Fractal zone plates

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Fractal zone plates (FZPs), i.e., zone plates with a fractal structure, are described. The focusing properties of this new type of zone plate are compared with those of conventional Fresnel zone plates. It is shown that the axial irradiance exhibited by the FZP has self-similarity properties that can be correlated to those of the diffracting aperture. © 2003 Optical Society of America

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Certain natural phenomena exhibit distinctive features that can be associated with the concept of fractal, and their study has become a matter of great interest for scientists in many fields.[1] In optics, the fractal structure of some optical wave fields and the diffraction patterns generated from various fractal apertures are examples of this trend.[2,3] Concretely, it was found recently that the modes of some unstable lasers have a fractal structure,[4] from which several parameters such as the fractal dimension of the intensity profile and self-similarity associated with magnification have been determined.[5,6] Diffraction from fractal structures, ranging from simple one-dimensional (1D) objects[7] to Cantor rings,[8] has been extensively studied. Interestingly, it was shown that even simple nonfractal diffracting structures such as Ronchi gratings exhibit transverse beam profiles that have fractal structures in almost all planes behind them.[9] In this Letter we present fractal zone plates as a novel family of diffracting two-dimensional objects with radial symmetry that show multiple foci with internal fractal properties along the optical axis.

Let us start by considering the irradiance at a given point on the optical axis that is provided by an optical system with a rotationally invariant pupil function described by $p(r)$. Within the Fresnel approximation, this magnitude is given, as a function of the axial distance R from the pupil plane, as

$$I(R) = \left(\frac{2\pi}{\lambda R}\right)^2 \left| \int_0^a p(r_0)\exp\left(-i\,\frac{\pi}{\lambda R}\,{r_0}^2\right) r_0 \mathrm{d}r_0 \right|^2, \quad (1)$$

where a is the maximum extent of the pupil function and λ is the wavelength of the light. For our purposes it is convenient to express the pupil transmittance as function of a new variable, defined as

$$\varsigma = (r_0/a)^2 - 0.5\,, \quad (2)$$

in such a way that $q(\varsigma) = p(r_0)$. By using the normalized axial coordinate $u = a^2/2\lambda R$, we can now express the irradiance along the optical axis as

$$I_0(u) = 4\pi^2 u^2 \left| \int_{-0.5}^{+0.5} q(\varsigma)\exp(-i2\pi u\varsigma)\mathrm{d}\varsigma \right|^2. \quad (3)$$

Let us now consider a system whose associated pupil function $q(\varsigma)$ holds a fractal structure. Inasmuch as the axial irradiance is expressed in terms of the Fourier transform of $q(\varsigma)$, from well-known properties of fractals[3] it is straightforward to conclude that such a system will provide an irradiance along the optical axis with a self-similar profile. We call these pupils fractal zone plates (FZPs) because, as we shall see, besides having a fractal structure they can be constructed from conventional zone plates.

To illustrate the properties of FZPs we centered our attention on binary pupils analogous to conventional Fresnel zone plates. As is well known, a Fresnel zone plate consists of alternately transparent and opaque zones whose radii are proportional to the square root of the natural numbers. By using Eq. (2) is easy to obtain that the function $q(\varsigma)$ for these pupils is a Ronchi-type periodic binary function with period p [see Fig. 1(a)] that can be written as

$$q(\varsigma) = q_{\mathrm{ZP}}(\varsigma, p) = \mathrm{rect}(\varsigma)\mathrm{rect}\{\mathrm{mod}[\varsigma + (p-1)/2, p]/p\}, \quad (4)$$

where the function mod(x, y) gives the remainder on division of x by y.

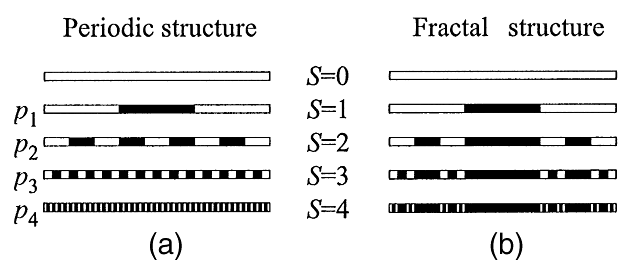

Fig. 1. Diagrams of the generation of binary function $q(\zeta)$ for (a) a Fresnel zone plate with periods $p_s = p(N, S)$ for $N = 2$ and several values of S and (b) its associated FZP. In this representation open and filled segments correspond to the values 1 and 0, respectively, of the generating binary function.

