# Fractional Brownian motion as a differentiable generalized Gaussian process

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**Abstract**

Brownian motion can be characterized as a generalized random process and, as such, has a generalized derivative whose covariance functional is the delta function. In a similar fashion, fractional Brownian motion can be interpreted as a generalized random process and shown to possess a generalized derivative. The resulting process is a generalized Gaussian process with mean functional zero and covariance functional that can be interpreted as a fractional integral or fractional derivative of the delta-function.

*Keywords:* Brownian motion, fractional Brownian motion, fractional derivative, covariance functional, delta function, generalized derivative, generalized Gaussian process

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# 1 Introduction

Fractional Brownian motion, like ordinary Brownian motion, has almost everywhere continuous sample paths of unbounded variation and ordinary derivatives of the process do not exist. Gel'fand and Vilenkin (1964) provided an alternative characterization of Brownian motion as a generalized Gaussian process defined as a random functional on a space of well behaved functions. Interpreted as a generalized random process, Brownian motion is differentiable.

A generalized Gaussian process is uniquely determined by its mean functional and the bivariate covariance functional. Correspondingly, the generalized derivative of a Gaussian process with zero mean functional is a generalized Gaussian process with zero mean functional and covariance functional that can be computed from the covariance functional of the original process. Gel'fand and Vilenkin provide a description of the generalized Gaussian process which represents the derivative of Brownian motion. This process has a covariance functional that can be interpreted in terms of the delta-function.

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The present paper considers fractional Brownian motion from the same perspective as a generalized process and shows how to characterize its generalized derivative. The resulting process is a generalized Gaussian process with mean functional zero and covariance functional that can be interpreted as a fractional integral or fractional derivative of the delta-function. Higher order derivatives can be similarly described.

## 2 Fractional Brownian motion as a generalized random process

The form of the fractional Brownian motion process considered here was introduced by Mandelbrot and Van Ness (1968). In Marinucci and Robinson(1999) it is called Type I fractional Brownian motion. This form of (standard) fractional Brownian motion for $0 < H < 1$ is represented in integral form as

$$B_H(r) = A(H)^{-1}\left[\int_{-\infty}^{r}(r-s)^{H-\frac{1}{2}}dB(s) - \int_{-\infty}^{0}(-s)^{H-\frac{1}{2}}dB(s)\right], \quad r \geq 0 \tag{2.1}$$

with $A(H) = \left[\frac{1}{2H} + \int_0^\infty \left\{(1+s)^{\ H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right\} ds\right]^{\frac{1}{2}}$ and where $B$ is standard Brownian motion and $H$ is the self similarity index. For $H = \frac{1}{2}$ the process coincides with Brownian motion. Samorodnitsky and Taqqu (1994, ch.7.2) give the 'moving average' representation (2.1) as well as an alternative harmonizable representation of the fractional Brownian motion process. Bhattacharya and Waymire (1990) provide some background discussion of the Hurst phenomenon and subsequent theoretical developments that led to the consideration of stochastic processes of this type.

The mean functional of (2.1) is $EB_H(r) = 0$ and the covariance kernel $V(r_1, r_2)$ is (Samorodnitsky and Taqqu, 1994)

$$V(r_1, r_2) = EB_H(r_1)B_H(r_2) = \frac{1}{2}\left[|r_1|^{2H} + |r_2|^{2H} - |r_2 - r_1|^{2H}\right].$$

Note that $B_H(0) = 0$ and for $r_1, r_2 > 0$ the covariance kernel becomes

$$V(r_1, r_2) = \frac{1}{2}\left[r_1^{2H} + r_2^{2H} - |r_2 - r_1|^{2H}\right]. \tag{2.2}$$

The usual covariance kernel of Brownian motion follows when $H = \frac{1}{2}$.

