Fractality field in the theory of scale relativity*

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Abstract
In the theory of scale relativity, space-time is considered to be a continuum that is not only curved, but also non-differentiable, and, as a consequence, fractal. The equation of geodesics in such a space-time can be integrated in terms of quantum mechanical equations. We show in this paper that the quantum potential is a manifestation of such a fractality of space-time (in analogy with Newton's potential being a manifestation of curvature in the framework of general relativity).

1 Introduction

The theory of scale relativity aims at describing a nondifferentiable continuous manifold by the building of new tools that implement Einstein's general relativity concepts in the new context (in particular, covariant derivative and geodesics equations). We refer the reader to Refs. [1, 2, 3, 4] for a detailed description of the construction of these tools. In the present short research note, we want to address a specific point of the theory, namely, the emergence of an additional potential energy which manifests the fractal and nondifferentiable geometry.

2 Non relativistic quantum mechanics

2.1 Quantum potential

In the scale relativity approach, one decomposes the velocity field on the geodesics bundle of a nondifferentiable space-time in terms of a classical, differentiable part, $\mathcal{V}$, and of a fractal, divergent, nondifferentiable part $\mathcal{W}$ of zero mean. Both velocity fields are complex due to a fundamental two-valuedness of the classical (differentiable) velocity issued from the nondifferentiability [1]. Then one builds a complex covariant total derivative that reads in the simplest case (spinless particle, nonrelativistic velocities and no external field) [1, 2, 3]

$$\frac{\acute{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\mathcal{D}\Delta. \tag{1}$$

*Reference: L. Nottale, *Progress in Physics* **1**, 12-16 (2005).

The constant $2\mathcal{D} =< d\xi^2 > /dt$ ($= \hbar/m$ in standard quantum mechanics) measures the amplitude of the fractal fluctuations. Note that it is possible to have a more complete construction in which the full velocity field $\mathcal{V} + \mathcal{W}$ intervenes in the covariant derivative [6]. In the same way as in general relativity, the geodesics equation can therefore be written, using this covariant derivative, in terms of a free, inertial motion-like equation,

$$\frac{\hat{d}\mathcal{V}}{dt} = 0. \tag{2}$$

Let us explicitly introduce the real and imaginary parts of the complex velocity $\mathcal{V} = V - iU$,

$$\frac{\hat{d}\mathcal{V}}{dt} = \left(\left\{ \frac{\partial}{\partial t} + V.\nabla \right\} - i \left\{ U.\nabla + \mathcal{D}\Delta \right\} \right) (V - iU) = 0. \tag{3}$$

We see in this expression that the real part of the covariant derivative, $\hat{d}_R/dt = \partial/\partial t + V.\nabla$, is the standard total derivative expressed in terms of partial derivatives, while the new terms are included in the imaginary part, $\hat{d}_I/dt = -(U.\nabla + \mathcal{D}\Delta)$. The field will find its origin in the consequences of these additional terms on the imaginary part of the velocity $-U$. Indeed, by separating the real and imaginary parts, equation (3) reads:

$$\left\{ \left(\frac{\partial}{\partial t} + V.\nabla \right) V - (U.\nabla + \mathcal{D}\Delta)U \right\} - i \left\{ (U.\nabla + \mathcal{D}\Delta)V + \left(\frac{\partial}{\partial t} + V.\nabla \right) U \right\} = 0. \tag{4}$$

Therefore the real part of this equation takes the form of an Euler-Newton equation of dynamics

$$\left(\frac{\partial}{\partial t} + V.\nabla \right) V = (U.\nabla + \mathcal{D}\Delta)U, \tag{5}$$

i.e.,

$$\frac{dV}{dt} = \frac{F}{m}, \tag{6}$$

where the total derivative of the velocity field V takes its standard form $dV/dt = (\partial/\partial t + V.\nabla)V$ and where the force F is given by $F = m(U.\nabla U + \mathcal{D}\Delta U)$.

