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FRACTIONAL FACTORIAL DESIGNS FOR EXPERIMENTS WITH FACTORS AT TWO AND THREE LEVELS

W. S. Connor, et al

National Bureau of Standards
Washington, D. C.

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UNITED STATES DEPARTMENT OF COMMERCE • Luther H. Hodges, *Secretary*
NATIONAL BUREAU OF STANDARDS • A. V. Astin, *Director*

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Preface

The designs presented in this publication are for experiments with some factors at two levels and other factors at three levels. The designs are constructed so that the grand mean, all main effects, and all two-factor interaction effects can be estimated without aliasing among them. It is assumed that all higher-order interaction effects are negligible, and their absence makes it possible to estimate the error variance.

These designs were developed in the Statistical Engineering Laboratory of the National Bureau of Standards under a program sponsored by the Bureau of Ships, Department of the Navy. Revision of the introductory material and of the analysis for six designs was carried out at The Research Triangle Institute under the sponsorship of the Office of Ordnance Research. The work was performed under the direction of W. S. Connor. Professor R. C. Bose served as consultant and contributed to the development of related theory. Shirley Young performed most of the work of constructing the designs and working out the corresponding estimates. Carroll Dannemiller devised an electronic computer program which was used to check the normal equations. A program previously developed by R. C. Burton was used to generate treatment combinations from 3^n factorials. Also, R. C. Burton participated in certain aspects of construction during the summer of 1958. Lola S. Deming supervised the preparation of the manuscript in final form.

OCTOBER 1960. A. V. ASTIN, *Director.*

Contents

List of designs

Fractional Factorial Designs for Experiments With Factors at Two and Three Levels

W. S. Connor and Shirley Young

1. Introduction

This catalog is the sequel to [1] [1] and [2]. It contains fractional factorial designs for use in experiments which investigate m factors at two levels and n factors at three levels. The grand mean μ, all main effects, and all two-factor interaction effects are estimated. All higher order interactions are assumed negligible and their absence allows estimation of the error variance. A design has been constructed for each of the 39 pairs (m,n) included from $m+n=5$ through $m+n=10$, $(m,n\neq 0)$. The design for (m,n) is designated DESIGN 2^m3^n.

It is believed that the method of construction described in section 2 is new. Morrison [3] published several designs which can be constructed by the present method, and his paper was an inspiration to the authors in formulating their method. The objective in forming these fractions was to keep the number of required treatment combinations small while retaining as much orthogonality among the estimates as possible. The designs are neither unique nor exhaustive, and there may be attractive alternative ways of fractionating the 2^m3^n complete factorials.

Section 3 contains a description of the mathematical model, in which it is assumed that all interactions between three or more factors are nonexistent, and of how to estimate the parameters contained in the model. Section 4 contains a discussion of how to test hypotheses and construct confidence intervals. A worked example is presented in section 5.

Section 6 is devoted to six designs, viz, 2^33^3, 2^53^3, 2^63^3, 2^53^4, 2^33^5, and 2^53^5, for which the interaction effects between factors at three levels are defined in a special way.

2. Construction of Designs

The designs are constructed by associating fractions $S_1, S_2, \ldots, S_t$ from the 2^m complete factorial with fractions $S_1', S_2', \ldots, S_t'$ from the 3^n complete factorial. The fractions S_i and S_i' $(i=1,2,\ldots,t)$ are obtained by conventional methods which have been described, for example, in [4,5]. Fractions are selected so that low order interaction effects, including main effects, are aliased with each other as little as possible. The association is such that every treatment combination in S_i is adjoined to every treatment combination in S_i', thus forming treatment combinations from the 2^m3^n complete factorial. The resulting fraction from the 2^m3^n complete factorial may be denoted by

$$S_1S_1' \quad S_2S_2' \quad \ldots S_tS_t'. \tag{2.1}$$

To illustrate, consider the 2^33^2 complete factorial, which contains 72 treatment combinations. The three factors with two levels will be denoted by A_1, A_2, and A_3, and the two factors with three levels by B_1 and B_2. One way of fractionating the 2^3 complete factorial into two distinct sets S_1 and S_2 is by finding the treatment combinations $(x_1x_2x_3)$, $(x_j=0,1;\ j=1,2,3)$ having x's which satisfy

$$x_1+x_2+x_3=0 \text{ and } x_1+x_2+x_3=1 \pmod 2, \tag{2.2}$$

respectively. These sets are as follows:

[1] Figures in brackets indicate the literature references on page 14.