Following Gel'fand and Vilenkin (1964), define the space $K$ of 'test functions' as follows. $K$ is the space of infinitely continuously differentiable functions $\phi$ with finite support on the real line $R$. The topology on this space is defined by convergence of sequences in $K$, where $\phi_n \to 0$ as $n \to \infty$ if all $\phi_n$ are defined on the same finite support and $\phi_n^{(i)} \to 0$ for the function $(i = 0)$ and all of its derivatives $(i = 1, ...)$[3]. For such $\phi \in K$ define

$$\langle B_H, \phi\rangle = \int B_H(r)\phi(r)dr. \tag{2.3}$$

[3]Other spaces of test functions can be chosen. For example, the space $S$ of infinitely

Integrals in linear functionals such as (2.3) are taken from 0 to $\infty$ and they are convergent due to the fact that all $\phi \in K$ have finite support. Test functions could differ at negative values of $r$ without affecting the value of the functional $\langle B_H, \phi \rangle$. Thus we can restrict ourselves to the subspace $K_+$ of $K$ of functions $\phi(r)$ with non-negative support. The representation (2.3) provides an interpretation of $B_H$ as a linear functional on the space $K_+$. It is easily seen that this functional is continuous in the topology on $K_+$. Since $E(B_H) = 0$, the mean functional is zero.

Next we derive the covariance functional of $B_H$. This functional, which we denote by $V_H[\phi, \psi]$ is given in terms of the covariance kernel $V(r_1, r_2)$ of the process $B_H$. For $\phi, \psi \in K_+$ we have

$$V_H[\phi, \psi] := \langle V, (\phi(t), \psi(s)) \rangle = \int \int V(t,s)\phi(t)\psi(s)dtds.$$

Substituting the expression for $V(t,s)$ from (2.2) we have

$$\begin{aligned} &2V_H[\phi, \psi] \\ &= \int_0^\infty \int_0^\infty \left[t^{2H} + s^{2H} - |t-s|^{2H}\right] \phi(t)\psi(s)dtds \\ &= \int_0^\infty \phi(t)dt \int_0^\infty s^{2H}\psi(s)ds + \int_0^\infty \psi(s)ds \int_0^\infty t^{2H}\phi(t)\psi dt \\ &- \int_0^\infty \phi(t) \left[\int_0^t (t-s)^{(2H+1)-1}\psi(s)ds\right] dt \\ &- \int_0^\infty \psi(s) \left[\int_0^s (t-s)^{(2H+1)-1}\phi(t)dt\right] ds. \end{aligned} \tag{2.4}$$

Denote the integral $\frac{1}{\Gamma(a)} \int_0^t (t-x)^{a-1} f(x)dx$ by $(I^a f)(t)$ for $a > 0$. This integral is the fractional integral (in the Liouville sense) of the function $f$. If $g(t) = (I^a f)(t)$ where $a > 0$, then $f$ is the fractional derivative of $g$ and we shall write $f(t) = (I^{-a} g)(t)$. We use these expressions to simplify (2.4) in what follows.

Start by noting that since $\left[t^{2H} \int_t^\infty \psi(s)ds\right]_0^\infty = 0$

$$\int_0^\infty s^{2H}\psi(s)ds = \left[t^{2H} \int_t^\infty \psi(s)ds\right]_0^\infty - \int_0^\infty s^{2H} (-\psi(s))ds,$$

which equals

$$\begin{aligned} &(2H) \int_0^\infty t^{2H-1} \int_t^\infty \psi(s)dsdt \\ &= (2H) \int_0^\infty t^{2H-1} \left[(I\psi)(\infty) - (I\psi)(t)\right] dt. \end{aligned}$$

differentiable functions that go to zero at infinity faster than any power, or spaces of functions that are not infinitely differentiable. The number of continuous derivatives that the test functions possess will determine the number of generalized derivatives of the process that can be defined on that space.

Use this expression in (2.4) to get

$$\begin{aligned}
&2V_H[\phi,\psi] \\
&= (2H)\,(I\phi)(\infty)\left[\int_0^\infty t^{2H-1}\left[(I\psi)(\infty)-(I\psi)(t)\right]dt\right] \\
&+ (2H)\,(I\psi)(\infty)\left[\int_0^\infty t^{2H-1}\left[(I\phi)(\infty)-(I\phi)(t)\right]dt\right] \\
&-\Gamma(2H+1)\left[\int_0^\infty \phi(t)(I^{2H+1}\psi)(t)dt+\int_0^\infty \psi(t)(I^{2H+1}\phi)(t)dt\right]. \quad (2.5)
\end{aligned}$$