Recall that, after one has introduced the wave function ψ from the complex action $\mathcal{S} = S_R + iS_I$, namely, $\psi = \exp(i\mathcal{S}/2m\mathcal{D}) = \sqrt{P}\exp(iS_R/2m\mathcal{D})$, equation (2) and its generalization including a scalar field, $m\,\hat{d}\mathcal{V}/dt = -\nabla\phi$ can be integrated under the form of a Schrödinger equation [1]

$$\mathcal{D}^2\Delta\psi + i\mathcal{D}\frac{\partial\psi}{\partial t} - \frac{\phi}{2m}\psi = 0. \tag{7}$$

Let us now show that the additional force derives from a potential. Indeed, the imaginary part of the complex velocity field is given, in terms of the modulus of ψ, by the expression:

$$U = \mathcal{D}\nabla \ln P. \tag{8}$$

The force becomes

$$F = m\mathcal{D}^2 \left[(\nabla \ln P.\nabla)(\nabla \ln P) + \Delta(\nabla \ln P) \right]. \tag{9}$$

Now, by intoducing $\sqrt{P}$ in this expression, one makes explicitly appear the remarkable identity that is already at the heart of the proof of the Schrödinger equation ([1], p.151), namely,

$$F = 2m\mathcal{D}^2 \left[2(\nabla \ln \sqrt{P}.\nabla)(\nabla \ln \sqrt{P}) + \Delta(\nabla \ln \sqrt{P}) \right] = 2m\mathcal{D}^2 \nabla \left(\frac{\Delta\sqrt{P}}{\sqrt{P}} \right). \tag{10}$$

Therefore the force F derives from a potential energy

$$Q = -2m\mathcal{D}^2 \frac{\Delta\sqrt{P}}{\sqrt{P}}, \tag{11}$$

which is nothing but the standard 'quantum potential', but here established as a mere manifestation of the nondifferentiable and fractal geometry intead of being deduced from a postulated Schrödinger equation.

The real part of the motion equation finally takes the standard form of the equation of dynamics in presence of a scalar potential,

$$\frac{dV}{dt} = \left(\frac{\partial}{\partial t} + V.\nabla\right) V = -\frac{\nabla Q}{m}, \tag{12}$$

while the imaginary part is the equation of continuity $\partial P/\partial t + \text{div}(PV) = 0$. The fact that the field equation is derived from the same remarkable identity that gives rise to the Schrödinger equation is also manifest in the similarity of its form with the free stationary Schrödinger equation, namely,

$$\mathcal{D}^2\Delta\sqrt{P} + \frac{Q}{2m}\sqrt{P} = 0 \quad \leftrightarrow \quad \mathcal{D}^2\Delta\psi + \frac{E}{2m}\psi = 0. \tag{13}$$

Now, the form (11) of the field equation means that the field can be known only after having solved the Schrödinger equation for the wave function. This is a situation somewhat different from that of general relativity, where, at least for test-particles, the description is reversed: given the energy-momentum tensor, one solves the Einstein field (i.e. space-time geometry) equations for the metric potentials, then one writes the geodesics equation in the space-time so determined and solve it for the motion of the particle. However, even in general relativity this case is an ideally simplified situation, since already in the two-body problem the motion of the bodies should be injected in the energy-momentum tensor, so that this is a looped system which has no exact analytical solution.

In the case of a quantum mechanical particle considered in scale relativity, the loop between the motion (geodesics) equation and the field equation is even more tight. Indeed, here the concept of test-particle loses its meaning. Even in the case of only one "particle", the space-time geometry is determined by the particle itself and by its motion, so that the field equation and the geodesics equation now participate of the same level of description. This explains why the motion/ geodesics equation, in its Hamilton-Jacobi form that takes the form of the Schrödinger equation, is obtained without having first written the field equation in an explicit way. Actually, the potential Q is implicitly contained in the Schrödinger form of the equations, and it is made explicit only when coming back to a fluid-like Euler-Newton representation. In the end, the particle is described by a wave function (which is constructed, in the scale relativity theory, from the geodesics), of which only the square of the modulus P is observable. Therefore one expects the "field" to be given by a function of P, which is exactly what is found.

2.2 Invariants and energy balance

Let us now make explicit the energy balance by accounting for this additional potential energy. This question has already been discussed in [7, 8] and in [9], but we propose here a different presentation. We shall express the energy equation in terms of the various equivalent variables which we use in scale relativity, namely, the wave function ψ, the complex velocity $\mathcal{V}$ or its real and imaginary parts V and $-U$.