Sets of Treatment Combinations from the 2^3

(2.3)	S_1			S_2		
	A_1	A_2	A_3	A_1	A_2	A_3
	0	0	0	1	1	1
	1	1	0	1	0	0
	1	0	1	0	1	0
	0	1	1	0	0	1

One way of fractionating the 3^2 complete factorial into three distinct sets S_1', S_2', and S_3' is by finding the treatment combinations (z_1z_2), $(z_k=0,1,2;\ k=1,2)$, having z's which satisfy

(2.4) $$z_1+z_2=0,\ z_1+z_2=1,\ z_1+z_2=2 \pmod 3,$$

respectively. These sets are as follows:

Sets of Treatment Combinations from the 3^2

(2.5)	S_1'		S_2'		S_3'	
	B_1	B_2	B_1	B_2	B_1	B_2
	0	0	1	0	2	0
	1	2	0	1	0	2
	2	1	2	2	1	1

DESIGN 2^33^2 appearing on page 20 of this catalog is a fractional design from the 2^33^2 complete factorial consisting of the following treatment combinations:

Treatment Combinations in the Fraction from the 2^33^2

	A_1	A_2	A_3	B_1	B_2	Response	A_1	A_2	A_3	B_1	B_2	Response	A_1	A_2	A_3	B_1	B_2	Response
	0	0	0	0	0	(85.9)	0	0	1	0	1	(88.9)	0	0	1	0	2	(139.0)
	0	1	1	0	0	(99.3)	0	1	0	0	1	(78.4)	0	1	0	0	2	(153.8)
	1	0	1	0	0	(119.8)	1	0	0	0	1	(42.0)	1	0	0	0	2	(180.0)
	1	1	0	0	0	(115.5)	1	1	1	0	1	(142.0)	1	1	1	0	2	(172.4)
	0	0	0	1	2	(118.3)	0	0	1	1	0	(94.9)	0	0	1	2	0	(184.0)
(2.6)	0	1	1	1	2	(115.4)	0	1	0	1	0	(110.4)	0	1	0	2	0	(93.0)
	1	0	1	1	2	(184.9)	1	0	0	1	0	(92.8)	1	0	0	2	0	(96.9)
	1	1	0	1	2	(161.7)	1	1	1	1	0	(167.2)	1	1	1	2	0	(172.7)
	0	0	0	2	1	(127.6)	0	0	1	2	2	(153.9)	0	0	1	1	1	(125.7)
	0	1	1	2	1	(166.8)	0	1	0	2	2	(184.3)	0	1	0	1	1	(102.7)
	1	0	1	2	1	(158.6)	1	0	0	2	2	(114.3)	1	0	0	1	1	(131.2)
	1	1	0	2	1	(138.6)	1	1	1	2	2	(199.9)	1	1	1	1	1	(223.7)

These treatment combinations may be denoted concisely by

(2.7) $S_1S'_1 \quad S_2S'_2 \quad S_2S'_3$.

The numbers in parentheses under the column headed "Response" are data that will be used subsequently for a numerical illustration.

This fractional factorial design involves 36 treatment combinations and is a one-half fraction of the complete factorial. In DESIGN $2^3 3^2$, the expression (2.7) is called the "Experimental Plan," and indicates how the sets S_i and S'_i, which are given under "Construction," are to be associated to form the treatment combinations (2.6).

This form is followed for all of the designs in the catalog, except that the actual formation of the treatment combinations as shown in the Experimental Plan is left to the reader.

3. Estimation of Effects

The response to the treatment combination $(x_1x_2 \ldots x_mz_1z_2 \ldots z_n)$ will be denoted by $Y^*(x_1x_2 \ldots x_mz_1z_2 \ldots z_n)$, which is a random variable with expected value $\eta(x_1x_2 \ldots x_mz_1z_2 \ldots z_n)$ and variance σ^2. It is assumed that the expected value of the response is expressible as a linear function of certain parameters which are the grand average, main effects, and two-factor interaction effects. It also is assumed that there are no higher order interaction effects.

In the linear function corresponding to a treatment combination, the coefficient of the grand average μ is 1, but the coefficients of the other parameters depend on the treatment combination. If the factor A is at level 0, then the coefficient of the main effect of A, also denoted by A, is -1; but if at level 1, then the coefficient is 1. The coefficient of the interaction parameter $A_jA_{j'}$ is the product of the coefficients of the component main effects A_j and $A_{j'}$ as is shown in the following table:

Coefficients of Pure A Effects

(3.1)

Factor levels		Coefficients		
		Main effects		Interaction
A_j	$A_{j'}$	A_j	$A_{j'}$	$A_jA_{j'}$
0	0	-1	-1	1
1	0	1	-1	-1
0	1	-1	1	-1
1	1	1	1	1

For a B factor there are two parameters which correspond to the main effect, viz, the linear effect B and the quadratic effect B^2. The terms "linear" and "quadratic" apply literally only to equally spaced levels but are formally useful in other cases too. For the levels 0, 1, and 2, the coefficients of B are -1, 0, and 1, respectively, and the coefficients of B^2 are 1, -2, and 1, respectively.