Now

$$\begin{aligned}
&\int_0^\infty \phi(t)(I^{2H+1}\psi)(t)dt \\
&= \left[(I\phi)(t)\,(I^{2H+1}\psi)(t)\right]_0^\infty - \int_0^\infty (I\phi)(t)(I^{2H}\psi)(t)dt \\
&= (I\phi)(\infty)\,(I^{2H+1}\psi)(\infty) - \int_0^\infty (I\phi)(t)(I^{2H}\psi)(t)dt \\
&= \int_0^\infty \left[(I\phi)(\infty)-(I\phi)(t)\right](I^{2H}\psi)(t)dt, \qquad (2.6)
\end{aligned}$$

and

$$\int_0^\infty t^{2H-1}\left[(I\psi)(\infty)-(I\psi)(t)\right]dt = \int_0^\infty t^{2H-1}\int_t^\infty \psi(s)dsdt. \qquad (2.7)$$

Using (2.6) and (2.7) in (2.5) gives the following expression for $2V_H[\phi,\psi]$,

$$\begin{aligned}
&(2H)\,(I\phi)(\infty)\left[\int_0^\infty t^{2H-1}\left[(I\psi)(\infty)-(I\psi)(t)\right]dt\right] \\
&+ (2H)\,(I\psi)(\infty)\left[\int_0^\infty t^{2H-1}\left[(I\phi)(\infty)-(I\phi)(t)\right]dt\right] \\
&-\Gamma(2H+1)\int_0^\infty \left[(I\phi)(\infty)-(I\phi)(t)\right](I^{2H}\psi)(t)dt \\
&-\Gamma(2H+1)\int_0^\infty \left[(I\psi)(\infty)-(I\psi)(t)\right](I^{2H}\phi)(t)dt \\
&= \int_0^\infty \left[(I\phi)(\infty)-(I\phi)(t)\right]\left[t^{2H-1}(2H)\,(I\psi)(\infty)-\Gamma(2H+1)(I^{2H}\psi)(t)\right]dt \\
&+ \int_0^\infty \left[(I\psi)(\infty)-(I\psi)(t)\right]\left[t^{2H-1}(2H)\,(I\phi)(\infty)-\Gamma(2H+1)(I^{2H}\phi)(t)\right]dt,
\end{aligned}$$

so that

$$\begin{aligned}
&V_H[\phi,\psi] = \\
&\frac{1}{2}\int_0^\infty \left[(I\phi)(\infty)-(I\phi)(t)\right]\left[t^{2H-1}(2H)\,(I\psi)(\infty)-\Gamma(2H+1)(I^{2H}\psi)(t)\right]dt + \\
&\frac{1}{2}\int_0^\infty \left[(I\psi)(\infty)-(I\psi)(t)\right]\left[t^{2H-1}(2H)\,(I\phi)(\infty)-\Gamma(2H+1)(I^{2H}\phi)(t)\right]dt. (2.8)
\end{aligned}$$

Setting $H = \frac{1}{2}$ in this expression, we find that (2.8) specializes to

$$V_{\frac{1}{2}}[\phi, \psi] = \int_0^\infty [(I\phi)(\infty) - (I\phi)(t)]\,[(I\psi)(\infty) - (I\psi)(t)]\,dt,$$

which is the covariance functional of Brownian motion as a generalized process, a formula given in Gel'fand and Vilenkin (1964, p. 259).

Thus, as a generalized random process, fractional Brownian motion is a generalized Gaussian process with mean functional zero and covariance functional given by (2.8). Observe that (2.8) is a bilinear functional involving fractional integrals of the test functions $\psi$ and $\phi$.

This alternative approach provides a new description of fractional Brownian motion. In the conventional manner, fractional Brownian motion can be described by its randomly selected sample paths, so that one can think about this process as being indexed by a random element in the probability space where the process lives. In contrast, the new description of fractional Brownian motion as a generalized process indexes the process by deterministic functions belonging to the class $K_+$. Its covariance properties are similarly indexed by these deterministic functions through the covariance functional $V_H[\phi, \psi]$.

