

Fraction Development in Children: Importance of Building Numerical Magnitude Understanding

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Abstract

This chapter situates fraction learning within the integrated theory of numerical development. We argue that the understanding of numerical magnitudes for whole numbers as well as for fractions is critical to fraction learning in particular and mathematics achievement more generally. Results from the Delaware Longitudinal Study, which examined domain-general and domain-specific predictors of fraction development between third and sixth grade, are highlighted. The findings support an approach to teaching fractions that centers on a number line. Implications for helping struggling learners are discussed.

Keywords: fractions; mathematics; numerical cognition; number line

Fraction Development in Children: Importance of Building Numerical Magnitude Understanding

Fractions are a crucial component of the U.S. mathematics curriculum in elementary and middle school (Council of Chief State School Officers & National Governors Association Center for Best Practices, 2010). Yet fractions and their magnitudes are sources of great confusion for many students. For example, about half of eighth graders on the National Assessment of Education Progress (NAEP; 2004) experienced difficulties when asked to order three fractions with unlike denominators. Moreover, only a third of eleventh graders can successfully translate decimals into fractions (Kloosterman, 2010) and many community college students still struggle when making fraction magnitude comparisons (Schneider & Siegler, 2010). Frequently students do not recognize that the numerator and denominator work together to form a fraction's magnitude. Persistent difficulties with fractions can have far-reaching consequences, both on professional and personal levels. Fraction knowledge supports learning of algebra (Booth & Newton, 2012), and algebra proficiency, in turn, prepares students for higher education and careers in science, technology, engineering, and mathematics (STEM) disciplines. Fraction knowledge also supports daily life functioning related to managing personal finances, understanding mortgage interest rates, doing home repairs, cooking, managing medical dosages, and so forth. Difficulties with fractions have even been suggested as a root cause of mathematics anxiety (Heitin, 2015).

This chapter has several aims. First, we situate fraction learning within the integrated theory of numerical development, proposed by Siegler, Thompson, and Schneider (2011). The integrated theory asserts that the unifying property of all real numbers is that they have magnitudes or numerical values that can be ordered on the number line (Siegler, Fuchs, Jordan, Gersten, & Ochsendorf, 2015). Next, we identify key areas of fraction knowledge and then chart

fraction development from early childhood through middle school. In particular, we discuss research findings from our large longitudinal study of fraction learning from third through sixth grade, supported by the U.S. Department of Education Institute of Education Sciences. The four-year study identified domain-general and domain-specific predictors and concomitants of fraction learning. Overall, we show that fraction development during this formal instructional period is fundamental to mathematics success more generally. Finally, we discuss implications for helping students who struggle with fractions, especially with respect to building numerical magnitude understandings.

Integrated theory of numerical development

Earlier theories of numerical development conceptualize whole number learning and fraction learning as two separate processes, with fractions secondary to whole numbers (Siegler et al., 2013; Siegler et al., 2011). Although whole numbers are characterized as being naturally acquired early in development, segmented theories assume fraction knowledge occurs much later and with great difficulty (e.g., Geary, 2004; 2006; Gelman & Williams, 1998). In fact, whole number understanding has been described as interfering with the learning of fractions (Geary, 2006; Vamakovoussi & Vosniadou, 2010). For example, children often overextend the principle of whole number ordinality (Ni & Zhou, 2005), identifying unit fractions with the larger denominator as having the larger magnitude (e.g., $1/6 > 1/5$ because $6 > 5$).

More recently, research has begun to emphasize how the numerical property of magnitude can unite number learning into a continuous process, one that includes fractions as well as whole numbers (Matthews, Lewis, & Hubbard, in press; Siegler & Lortie-Forgues, 2014). Properties of whole numbers, however, are not invariant across all real numbers, and, in particular, differ from fractions in important ways. Whole numbers have unique successors,

which are expressed as a single symbol and are finite within a given interval; they increase or stay the same with addition and multiplication and decrease or stay the same with subtraction and division. Properties of fractions, on the other hand, sometimes behave in the opposite manner. However, the one property that unites all real numbers is that they have magnitudes that can be located on a number line (Siegler, Fazio, Bailey, & Zhou, 2013; Siegler & Lortie-Forgues, 2014; Siegler et al., 2011).