For two factors $B_kB_{k'}$, there are four interaction parameters, viz, $B_kB_{k'}$, $B_kB^2_{k'}$, $B^2_kB_{k'}$, and $B^2_kB^2_{k'}$. The coefficients of these parameters are the products of the coefficients of the component main effects, as follows:

Coefficients of Pure B Effects

(3.2)

Factor levels		Coefficients							
		Main effects				Interactions			
B_k	$B_{k'}$	B_k	$B_{k'}$	B_k^2	$B_{k'}^2$	$B_kB_{k'}$	$B_kB_{k'}^2$	$B_k^2B_{k'}$	$B_k^2B_{k'}^2$
0	0	−1	−1	1	1	1	−1	−1	1
1	0	0	−1	−2	1	0	0	2	−2
2	0	1	−1	1	1	−1	1	−1	1
0	1	−1	0	1	−2	0	2	0	−2
1	1	0	0	−2	−2	0	0	0	4
2	1	1	0	1	−2	0	−2	0	−2
0	2	−1	1	1	1	−1	−1	1	1
1	2	0	1	−2	1	0	0	−2	−2
2	2	1	1	1	1	1	1	1	1

For six of the designs, viz, $2^3 3^3$, $2^5 3^3$, $2^6 3^3$, $2^5 3^4$, $2^3 3^5$, and $2^5 3^5$, the coefficients of the interaction effects will be defined in a different way. These definitions will be presented in section 6.

For two factors A and B, there are two interaction parameters, viz, AB and AB^2. The coefficients of these parameters, too, are the products of the component main effects, thus:

Coefficients of Mixed A, B Effects

(3.3)

Factor levels		Coefficients				
		Main effects			Interactions	
A	B	A	B	B^2	AB	AB^2
0	0	−1	−1	1	1	−1
1	0	1	−1	1	−1	1
0	1	−1	0	−2	0	2
1	1	1	0	−2	0	−2
0	2	−1	1	1	−1	−1
1	2	1	1	1	1	1

These rules will be illustrated for DESIGN $2^3 3^2$, which is a one-half fraction of the $2^3 3^2$ complete factorial. In (3.4) the expected responses, $\eta(x_1 x_2 x_3 z_1 z_2)$, for all 36 treatment combinations are expressed as linear functions of 27 parameters. The column vector τ contains the following elements in the order given:

μ, A_1, A_2, A_3, A_1A_2, A_1A_3,

A_2A_3, B_1, B_1^2, B_2, B_2^2, B_1B_2,

$B_1B_2^2$, $B_1^2B_2$, $B_1^2B_2^2$, A_1B_1, $A_1B_1^2$, A_1B_2,

$A_1B_2^2$, A_2B_1, $A_2B_1^2$, A_2B_2, $A_2B_2^2$, A_3B_1,

$A_3B_1^2$, A_3B_2, $A_3B_2^2$.

Expected Responses of the Treatment Combinations in Design $2^3 3^2$ Expressed as Linear Functions of the Grand Average and Main and Interaction Effects

