis of bacteriorhodopsin in material science.

4.5. Alpha-Stable Processes

As previously mentioned, two-parameter models are useful as they can separately characterize the local irregularity and global persistence. The CG process is one of such models and it is Gaussian. In some applications, for example, network traffic at small scales, a series is nonGaussian; see, for example, Scherrer et al. [97]. One type of models that are of two-parameter and nonGaussian in general is α-stable process.

Stable distributions imply a family of distributions. They are defined by their characteristic functions given by [75, page 5], for a random variable Y,

$$\Phi(\theta) = E\left(e^{j\theta Y}\right) = \begin{cases} \exp\left\{ j\mu\theta - |\sigma\theta|^{\alpha}\left[1 + j\beta\,\mathrm{sign}(\theta)\tan\left(\frac{\pi\alpha}{2}\right)\right]\right\}, & \alpha \neq 1, \\ \exp\{ j\mu\theta - |\sigma\theta|\left[1 + j\beta\,\mathrm{sign}(\theta)\ln(|\theta|)\right]\}, & \alpha = 1. \end{cases} \tag{4.37}$$

The expression $Y \sim S_{\sigma,\beta,\mu}^{(\alpha)}$ implies that Y follows $\Phi(\theta)$.

The parameters in $\Phi(\theta)$ are explained as follows.

(i) The parameter $0 < \alpha \leq 2$ is characteristic exponent. It specifies the level of local roughness in the distribution, that is, the weight of the distribution tail.

(ii) The parameter $-1 \leq \beta \leq 1$ specifies the skewness. Its positive values correspond to the right tail while negative ones to the left.

(iii) The parameter $\sigma \geq 0$ is a scale factor, implying the dispersion of the distribution.

(iv) $\mu \in \mathbb{R}$ is the location parameter, expressing the mean or median of the distribution.

Note 18. The family of α-stable distributions does not have a closed form of expressions in general. A few exceptions are the Cauchy distribution and the Levy one.

Note 19. The property of heavy tail is described as follows. $\mathrm{E}[|Y|^p] < \infty$ for $p \in (0, \alpha)$, and $\mathrm{E}[|Y|^p] = \infty$ for $p \geq \alpha$.

When $\alpha = 2$, the characteristic function (4.37) reduces to that of the Gaussian distribution with the mean denoted by μ and the variance denoted by $2\sigma^2$. That is,

$$\Phi(\theta) = \mathrm{E}\left(e^{j\theta Y}\right) = \exp\left[j\mu\theta - (\sigma\theta)^2\right]. \tag{4.38}$$

In this case, the PDF of Y is symmetric about the mean.

Alpha-stable processes are in general nonGaussian. They include two. One is linear fractional stable noise (LFSN) and the other log-fractional stable noise (Log-FSN).

The model of linear fractional stable motion (LFSM) is defined by the following stochastic integral [75, page 366]. Denote by $L_{\alpha,H}(t)$ the LFSM. Then,

$$L_{\alpha,H}(t) = \int_{-\infty}^{\infty} \left\{ a\left[(t-u)_+^{H-1/\alpha} - (-u)_+^{H(t)-1/\alpha}\right] + b\left[(t-u)_-^{H-1/\alpha} - (-u)_-^{H(t)-1/\alpha}\right] \right\} M du, \tag{4.39}$$

where a and b are arbitrary constants, $M \in \mathbb{R}$ is a random measure, and H the Hurst parameter. The range of H is given by

$$H = \begin{cases} (0,1], & \alpha \geq 1, \\ \left(0, \dfrac{1}{\alpha}\right], & \alpha < 1. \end{cases} \tag{4.40}$$

Denote by $Lo_{\alpha,H}(t)$ the Log-FSM. Then,

$$Lo_{\alpha}(t) = \int_{-\infty}^{\infty} [\ln|t-Y| - \ln|Y|] M du. \tag{4.41}$$

LSFN is the increments process of LSFM while Log-FSN is the increment process of Log-FSM. Denote the LSFN and Log-FSN respectively by $N_{\alpha,H}(i)$ and $NLo_{\alpha,H}(i)$. Then,

$$\begin{aligned} N_{\alpha,H}(i) &= L_{\alpha,H}(i+1) - L_{\alpha,H}(i), \quad i \in \mathbb{Z}, \\ NLo_{\alpha,H}(i) &= Lo_{\alpha}(i+1) - Lo_{\alpha}(i), \quad i \in \mathbb{Z}. \end{aligned} \tag{4.42}$$

LSFN is nonGaussian except α =2. It is stationary self-similar with the self-similarity measured by H and the local roughness characterized by α [75]. However, two parameters are not independent because the LRD condition ([75], Karasaridis and Hatzinakos [98]) relates them by

$$\alpha H > 1. \tag{4.43}$$

4.6. Ornstein-Uhlenbeck (OU) Processes and Their Generalizations

In the above subsections, the series may be LRD. We now turn to a type of SRD fractal time series called OU processes.

