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*Research Article*

# Fractional Complex Transform and exp-Function Methods for Fractional Differential Equations

**Ahmet Bekir,[1] Özkan Güner,[2] and Adem C. Cevikel[3]**

[1] *Eskisehir Osmangazi University, Art-Science Faculty, Department of Mathematics and Computer Science, 26480 Eskişehir, Turkey*
[2] *Dumlupınar University, School of Applied Sciences, Department of Management Information Systems, 43100 Kütahya, Turkey*
[3] *Yildiz Technical University, Education Faculty, Department of Mathematics Education, 34220 Istanbul, Turkey*

Correspondence should be addressed to Adem C. Cevikel; acevikel@yildiz.edu.tr

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The exp-function method is presented for finding the exact solutions of nonlinear fractional equations. New solutions are constructed in fractional complex transform to convert fractional differential equations into ordinary differential equations. The fractional derivatives are described in Jumarie's modified Riemann-Liouville sense. We apply the exp-function method to both the nonlinear time and space fractional differential equations. As a result, some new exact solutions for them are successfully established.

## 1. Introduction

Fractional differential equations (FDEs) are viewed as alternative models to nonlinear differential equations. Varieties of them play important roles and serve as tools not only in mathematics but also in physics, biology, fluid flow, signal processing, control theory, systems identification, and fractional dynamics to create the mathematical modeling of many nonlinear phenomena. Besides, they are employed in social sciences such as food supplement, climate, finance, and economics. Oldham and Spanier first considered the fractional differential equations arising in diffusion problems [1]. The fractional differential equations have been investigated by many authors [2–4].

In recent decades, some effective methods for fractional calculus appeared in open literature, such as the exp-function method [5], the fractional subequation method [6–8], the $(G'/G)$-expansion method [9, 10], and the first integral method [11].

The fractional complex transform [12, 13] is the simplest approach; it is to convert the fractional differential equations into ordinary differential equations, making the solution procedure extremely simple. Recently, the fractional complex transform has been suggested to convert fractional-order differential equations with modified Riemann-Liouville derivatives into integer order differential equations, and the reduced equations can be solved by symbolic computation. The exp-function method [14–20] can be used to construct the exact solutions for fractional differential equations. The present paper investigates the applicability and efficiency of the exp-function method on fractional nonlinear differential equations. The aim of this paper is to extend the application of the exp-function method to obtain exact solutions to some fractional differential equations in mathematical physics and biology.

This paper is organized as follows. In Section 2, some basic properties of Jumarie's modified Riemann-Liouville derivative are given. The main steps of the exp-function method are given in Section 3. In Sections 4–6, we construct the exact solutions of the fractional-order biological population model, fractional Burgers equation, and fractional Cahn-Hilliard equation via this method. Some conclusions are shown in Section 7.

## 2. Modified Riemann-Liouville Derivative

In the last few decades, in order to improve the local behavior of fractional types, a few local versions of fractional derivatives have been proposed, that is, Caputo's fractional derivative [21], Grünwald-Letnikov's fractional derivative [22], the Riemann-Liouville derivative [22], Jumarie's modified Riemann-Liouville derivative [23, 24]. Jumarie's derivative is defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}\left[f(\xi)-f(0)\right]d\xi, \quad 0<\alpha<1, \tag{1}$$

where $f : R \to R, t \to f(t)$ denotes a continuous (but not necessarily first-order-differentiable) function. We list some important properties for the modified Riemann-Liouville derivative as follows.

(1) Assume that $f(t)$ denotes a continuous $R \to R$ function. We use the following equality for the integral with respect to $(dt)^\alpha$:

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1} f(\xi)\,d\xi, \qquad I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha+1)}\int_0^t f(\xi)(dt)^\alpha, \quad 0<\alpha\le 1. \tag{2}$$

(2) Some useful formulas include

$$f^{(\alpha)}\left[g(t)\right] = \frac{df}{dt}g^{(\alpha)}(t), \tag{3}$$

$$D_t^\alpha t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}t^{\gamma-\alpha}, \tag{4}$$

$$\int (dt)^\beta = t^\beta. \tag{5}$$

(3) Let $u(t)$ and $v(t)$ satisfy the definition of the modified Riemann-Liouville derivative, and let $f(t)$ be an $\alpha$-order-differentiable function:

$$D_t^\alpha (u(t)v(t)) = v(t)D_t^\alpha u(t) + u(t)D_t^\alpha v(t), \qquad D_t^\alpha f[u(t)] = f_u'[u(t)]D_t^\alpha u(t) = D_u^\alpha f[u(t)]\left(u'(t)\right)^\alpha. \tag{6}$$

Function $f(t)$ should be differentiable with respect to $g(t)$, and $g(t)$ is fractional differentiable in (3). The previous results are employed in the following sections.