According to the integrated theory, number learning involves continually expanding the size and type of number whose magnitudes can be accurately represented (Siegler & Lortie-Forgues, 2014). Children typically begin with non-symbolic magnitude representations. The ability to discriminate between sets of one to four objects (the object tracking system) and more than four objects (the approximate number system) emerges in infancy (Feigenson, Dehaene, & Spelke, 2004). For example, even infants can recognize the difference between eight and 16 dots, a 2:1 ratio (Xu & Spelke, 2000). Because infants prefer to look at novel objects, their looking time decreases, or habituates, to the repeated presentation of the same image (e.g., Groves & Thompson, 1970). In Xu and Spelke's (2000) study, after being habituated to an array of dots that either contained eight or 16 dots, infants looked longer at an array of dots with the opposite amount (i.e., either eight or 16 dots). Moreover, there is evidence that sensitivity to ratios is also present in infants (McCrink & Wynn, 2007). Matthews et al. (in press) argue there is a core cognitive system that processes non-symbolic ratios, which is referred to as the "ratio processing system" (RPS). Analogous to whole number development, in which both formal and informal learning links non-symbolic magnitudes (e.g., three dots) to symbolic representations (e.g., the numeral "3"), children's non-symbolic understanding of ratios (e.g., the ability to recognize that

the proportion of red to blue in a bar is the same or different as the ratio of red to blue in another bar) may underpin the learning of conventional fraction symbols (Boyer & Levine, 2012).

The ability to link non-symbolic to symbolic magnitude representations develops gradually (Siegler & Lortie-Forgues, 2014). Comprehension of numerical magnitudes can be seen on a number line task, where children are presented with an image of a horizontal line flanked with two values (e.g., 0 on the left and 100 on the right) and asked to locate the position of a given number on the line (Siegler & Opfer, 2003). Accuracy on the number line task undergoes a developmental shift, with children becoming increasingly closer to the target number with age. On unfamiliar scales, children overestimate relatively smaller numbers and underestimate relatively larger magnitudes (Siegler & Booth, 2004; Siegler & Opfer, 2003). Magnitudes on a scale from 0-10 are typically understood linearly by first grade (Bertelletti, Lucangeli, Piazza, Dehaene, & Zorzi, 2010; Siegler & Booth, 2004), with 0-100 understood by second grade (Booth & Siegler, 2006), 0-1,000 by fourth grade (Siegler & Opfer, 2003), and 0-100,000 by sixth grade (Opfer & Siegler, 2007; Opfer & Thompson, 2008).

Eventually, children learn to represent fractions as well as whole numbers on the number line (Siegler & Lortie-Forgues, 2014). Children are first able to locate proper fractions (a/b , where b is larger than a) and then mixed numbers (a combination of a whole number plus a proper fraction) on a number line (e.g., 0 to 2), and later learn the magnitudes of improper fractions (b/a , where b is larger than a ; Resnick et al., 2015). The integrated theory of numerical development emphasizes that fractions are a pivotal part of number learning more generally, with fractions representing the first opportunity for children to learn properties of whole numbers are not always true of all numbers (Siegler & Lortie-Forgues, 2014; Siegler et al., 2011).

There is growing evidence that magnitude understanding is foundational to mathematics learning. Accuracy in whole number line estimation (e.g., Booth & Siegler, 2006; 2008; Holloway & Ansari, 2008; Jordan et al., 2013; Sasanguie, Göbel, Moll, Smets, & Reynvoet, 2013) and fraction number line estimation (Bailey et al., 2012; Resnick et al., 2015; Siegler et al., 2012; Siegler & Pyke, 2013; Siegler et al., 2011) both predict overall mathematics achievement.

