FRACTIONAL DIFFERENTIATION OF FUNCTIONS WITH LACUNARY FOURIER SERIES[1]

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1. This paper gives some results of a special nature for functions which have lacunary Fourier series. Before stating these, we begin with some definitions and comments.

Let f: $(0, 2\pi)\to R$ be integrable and assume it has the Fourier series

$$S(f) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos n_k x + b_k \sin n_k x. \tag{1}$$

f is said to have a lacunary Fourier series if there is a $\lambda > 1$ such that for $k=1, 2, \cdots$ we have $n_{k+1} > \lambda n_k$. We will assume $a_0 = 0$. Sometimes we will prefer to write $S(f)$ in its complex form

$$S(f) = \sum_{-\infty}^{+\infty}{}' C_k \exp(i n_k x), \tag{1'}$$

where $\sum'$ indicates that the sum excludes the term $k=0$.

Let $0 \leqq k < \alpha < k+1$ and $\alpha + \beta = k+1$, with k an integer, be given. We define the βth integral of f by

$$F(x) = f_\beta(x) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^{x} f(t)(x-t)^{\beta-1} dt. \tag{2}$$

A discussion of this integral is given in [6, page 133]. There it is shown that

$$S(f_\beta) = \sum_{-\infty}^{\infty}{}' \frac{C_k}{(i n_k)^\beta} \exp(i n_k x),$$

and it is pointed out that the convergence of the integral is bound up with the fact that $a_0 = 0$. From this it is clear that F has a lacunary Fourier series if f does. f is said to have an αth derivative if F has a $k+1$ Peano derivative at x_0, i.e., there exists a polynomial $P_{x_0}(t)$ of degree $\leqq k+1$ such that

$$F(x_0 + t) - P_{x_0}(t) = o(|t|^{k+1}), \qquad t \to 0. \tag{3}$$

If

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$$\left\{\frac{1}{\rho}\int_{-\rho}^{\rho} | F(x_0+t) - P_{x_0}(t) |^p dt\right\}^{1/p} = o(\rho^{k+1}),$$

$$\rho \to 0 \quad (1 \leqq p < \infty),$$

f is said to have an αth derivative in the L^p sense. If there exists a polynomial $Q_{x_0}(t)$ of degree $\leqq k$ such that $R_{x_0}(t) = f(x_0+t) - Q_{x_0}(t) = O(|t|^\alpha)$ as $t \to 0$ then f is said to satisfy the condition Λ_α and if

$$\left\{\frac{1}{\rho}\int_{-\rho}^{\rho} | R_{x_0}(t) |^p dt\right\}^{1/p} = O(\rho^\alpha), \qquad \rho \to 0 \quad (1 \leqq p < \infty),$$

f is said to satisfy the condition Λ_α^p. f is said to satisfy the condition N_α^p if for some $\rho > 0$

$$\frac{1}{\rho}\int_{-\rho}^{\rho} \frac{| R_{x_0}(t) |^p}{| t |^{1+p\alpha}} dt < \infty.$$

If $\alpha = 1$, the condition Λ_1 is replaced by the condition Λ_*, which requires that $\Delta^2 f(x_0, t) = f(x_0+t) + f(x_0-t) - 2f(x_0) = O(t)$ as $t \to 0$. Finally, f is said to have an approximate derivative at x_0, equal to $f'_{ap}(x_0)$, if $|f(x_0+t) - f(x_0) - f'_{ap}(x_0)t| = o(t)$ as $t \to 0$ through a set having zero as a point of density.

Throughout, constants depending only on the parameters of the problem will be denoted by D_k, D, M, C etc. These will not necessarily denote the same constant at each point.

The results we will state were motivated by the main result in [4] which gives the following theorem for the case $p = 2$.

THEOREM A. *The necessary and sufficient condition for f to satisfy the conditions N_α^p, N_α^2 $(2 \leqq p < \infty)$ almost everywhere in a measurable set E is that f satisfy the condition Λ_α^p and have an αth derivative in the L^p sense almost everywhere in the set E.*

The proof of this theorem follows the proof of the theorem in [4] with $p = 2$ and a few minor modifications. The results in [4] parallel the results of [3] on differentiability of functions.