The first and main form of the energy equation is the Schrödinger equation itself, that we have derived as a prime integral of the geodesics equation. The Schrödinger equation is therefore the quantum equivalent of the metric form (i.e., of the equation of conservation of the energy). It may be written in the free case under the form

$$\mathcal{D}^2 \,\frac{\Delta\psi}{\psi} = -i\mathcal{D}\,\frac{\partial \ln\psi}{\partial t}. \tag{14}$$

In the stationary case with given energy E, it becomes:

$$E = -2m\mathcal{D}^2 \,\frac{\Delta\psi}{\psi}. \tag{15}$$

Now we can use the fundamental remarkable identity $\Delta\psi/\psi = (\nabla \ln\psi)^2 + \Delta \ln\psi$. Re-introducing the complex velocity field $\mathcal{V} = -2i\mathcal{D}\nabla \ln\psi$ in this expression we finally obtain the correspondence:

$$E = -2m\mathcal{D}^2 \,\frac{\Delta\psi}{\psi} = \frac{1}{2} m \left(\mathcal{V}^2 - 2i\mathcal{D}\nabla.\mathcal{V}\right). \tag{16}$$

(Note that when a potential term is present, all these relations remain true by replacing E by $E-\phi$). This is the non-relativistic equivalent of Pissondes' relation [8] in the relativistic case, $\mathcal{V}^\mu \mathcal{V}_\mu + i\lambda\partial^\mu \mathcal{V}_\mu = 1$ (see also hereafter). Therefore the form of the energy $E = (1/2)mV^2$ is not conserved: this is precisely due to the existence of the additional potential energy of geometric origin. Let us prove this statement.

From equation (16) we know that the imaginary part of $(\mathcal{V}^2 - 2i\mathcal{D}\nabla.\mathcal{V})$ is zero. By writing its real part in terms of the real velocities U and V, we find:

$$E = \frac{1}{2} m \left(\mathcal{V}^2 - 2i\mathcal{D}\nabla.\mathcal{V}\right) = \frac{1}{2} m (V^2 - U^2 - 2\mathcal{D}\nabla.U). \tag{17}$$

Now we can express the potential energy Q given in equation (11) in terms of the velocity field U:[1]

$$Q = -\frac{1}{2} m \,(U^2 + 2\mathcal{D}\nabla.U), \tag{18}$$

so that we finally write the energy balance under the three equivalent forms:

$$E = -2m\mathcal{D}^2 \,\frac{\Delta\psi}{\psi} = \frac{1}{2} m \left(\mathcal{V}^2 - 2i\mathcal{D}\nabla.\mathcal{V}\right) = \frac{1}{2} m V^2 + Q. \tag{19}$$

More generally, in presence of an external potential energy ϕ and in the non-stationary case, it reads:

$$-\frac{\partial S_R}{\partial t} = \frac{1}{2} m V^2 + Q + \phi, \tag{20}$$

where S_R is the real part of the complex action (i.e., $S_R/2m\mathcal{D}$ is the phase of the wave function).

[1]A misprint in the published version concerning the sign of the second term has been corrected here.

3 Relativistic quantum mechanics

3.1 Quantum potential

All the above description can be directly generalized to relativistic QM and the Klein-Gordon equation [10, 2, 3]. The geodesics equation still reads in this case:

$$\frac{\acute{d}\,\mathcal{V}_\alpha}{ds} = 0, \tag{21}$$

where the total derivative is given by [10, 3]

$$\frac{\acute{d}}{ds} = \left(\mathcal{V}^\mu + i\,\frac{\lambda}{2}\,\partial^\mu\right)\partial_\mu\ . \tag{22}$$

The complex velocity field $\mathcal{V}_\alpha$ reads in terms of the wave function

$$\mathcal{V}_\alpha = i\lambda\,\partial_\alpha \ln\psi. \tag{23}$$

The relation between the non-relativistic fractal parameter $\mathcal{D}$ and the relativistic one λ is simply $2\mathcal{D} = \lambda c$. In particular, in the standard QM case, λ is the Compton length of the particle, $\lambda = \hbar/mc$, and we recover $\mathcal{D} = \hbar/2m$.