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In a similar way, one constructs FZPs by replacing the 1D periodic function described above with a 1D binary function with a fractal profile. Consider, for example, the regular Cantor-fractal, the procedure for whose construction is shown in Fig. 1(b). In the first stage ($S = 1$) the initial segment is divided into an odd number of segments, $2N - 1$, and the segments in the even positions are removed (in the figure a triadic Cantor-set was assumed; thus $2N - 1 = 3$). For the remaining N segments at the first stage this slicing-and-removing process is repeated in the second stage, and so on. In mathematical terms, the FZP transmittance function, developed up to a certain growing stage S, can be expressed as the product of the periodic functions $q(\varsigma)$ in Eq. (4) as

$$q(\varsigma) = q_{\mathrm{FZP}}(\varsigma, N, S) = \prod_{i=0}^{S} q_{\mathrm{ZP}}\left[\varsigma, \frac{2}{(2N-1)^i}\right]. \quad (5)$$

It is instructive to note that the FZP in Eq. (5) can be understood as an associated Fresnel zone plate $q_{\mathrm{ZP}}[\varsigma, p(N, S)]$, with period

$$p(N, S) = \frac{2}{(2N-1)^S} \quad (6)$$

but with some missing clear zones [cf. Figs. 1(a) and 1(b)]. Figure 2 shows a FZP generated from a triadic Cantor-set, up to $S = 3$, and the corresponding Fresnel zone plate with period $p(2, 3)$.

To compare the axial behavior of a FZP with its associated Fresnel zone plate, we obtain the axial irradiance distributions I_0^{FZP} and I_0^{ZP}, respectively, analytically in both cases. For the first case, from the recursive building procedure of the FZP and by proper use of the convolution theorem for the Fourier transform in Eq. (3) it is easy to obtain

$$I_0^{\mathrm{FZP}}(u, N, S) = 4\sin^2\left[\frac{\pi u}{(2N-1)^S}\right] \times \prod_{i=1}^{S} \frac{\sin^2[2\pi N u/(2N-1)^i]}{\sin^2[2\pi u/(2N-1)^i]}. \quad (7)$$

For the associated Fresnel zone plate, Eq. (3) leads to the well-known result[10]

$$I_0^{\mathrm{ZP}}(u, N, S) = 4\sin^2\left[\frac{\pi u}{(2N-1)^S}\right] \times \frac{\sin^2\{\pi[(2N-1)^S + 1u/(2N-1)^S]u\}}{\sin^2[2\pi u/(2N-1)^S]}. \quad (8)$$

The axial irradiance of the FZP computed for different stages of growth S and for $N = 2$ is shown in Fig. 3 (top). The irradiance of the associated Fresnel zone plate is shown in the same figure (bottom) for comparison. Note that the scale for the axial coordinate in each step is a version of that in the previous step that has been demagnified by a factor of $2N - 1 = 3$. It can be seen that the axial positions of the central lobes of the foci coincide with those of the associated Fresnel zone plate. It is clear that, whereas the internal structure of each focus in the Fresnel zone plate vanishes progressively, the axial response for the FZP exhibits its characteristic fractal profile. In fact, the four patterns at the top of Fig. 3 are self-similar. This scaling property along the optical axis, which holds for any N, was to our knowledge never reported previously and may be called the axial scale property. This means that the axial irradiance reproduces the self-similarity of the FZP.

One can see the fractal behavior of the internal structure of the FZP foci better by noting that the irradiances in Eqs. (6) and (7) are periodic functions of

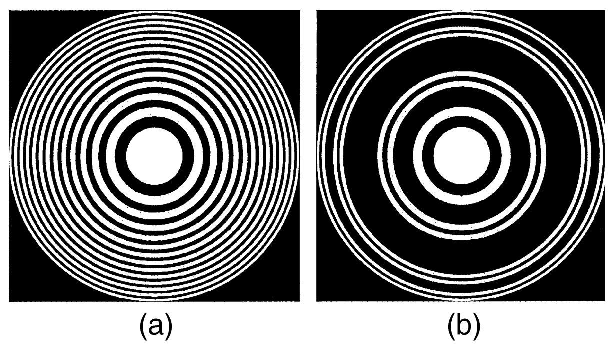

Fig. 2. (a) Fresnel zone plate and (b) the associated FZP generated from the 1D functions in Fig. 1 for $S = 3$. The generating process consists in rotating the whole structure around one extreme after the change of variables in Eq. (2).