## 3. Fractional Complex Transform and exp-Function Method

We consider the following nonlinear FDE of the type

$$F\left(u, D_t^\alpha u, D_x^\beta u, D_t^\alpha D_t^\alpha u, D_t^\alpha D_x^\beta u, D_x^\beta D_x^\beta u, \ldots\right) = 0, \quad 0<\alpha,\ \beta<1, \tag{7}$$

where $u$ is an unknown function and $F$ is a polynomial of $u$ and its partial fractional derivatives, in which the highest order derivatives and the nonlinear terms are involved. In the following, we give the main steps of the exp-function method.

*Step 1.* Li and He [25, 26] suggested a fractional complex transform to convert fractional differential equations into ordinary differential equations, so all analytical methods devoted to the advanced calculus can be easily applied to the fractional calculus. The complex wave variable was as follows:

$$u(x,t) = U(\xi), \qquad \xi = \frac{\tau x^\beta}{\Gamma(1+\beta)} + \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}, \tag{8}$$

where $\tau$ and $\lambda$ are nonzero arbitrary constants; we can rewrite (7) in the following nonlinear ordinary differential equation:

$$Q\left(U, U', U'', U''', \ldots\right) = 0, \tag{9}$$

where the prime denotes the derivation with respect to $\xi$. If possible, we should integrate (9) term by term one or more times.

*Step 2.* According to exp-function method, which was developed by He and Wu [14], we assume that the wave solution can be expressed in the following form:

$$U(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp[n\xi]}{\sum_{m=-p}^{q} b_m \exp[m\xi]}, \tag{10}$$

where $p$, $q$, $c$, and $d$ are positive integers which are known to be further determined and $a_n$ and $b_m$ are unknown constants. We can rewrite (10) in the following equivalent form:

$$U(\xi) = \frac{a_{-c}\exp[-c\xi] + \cdots + a_d \exp[d\xi]}{b_{-p}\exp[-p\xi] + \cdots + b_q \exp[q\xi]}. \tag{11}$$

*Step 3.* This equivalent formulation plays a significant and fundamental part for finding the exact solution of mathematical problems. To determine the values of $c$ and $p$, we balance the linear term of highest order of (9) with the highest order nonlinear term. Similarly, to determine the value of $d$ and $q$, we balance the linear term of lowest order of (9) with lowest order nonlinear term [27–29].

In the following sections, we present three examples to illustrate the applicability of the exp-function method and fractional complex transform to solve nonlinear fractional differential equations.

## 4. Fractional-Order Biological Population Model

We consider a time fractional biological population model of the form [30, 31]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2}{\partial x^2}\left(u^2\right) + \frac{\partial^2}{\partial y^2}\left(u^2\right) + h\left(u^2 - r\right), \quad t>0, \quad 0<\alpha\le 1,\ x, y \in R, \tag{12}$$

where $u$ denotes the population density $h(u^2 - r)$ represents the population supply due to births and deaths, and $h$, $r$ are constants.

For our goal, we present the following transformation:

$$u(x, y, t) = U(\xi), \quad \xi = \nu x + i\nu y - \frac{ct^{\alpha}}{\Gamma(1+\alpha)}, \tag{13}$$

where $c$ and $\nu$ are constants and $i^2 = -1$.

Then by the use of (13), (12) can be turned into an ODE:

$$cU' + hU^2 - hr = 0, \tag{14}$$

where "$U'$" $= dU/d\xi$.

Balancing the order of $U'$ and $U^2$ in (14), we get

$$U' = \frac{c_1 \exp[-(c+p)\xi] + \cdots}{c_2 \exp[-2p\xi] + \cdots}, \qquad U^2 = \frac{c_3 \exp[-2c\xi] + \cdots}{c_4 \exp[-2p\xi] + \cdots}, \tag{15}$$

where $c_i$ are determined coefficients only for simplicity. Balancing highest order of exp-function in (15), we obtain

$$-(p+c) = -2c, \tag{16}$$

which leads to the result that

$$p = c. \tag{17}$$

In the same way to determine the values of $d$ and $q$, we balance the linear term of the lowest order in (14):

$$U' = \frac{\cdots + d_1 \exp[(q+d)\xi]}{\cdots + d_2 \exp[2q\xi]}, \qquad U^2 = \frac{\cdots + d_3 \exp[2d\xi]}{\cdots + d_4 \exp[2q\xi]}, \tag{18}$$