We are now ready to state the two theorems of this paper.

THEOREM 1. *Let f have a lacunary Fourier series and suppose f has an αth derivative at some point x_0 $(0 \leqq k < \alpha < k+1)$. Then f satisfies the condition Λ_α everywhere.*

THEOREM 2. *Let F be a function with a lacunary Fourier series. If $\sum (a_j^2 + b_j^2) n_j^2 < \infty$ then F has a derivative almost everywhere.*

The first of these is proved by a method similar to one in [5] which

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is adapted to higher orders of differentiation by means of Taylor's theorem. The second is a corollary of the main theorem in [3]. It is included here because of some if its corollaries.

2. We start with the proof of Theorem 1. We may assume $x_0=0$ and, under the assumption that the polynomial in (3) is identically zero, we have $F(x_0+t)=F(t)=R(t)$ where $R(t)=O(|t|^{k+1})$ as $t\to 0$. Let $N=N_j=\min\{n_j-n_{j-1},\ n_{j+1}-n_j\}$. We can choose an integer $p>0$ such that $N-1<p(k+2)\leqq N+k+1$. Let

$$K^j(t) = K(t) = B_k^{-1}\left(\frac{\sin(pt/2)}{p\sin(t/2)}\right)^{2(k+2)},$$

where

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{\sin(pt/2)}{p\sin(t/2)}\right)^{2(k+2)}dt.$$

$K(t)$ is a trigonometric polynomial of degree $(p-1)(k+2)$ and $(1/\pi)\int_{-\pi}^{\pi}K(t)dt=1$. Hence we may write $K(t)=1/2+\sum_{\sigma=1}^{N-1}d_\sigma\cos\sigma t$ where

$$d_\sigma = \frac{1}{\pi}\int_{-\pi}^{\pi}K(t)\cos\sigma t dt \qquad \sigma = 1, 2, \cdots, N-1.$$

Because

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{\sin(pt/2)}{p\sin(t/2)}\right)^{2(k+2)}dt > \frac{2}{\pi}\int_{0}^{1/p}\left(\frac{\sin(pt/2)}{p\sin(t/2)}\right)^{2k+4}dt$$
$$\geqq \frac{1}{p}\left(\frac{2}{\pi}\right)^{2k+5} = \frac{1}{p}D_k,$$

it can be shown that

$$\begin{aligned} K(t) &\leqq D_k' p, \qquad 0\leqq |t| \leqq 1/N,\\ &\leqq D_k'(1/p^{k+2}|t|^{k+3}), \qquad 1/N\leqq |t| \leqq \pi, \end{aligned}$$

where we recall $k<\alpha<k+1$ is fixed.

It is easy to show that

$$2K(t)\cos n_jt = \cos n_jt + \sum_{\sigma=1}^{N-1}d_\sigma[\cos(n_j+\sigma)t+\cos(n_j-\sigma)t],$$

$$2K(t)\sin n_jt = \sin n_jt + \sum_{\sigma=1}^{N-1}d_\sigma[\sin(n_j+\sigma)t+\sin(n_j-\sigma)t].$$

Since $n_{j-1}<n_j-\sigma<n_j+\sigma<n_{j+1}$ for $\sigma=1, 2, \cdots, N-1$, we have

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$$(4) \qquad (a_j, b_j) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)2K(t)(\cos n_j t, \sin n_j t)dt,$$

where a_j, b_j are the Fourier coefficients of F.

Let $\epsilon>0$ be given. Then there exists a $\delta>0$ such that for $|x|<\delta$ $|F(x)|<\epsilon|x|^{k+1}$ since f has an αth derivative at $x=0$. Assuming $1/N<\delta$ and applying formula (4) we obtain