Understanding of fractions involves both conceptual and procedural knowledge

Any mathematical domain involves both concepts and procedures (Geary, 2004). Knowledge of mathematical concepts is often termed *conceptual knowledge*; knowing how to execute steps to reach a correct answer is termed *procedural knowledge*. In the domain of fractions, measures of conceptual understanding typically assess students' ability to estimate fraction magnitudes on a number line (as discussed previously), compare fraction magnitudes (e.g., Hecht, Close, & Santisi, 2003), find equivalent fractions, and recognize parts of a whole and parts of a set (e.g., Jordan et al., 2013). Measures of procedural knowledge require students to solve fraction arithmetic problems, with variations of addition, subtraction, multiplication, and division items (e.g., Siegler et al., 2011). Procedural knowledge may also be used to help students compare magnitudes and find equivalent fractions (e.g., cross multiplying the numerators with the denominators).

Facility with both fraction concepts and procedures is needed for students to develop fraction proficiency. Imagine, for example, a student who demonstrates near perfect accuracy when estimating fraction magnitudes on a number line. However, when given the addition problem $\frac{1}{2} + \frac{1}{4}$, the student does not know the procedural steps for finding a common denominator (e.g., $\frac{1}{2} = \frac{2}{4}$; $\frac{2}{4} + \frac{1}{4} = \frac{3}{4}$). The student may nevertheless reach the correct answer by using his or her strong grasp of fraction magnitudes. That is, the student visualizes a

mental number line and sees that adding $1/2$ and $1/4$ results in a location on the number line that is $3/4$ of the whole number one. In this case, the student compensates for low procedural knowledge with high conceptual knowledge. However, the same student may struggle with the problem $2/7 + 4/21$. Although he or she understands the conceptual underpinnings of the problem, the mental number line provides limited help for finding exact sums of two fractions with greater numerators and denominators. To solve this problem, the procedure of identifying a common denominator, which involves multiplicative skills, is more advantageous and accessible than a mental number line ($2/7 = 6/21$; $6/21 + 4/21 = 10/21$), although magnitude understanding can help students evaluate the plausibility of their solutions (Bailey et al., 2015; Siegler & Lortie-Forgues, 2014). Conceptual knowledge of magnitudes is necessary but not sufficient for achieving with fractions. Likewise, students who execute procedures for solving fraction arithmetic problems without magnitude understanding will encounter limited success on problem solving tasks that require deeper knowledge.

Fraction concepts and procedures ideally develop hand over hand (Hansen et al., 2015b). Recent analyses exploring how each type of knowledge influences the other indicate that the relation between fraction concepts and procedures is bidirectional; knowledge of fraction concepts impacts the development of fraction procedures and in turn, knowledge of procedures affects concepts, although likely to a lesser extent (Bailey et al., 2015; Fuchs et al., 2013; Fuchs et al., 2014; Hecht & Vagi, 2010). Overall, examination of the development of both fraction concepts and fraction procedures provides a more accurate picture of students' fraction development.

Fraction development in early childhood

Before formal instruction, most preschoolers have a basic understanding of fractions. Young children successfully solve non-symbolic calculations with fractions, make sense of fractions through equal sharing, and show an early sense of proportionality and partitioning. Despite these early foundational abilities, children also hold misconceptions about fractions that can persist throughout elementary school. Each of these areas is discussed in this section.

Early fraction calculation ability. To assess fraction calculation in three- to seven-year-old children, Mix, Levine, and Huttenlocher (1999) modified a previously used nonverbal calculation task where children are asked to view an array of black dots, which is then covered and transformed by adding dots to or subtracting them from the covered array; the child's task is point at a picture of an equivalent array of dots after the transformation (Jordan, Levine, & Huttenlocher, 1994). On Mix and colleagues' nonverbal fraction calculation task, an amount representing the first term of a fraction problem (represented as quarters of a circular sponge) was placed into a shallow hole (e.g., two quarters of the sponge). Then, the hole was hidden, and parts of the circular sponge (e.g., one quarter of the sponge) was either added to or subtracted from the amount that was hidden, but the child could not see the outcome (e.g., a quarter of the sponge; $1/2 - 1/4 = 1/4$). The experimenter then asked the child to choose the correct amount from an array of four pictures showing different fractions (pieces) of a circle. The researchers found that while nonverbal whole-number calculation skill emerges about a year earlier than nonverbal fraction calculation skill, children as young as four years of age perform significantly above chance on fraction calculations with totals less than or equal to one. Mix and colleagues suggest that this later development of nonverbal fraction calculation skills may be related to the complexity of the spatial skills required for fraction versus whole-number calculation. That is, whole number calculations simply require movements in and out of space whereas the fractions