where $d_i$ are determined coefficients only for simplicity. From (18), we have

$$q + d = 2d, \tag{19}$$

and this gives

$$q = d. \tag{20}$$

For simplicity, we set $p = c = 1$ and $q = d = 1$, so (11) reduces to

$$U(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \tag{21}$$

Substituting (21) into (14) and by the help of symbolic computation, we have

$$\frac{1}{A}\left[R_2 \exp(2\xi) + R_1 \exp(\xi) + R_0 + R_{-1} \exp(-\xi) + R_{-2} \exp(-2\xi)\right] = 0, \tag{22}$$

where

$$\begin{aligned} A &= \left(b_{-1} \exp(-\xi) + b_0 + b_1 \exp(\xi)\right)^2, \\ R_2 &= ha_1^2 - hrb_1^2, \\ R_1 &= ca_1 b_0 - ca_0 b_1 + 2ha_1 a_0 - 2hrb_1 b_0, \\ R_0 &= -2ca_{-1} b_1 + 2ca_1 b_{-1} - 2hrb_1 b_{-1} - hrb_0^2 \\ &\quad + 2ha_1 a_{-1} + ha_0^2, \\ R_{-1} &= -ca_{-1} b_0 + ca_0 b_{-1} + 2ha_0 a_{-1} - 2hrb_0 b_{-1}, \\ R_{-2} &= ha_{-1}^2 - hrb_{-1}^2. \end{aligned} \tag{23}$$

Solving this system of algebraic equations by using symbolic computation, we get the following results.

*Case 1.* Consider

$$\begin{aligned} a_1 &= \frac{rb_0^2 - a_0^2}{4\sqrt{r} b_{-1}}, \qquad a_0 = a_0, \qquad a_{-1} = -\sqrt{r} b_{-1}, \\ b_1 &= \frac{rb_0^2 - a_0^2}{4rb_{-1}}, \qquad b_0 = b_0, \qquad b_{-1} = b_{-1}, \\ c &= 2h\sqrt{r}, \end{aligned} \tag{24}$$

where $a_0$, $b_0$, and $b_{-1}$ are free parameters which exist provided that $b_{-1} \neq 0$. Substituting these results into (21), we get the following exact solution:

$$\begin{aligned} u(x, y, t) &= \frac{rb_0^2 - a_0^2}{4\sqrt{r} b_{-1}} \exp\left(\nu x + i\nu y - \frac{2h\sqrt{r} t^{\alpha}}{\Gamma(1+\alpha)}\right) \\ &\quad + a_0 - \sqrt{r} b_{-1} \exp\left(-\left(\nu x + i\nu y - \frac{2h\sqrt{r} t^{\alpha}}{\Gamma(1+\alpha)}\right)\right) \\ &\quad \times \left(\frac{rb_0^2 - a_0^2}{4rb_{-1}} \exp\left(\nu x + i\nu y - \frac{2h\sqrt{r} t^{\alpha}}{\Gamma(1+\alpha)}\right)\right. \\ &\quad \left. + b_0 + b_{-1} \exp\left(-\left(\nu x + i\nu y - \frac{2h\sqrt{r} t^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}. \end{aligned} \tag{25}$$

*Case 2.* Consider

$$\begin{aligned} a_1 &= \frac{a_0^2}{4\sqrt{r} b_{-1}}, \qquad a_0 = a_0, \qquad a_{-1} = -\sqrt{r} b_{-1}, \\ b_1 &= -\frac{a_0^2}{4rb_{-1}}, \qquad b_0 = 0, \qquad b_{-1} = b_{-1}, \\ c &= 2h\sqrt{r}, \end{aligned} \tag{26}$$

where $a_0$ and $b_{-1}$ are free parameters, which exist provided that $b_{-1} \neq 0$. Substituting these results into (21), we obtain the following exact solution:

$$\begin{aligned} u(x,y,t) &= \frac{a_0^2}{4\sqrt{r}b_{-1}}\exp\left(\nu x + i\nu y - \frac{2h\sqrt{r}t^{\alpha}}{\Gamma(1+\alpha)}\right) + a_0 \\ &\quad - \sqrt{r}b_{-1}\exp\left(-\left(\nu x + i\nu y - \frac{2h\sqrt{r}t^{\alpha}}{\Gamma(1+\alpha)}\right)\right) \\ &\quad \times\left(-\frac{a_0^2}{4rb_{-1}}\exp\left(\nu x + i\nu y - \frac{2h\sqrt{r}t^{\alpha}}{\Gamma(1+\alpha)}\right)\right. \\ &\quad \left. + b_{-1}\exp\left(-\left(\nu x + i\nu y - \frac{2h\sqrt{r}t^{\alpha}}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}. \end{aligned} \tag{27}$$