$$\begin{aligned} |a_j| &\leqq \frac{2}{\pi}\int_0^{\pi} |F(x)|\, K(x)dx \leqq D_k' N\epsilon \int_{0<|x|\leqq 1/N} |x|^{k+1}dx \\ &\quad + D_k' N^{-(k+2)}\epsilon \int_{1/N\leqq |x|\leqq\delta} |x|^{-2}dx \\ &\quad + D_k' N^{-(k+2)} \int_{\delta\leqq|x|\leqq\pi} |F(x)|\cdot|x|^{-(k+3)}dx \\ &\leqq M_1\epsilon N^{-(k+1)} + M_2\delta^{-(k+3)}N^{-(k+2)}\int_{-\pi}^{\pi}|F(x)|\,dx \\ &\leqq M_1(L)^{k+1}\epsilon(n_j)^{-(k+1)} + M_3\frac{\delta^{-(k+3)}}{n_j}(L)^{k+2}\left(\frac{1}{n_j}\right)^{k+1}, \end{aligned}$$

where M_1, M_2 and M_3 are constants and $L=\min\{\lambda-1, 1-1/\lambda\}$. Since M_1, M_2, M_3 and L are all fixed constants and we can choose j so large that $M_3\delta^{-(k+3)}<n_j\epsilon$, we see that $a_j=o(n_j^{-(k+1)})$ as $j\to\infty$. Similarly, $b_j=o(n_j^{-(k+1)})$ as $j\to\infty$.

If we denote by $c_j(F)$ and $c_j(f)$ the Fourier coefficients of F and f respectively we see that $c_j(F)=o(n_j^{-(k+1)})$ as $|j|\to\infty$ and hence that $c_j(f)=o(n_j^{-(k+1)+\beta})=o(n_j^{-\alpha})$ as $|j|\to\infty$. A result in [2] shows that for $0<\alpha<1$ we have f satisfies the condition Λ_α if and only if $c_j=O(n_j^{-\alpha})$.

Suppose that $k<\alpha<k+1$. Then the series

$$\sum_{-\infty}^{\infty} c_j(in_j)^k \exp(in_j t)$$

converges uniformly. To see this, let $\epsilon>0$ be given. Then there exists $M>0$ such that $|c_j|\leqq Mn_j^{-\alpha}$, $j=\mp 1, \mp 2, \cdots$. Choose N such that for $|j|\geqq N$, $|n_j|^{\beta-1}\leqq C\epsilon$, where $C^{-1}=2M\sum_1^\infty(\lambda^{\beta-1})^j$. Then we see that

$$\begin{aligned} \sum_{|j|\geqq N} |c_j|\cdot|n_j|^k &\leqq \sum_{|j|\geqq N} M\cdot|n_j|^{-\alpha}\cdot|n_j|^k \\ &\leqq 2M|n_N|^{\beta-1}\sum_{=N}(\lambda^{\beta-1})^j \leqq C^{-1}C\epsilon = \epsilon. \end{aligned}$$

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This shows that f has k derivatives and that the kth derivative has coefficients which are $o(n_j^{\beta-1})$ as $j\to\infty$, and hence $f^k(x)$ satisfies the condition $\Lambda_{1-\beta}$.

We now apply Taylor's theorem with remainder to see that

$$f(x+t) = \sum_{j=1}^{k} f^j(x)\,\frac{t^j}{j!} + R(t)$$

where

$$R(t) = \frac{1}{(k-1)!}\int_0^t (t-u)^{k-1}[f^k(x+u) - f^k(u)]du.$$

Since $f^k(x)$ satisfies $\Lambda_{1-\beta}$, it follows that $f(x)$ is in Λ_α because

$$R(t) = O\left(\int_0^t (t-u)^{k-1}u^{1-\beta}du\right) = O(|t|^{k+1-\beta}) = O(|t|^{\alpha}), \qquad t\to 0.$$

Finally, suppose that the polynomial in (3) is not identically zero, then we can find a C_0^∞ function g which has this polynomial as its Taylor's expansion up to the $(k+1)$th term at the point $x=0$ and with support in $(-\pi, \pi)$. It is easy to see that g has Fourier coefficients which are $O(n^{-(k+3)})$. Let $G=F-g$ and let F be replaced by G in the integral formula (4). Then $|a_j|$ is dominated by that integral $+O(\sum_{n_j}^{n_{j+1}}|a_j(g)|)$ where $a_j(g)$ are the Fourier coefficients of g. The integral is $O(n_j^{-(k+1)})$ as before and

$$\sum_{n_j}^{n_{j+1}} |a_j(g)| = O\left(\sum_{n_j}^{\infty} n_j^{-(k+3)}\right) = O\left(n_j^{-(k+1)}\sum_{n_j} n^{-2}\right)$$
$$= o(n_j^{-(k+1)}), \qquad j\to\infty.$$

Hence $|a_j| = o(n_j^{-(k+1)})$. This completes the proof of Theorem 1.