$$
(3.4)\qquad
\begin{bmatrix}
00000\\ 01100\\ 10100\\ 11000\\ \\
00012\\ 01112\\ 10112\\ 11012\\ \\
00021\\ 01121\\ 10121\\ 11021\\ \\
00101\\ 01001\\ 10001\\ 11101\\ \\
00110\\ 01010\\ 10010\\ 11110\\ \\
00122\\ 01022\\ 10022\\ 11122\\ \\
00102\\ 01002\\ 10002\\ 11102\\ \\
00120\\ 01020\\ 10020\\ 11120\\ \\
00111\\ 01011\\ 10011\\ 11111
\end{bmatrix}\eta
=
\begin{bmatrix}
1&-1&-1&-1&1&1&1&-1&1&-1&1&1&-1&-1&1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1\\
1&-1&1&1&-1&-1&1&-1&1&-1&1&1&-1&-1&1&1&-1&1&-1&-1&1&-1&1&-1&1&-1&1\\
1&1&-1&1&-1&1&-1&-1&1&-1&1&1&-1&-1&1&-1&1&-1&1&1&-1&1&-1&-1&1&-1&1\\
1&1&1&-1&1&-1&-1&-1&1&-1&1&1&-1&-1&1&-1&1&-1&1&-1&1&-1&1&1&-1&1&-1\\
\\
1&-1&-1&-1&1&1&1&0&-2&1&1&0&0&-2&-2&0&2&-1&-1&0&2&-1&-1&0&2&-1&-1\\
1&-1&1&1&-1&-1&1&0&-2&1&1&0&0&-2&-2&0&2&-1&-1&0&-2&1&1&0&-2&1&1\\
1&1&-1&1&-1&1&-1&0&-2&1&1&0&0&-2&-2&0&-2&1&1&0&2&-1&-1&0&-2&1&1\\
1&1&1&-1&1&-1&-1&0&-2&1&1&0&0&-2&-2&0&-2&1&1&0&-2&1&1&0&2&-1&-1\\
\\
1&-1&-1&-1&1&1&1&1&1&0&-2&0&-2&0&-2&-1&-1&0&2&-1&-1&0&2&-1&-1&0&2\\
1&-1&1&1&-1&-1&1&1&1&0&-2&0&-2&0&-2&-1&-1&0&2&1&1&0&-2&1&1&0&-2\\
1&1&-1&1&-1&1&-1&1&1&0&-2&0&-2&0&-2&1&1&0&-2&-1&-1&0&2&1&1&0&-2\\
1&1&1&-1&1&-1&-1&1&1&0&-2&0&-2&0&-2&1&1&0&-2&1&1&0&-2&-1&-1&0&2\\
\\
1&-1&-1&1&1&-1&-1&-1&1&0&-2&0&2&0&-2&1&-1&0&2&1&-1&0&2&-1&1&0&-2\\
1&-1&1&-1&-1&1&-1&-1&1&0&-2&0&2&0&-2&1&-1&0&2&-1&1&0&-2&1&-1&0&2\\
1&1&-1&-1&-1&-1&1&-1&1&0&-2&0&2&0&-2&-1&1&0&-2&1&-1&0&2&1&-1&0&2\\
1&1&1&1&1&1&1&-1&1&0&-2&0&2&0&-2&-1&1&0&-2&-1&1&0&-2&-1&1&0&-2\\
\\
1&-1&-1&1&1&-1&-1&0&-2&-1&1&0&0&2&-2&0&2&1&-1&0&2&1&-1&0&-2&-1&1\\
1&-1&1&-1&-1&1&-1&0&-2&-1&1&0&0&2&-2&0&2&1&-1&0&-2&-1&1&0&2&1&-1\\
1&1&-1&-1&-1&-1&1&0&-2&-1&1&0&0&2&-2&0&-2&-1&1&0&2&1&-1&0&2&1&-1\\
1&1&1&1&1&1&1&0&-2&-1&1&0&0&2&-2&0&-2&-1&1&0&-2&-1&1&0&-2&-1&1\\
\\
1&-1&-1&1&1&-1&-1&1&1&1&1&1&1&1&1&-1&-1&-1&-1&-1&-1&-1&-1&1&1&1&1\\
1&-1&1&-1&-1&1&-1&1&1&1&1&1&1&1&1&-1&-1&-1&-1&1&1&1&1&-1&-1&-1&-1\\
1&1&-1&-1&-1&-1&1&1&1&1&1&1&1&1&1&1&1&1&1&-1&-1&-1&-1&-1&-1&-1&-1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
\\
1&-1&-1&1&1&-1&-1&-1&1&1&1&-1&-1&1&1&1&-1&-1&-1&1&-1&-1&-1&-1&1&1&1\\
1&-1&1&-1&-1&1&-1&-1&1&1&1&-1&-1&1&1&1&-1&-1&-1&-1&1&1&1&1&-1&-1&-1\\
1&1&-1&-1&-1&-1&1&-1&1&1&1&-1&-1&1&1&-1&1&1&1&1&-1&-1&-1&1&-1&-1&-1\\
1&1&1&1&1&1&1&-1&1&1&1&-1&-1&1&1&-1&1&1&1&-1&1&1&1&-1&1&1&1\\
\\
1&-1&-1&1&1&-1&-1&1&1&-1&1&-1&1&-1&1&-1&-1&1&-1&-1&-1&1&-1&1&1&-1&1\\
1&-1&1&-1&-1&1&-1&1&1&-1&1&-1&1&-1&1&-1&-1&1&-1&1&1&-1&1&-1&-1&1&-1\\
1&1&-1&-1&-1&-1&1&1&1&-1&1&-1&1&-1&1&1&1&-1&1&-1&-1&1&-1&-1&-1&1&-1\\
1&1&1&1&1&1&1&1&1&-1&1&-1&1&-1&1&1&1&-1&1&1&1&-1&1&1&1&-1&1\\
\\
1&-1&-1&1&1&-1&-1&0&-2&0&-2&0&0&0&4&0&2&0&2&0&2&0&2&0&-2&0&-2\\
1&-1&1&-1&-1&1&-1&0&-2&0&-2&0&0&0&4&0&2&0&2&0&-2&0&-2&0&2&0&2\\
1&1&-1&-1&-1&-1&1&0&-2&0&-2&0&0&0&4&0&-2&0&-2&0&2&0&2&0&2&0&2\\
1&1&1&1&1&1&1&0&-2&0&-2&0&0&0&4&0&-2&0&-2&0&-2&0&-2&0&-2&0&-2
\end{bmatrix}\tau
$$

The first equation is read as

$$\eta(00000)=\mu-A_1-A_2-A_3+A_1A_2+A_1A_3+A_2A_3$$

$$-B_1+B_1^2-B_2+B_2^2+B_1B_2-B_1B_2^2-B_1^2B_2+B_1^2B_2^2$$

(3.5) $$+A_1B_1-A_1B_1^2+A_1B_2-A_1B_2^2+A_2B_1-A_2B_1^2+A_2B_2-A_2B_2^2$$

$$+A_3B_1-A_3B_1^2+A_3B_2-A_3B_2^2$$

and the other equations are read similarly.

For the following discussion it is assumed that the responses $Y^*(x_1 \ldots x_m z_1 \ldots z_n)$ have variance σ^2 and are uncorrelated.