# 3 The generalized derivative of the fractional Brownian motion process

One advantage of this new description of fractional Brownian motion is that it is differentiable, and the process representing the derivative is also a generalized Gaussian process. The mean functional is zero for the derivative process and, according to Gel'fand and Vilenkin (1964, p. 257), its covariance functional $V_H'[\phi, \psi]$ satisfies

$$V_H'[\phi, \psi] = V_H[\phi', \psi'].$$

Substituting $\phi', \psi'$ for $\phi$ and $\psi$, respectively in (2.8), we get the expression

$$\begin{aligned}
&V_H[\phi', \psi'] \\
&= \frac{1}{2}\int_0^\infty [\phi(\infty) - \phi(t)]\left[t^{2H-1}(2H)\psi(\infty) - \Gamma(2H+1)(I^{2H}\psi')(t)\right]dt \\
&+ \frac{1}{2}\int_0^\infty [\psi(\infty) - \psi(t)]\left[t^{2H-1}(2H)\phi(\infty) - \Gamma(2H+1)(I^{2H}\phi')(t)\right]dt \\
&= \frac{\Gamma(2H+1)}{2}\left\{\int_0^\infty \phi(t)(I^{2H-1}\psi)(t)dt + \int_0^\infty \psi(t)(I^{2H-1}\phi)(t)dt\right\} \qquad (3.1)
\end{aligned}$$

since $(I^{a+1}f')(t) = (I^a f)(t)$ and $\phi(\infty) = \psi(\infty) = 0$, in view of the finite support of the test functions.

Next we interpret the bilinear functional $V_H'$. First, for ordinary Brownian motion ($H = \frac{1}{2}$) the functional $V_H'[\phi, \psi]$ has the simple form

$$V_{\frac{1}{2}}'[\phi, \psi] = \int_0^\infty \phi(t)\psi(t)dt,$$

which can be interpreted in terms of the delta-function $\delta(w)$, i.e.,

$$\begin{aligned}
V'_{\frac{1}{2}}[\phi,\psi] &= \int_0^\infty \phi(t)\psi(t)dt. \\
&= \int_0^\infty \int_{-\infty}^\infty \delta(w)\phi(t)\psi(t+w)dtdw \\
&= \int_0^\infty \int_{-\infty}^\infty \delta(s-t)\phi(t)\psi(s)dtds \\
&= \int_0^\infty \int_0^\infty \delta(s-t)\phi(t)\psi(s)dtds. \qquad (3.2)
\end{aligned}$$

Thus, the covariance kernel of the derivative of standard Brownian motion is the delta function, as shown in Gel'fand and Vilenkin (1964, p. 260).

Similarly in the fractional case we can interpret $V'_H$ in terms of a generalized fractional integral/derivative of the delta-function. Treating $w(t) = (I^a f)(t)$ as a generalized function on $K$, the functional $\langle w, \phi\rangle = \int w(t)\phi(t)dt$ is differentiable as a generalized function with derivative $\langle w', \phi\rangle = \int w'(t)\phi(t)dt = -\int w(t)\phi'(t)dt$ by definition of a generalized derivative (Gel'fand and Shilov, 1964). Using this relation in the expression for $V'_H[\phi,\psi]$ gives

$$V'_H[\phi,\psi] = \frac{\Gamma(2H+1)}{2}\left\{\int_0^\infty \phi(t)(I^{2H-1}\psi)(t)dt + \int_0^\infty \psi(t)(I^{2H-1}\phi)(t)dt\right\}.$$

As we see in what follows, this expression can be written in the form

$$V'_H[\phi,\psi] = \Gamma(2H+1)\int_0^\infty \int_0^\infty \left(I^{2H-1}\delta\right)(s-t)\phi(t)\psi(s)dtds. \qquad (3.3)$$

extending the representation (3.2) for the covariance functional of the first derivative of Brownian motion. So the covariance kernel of the derivative of fractional Brownian motion (treated as a generalized process) is the fractional derivative/integral $(I^{2H-1}\delta)$ of the delta function. For $H > \frac{1}{2}$ this is a fractional integral, while for $H < \frac{1}{2}$ it is a fractional derivative. We examine the two cases separately.