3. To begin the proof of Theorem 2 we note that because of the main result in [3] we only need to show that f satisfies the condition Λ_* and that the integral

$$\int_{-\pi}^{\pi} \frac{|\Delta^2 f(x,t)|^2}{|t|^3}\,dt < \infty \qquad \text{for almost all } x. \tag{5}$$

However, since $a_j,\ b_j = O(n_j)$ as $j\to\infty$, it can be shown using the technique of [6, Volume I, page 47] that f satisfies the condition Λ_*.

We show the integral (5) is finite for almost every x in $[-\pi, \pi]$ by showing the following integral is finite

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(6) $$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|\Delta^2 f(x,t)|^2}{|t|^3}\,dtdx.$$

This is equal to

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1}{|t|^3}\left(-\sum\,[a_k\cos n_kx+b_k\sin n_kx]4\sin^2\left(\frac{n_kt}{2}\right)\right)^2dtdx.$$

By Parseval's equality this is

$$16\int_{-\pi}^{\pi}\left(\sum\,(a_k^2+b_k^2)\sin^4\left(\frac{n_kt}{2}\right)\right)\frac{dt}{|t|^3}\,.$$

We have

$$\begin{aligned}\int_{-\pi}^{\pi}\sin^4\left(\frac{n_kt}{2}\right)\frac{dt}{|t|^3}&=2\int_0^{\pi/nk}\sin^4\left(\frac{n_kt}{2}\right)\frac{dt}{t^3}+2\int_{\pi/nk}^{\pi}\sin^4\left(\frac{n_kt}{2}\right)\frac{dt}{t^3}\\&\leqq 2\int_0^{\pi/nk}\left(\frac{n_kt}{2}\right)^4\frac{dt}{t^3}+2\int_{\pi/nk}^{\pi}\frac{dt}{t^3}\\&=\left(\frac{n_k}{2}\right)^4\cdot\left(\frac{\pi}{n_k}\right)^2+\left(\frac{n_k^2}{2}-\frac{1}{\pi^2}\right)=O(n_k^2),\end{aligned}$$

and hence the integral (6) is bounded by $16D\sum_k(a_k^2+b_k^2)n_k^2<\infty$. This completes the proof of the theorem.

We add some corollaries.

COROLLARY 1. *Suppose f has a lacunary Fourier series. Then if f has an approximate derivative in a set of positive measure f has a derivative almost everywhere.*

PROOF. This follows from the fact (see e.g. [1, Volume II, page 263]) that if a function f has a lacunary Fourier series which converges on a set of positive measure to a function which can be extended to an absolutely continuous one, then $\sum_j(a_j^2+b_j^2)n_j^2<\infty$.

COROLLARY 2. *Let $0<\alpha<1$, $\alpha+\beta=1$, and f have a lacunary Fourier series. Then f has an αth derivative in a set E of positive measure and hence almost everywhere if and only if for almost every x in E there is an $\eta_x=\eta>0$ such that*

(7) $$\int_{-\eta}^{\eta}\frac{R_x(t)^2}{|t|^{1+2\alpha}}\,dt<\infty.$$

PROOF. Let $F(x)=f_\beta(x)$. Assuming (7), we have $F(x)$ has a derivative in the L^2 sense and hence an approximate derivative almost

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everywhere in E. By Corollary 1, $F(x)$ has an ordinary derivative almost everywhere. On the other hand, if f has an αth derivative almost everywhere in E, then $F(x)$ has an ordinary derivative almost everywhere again by Corollary 1. By Theorem 1, f satisfies the condition Λ_α almost everywhere. Hence, by Theorem A, (7) holds almost everywhere.

This corollary extends to the case $\alpha > 1$ by applying Taylor's theorem as in Theorem 1.

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