The normal equations are formed from the equations of expectation in the usual way. Let the column vector of expected responses be denoted by η, the matrix of coefficients by X, and the column vector of parameters by τ. Then the equations of expectation may be written concisely as

(3.6) $$\eta=X\tau.$$

Letting y denote the column vector of observed responses, and $\hat{}$ the column vector of estimates, the normal equations are

(3.7) $$Y(\tau)=X'y=X'X\hat{\tau}=C\hat{\tau},$$

for $C=X'X$. The equations may be solved for $\hat{\tau}$ as follows:

(3.8) $$\hat{\tau}=C^{-1}X'y$$

The designs in this catalogue have been constructed so that C is nonsingular and there are not many nonzero elements in C^{-1}. Indeed, for some of the designs C^{-1} is diagonal. Letting f denote the number of treatment combinations in the design, the elements in the principal diagonal can be calculated from (3.9), except when there is a nonzero element off the diagonal in the same row as the element under consideration. In that event special formulas are required. If all of the off-diagonal elements are zero, then the analysis is termed "Completely Orthogonal." The formulas for $B_kB_{k'}$, $B_kB_{k'}^2$, $B_k^2B_{k'}$, and $B_k^2B_{k'}^2$ do not apply to the six designs which are discussed in section 6.

Elements in the Main Diagonal of the Inverse Matrix
(except as explained above)

(3.9)

Parameter	Element
A, $A_jA_{j'}$	$1/f$
AB, B	$3/2f$
AB^2, B^2	$1/2f$
$B_kB_{k'}$	$9/4f$
$B_kB_{k'}^2$, $B_k^2B_{k'}$	$3/4f$
$B_k^2B_{k'}^2$	$1/4f$

For DESIGN $2^3 3^2$, appearing on page 20 there are 36 treatment combinations, so that these elements, excluding the first, are 1/24, 1/72, 1/16, 1/48, and 1/144, respectively. To estimate the effects of A_1, A_2, A_3, A_1A_2, A_1A_3, and A_2A_3 requires special formulas, which are given under the heading "Analysis." It is stated that "the matrix $\frac{1}{96}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_3 \\ A_1A_2 \end{bmatrix}$, $\begin{bmatrix} A_2 \\ A_1A_3 \end{bmatrix}$, $\begin{bmatrix} A_1 \\ A_2A_3 \end{bmatrix}$," by which is meant that the following matric equations are formed:

$$
(3.10)\qquad
\begin{gathered}
\begin{bmatrix} \hat{A}_1 \\ \widehat{A_2A_3} \end{bmatrix} = \frac{1}{96}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} Y(A_1) \\ Y(A_2A_3) \end{bmatrix}, \\
\begin{bmatrix} \hat{A}_2 \\ \widehat{A_1A_3} \end{bmatrix} = \frac{1}{96}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} Y(A_2) \\ Y(A_1A_3) \end{bmatrix}, \\
\begin{bmatrix} \hat{A}_3 \\ \widehat{A_1A_2} \end{bmatrix} = \frac{1}{96}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} Y(A_3) \\ Y(A_1A_2) \end{bmatrix}.
\end{gathered}
$$

The values of the estimates are as given in (3.11).

Estimates of the Parameters

$$\hat{\mu}=\frac{1}{36}Y(\mu) \qquad \hat{B}_1=\frac{1}{24}Y(B_1)$$

$$\hat{A}_1=\frac{1}{96}\,[3Y(A_1)-Y(A_2A_3)] \qquad \hat{B}_1^2=\frac{1}{72}Y(B_1^2)$$

$$\hat{A}_2=\frac{1}{96}\,[3Y(A_2)-Y(A_1A_3)] \qquad \hat{B}_2=\frac{1}{24}Y(B_2)$$

$$\hat{A}_3=\frac{1}{96}\,[3Y(A_3)-Y(A_1A_2)] \qquad \hat{B}_2^2=\frac{1}{72}Y(B_2^2)$$

$$\widehat{A_1A_2}=\frac{1}{96}\,[-Y(A_3)+3Y(A_1A_2)] \qquad \widehat{B_1B_2}=\frac{1}{16}Y(B_1B_2)$$

$$\widehat{A_1A_3}=\frac{1}{96}\,[-Y(A_2)+3Y(A_1A_3)] \qquad \widehat{B_1B_2^2}=\frac{1}{48}Y(B_1B_2^2)$$

$$(3.11)\qquad \widehat{A_2A_3}=\frac{1}{96}\,[-Y(A_1)+3Y(A_2A_3)] \qquad \widehat{B_1^2B_2}=\frac{1}{48}Y(B_1^2B_2)$$

$$\widehat{B_1^2B_2^2}=\frac{1}{144}Y(B_1^2B_2^2)$$

$$\widehat{A_1B_1}=\frac{1}{24}Y(A_1B_1) \qquad \widehat{A_2B_1}=\frac{1}{24}Y(A_2B_1) \qquad \widehat{A_3B_1}=\frac{1}{24}Y(A_3B_1)$$