problems also required children to rotate, separate, and recombine amounts. Overall, the findings suggest that well before formal instruction in fractions begins, children can meaningfully calculate with fractions, most likely through their experience with parts and wholes in the environment as well as -- to some extent -- through their fundamental non-symbolic understandings of number and proportions.

Equal sharing. One way children make sense of fractions is through equal sharing experiences (Davis & Pepper, 1992; Empson, 1999). Children learn about fractions through real-world experiences with sharing snacks and toys, and perhaps unsurprisingly, many become concerned with the “fairness” or “evenness” of the sizes of the fractions when working on equal sharing problems (Brizuela, 2006; Empson, 1999). In a descriptive study of fraction learning in a first-grade classroom, Empson (1999) showed that children develop fraction concepts using their own representations in equal sharing situations. Before instruction, the majority of children successfully solved equal sharing problems that involved halving or repeated halving (e.g., when asked to share nine apples equally between two horses, children understood that both horses will receive four apples and the remaining apple must be split into two equal parts). By the end of the study period, nearly all of the children showed knowledge of at least one fraction other than one-half and could solve equal sharing problems that involved partitioning other than repeated halving after instruction based on children’s informal understanding of fractions.

Early knowledge of proportionality. Proportional reasoning is foundational to understanding fractions (Boyer & Levine, 2012). Building on the idea that children have intuitive understanding of proportionality that is not dependent on formal instruction, Singer-Freeman and Goswami (2001) administered a proportional analogy task to preschool children. Two models of familiar foods were used in the task: one represented a continuous quantity (pizza) and the other

a discontinuous quantity (box of chocolates). During each trial, the experimenter presented two models to the child (e.g., two pizzas) and then removed a proportion from one of the models (e.g., one-half of a pizza). Children were asked to take away the same amount from the other model. Proportional reasoning with continuous quantities (e.g., pizzas) was relatively easy for the young children. On the other hand, children encountered greater difficulty reasoning about proportions when the task involved discrete quantities (e.g., individual chocolates), most likely because the presence of discrete, countable quantities encourages counting (Boyer, Levine, & Huttenlocher, 2008; Singer-Freeman & Goswami, 2001). Rather than thinking of a fraction as one magnitude (e.g., “one-third of the candy” or “less than half of the candy”), children may focus only on the size of the numerator or denominator or the total amount (e.g., “one piece of candy” or “three pieces total”).

Early misconceptions. The early fraction knowledge that young children demonstrate is not without misunderstandings. Children even struggle with the concept of “half.” Although children can solve concrete problems that involve halving (e.g., half of the pizza), they do not seem to have a complete understanding of the term. Some kindergartners and first graders see “half” as any fraction, regardless of whether they were halves, thirds, or quarters (Brizuela, 2006; Empson, 1999). Still other children are uncomfortable naming fractions, instead preferring to refer to fractions of pizzas or cookies as “pieces” (Brizuela, 2006; Mack, 1990). Some children do not understand what a half is. For example, when asked to identify “half”, they may point to a partitioning line that separates a figure and call that “the half”, rather than one of the pieces themselves (Brizuela, 2006). Interestingly, young students sometimes assert that half means “a little more than,” particularly in the context of age. For example, children might assert that on their next birthday, they will be seven, and on the day after the birthday, they will be seven and a

half (Brizuela, 2006). These conflicting sets of beliefs indicate that young children are thinking about fractions; however, full understanding of seemingly simple concepts, such as half, develops later. Because children live in a world that uses fractions in colloquial language, they pick up on aspects of fractions that are salient to them and, as a result, may hold several differing, seemingly conflicting understandings of fractions (Brizuela, 2006).