## 5. Time Fractional Burgers Equation

We consider the one-dimensional time fractional Burgers equation with the value problem [32]

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + \varepsilon u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \quad t > 0,\ 0 < \alpha \le 1, \tag{28}$$

$$u(x,0) = g(x), \tag{29}$$

where $\alpha$ is a parameter describing the order of the fractional time derivative. The function $u(x,t)$ is assumed to be a causal function of time.

For our purpose, we introduce the following transformations:

$$u(x,t) = U(\xi), \quad \xi = \lambda x - \frac{ct^{\alpha}}{\Gamma(1+\alpha)}, \tag{30}$$

where $\lambda$ and $c$ are nonzero constants.

Substituting (30) into (28), we can show that (28) reduced into the following ODE:

$$-cU' + \lambda\varepsilon UU' - \lambda^2\nu U'' = 0, \tag{31}$$

where "$U'$" $= dU/d\xi$.

Integrating (31) with respect to $\xi$ yields

$$-cU + \frac{\lambda\varepsilon}{2}U^2 - \lambda^2\nu U' + \xi_0 = 0, \tag{32}$$

where $\xi_0$ is a constant of integration.

By the same procedure as illustrated in Section 3, we can determine values of $c$ and $p$ by balancing terms $U^2$ and $U'$ in (32). By symbolic computation, we have

$$\begin{aligned} U' &= \frac{c_1\exp[-(c+p)\xi] + \cdots}{c_2\exp[-2p\xi] + \cdots}, \\ U^2 &= \frac{c_3\exp[-2c\xi] + \cdots}{c_4\exp[-2p\xi] + \cdots}, \end{aligned} \tag{33}$$

where $c_i$ are determined coefficients only for simplicity. Balancing the highest order of exp-function in (33), we have

$$-(p+c) = -2c, \tag{34}$$

which leads to the result that

$$p = c. \tag{35}$$

Similarly, to determine the values of $d$ and $q$, we balance the linear term of the lowest order in (32):

$$\begin{aligned} U' &= \frac{\cdots + d_1\exp[(q+d)\xi]}{\cdots + d_2\exp[2q\xi]}, \\ U^2 &= \frac{\cdots + d_3\exp[2d\xi]}{\cdots + d_4\exp[2q\xi]}, \end{aligned} \tag{36}$$

where $d_i$ are determined coefficients only for simplicity. From (36), we obtain

$$q + d = 2d, \tag{37}$$

and this gives

$$q = d. \tag{38}$$

For simplicity, we set $p = c = 1$ and $q = d = 1$, so (11) reduces to

$$U(\xi) = \frac{a_1\exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{b_1\exp(\xi) + b_0 + b_{-1}\exp(-\xi)}. \tag{39}$$

Substituting (39) into (32) and by the help of symbolic computation, we have

$$\begin{aligned} &\frac{1}{A}\left[R_2\exp(2\xi) + R_1\exp(\xi) + R_0\right. \\ &\quad \left. + R_{-1}\exp(-\xi) + R_{-2}\exp(-2\xi)\right] = 0, \end{aligned} \tag{40}$$

where

$$\begin{aligned} A &= \left(b_{-1}\exp(-\xi) + b_0 + b_1\exp(\xi)\right)^2, \\ R_2 &= \xi_0 b_1^2 - ca_1b_1 + \frac{1}{2}\lambda\varepsilon a_1^2, \\ R_1 &= -\lambda^2\nu a_1b_0 + \lambda\varepsilon a_1a_0 + \lambda^2\nu a_0b_1 - ca_0b_1 - ca_1b_0 \\ &\quad + 2\xi_0b_1b_0, \\ R_0 &= -2\lambda^2\nu a_1b_{-1} + 2\lambda^2\nu a_{-1}b_1 - ca_0b_0 + \lambda\varepsilon a_1a_{-1} \\ &\quad + \xi_0b_0^2 - ca_1b_{-1} - ca_{-1}b_1 + \frac{1}{2}\lambda\varepsilon a_0^2 + 2\xi_0b_1b_{-1}, \\ R_{-1} &= \lambda\varepsilon a_0a_{-1} + \lambda^2\nu a_{-1}b_0 - \lambda^2\nu a_0b_{-1} + 2\xi_0b_0b_{-1} \\ &\quad - ca_0b_{-1} - ca_{-1}b_0, \\ R_{-2} &= \xi_0b_{-1}^2 - ca_{-1}b_{-1} + \frac{1}{2}\lambda\varepsilon a_{-1}^2 \end{aligned} \tag{41}$$

Solving this system of algebraic equations by using symbolic computation, we obtain the following results.