The calculations are similar to the non-relativistic case. We decompose the complex velocity in terms of its real and imaginary parts, $\mathcal{V}_\alpha = V_\alpha - i\,U_\alpha$, so that the geodesics equation becomes

$$\left\{V^\mu - i\left(U^\mu - \frac{\lambda}{2}\,\partial^\mu\right)\right\}\partial_\mu\left(V_\alpha - i\,U_\alpha\right) = 0, \tag{24}$$

i.e.,

$$\left\{V^\mu\partial_\mu V_\alpha - \left(U^\mu - \frac{\lambda}{2}\,\partial^\mu\right)\partial_\mu U_\alpha\right\} - i\left\{\left(U^\mu - \frac{\lambda}{2}\,\partial^\mu\right)\partial_\mu V_\alpha + V^\mu\partial_\mu U_\alpha\right\} = 0. \tag{25}$$

The real part of this equation takes the form of a relativistic Euler-Newton equation of dynamics:

$$\frac{dV_\alpha}{ds} = V^\mu\partial_\mu V_\alpha = \left(U^\mu - \frac{\lambda}{2}\,\partial^\mu\right)\partial_\mu U_\alpha. \tag{26}$$

Therefore the relativistic case is similar to the non-relativistic one, since a generalized force also appears in the right-hand side of this equation. Let us now prove that it also derives from a potential. Using the expression for U_α in terms of the modulus $\sqrt{P}$ of the wave function,

$$U_\alpha = -\lambda\,\partial_\alpha \ln\sqrt{P}, \tag{27}$$

we may write the force under the form

$$\begin{aligned}\frac{F_\alpha}{m} &= -\lambda\,\partial^\mu \ln\sqrt{P}\;\partial_\mu(-\lambda\,\partial_\alpha \ln\sqrt{P}) + \frac{\lambda^2}{2}\partial^\mu\partial_\mu\partial_\alpha \ln\sqrt{P}\\ &= \lambda^2\left(\partial^\mu \ln\sqrt{P}\;\partial_\mu\partial_\alpha \ln\sqrt{P} + \frac{1}{2}\partial^\mu\partial_\mu\partial_\alpha \ln\sqrt{P}\right).\end{aligned} \tag{28}$$

Since $\partial^\mu\partial_\mu\partial_\alpha = \partial_\alpha\partial^\mu\partial_\mu$ commutes and since $\partial_\alpha(\partial^\mu \ln f\;\partial_\mu \ln f) = 2\;\partial^\mu \ln f\;\partial_\alpha\partial^\mu \ln f$, we obtain

$$\frac{F_\alpha}{m} = \frac{1}{2}\,\lambda^2\,\partial_\alpha\left(\partial^\mu \ln\sqrt{P}\;\partial_\mu \ln\sqrt{P} + \partial^\mu\partial_\mu \ln\sqrt{P}\right). \tag{29}$$

We can now make use of the remarkable identity (that generalizes to four dimensions the one which is also at the heart of the non-relativistic case)

$$\partial^\mu \ln\sqrt{P}\,\partial_\mu \ln\sqrt{P} + \partial^\mu\partial_\mu \ln\sqrt{P} = \frac{\partial^\mu\partial_\mu\sqrt{P}}{\sqrt{P}}, \tag{30}$$

and we finally obtain

$$\frac{dV_\alpha}{ds} = \frac{1}{2}\,\lambda^2\,\partial_\alpha\left(\frac{\partial^\mu\partial_\mu\sqrt{P}}{\sqrt{P}}\right). \tag{31}$$

Therefore, as in the non-relativistic case, the force derives from a potential energy

$$Q_R = \frac{1}{2}mc^2\,\lambda^2\,\frac{\partial^\mu\partial_\mu\sqrt{P}}{\sqrt{P}}, \tag{32}$$

that can also be expressed in terms of the velocity field U as

$$Q_R = \frac{1}{2}mc^2\left(U^\mu U_\mu - \lambda\,\partial^\mu U_\mu\right). \tag{33}$$

At the non-relativistic limit ($c \to \infty$), the Dalembertian $\partial^\mu\partial_\mu = (\partial^2/c^2\partial t^2 - \Delta)$ is reduced to $-\Delta$, and since $\lambda = 2\mathcal{D}/c$, we recover the nonrelativistic potential energy $Q = -2m\mathcal{D}^2\Delta\sqrt{P}/\sqrt{P}$. Note the correction to the potential introduced by Pissondes [7] which is twice this potential and therefore cannot agree with the nonrelativistic limit.