Although children show early knowledge of fraction calculation, equal sharing, and proportional reasoning, this knowledge does not translate easily to symbolic fractions and their magnitudes. When children use written arithmetic, they become confused because the same symbols that they have used for whole numbers are used in a new way to represent fractions (Sophian, Garyantes, & Chang, 1997). For example, children have seen the number “1” used to represent one object and “2” used to represent two objects, but the fraction “ $1/2$ ” can have multiple meanings (half of one object, or three out of six objects).

Fraction development between third and sixth grade: Findings from the Delaware Longitudinal Study

The goal of our longitudinal project was to identify predictors of fraction learning from third through sixth grade. We examined the development of fraction concepts and procedures over multiple time points: before, during, and right after formal fraction instruction in school. We uncovered empirically distinct growth trajectory classes for fraction learning and analyzed student performance to help explain why so many children struggle with fractions.

We started the project with about 500 children in third grade. Participants were drawn from schools that primarily followed a curriculum sequence based on the U.S. Common Core State Standards-Mathematics (Council of Chief State School Officers & National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Although

students represented a diverse range of ethnicities, SES, and ability levels, we oversampled in higher risk schools serving low-income communities. Based on the aforementioned integrated theory of numerical development, we hypothesized that accurate representations of numerical magnitudes would be uniquely important for acquisition of fraction knowledge in particular and mathematics skills more generally. To test this premise, we examined the extent to which general and number specific cognitive processes predict fraction outcomes.

Domain-general predictors included measures of classroom attention, working memory, language, nonverbal reasoning, and reading fluency. Our mathematics specific predictors included whole number line estimation, non-symbolic proportional reasoning, and whole number calculation skills (i.e., addition and multiplication fluency and long division). Non-symbolic proportional reasoning supports thinking about multiplicative relations, a crucial component of fractions (Boyer & Levine, 2012). Most of the predictor measures were administered in third and fifth grades, with the exception of proportional reasoning, multiplication, and long division, which were administered only fifth grade. Fraction outcomes, which were assessed over multiple time points between fourth and sixth grades, included a fraction number line estimation task, a curriculum-based fraction concepts measure using released items from the U.S. National Assessment of Educational Progress (NAEP; 2007, 2009), and a measure of fraction arithmetic procedures. In all of our main analyses we controlled for the influence of background factors (i.e., gender, income status, and age).

Predictors of fraction knowledge. Overall, multiple regression analyses showed that both domain-specific knowledge and domain-general knowledge shape change in fraction learning, with numerical magnitude understanding being especially important (Jordan et al., 2013; Hansen et al 2015a). The standardized beta coefficients (measure of effect size or the

expected increase or decrease of the dependent variable in standard deviation units) of predictors, by grade, are presented in Table 1.

(1) *Third grade predictors of fourth grade fraction outcomes* (Jordan et al., 2013). For fourth-grade fraction concepts, number line estimation, attentive behavior, addition fluency, language, nonverbal ability, and reading fluency, all made significant and unique contributions. For fourth-grade fraction procedures, number line estimation, working memory, attentive behavior, and addition fluency contributed reliably and independently to the model. Overall, the complete set of number-related and general processes predicted fourth graders' fraction concepts better than fraction procedures (56% versus 30% of explained variance).

(2) *Fifth grade predictors of sixth grade fraction outcomes* (Hansen et al., 2015a). Whole number line estimation, non-symbolic proportional reasoning, attentive behavior, long division, and working memory all contributed uniquely to fraction concepts in sixth grade. On a sixth-grade measure of fraction procedures, attentive behavior, whole number line estimation, multiplication fluency, and long division made unique and significant contributions. Overall, the entire set of fifth grade predictors accounted for 58% of the explained variance for fraction concepts and 40% for fraction procedures in sixth grade. The lower overall predictability of our measures for fraction procedures compared to fraction concepts could indicate that instruction in rote computation algorithms may be used more in current practice than instruction that emphasizes understanding of what fractions are.