*Case 1.* Consider

$$a_1 = \frac{b_1\left(\varepsilon a_{-1} - 4\lambda\nu b_{-1}\right)}{\varepsilon b_{-1}}, \qquad a_0 = 0, \qquad a_{-1} = a_{-1},$$
$$b_1 = b_1, \qquad b_0 = 0, \qquad b_{-1} = b_{-1},$$
$$\lambda = \lambda, \qquad \varepsilon = \varepsilon, \qquad \nu = \nu,$$
$$\xi_0 = \frac{\lambda a_{-1}\left(\varepsilon a_{-1} - 4\lambda\nu b_{-1}\right)}{2b_{-1}^2}, \qquad c = \frac{\lambda\left(\varepsilon a_{-1} - 2\lambda\nu b_{-1}\right)}{b_{-1}}, \tag{42}$$

where $a_{-1}$ and $b_{-1}$ are free parameters which exist provided that $b_{-1} \neq 0$ and $\varepsilon a_{-1} - 2\lambda\nu b_{-1} \neq 0$. Substituting these results into (39), we obtain the following exact solution:

$$\begin{aligned} u(x,t) &= \frac{b_1\left(\varepsilon a_{-1} - 4\lambda\nu b_{-1}\right)}{\varepsilon b_{-1}} \exp\left(\lambda x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right) \\ &\quad + a_{-1}\exp\left(-\left(\lambda x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right) \\ &\quad \times \left(b_1 \exp\left(\lambda x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right. \\ &\quad \left. + b_{-1}\exp\left(-\left(\lambda x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}. \end{aligned} \tag{43}$$

*Case 2.* Consider

$$\begin{aligned} a_1 &= -\left(\varepsilon a_{-1}^2 b_0^2 - 2a_{-1}b_0^2\lambda\nu b_{-1} - 2\varepsilon a_0 b_0 b_{-1} a_{-1}\right. \\ &\quad \left. + 2\lambda\nu a_0 b_0 b_{-1}^2 + \varepsilon a_0^2 b_{-1}^2\right) \\ &\quad \times \left(\varepsilon a_{-1} - \lambda\nu b_{-1}\right) \\ &\quad \times \left(4b_{-1}^2\lambda^2\nu^2\right)^{-1}, \\ b_1 &= -\varepsilon\left(\varepsilon a_{-1}^2 b_0^2 - 2a_{-1}b_0^2\lambda\nu b_{-1} - 2\varepsilon a_0 b_0 b_{-1} a_{-1}\right. \\ &\quad \left. + 2\lambda\nu a_0 b_0 b_{-1}^2 + \varepsilon a_0^2 b_{-1}^2\right) \\ &\quad \times \left(4b_{-1}^3\lambda^2\nu^2\right)^{-1}, \\ a_0 &= a_0, \qquad a_{-1} = a_{-1}, \qquad b_0 = b_0, \\ b_{-1} &= b_{-1}, \qquad \lambda = \lambda, \qquad \varepsilon = \varepsilon, \qquad \nu = \nu, \\ \xi_0 &= \frac{\lambda a_{-1}\left(\varepsilon a_{-1} - 2\lambda\nu b_{-1}\right)}{2b_{-1}^2}, \qquad c = \frac{\lambda\left(\varepsilon a_{-1} - \lambda\nu b_{-1}\right)}{b_{-1}}, \end{aligned} \tag{44}$$

where $a_{-1}$, $a_0$, $b_{-1}$, and $b_0$ are free parameters which exist provided that $b_{-1} \neq 0$ and $\varepsilon a_{-1} - 2\lambda\nu b_{-1} \neq 0$. Substituting these results into (39), we get the following exact solution:

$$\begin{aligned} u(x,t) &= -\left(\varepsilon a_{-1}^2 b_0^2 - 2a_{-1}b_0^2\lambda\nu b_{-1} - 2\varepsilon a_0 b_0 b_{-1} a_{-1}\right. \\ &\quad \left. + 2\lambda\nu a_0 b_0 b_{-1}^2 + \varepsilon a_0^2 b_{-1}^2\right)\left(\varepsilon a_{-1} - \lambda\nu b_{-1}\right) \\ &\quad \times \left(4b_{-1}^2\lambda^2\nu^2\right)^{-1} \\ &\quad \times \exp\left(\lambda x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right) + a_0 \\ &\quad + a_{-1}\exp\left(-\left(\lambda x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right) \\ &\quad \times \left(-\varepsilon\left(\varepsilon a_{-1}^2 b_0^2 - 2a_{-1}b_0^2\lambda\nu b_{-1} - 2\varepsilon a_0 b_0 b_{-1} a_{-1}\right.\right. \\ &\quad \left. + 2\lambda\nu a_0 b_0 b_{-1}^2 + \varepsilon a_0^2 b_{-1}^2\right) \\ &\quad \times \left(4b_{-1}^3\lambda^2\nu^2\right)^{-1} \\ &\quad \times \exp\left(\lambda x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right) + b_0 \\ &\quad \left. + b_{-1}\exp\left(-\left(\lambda x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}. \end{aligned} \tag{45}$$

## 6. Space-Time Fractional Cahn-Hilliard Equation

We consider the space-time fractional Cahn-Hilliard equation [33]

$$D_t^\alpha u - \gamma D_x^\alpha u - 6u\left(D_x^\alpha u\right)^2 - \left(3u^2 - 1\right) D_x^{2\alpha} u + D_x^{4\alpha} u = 0, \tag{46}$$

where $0 < \alpha \leq 1$ and $u$ is the function of $(x,t)$. For the case corresponding to $\alpha = 1$, this equation is related to a number of interesting physical phenomena like the spinodal decomposition, phase separation, and phase ordering dynamics. Moreover, it becomes important in material sciences [34]. Nevertheless we notice that this equation is very difficult to be solved and several articles investigated it [35].

Firstly, we consider the following transformations:

$$u(x,t) = U(\xi), \qquad \xi = \frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}, \tag{47}$$

where $c$ is a nonzero constant.

Substituting (47) into (46), we can know this equation reduced into an ODE:

$$-cU' - \gamma U' - 6U\left(U'\right)^2 - 3U^2U'' + U'' + U'''' = 0, \tag{48}$$

where "$U'$" $= dU/d\xi$.

Integrating (48) with respect to $\xi$ yields

$$-cU - \gamma U - 3U^2U' + U' + U''' + \xi_0 = 0, \tag{49}$$

where $\xi_0$ is a constant of integration.

Here take notice of nonlinear term in (49), and we can balance $U^2U'$ and $U'''$ by the idea of the exp-function method

[14] to determine the values of $p, q, c$, and $d$. By simple calculation, we have

$$\begin{aligned} U''' &= \frac{c_1 \exp\left[(-7p-c)\xi\right] + \cdots}{c_2 \exp\left[-8p\xi\right] + \cdots} \\ &= \frac{c_1 \exp\left[-(7p+c)\xi\right] + \cdots}{c_2 \exp\left[-8p\xi\right] + \cdots}, \\ U^2U' &= \frac{c_3 \exp\left[(-p-3c)\xi\right] + \cdots}{c_4 \exp\left[-4p\xi\right] + \cdots} \\ &= \frac{c_3 \exp\left[-(5p+3c)\xi\right] + \cdots}{c_4 \exp\left[-8p\xi\right] + \cdots}, \end{aligned} \tag{50}$$

where $c_i$ are determined coefficients only for simplicity. Balancing the highest order of exp-function in (50), we have

$$-(7p+c) = -(5c+3p), \tag{51}$$

which leads to the result that

$$p = c. \tag{52}$$

Similarly, to determine the values of $d$ and $q$, we balance the linear term of lowest order in (49)

$$\begin{aligned} U''' &= \frac{\cdots + d_1 \exp\left[(7q+d)\xi\right]}{\cdots + d_2 \exp\left[8q\xi\right]}, \\ U^2U' &= \frac{\cdots + d_3 \exp\left[(2d+6q)\xi\right]}{\cdots + d_4 \exp\left[8q\xi\right]}, \end{aligned} \tag{53}$$

where $d_i$ are determined coefficients only for simplicity. From (53), we obtain

$$(6q+2d) = (d+7q), \tag{54}$$

and this gives

$$q = d. \tag{55}$$

For simplicity, we set $p = c = 1$ and $q = d = 1$, so (11) reduces to

$$u(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1}\exp(-\xi)}. \tag{56}$$