3.2 Invariants and energy balance

As shown by Pissondes [7, 8], the four-dimensional energy equation $u^\mu u_\mu = 1$ is generalized in terms of the complex velocity under the form $\mathcal{V}^\mu\mathcal{V}_\mu + i\lambda\partial^\mu\mathcal{V}_\mu = 1$. Let us show that the additional term is a manifestation of the new scalar field Q which takes its origin in the fractal and nondifferentiable geometry. Start with the geodesics equation

$$\frac{\hat{d}\,\mathcal{V}_\alpha}{ds} = \left(\mathcal{V}^\mu + i\,\frac{\lambda}{2}\,\partial^\mu\right)\partial_\mu\,\mathcal{V}_\alpha = 0. \tag{34}$$

Then, after introducing the wave function by using the relation $\mathcal{V}_\alpha = i\lambda\,\partial_\alpha \ln\psi$, after calculations similar to the above ones (now on the full function ψ instead of only its modulus $\sqrt{P}$), the geodesics equation becomes:

$$\frac{\hat{d}\,\mathcal{V}_\alpha}{ds} = -\frac{\lambda^2}{2}\,\partial_\alpha\left(\partial^\mu \ln\psi\,\partial_\mu \ln\psi + \partial^\mu\partial_\mu \ln\psi\right) = \frac{1}{2}\,\partial_\alpha\left(-\lambda^2\,\frac{\partial^\mu\partial_\mu\psi}{\psi}\right) = 0. \tag{35}$$

Under its right-hand form, this equation is integrated in terms of the Klein-Gordon equation,

$$\lambda^2\,\partial^\mu\partial_\mu\psi + \psi = 0. \tag{36}$$

Under its left hand form, the integral writes

$$-\lambda^2(\partial^\mu \ln\psi\,\partial_\mu \ln\psi + \partial^\mu\partial_\mu \ln\psi) = 1. \tag{37}$$

It becomes in terms of the complex velocity [8]

$$\mathcal{V}^{\mu}\mathcal{V}_{\mu} + i\lambda\partial^{\mu}\mathcal{V}_{\mu} = 1, \tag{38}$$

which is therefore but another form taken by the KG equation (as expected from the fact that the KG equation is the quantum equivalent of the Hamilton-Jacobi equation). Let us now separate the real and imaginary parts of this equation. One obtains:

$$V^{\mu}V_{\mu} - (U^{\mu}U_{\mu} - \lambda\,\partial^{\mu}U_{\mu}) = 1, \qquad 2\,V^{\mu}U_{\mu} - \lambda\,\partial^{\mu}V_{\mu} = 0. \tag{39}$$

Then the energy balance writes, in terms of the additional potential energy Q_R

$$V^{\mu}V_{\mu} = 1 + 2\,\frac{Q_R}{mc^2}. \tag{40}$$

Let us show that we actually expect such a relation for the quadratic invariant in presence of an external potential ϕ. The energy relation writes in this case $(E-\phi)^2 = p^2c^2 + m^2c^4$, i.e. $E^2 - p^2c^2 = m^2c^4 + 2E\phi - \phi^2$. Introducing the rest frame energy by writing $E = mc^2 + E'$, we obtain

$$V^{\mu}V_{\mu} = \frac{E^2 - p^2c^2}{m^2c^4} = 1 + 2\,\frac{\phi}{mc^2} + \left[2\,\frac{E'}{mc^2}\,\frac{\phi}{mc^2} - \frac{\phi^2}{m^2c^4}\right]. \tag{41}$$

This justifies the relativistic factor 2 in equation (40) and supports the interpretation of Q_R in terms of a potential, at least at the level of the leading terms.

Now, concerning the additional terms, it should remain clear that this is only an approximate description in terms of field theory of what are ultimately (in this framework) the manifestations of the fractal and nondifferentiable geometry of space-time. Therefore we expect the field theory description to be a first order approximation in the same manner as, in general relativity, the description in terms of Newtonian potential.

In particular, in the non-relativistic limit $c \to \infty$ the last two terms of equation (41) vanish and we recover the energy equation (19) which is therefore exact in this case.

4 Conclusion

Placing ourselves in the framework of the scale-relativity theory, we have shown in a detailed way that the quantum potential, whose origin remained mysterious in standard quantum mechanics, is a manifestation of the nondifferentiability and fractality of space-time in the new approach.

This result is expected to have many applications, as well in physics as in other sciences, including biology [4]. It has been used, in particular, to suggest a new solution to the problem of ‘dark matter’ in cosmology [11, 5], based on the proposal that chaotic gravitational system can be described on long time scales (longer than their horizon of predictibility) by the scale-relativistic equations and therefore by a macroscopic Schrödinger equation [12]. In this case there would be no need for additional non baryonic dark matter, since the various observed non-Newtonian dynamical effects (that the hypothesis of dark matter wants to explain despite the check of all attempts of detection) would be readily accounted for by the new scalar field that manifests the fractality of space.

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