4.6.1. Ordinary OU Process

Following the idea addressed by Uhlenbeck and Ornstein [99], the ordinary OU process is regarded as the solution to the Langevin equation (see, e.g., [91, 92], Lu [100], Valdivieso et al. [101]), which is a stochastic differential equation given by

$$\left(\frac{d}{dt} + \lambda\right) X(t) = w(t),$$
$$X(0) = X_0, \tag{4.44}$$

where λ is a positive parameter, $w(t)$ is the white noise with zero mean, and X_0 is a random variable independent of the standard Brownian motion $B(t)$. The stationary solution to the above equation is given by

$$X(t) = X_0 e^{-\lambda t} + \int_{-\infty}^{t} e^{\lambda u} w(u) du. \tag{4.45}$$

Denote the Fourier transforms of $w(t)$ and $X(t)$, respectively, by $W(\omega)$ and $X(\omega)$. Note that the system function of (4.44) in the frequency domain is given by

$$G_{\mathrm{OU}}(\omega) = \frac{1}{\lambda + j\omega}. \tag{4.46}$$

Then, according to the convolution theorem, one has

$$X(\omega) = G_{\mathrm{OU}}(\omega) W(\omega). \tag{4.47}$$

Since the PSD of the normalized $w(t)$ equals to 1, that is, $|W(\omega)|^2 = 1$, we immediately obtain the PSD of the OU process given by

$$S_{\mathrm{OU}}(\omega) = \frac{1}{\lambda^2 + \omega^2}. \tag{4.48}$$

Consequently, the ACF of the OU process is given by

$$\mathrm{E}[X(t)X(t+\tau)] = \mathrm{F}^{-1}[S_{\mathrm{OU}}(\omega)] = \frac{e^{-\lambda|\tau|}}{2\lambda}, \tag{4.49}$$

where F^{-1} is the operator of the inverse Fourier transform.

The ordinary OU process is obviously SRD. It is one-dimensional. What interests people in the field of fractal time series is the generalized OU processes described hereinafter.

4.6.2. Generalized Version I of the OU process

Consider the following fractional Langevin equation with a single parameter $\beta > 0$:

$$\left(\frac{d}{dt}+\lambda\right)^{\beta} X_1(t) = w(t). \tag{4.50}$$

Denote by $g_{X_1}(t)$ the impulse response function of the above system. Then, it is the solution to the following equation:

$$\left(\frac{d}{dt}+\lambda\right)^{\beta} g_{X_1}(t) = \delta(t), \tag{4.51}$$

where $\delta(t)$ is the Dirac-δ function. Doing the Fourier transforms on the both sides on the above equation yields

$$G_{X_1}(\omega) = \frac{1}{(\lambda - j\omega)^{\beta}}, \tag{4.52}$$

where $G_{X_1}(\omega)$ is the Fourier transform of $g_{X_1}(t)$.

Note that the PSD of $X_1(t)$ is equal to

$$S_{X_1}(\omega) = G_{X_1}(\omega)[G_{X_1}(\omega)]^{*}, \tag{4.53}$$

where $[G_{X_1}(\omega)]^{*}$ is the complex conjugate of $G_{X_1}(\omega)$. Then,

$$S_{X_1}(\omega) = \frac{1}{(\lambda^2 + \omega^2)^{\beta}}, \tag{4.54}$$

which is the solution to (4.50) in the frequency domain. The solution to (4.50) in the time domain, therefore, is given by

$$C_{X_1}(\tau) = \mathrm{E}[X_1(t)X_1(t+\tau)] = \mathrm{F}^{-1}[S_{X_1}(\omega)] = \frac{\lambda^{-2v}}{2^{v}\sqrt{\pi}\Gamma(v+1/2)}|\lambda\tau|^{v}K_v(|\lambda\tau|), \tag{4.55}$$

where $v = \beta - 1/2$ and K_v is the modified Bessel function of the second kind of order v [29, 91].

Let $v = H \in (0,1)$. Then, one has

$$S_{X_1}(\omega) = \frac{1}{(\lambda^2 + \omega^2)^{H+1/2}}, \tag{4.56}$$

which exhibits that $X_1(t)$ is SRD because its PSD is convergent for $\omega \to 0$.