In the aforementioned study of sixth grade predictors (Hansen et al., 2015a), number line estimation is more important to fraction concepts than to procedures, which differs from our earlier finding that third-grade number line estimation is equally predictive of fraction concepts and procedures at fourth grade (Jordan et al., 2013). A possible explanation for this

developmental difference might be that fourth grade fraction computation involves relatively simple procedures involving like denominators for addition and subtractions. In later grades, students must be facile with multiple procedures and operations. Attention and whole number computational abilities (multiplication fluency and division) emerge as key predictors of sixth grade fraction procedures.

Interestingly, our non-symbolic proportional reasoning measure contributed independently to students' proficiency with fraction concepts, over and above symbolic math-related and general predictors. The data are in keeping with Matthews et al.'s (in press) conjecture, discussed earlier, that a core cognitive system that processes non-symbolic ratios underpins fraction learning. Although proportional reasoning and whole number line estimation are moderately correlated, the independent contribution of proportional reasoning shows that fundamental understandings of scale relations and multiplicative reasoning (e.g., $1/3$ is the same as $3/9$; Boyer & Levine, 2012; Gunderson, Ramirez, Beilock, & Levine, 2012) are important for learning fraction concepts, particularly in the context of area models for representing fractions that are widely used in the United States.

We also examined students' performance on individual items on our fraction concepts and procedures measures in sixth grade (Hansen et al., 2015b). There were substantial gaps between lower and higher performing children on items requiring them to compare and order fractions. Lower performing students struggled relative to their higher performing peers on tasks that required flexible thinking, such as ones where the denominator of the fraction did not correspond directly to the pieces shown (e.g., "Shade $4/5$ of 10 circles"). It should be noted that almost all students experienced difficulties when presented with fraction estimation problems (e.g., "Estimate the sum: $7/8 + 12/13$ "), suggesting that generally more attention should be

devoted to this instructional topic. On procedures, we found that low performing sixth graders tend to misapply whole number strategies when adding and subtracting fractions with unlike denominators (e.g., $2/5 + 3/4 = 5/9$). This approach reflects weak understanding or attention to the bipartite structure of fractions, that is, that the numerator and denominator of the fraction together represent one magnitude.

Growth in fraction magnitude understanding. We assessed the development of children's fraction number line estimation over five time points between fourth and sixth grade (Resnick et al., 2015). No previous studies had examined the development of fraction magnitude estimation during this time, which is when the majority of fraction instruction takes place in American schools. At each time point, students were asked to estimate where fractions should be placed on number lines ranging from 0 to 1 and 0 to 2. See Figure 1 for examples of task items.

Over the course of the study, most students increased their estimation accuracy. However, latent class growth analyses revealed three empirically distinct growth trajectory classes: (1) students who were highly accurate from the start and became even more accurate; (2) students who were inaccurate at first but showed steep growth; and (3) students who started inaccurate and showed minimal growth. Disturbingly, about 42% of students left sixth grade with only a minimal grasp of fraction magnitudes. Another 26% of students seemed to benefit a great deal from their fractions instruction in later elementary and early middle school but still were not as accurate as roughly one third of the students who started the study with considerable strength in estimating fraction magnitudes. Growth trajectory class membership was highly predictive of performance on a high-stakes, statewide mathematics test at the end of sixth grade. That is, 67% of the inaccurate, minimal growth group did *not* meet state mathematics standards vs. only 5% of the highly accurate group and 17% of the steep growth group.

Students with poor multiplication fluency, weak classroom attention, and inaccurate whole number line estimation skill at the start of the study in fourth grade were much more likely to fall into the low-accuracy, minimal growth group. Analyses of performance showed that younger grades and low-accuracy/minimal-growth students typically estimated both proper and improper fractions as being less than one, indicating they were not attending to the relation between numerator and denominator. These students seemed to define fractions as “really small” or “less than one”. Emphasis on proper fractions in early fraction instruction may lead students to view all fractions (a/b) as numbers between zero and one (Vosniadou, Vamvakoussi, & Skopeliti, 2008).