$$\widehat{A_1B_1^2}=\frac{1}{72}Y(A_1B_1^2) \qquad \widehat{A_2B_1^2}=\frac{1}{72}Y(A_2B_1^2) \qquad \widehat{A_3B_1^2}=\frac{1}{72}Y(A_3B_1^2)$$

$$\widehat{A_1B_2}=\frac{1}{24}Y(A_1B_2) \qquad \widehat{A_2B_2}=\frac{1}{24}Y(A_2B_2) \qquad \widehat{A_3B_2}=\frac{1}{24}Y(A_3B_2)$$

$$\widehat{A_1B_2^2}=\frac{1}{72}Y(A_1B_2^2) \qquad \widehat{A_2B_2^2}=\frac{1}{72}Y(A_2B_2^2) \qquad \widehat{A_3B_2^2}=\frac{1}{72}Y(A_3B_2^2)$$

4. Tests of Significance and Confidence Intervals

In this section it is assumed that the responses $Y^*(x_1 \ldots x_m z_1 \ldots z_n)$ are normally and independently distributed, with expected values $\eta(x_1 \ldots x_m z_1 \ldots z_n)$ and common variance σ^2. The estimates will, in general, tend to be normal even if the responses are not very normal. If all of the estimates for a design are least squares estimates, then the estimate s of σ is obtained according to the usual theory: the sum of squares for error, S_e, is the total sum of squares, S_t, minus the sum of squares for parameters, S_p. Then s is the square root of $S_e/(f-q)$, where f is the number of treatment combinations in the design and q is the number of parameters to be estimated. The expected value of s^2 is σ^2. The quantity S_t is Σy^2, where the summation runs over all observed responses; and the quantity S_p is $\Sigma\hat{\rho}Y(\rho)$, where the summation is over all parameters ρ in τ

To test the null hypothesis H_0 that the parameter ρ is zero, H_0: $\rho=0$, against the alternative hypothesis H_1: $\rho\neq0$, Student's t with $(f-q)$ degrees of freedom is used as follows:

$$t=\hat{\rho}/\sqrt{\hat{v}(\hat{\rho})}, \text{ with } (f-q) \ d.f. \tag{4.1}$$

where $\hat{v}(\hat{\rho})$ is the estimated variance of $\hat{\rho}$. For a least squares estimate $\hat{\rho}$, the variance $V(\hat{\rho})$ is σ^2 times the appropriate element in the main diagonal of the inverse matrix, C^{-1}. For some estimates, this can be calculated from (3.9), and for others, read from the matrices which are presented under "Analysis." For example, in DESIGN $2^3 3^2$ the variances of A_1, A_2, A_3, A_1A_2, and A_1A_3 are $3/96\ \sigma^2$ where 3/96 is read from the matrices. The estimated variance $\hat{v}(\hat{\rho})$ is obtained from $V(\hat{\rho})$ by replacing σ^2 by s^2. A two-sided confidence interval with confidence coefficient $1-\alpha$ for ρ is defined by the following limits:

$$\hat{\rho}\pm t_{1-\frac{1}{2}\alpha}\sqrt{\hat{v}(\hat{\rho})}, \text{ where } t_{1-\frac{1}{2}\alpha} \text{ has } (f-q) \ d.f. \tag{4.2}$$

If this interval includes zero, then the hypothesis H_0 is accepted; otherwise H_0 is rejected. It should be noted that all values in the interval are consistent with the data.

It may be desired to carry out a test for several estimates simultaneously. For example, it may be desired to test that all two-factor interactions for the A factors are zero; or that the linear and quadratic effects for some factor B are zero. This can be done by an F-test.

5. An Example

Some data corresponding to DESIGN $2^3 3^2$ are given in (2.6). They are taken from a publication by W. J. Youden [6]. Youden was concerned with comparing various methods of producing tomato plant seedlings prior to transplanting in the field. Comparison was made by planting in the field and then weighing the ripe produce. Thus, the observations were pounds of tomatoes.

Although Youden used five methods of production, we shall select only three: flats, fibre pots, and fibre pots soaked in one percent sodium nitrate solution. Other factors considered were different soil conditions, different sizes of pots, different varieties of tomato, and different locations on the field. The factors and their levels are recorded below:

Factors and Levels for a Tomato Experiment

Factors	*Levels*	
Soil condition, A_1	Field soil	0
	Plus fertilizer	1
Size of pot, A_2	Three-inch	0
	Four-inch	1
Variety of tomato, A_3	Bonny Best	0
	Marglobe	1
Method of production, B_1	Flat	0
	Fibre	1
	Fibre+NO_3	2
Location on field, B_2		0, 1, 2

The object of the experiment was to evaluate the effects of these factors on the yield of tomatoes.