Keep in mind that the Langevin equation is in the sense of generalized functions since we take $w(t)$ as the differential of the standard Brownian motion $B(t)$, which is differentiable if it is regarded as a generalized function only. In the domain of generalized functions and following [17, page 278], there is a generalized limit given by

$$\lim_{\omega \to \infty} \cos \omega t = 0. \tag{4.57}$$

Therefore, the PSD of the fBm of the Weyl type (see (4.5)) has the following asymptotic property:

$$\lim_{\omega \to \infty} S_{B_H,W}(t,\omega) \sim \frac{1}{|\omega|^{2H+1}} \quad \text{for } \omega \longrightarrow \infty. \tag{4.58}$$

On the other hand, from (4.56), we see that the PSD of $X_1(t)$ has the asymptotic expression given by

$$S_{X_1}(\omega) \sim \frac{1}{|\omega|^{2H+1}} \quad \text{for } \omega \gg \lambda. \tag{4.59}$$

Therefore, we see that $S_{X_1}(\omega)$ has the approximation given by

$$S_{X_1}(\omega) \sim S_{B_H,W}(t,\omega) \quad \text{for } \omega \longrightarrow \infty. \tag{4.60}$$

Hence, we have Note 20.

Note 20. The generalized OU process governed by (4.50) can be taken as the locally stationary counterpart of fBm.

According to (3.5), we have

$$\mathrm{F}^{-1}\left(\frac{1}{|\omega|^{2H+1}}\right) \sim |\tau|^{2H}. \tag{4.61}$$

Therefore, we obtain

$$C_{X_1}(\tau) \sim c_{X_1}|\tau|^{2H} \quad \text{for } \tau \longrightarrow 0, \tag{4.62}$$

where c_{X_1} is a constant. Following (3.11) and (3.12), we have the fractal dimension of $X_1(t)$ given by

$$D_{X_1} = 2 - H. \tag{4.63}$$

4.6.3. Generalized Version II of the OU Process (Lim et al. [37])

We now further extend the Langevin equation by indexing it with two fractions $\alpha, \beta > 0$ so that

$$\left({}_{-\infty}D_t^{\alpha} + \lambda\right)^{\beta} X_2(t) = w(t), \tag{4.64}$$

where ${}_{-\infty}D_t^{\alpha}$ is the operator of the Weyl fractional derivative. Denote by $g_{X_2}(t)$ the impulse response function of the above system. Then,

$$\left({}_{-\infty}D_t^{\alpha} + \lambda\right)^{\beta} g_{X_2}(t) = \delta(t). \tag{4.65}$$

The Fourier transform of $g_{X_2}(t)$, which is denoted by $G_{X_2}(\omega)$, is given by

$$G_{X_2}(\omega) = \frac{1}{\left(\lambda + (-j\omega)^{\alpha}\right)^{\beta}}. \tag{4.66}$$

Therefore, the PSD of $X_2(t)$ is given by

$$S_{X_2}(\omega) = G_{X_2}(\omega)\left[G_{X_2}(\omega)\right]^* = \frac{1}{\left|\lambda + (j\omega)^{\alpha}\right|^{2\beta}}. \tag{4.67}$$

Note that

$$S_{X_2}(\omega) \sim \frac{1}{\omega^{2\alpha\beta}} \quad \text{for } \omega \longrightarrow \infty. \tag{4.68}$$

In addition,

$$\mathrm{F}^{-1}\left(\frac{1}{|\omega|^{2\alpha\beta}}\right) \sim |\tau|^{2\alpha\beta-1}. \tag{4.69}$$

Thus, the ACF of $X_2(t)$ has the asymptotic expression given by

$$C_{X_2}(\tau) \sim c_{X_2}|\tau|^{2\alpha\beta-1} \quad \text{for } \tau \longrightarrow 0, \tag{4.70}$$

where c_{X_2} is a constant. Hence, the fractal dimension of $X_2(t)$ is given by

$$D_{X_2} = \frac{5}{2} - \alpha\beta. \tag{4.71}$$

In the above, $1/2 < \alpha\beta < 3/2$, which is a condition to assure $1 < D_{X_2} < 2$.

Note 21. The local irregularity of series relies on the fractal dimension instead of the statistical dependence. The local irregularity of an SRD series may be strong if its fractal dimension is large.

5. Conclusions

The concepts, such as power law in PDF, ACF, and PSD in fractal time series, have been discussed. Both LRD and SRD series have been explained. Several models, fBm, fGn, the GC process, alpha-stable processes, and generalized OU processes have been interpreted. Note that several models revisited above are a few in the family of fractal time series. There are others; see, for example, [78, 102–112]. As a matter of fact, the family of fractal time series is affluent but those revisited might yet be adequate to describe the fundamental of fractal time series from the point of view of engineering in the tutorial sense.

Acknowledgment

This work was partly supported by the National Natural Science Foundation of China (NSFC) under the project Grant nos. 60573125 and 60873264.

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