Overall analysis by fraction type revealed that students’ estimation skills with improper fractions were less accurate and developed later than their skill with proper fractions and mixed numbers. This finding supports the integrated theory of numerical development, described earlier, which asserts that numerical development involves increasingly widening the range and type of number understood as magnitudes that can be accurately located on a number line (Siegler & Lortie-Forgues, 2014).

In sum, findings to date from the Delaware Longitudinal Study show that although a range of domain-general and number-specific competencies predict fraction outcomes, the ability to estimate numerical magnitudes on a number line is a key marker of fraction and mathematics success. Children with mathematics difficulties may have fundamental problems related to whole number magnitude representations that are further complicated by the introduction of fractions into the curriculum. In addition, attentive behavior, non-symbolic proportional reasoning, and calculation fluency all were consistently important predictors of fraction learning and growth.

Helping students who struggle with fractions

According to a recent report by the U.S. Department of Education National Center for Educational Statistics, only 32% of eighth graders nationwide scored at or above proficiency on the 2015 nationwide NAEP assessments (NAEP; 2015). Moreover, the situation is most urgent for minority and low-income students, who generally perform more poorly than their non-minority and non-low income peers. As such, current research findings on the importance of numerical magnitude understanding for developing fraction knowledge in particular, and to mathematics achievement more generally, should have particular resonance for educators and policymakers.

Research findings strongly support an approach to teaching fractions as well as whole numbers that centers on a number line. Students who develop an understanding that all real numbers, including fractions, are assigned to their own location on a number line have an advantage not only in learning fractions but also in learning algebraic concepts (Booth, Newton, & Twiss-Garrity, 2014). Until recently, a part-whole interpretation of fractions has been dominant in the U.S. mathematics curriculum (Siegler et al., 2015). Yet instructional activities using number lines successfully support student learning of whole numbers (Ramani & Siegler, 2008) and fractions (Fuchs et al., 2013; Fuchs et al., 2014).

Part-whole instruction defines a/b as a parts out of b parts. Even low-performing students are successful on part-whole problems where the denominator matches the number of partitions in the whole (Hansen et al., 2015b); while low-performing students can shade $3/4$ of a rectangle when the rectangle is separated into four equal parts, they are unable to when it is partitioned into 8 equal parts. These same students, however, may be able to show that $3/4$ is equivalent to $6/8$ by performing the operation: $3/4 \times 2/2 = 6/8$. Such disconnects between conceptual and procedural knowledge makes it difficult for students to use fraction procedures in every day situations

because they do not understand the concepts that support all procedures. A number line approach has the potential to remedy some of these difficulties by emphasizing that fractions are numbers that can be ordered and placed on the number line (Fuchs et al., 2013; Fuchs et al., 2014). For example, a number line could be used to illustrate fraction equivalence ($\frac{3}{4}$ is the same as $\frac{6}{8}$ because they are actually the same location on a number line), the inversion property of fractions (fractions with the same numerator become smaller as the denominators increase), and the density of fractions (between any two fractions, there are an infinite number of fractions).

To promote learning of fraction magnitudes, interventions should also address students' attention. Attentive behavior allows students to stay on task and to acquire relevant knowledge and skills in their mathematics classrooms. Attentive behavior also facilitates effective strategy application on number line tasks; in order to understand the magnitude of a given fraction, students need to attend to the numerator and denominator simultaneously and to inhibit ineffective whole number strategies (Bonato, Fabri, Umilta, & Zorzi, 2007; Meert, Gregoire, & Noel, 2010). Further, students must attend to the varying end points of the number line (e.g., 0 to 1, 0 to 2, 0 to 5) when placing fractions; low-performing students tend to always place $\frac{1}{2}$ in the midpoint of the line rather than the absolute value of $\frac{1}{2}$ (Carrique, Hansen et al., submitted).