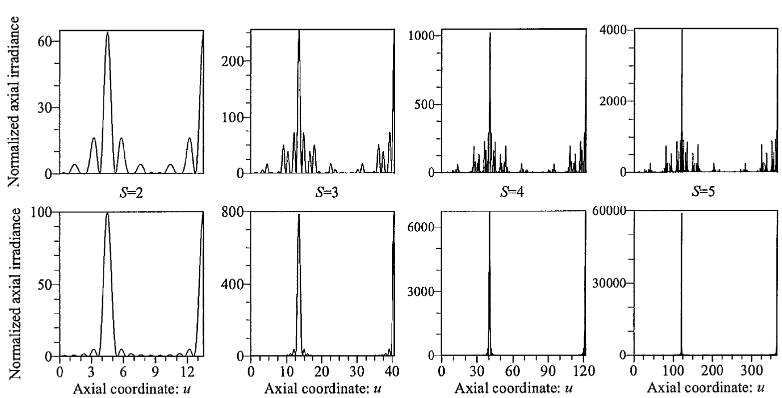

Fig. 3. Normalized irradiance versus axial coordinate u obtained top for a FZP at four stages of growth and bottom, for its associated Fresnel zone plate. In all cases $N = 3$.

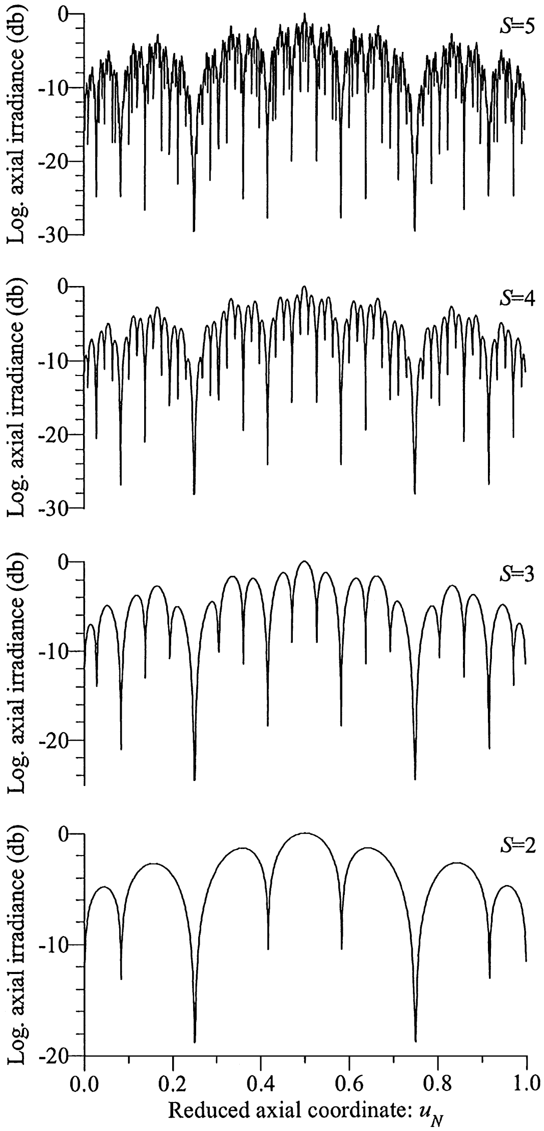

Fig. 4. Log plot of axial irradiances versus reduced axial coordinate u_N obtained from the upper part of Fig. 3.

u, with period $u_p = (2N - 1)^S$. On reducing u to the fraction of this period through the change of variables $u_N = u/u_p$, one achieves

$$I_{0N}{}^{\mathrm{FZP}}(u_N, N, S) = 4\sin^2(\pi u_N) \times \prod_{i=1}^{S} \frac{\sin^2[2\pi N(2N-1)^{S-i}u_N]}{\sin^2[2\pi(2N-1)^{S-i}u_N]}. \quad (9)$$

The result of Eq. (9) for the FZP is shown in Fig. 4. It can be seen that the axial irradiance for a given stage S is a modulated version of that associated with the previous stage. In our case, as S becomes larger, an increasing number of zeros and maxima are encountered, which are scale invariant over dilations of factor $2N - 1 = 3$, as corresponds to a self-similar structure. Similarly to that of a Fresnel zone plate, the axial irradiance behavior of the FZP can easily be interpreted as the interference between the successive rings over the pupil.

Summarizing, a new type of radially symmetric pupil, which we have named a fractal zone plate, has been introduced. These pupils, which have fractal structure along the square of the radial coordinate, can be understood as conventional zone plates with some missing clear zones. We have shown that the irradiance along the optical axis produced by these pupils shows a characteristic fractal profile. Moreover, provided that there is a theoretical relation between the transmittance of a FZP and its axial response, syntheses of fractal axial irradiances are now possible with this kind of zone plate. In other words, if a desired axial distribution has a fractal behavior, the generating FZP can be readily obtained by a 1D inverse Fourier transform by use of Eq. (3).

Aside from their theoretical interest, FZPs can be used in other regions of the electromagnetic spectrum, such as microwaves and x rays and even with slow neutrons, for which Fresnel zone plates were successfully applied. In optics, the effects on the focal properties of such zone plates that are imposed by various construction parameters (such as the lacunarity) and the influence of the optical aberrations on their axial response are subjects for continuing study.

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