Substituting (56) into (49) and by the help of symbolic computation, we obtain

$$\begin{aligned} &\frac{1}{A}\left[R_4 \exp(4\xi) + R_3 \exp(3\xi) + R_2 \exp(2\xi) + R_1 \exp(\xi)\right. \\ &\quad + R_0 + R_{-1}\exp(-\xi) + R_{-2}\exp(-2\xi) + R_{-3}\exp(-3\xi) \\ &\quad \left. + R_{-4}\exp(-4\xi)\right] = 0, \end{aligned} \tag{57}$$

where

$$\begin{aligned} A &= \left(b_{-1}\exp(-\xi) + b_0 + b_1 \exp(\xi)\right)^4, \\ R_4 &= \xi_0 b_1^4 - c a_1 b_1^3 - \gamma a_1 b_1^3, \\ R_3 &= -3a_1^3 b_0 - 2a_0 b_1^3 - c a_0 b_1^3 - \gamma a_0 b_1^3 + 2a_1 b_1^2 b_0 \\ &\quad + 3a_1^2 a_0 b_1 + 4\xi_0 b_1^3 b_0 - 3\gamma a_1 b_1^2 b_0 - 3c a_1 b_1^2 b_0, \\ R_2 &= -10a_{-1} b_1^3 - 6a_1^3 b_{-1} - c a_{-1} b_1^3 - \gamma a_{-1} b_1^3 \\ &\quad + 10a_1 b_1^2 b_{-1} + 6a_1^2 a_{-1} b_1 - 2a_1 b_1 b_0^2 + 4\xi_0 b_1^3 b_{-1} \\ &\quad + 2a_0 b_1^2 b_0 - 6a_1^2 a_0 b_0 + 6a_1 a_0^2 b_1 + 6\xi_0 b_1^2 b_0^2 \\ &\quad - 3c a_1 b_1 b_0^2 - 3c a_0 b_1^2 b_0 - 3\gamma a_1 b_1 b_0^2 - 3\gamma a_0 b_1^2 b_0 \\ &\quad - 3c a_1 b_1^2 b_{-1} - 3\gamma a_1 b_1^2 b_{-1}, \\ R_1 &= -3c a_0 b_1 b_0^2 - 3\gamma a_0 b_1 b_0^2 - 2a_0 b_1 b_0^2 - c a_1 b_0^3 \\ &\quad - \gamma a_1 b_0^3 - 3a_0^2 a_1 b_0 + 4\xi_0 b_1 b_0^3 + 2a_1 b_0^3 + 3a_0^3 b_1 \\ &\quad - 6c a_1 b_1 b_0 b_{-1} - 6\gamma a_1 b_1 b_0 b_{-1} + 22a_0 b_1^2 b_{-1} \\ &\quad - 10a_{-1} b_1^2 b_0 - 3a_1^2 a_{-1} b_0 - 15a_1^2 a_0 b_{-1} \\ &\quad - 12a_1 b_1 b_0 b_{-1} - 3c a_0 b_1^2 b_{-1} - 3c a_{-1} b_1^2 b_0 \\ &\quad - 3\gamma a_0 b_1^2 b_{-1} - 3\gamma a_{-1} b_1^2 b_0 + 18a_1 b_1 a_0 a_{-1} \\ &\quad + 12\xi_0 b_1^2 b_0 b_{-1}, \\ R_0 &= -3c a_{-1} b_1 b_0^2 - 6\gamma a_0 b_1 b_0 b_{-1} + \xi_0 b_0^4 - c a_0 b_0^3 \\ &\quad + 28a_{-1} b_1^2 b_{-1} - 28a_1 b_{-1}^2 b_1 + 8a_1 b_{-1} b_0^2 \\ &\quad - 6c a_0 b_1 b_0 b_{-1} - 8a_{-1} b_1 b_0^2 \\ &\quad - 3c a_1 b_{-1}^2 b_1 - 3c a_1 b_0^2 b_{-1} - 3c a_{-1} b_1^2 b_{-1} \\ &\quad - \gamma a_0 b_0^3 - 3\gamma a_1 b_{-1}^2 b_1 - 3\gamma a_1 b_0^2 b_{-1} - 3\gamma a_{-1} b_1^2 b_{-1} \\ &\quad - 3\gamma a_{-1} b_0^2 b_1 + 12\xi_0 b_1 b_0^2 b_{-1} - 12a_1 a_0^2 b_{-1} \\ &\quad - 12a_1^2 b_{-1} a_{-1} + 12a_1 a_{-1}^2 b_1 + 12b_1 a_0^2 a_{-1}, \\ R_{-1} &= -3c a_0 b_{-1} b_0^2 - 3\gamma a_0 b_{-1} b_0^2 + 2a_0 b_{-1} b_0^2 \\ &\quad - c a_{-1} b_0^3 - \gamma a_{-1} b_0^3 + 3a_0^2 a_{-1} b_0 \\ &\quad + 4\xi_0 b_0^3 b_{-1} - 2a_{-1} b_0^3 - 3a_0^3 b_{-1} - 6c a_{-1} b_1 b_0 b_{-1} \\ &\quad - 6\gamma a_{-1} b_1 b_0 b_{-1} + 10a_1 b_{-1}^2 b_0 - 22a_0 b_{-1}^2 b_1 \\ &\quad + 3a_1 a_{-1}^2 b_0 + 15a_0 a_{-1}^2 b_1 - 12a_{-1} b_1 b_0 b_{-1} \\ &\quad - 3c a_1 b_0 b_{-1}^2 - 3c a_0 b_1 b_{-1}^2 - 3\gamma a_1 b_{-1}^2 b_0 \\ &\quad - 3\gamma a_0 b_1 b_{-1}^2 - 18a_1 a_{-1} a_0 b_{-1} + 12\xi_0 b_{-1}^2 b_0 b_1, \end{aligned}$$