In the case of a fractional integral with $a = 2H - 1 > 0$ and $t > 0$ we have

$$\left(I^a\delta\right)(t) = \frac{1}{\Gamma(a)}\int_0^t (t-x)^{a-1}\delta(x)dx = \frac{t^{a-1}}{\Gamma(a)}. \qquad (3.4)$$

Then

$$\begin{aligned}
&\int_0^\infty \phi(t)(I^a\psi)(t)dt \\
&= \int_0^\infty \phi(t)\left[\frac{1}{\Gamma(a)}\int_0^t (t-x)^{a-1}\psi(x)dx\right]dt \\
&= \int_0^\infty \phi(t)\left[\int_0^t \left(I^a\delta\right)(t-x)\psi(x)dx\right]dt \\
&= \int_0^\infty \phi(t)\left[\int_0^t \left(I^a\delta\right)(w)\psi(t-w)dw\right]dt \\
&= \int_0^\infty \int_0^\infty \left(I^a\delta\right)(t-s)\phi(t)\psi(s)dsdt,
\end{aligned}$$

and similarly

$$\int_0^\infty \psi(t)(I^a\phi)(t)dt = \int_0^\infty \int_0^\infty (I^a\delta)(t-s)\phi(t)\psi(s)dsdt,$$

so that

$$\begin{aligned} V_H'[\phi,\psi] &= \frac{\Gamma(2H+1)}{2}\left\{\int_0^\infty \phi(t)(I^{2H-1}\psi)(t)dt + \int_0^\infty \psi(t)(I^{2H-1}\phi)(t)dt\right\} \\ &= \Gamma(2H+1)\int_0^\infty\int_0^\infty \left(I^{2H-1}\delta\right)(t-s)\phi(t)\psi(s)dtds, \end{aligned}$$

giving the result (3.3).

In the case of a fractional derivative with $a = 2H-1 < 0$ $(0 < H < \frac{1}{2})$ we write $I^{2H-1}f = I^{2H}f'$ and then

$$\begin{aligned} &\int_0^\infty \phi(t)(I^a\psi')(t)dt \\ &= \int_0^\infty \phi(t)\left[\frac{1}{\Gamma(a)}\int_0^t (t-x)^{a-1}\psi'(x)dx\right]dt \\ &= \int_0^\infty \phi(t)\left[\int_0^t (I^a\delta)(t-x)\psi'(x)dx\right]dt \\ &= \int_0^\infty \phi(t)\left[\int_0^t (I^a\delta)(w)\psi'(t-w)dw\right]dt \\ &= \int_0^\infty \phi(t)\left[\int_0^t \left(I^{a-1}\delta\right)(w)\psi(t-w)dw\right]dt \\ &= \int_0^\infty\int_0^\infty \left(I^{a-1}\delta\right)(t-s)\phi(t)\psi(s)dsdt, \end{aligned}$$

with a similar result for $\int_0^\infty \phi(t)(I^a\phi')(t)dt$. It follows that

$$\begin{aligned} V_H'[\phi,\psi] &= \frac{\Gamma(2H+1)}{2}\left\{\int_0^\infty \phi(t)(I^{2H-1}\psi)(t)dt + \int_0^\infty \psi(t)(I^{2H-1}\phi)(t)dt\right\} \\ &= \frac{\Gamma(2H+1)}{2}\left\{\int_0^\infty \phi(t)(I^{2H}\psi')(t)dt + \int_0^\infty \psi(t)(I^{2H}\phi')(t)dt\right\} \\ &= \Gamma(2H+1)\int_0^\infty\int_0^\infty \left(I^{2H-1}\delta\right)(t-s)\phi(t)\psi(s)dsdt, \end{aligned}$$

as required for (3.3).

Clearly, one can proceed with further differentiation of the fractional process. Subsequent $m$-th order derivatives will provide generalized Gaussian processes with mean functional zero and covariance functional expressed in terms of the generalized function $\left(I^{2H-m}\delta\right)(t-s)$.

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# Bibliography

[1] Bhattacharya, R. N. and E. C. Waymire (1990). *Stochastic Processes with Applications.* New York: John Wiley.

[2] Gel'fand I M. and G. E. Shilov (1964). *Generalized Functions, Vol.4.* New York: Academic Press.

[3] Gel'fand I M. and N. Ya. Vilenkin (1964). *Generalized Functions, Vol.1.* New York: Academic Press.

[4] Mandelbrot, B.B. and J. W. Van Ness (1968). "Fractional Brownian Motions, Fractional Noises and Applications". *SIAM Review,* 10, 422-437.

[5] Marinucci, D. and P. M. Robinson (1999). "Alternative Forms of Fractional Brownian Motion". *Journal of Statistical Planning and Inference,* 80, 111-122.

[6] Samorodnitsky, G. and M. S. Taqqu (1994). *Stable Non-Gaussian Random Processes.* London: Chapman & Hall.