The importance of multiplication fluency should not be underestimated for fraction learning. Fast and accurate calculation skills facilitate reasoning about fractions and their (Seethaler, Fuchs, Star, & Bryant, 2011; Hecht et al., 2003). For example, it helps students see, with minimal cognitive effort, multiplicative relations between equivalent fractions ($\frac{1}{4}$ is the same as $\frac{2}{8}$) as well as between improper fractions and mixed numbers ($\frac{6}{4}$ is the same as $6 \times \frac{1}{4}$, or $1 \frac{1}{2}$). Unfortunately, many U.S. curricula do not support multiplication fluency practice after fifth grade.

Additionally, as noted earlier in this chapter, typical instruction focuses on proper fractions (i.e., fractions less than 1; Vosniadou et al., 2008), even if a number line approach is used. Focusing on proper fractions facilitates to the common misunderstanding that all fractions are small numbers less than one. This type of reasoning makes sense given the general use of the word “fraction” to mean a tiny part, such as a fraction of a second or a fraction of an inch. Children should work with number lines showing more than one from the start, although this assertion requires further investigation. Placing proper and improper fractions on number lines from 0 to 2 encourages reasoning about the relationship between the numerator and the denominator as students move from 0 to 1 and beyond 1. Number line approaches help children see when $a < b$, the fraction is less than 1; when $a = b$, the fraction is equal to 1; and when $a > b$, the fraction is greater than one.

According to the U.S. Department What Works Clearinghouse Practice Guide on assisting students who are struggling with mathematics, in depth treatment of rational number should be the focus of middle school math interventions (Gersten et al., 2009). Research supported instruction emphasizing fraction magnitudes and centered on a number line must be implemented to stem cascading difficulties when algebra becomes a primary focus (Siegler et al., 2012; Siegler & Pyke, 2013).

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Table 1

Multiple Regression Standardized Beta Coefficients of Predictors of Fraction Concepts and Procedures by Grade

Variable	Fourth grade		Sixth grade	
	Concepts	Procedures	Concepts	Procedures
Whole number line estimation ¹	-.271***	-.237***	-.361***	-.183***
Attention	.225***	.135*	.170***	.254***
Language	.196***	.081	---	---
Addition fluency	.169***	.126*	---	---
Nonverbal reasoning	.111**	.097	---	---
Reading fluency	.086*	-.037	.048	.011
Working memory	.044	.137**	.114**	.011
Non-symbolic proportional reasoning	---	---	.198***	.080
Multiplication fluency	---	---	.037	.171**
Long division	---	---	.113*	.171**

Note. Predictors of fourth-grade outcomes were assessed in third grade; predictors of sixth-grade outcomes were assessed in fifth grade. Empty cells indicate variables that were not included in the multiple regression analysis.

* $p < .05$.

** $p < .01$.

*** $p < .001$.

¹ Percentage of absolute error with lower score indicating more accurate estimation

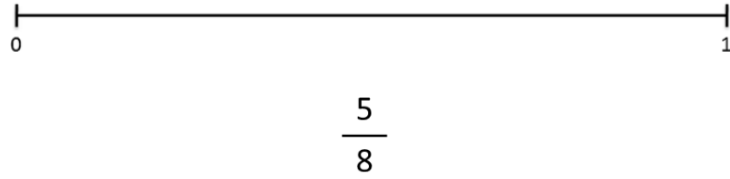
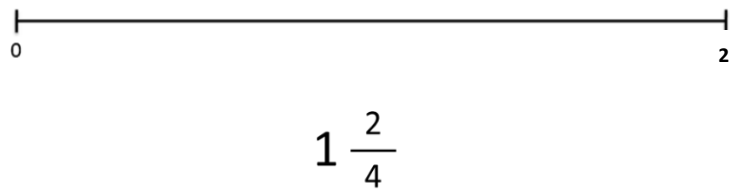
A.**B.**

Figure 1. Example items for (A) 0-1 and (B) 0-2 fraction number line estimation tasks. Students are asked to mark where the fraction should be placed on the line.