The Y's are the inner products of the column vector y with the column vectors of X. They can be conveniently calculated by forming summary tables of the kind often used in analyzing complete factorials. For example, $Y(A_1)$, $Y(A_2)$, and $Y(A_1A_2)$ are obtained from the following table, which contains sums of nine responses:

A Summary Table

		Size of Pot (A_2) 0	1	Total
Soil Condition (A_1)	0:	1118. 2	1104. 1	2222. 3
	1:	1120. 5	1493. 7	2614. 2
	Total:	2238. 7	2597. 8	4836. 5
	Diagonal totals:	2611. 9,	2224. 6	

From the entries in this table we find

$$Y(A_1)=2614.2-2222.3=391.9$$
$$Y(A_2)=2597.8-2238.7=359.1$$
$$Y(A_1A_2)=2611.9-2224.6=387.3$$

The complete list of 27 distinct Y's is given below:

Values of the $Y(\rho)$'s

$Y(\mu)=4836.5$	$Y(B_1)=373.6$
$Y(A_1)=391.9$	$Y(B_1^2)=-50.2$
$Y(A_2)=359.1$	$Y(B_2)=445.5$
$Y(A_3)=581.7$	$Y(B_2^2)=257.9$
$Y(A_1A_2)=387.3$	$Y(B_1B_2)=-118.9$
$Y(A_1A_3)=354.7$	$Y(B_1B_2^2)=-347.3$
$Y(A_2A_3)=60.3$	$Y(B_1^2B_2)=100.5$
	$Y(B_1^2B_2^2)=620.9$
$Y(A_1B_1)=-155.0$	$Y(A_2B_2)=13.3$
$Y(A_1B_1^2)=-490.4$	$Y(A_2B_2^2)=-175.5$
$Y(A_1B_2)=51.1$	$Y(A_3B_1)=175.4$
$Y(A_1B_2^2)=-46.1$	$Y(A_3B_1^2)=-2.4$
$Y(A_2B_1)=14.2$	$Y(A_3B_2)=-190.3$
$Y(A_2B_1^2)=-40.8$	$Y(A_3B_2^2)=-273.9$

From the Y's the estimates are calculated as indicated in (3.11). The estimates are given below bracketed by their 0.95 confidence interval limits.

Estimates and Confidence Limits

124.9,	$\hat{\mu}=134.3$, 143.8	3.9,	$\hat{B}_1=15.6$, 27.2
1.6,	$\hat{A}_1=11.6$, 21.7	−7.4,	$\hat{B}_1^2=-0.7$, 6.0
−2.5,	$\hat{A}_2=7.5$, 17.6	6.9,	$\hat{B}_2=18.6$, 30.2
4.1,	$\hat{A}_3=14.1$, 24.2	−3.1,	$\hat{B}_2^2=3.6$, 10.3
−4.0,	$\widehat{A_1A_2}=6.0$, 16.1	−21.7,	$\widehat{B_1B_2}=-7.4$, 6.8
−2.7,	$\widehat{A_1A_3}=7.3$, 17.4	−15.5,	$\widehat{B_1B_2^2}=-7.2$, 1.0
−12.3,	$\widehat{A_2A_3}=-2.2$, 7.9	−6.1,	$\widehat{B_1^2B_2}=2.1$, 10.3
		−0.4,	$\widehat{B_1^2B_2^2}=4.3$, 9.1
−18.1,	$\widehat{A_1B_1}=-6.5$, 5.2	−11.1,	$\widehat{A_2B_2}=0.6$, 12.2
−13.5,	$\widehat{A_1B_1^2}=-6.8$, −0.1	−9.1,	$\widehat{A_2B_2^2}=-2.4$, 4.3
−9.5,	$\widehat{A_1B_2}=2.1$, 13.7	−4.3,	$\widehat{A_3B_1}=7.3$, 18.9
−7.3,	$\widehat{A_1B_2^2}=-0.6$, 6.1	−6.7,	$\widehat{A_3B_1^2}=-0.03$, 6.7
−11.0,	$\widehat{A_2B_1}=0.6$, 12.2	−19.5,	$\widehat{A_3B_2}=-7.9$, 3.7
−7.3,	$\widehat{A_2B_1^2}=-0.6$, 6.1	−10.5,	$\widehat{A_3B_2^2}=-3.8$, 2.9

The analysis of variance is as follows:

Source of Variation	D.F.	Sum of Squares	Mean Square	F
Parameters	26	50404	1939	3.06
Error	9	5708	634	
Total	35	56112		

This value of F may be compared with the upper 0.95 point of the F(26, 9) distribution which is 2.89.

Until now nothing has been said of the fact that there are both qualitative and quantitative factors in this example. The analysis has been carried out, effects estimated, confidence intervals placed on the estimated effects, and an analysis of variance test performed, all ignoring the fact that we have both qualitative and quantitative factors. This has been done to illustrate the calculations.