$$
\begin{aligned}
R_{-2} &= 10a_1b_{-1}^3 + 6a_{-1}^3b_1 - ca_1b_{-1}^3 - \gamma a_1b_{-1}^3 \\
&\quad - 10a_{-1}b_1b_{-1}^2 - 6a_1a_{-1}^2b_{-1} + 4\xi_0b_{-1}^3b_1 - 2a_0b_0b_{-1}^2 \\
&\quad + 2a_{-1}b_0^2b_{-1} + 6a_0a_{-1}^2b_0 - 6a_{-1}a_0^2b_{-1} + 6\xi_0b_{-1}^2b_0^2 \\
&\quad - 3ca_0b_0b_{-1}^2 - 3ca_{-1}b_{-1}b_0^2 - 3\gamma a_0b_0b_{-1}^2 \\
&\quad - 3\gamma a_{-1}b_0^2b_{-1} - 3ca_{-1}b_{-1}^2b_1 - 3\gamma a_{-1}b_{-1}^2b_1, \\
R_{-3} &= 2a_0b_{-1}^3 + 3a_{-1}^3b_0 - ca_0b_{-1}^3 - \gamma a_0b_{-1}^3 - 2a_{-1}b_0b_{-1}^2 \\
&\quad - 3a_{-1}^2a_0b_{-1} + 4\xi_0b_0b_{-1}^3 - 3ca_{-1}b_0b_{-1}^2 \\
&\quad - 3\gamma a_{-1}b_0b_{-1}^2, \\
R_{-4} &= \xi_0b_{-1}^4 - ca_{-1}b_{-1}^3 - \gamma a_{-1}b_{-1}^3.
\end{aligned} \tag{58}
$$

Solving this system of algebraic equations by using symbolic computation, we obtain the following results:

$$
\begin{gathered}
a_1 = \frac{a_0^2\sqrt{6}}{8b_{-1}}, \qquad a_0 = a_0, \qquad a_{-1} = b_{-1}\sqrt{\frac{2}{3}}, \\
b_1 = \frac{3a_0^2}{8b_{-1}}, \qquad b_0 = -a_0\sqrt{6}, \qquad b_{-1} = b_{-1}, \\
c = c, \qquad \gamma = -c, \qquad \xi_0 = 0,
\end{gathered} \tag{59}
$$

where $a_0$ and $b_{-1} \neq 0$ are free parameters.

From (59), substituting these results into (56), we obtain the following exact solution:

$$
\begin{aligned}
&u(x,t) \\
&= \frac{a_0^2\sqrt{6}}{8b_{-1}}\exp\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right) + a_0 \\
&+ b_{-1}\sqrt{\frac{2}{3}}\exp\left(-\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right) \\
&\times \left(\frac{3a_0^2}{8b_{-1}}\exp\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right) - a_0\sqrt{6}\right. \\
&\qquad \left. + b_{-1}\exp\left(-\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)^{-1}.
\end{aligned} \tag{60}
$$

## 7. Conclusion

In this paper, we have successfully developed fractional complex transform with the help of exp-function method to obtain exact solution of some fractional differential equations. The fractional complex transform and exp-function methods are extremely simple but effective and powerful for solving fractional differential equations. These methods are accessible to solve other similar nonlinear equations in fractional calculus. To our knowledge, these new solutions have not been reported in former literature; they may be of significant importance for the explanation of some special physical phenomena.

## Conflict of Interests

The authors declare that there is no conflict of interests in this paper.

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