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The breakdown of the total sum of squares by sources of variation is given in the following analysis of variance table:

Source	D.F.	S.S.	M.S.	F
Pure A Effects	6	20297	3383	5.33**
Mixed Effects				
A_1B_1, $A_1B_1^2$	2	4341	2171	
A_1B_2, $A_1B_2^2$	2	138	69	
A_2B_1, $A_2B_1^2$	2	32	16	
A_2B_2, $A_2B_2^2$	2	435	218	
A_3B_1, $A_3B_1^2$	2	1282	641	
A_3B_2, $A_3B_2^2$	2	2551	1275	
Pure B Effects:				
B_1,B_1^2	2	5851	2925	4.61*
B_2,B_2^2	2	9193	4597	7.25*
B_1B_2, $B_1B_2^2$, $B_1^2B_2$, $B_1^2B_2^2$	4	6284	1571	
Error	9	5708	634	
Total	35	56112		

*Significant at 0.05 **Significant at 0.01

From these analyses, all main effects are significant, except for the size of the pot. No two-factor interaction effects are significant.

6. Six Special Designs

Designs $2^3 3^3$, $2^5 3^3$, $2^6 3^3$, $2^5 3^4$, $2^3 3^5$, and $2^5 3^5$, have been treated differently from the other designs in this catalogue. The pure B effects have been defined differently from the definitions in section 3. They are denoted by L(B), Q(B), $L(B_k B_{k'})$, $Q(B_k B_{k'})$, $L(B_k B_{k'}^2)$, and $Q(B_k B_{k'}^2)$. In the linear function, which is the expected response to a treatment combination, the coefficients of the B effects depend on the levels of the B factors as follows:

(6.1) Coefficients of Pure B Effects

Factor levels		Coefficients							
		Main effects				Interaction effects			
B_k	$B_{k'}$	$L(B_k)$	$Q(B_k)$	$L(B_{k'})$	$Q(B_{k'})$	$L(B_k B_{k'})$	$Q(B_k B_{k'})$	$L(B_k B_{k'}^2)$	$Q(B_k B_{k'}^2)$
0	0	−1	1	−1	1	−1	1	−1	1
1	0	0	−2	−1	1	0	−2	0	−2
2	0	1	1	−1	1	1	1	1	1
0	1	−1	1	0	−2	0	−2	1	1
1	1	0	−2	0	−2	1	1	−1	1
2	1	1	1	0	−2	−1	1	0	−2
0	2	−1	1	1	1	1	1	0	−2
1	2	0	−2	1	1	−1	1	1	1
2	2	1	1	1	1	0	−2	−1	1

The main effects are the same as before, i.e., L(B)=B and Q(B)=B^2. However, the interaction effects are different from the effects of section 3, but are related to them by the following matric equation:

(6.2)
$$\begin{bmatrix} L(B_k B_{k'}) \\ Q(B_k B_{k'}) \\ L(B_k B_{k'}^2) \\ Q(B_k B_{k'}^2) \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -3 & 3 & 3 & 9 \\ -1 & -3 & -3 & 3 \\ -3 & 3 & -3 & -9 \\ 1 & 3 & -3 & 3 \end{bmatrix} \begin{bmatrix} B_k B_{k'} \\ B_k B_{k'}^2 \\ B_k^2 B_{k'} \\ B_k^2 B_{k'}^2 \end{bmatrix}.$$

The inverse equation is

(6.3)
$$\begin{bmatrix} B_k B_{k'} \\ B_k B_{k'}^2 \\ B_k^2 B_{k'} \\ B_k^2 B_{k'}^2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -3 & -3 & -3 & 3 \\ 1 & -3 & 1 & 3 \\ 1 & -3 & -1 & -3 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} L(B_k B_{k'}) \\ Q(B_k B_{k'}) \\ L(B_k B_{k'}^2) \\ Q(B_k B_{k'}^2) \end{bmatrix};$$

The reason for introducing these new interaction effects is that there is more orthogonality among them than among the effects of section 3. The normal equations are easier to solve. If it is desired to estimate $B_kB_{k'}$, . . ., $B_k^2B_{k'}^2$, this can be done by using (6.3).

The elements in the main diagonal of the inverse matrix can be calculated from (6.4), except when there is a nonzero element off the diagonal in the same row as the element under consideration. In that event, special formulas are required.

Elements in the Main Diagonal of the Inverse Matrix (except as explained above)

	Parameter	*Element*
	A, $A_jA_{j'}$	$1/f$
(6.4)	AB, L(B), $L(B_kB_{k'})$, $L(B_kB_{k'}^2)$	$3/2f$
	AB^2, Q(B), $Q(B_kB_{k'})$, $Q(B_kB_{k'}^2)$	$1/2f$

As before, f denotes the number of treatment combinations in the design.

7. References

[1] National Bureau of Standards, Fractional factorial experiment designs for factors at two levels, NBS Applied Mathematics Series 48 (U.S. Government Printing Office, Washington 25, D.C., 1957)

[2] W. S. Connor and Marvin Zelen, Fractional factorial experiment designs for factors at three levels, NBS Applied Mathematics Series 54 (U.S. Government Printing Office, Washington 25, D.C., 1959)

[3] Milton Morrison, Fractional replication for mixed series, Biometrics **12**, 1–19 (1956)

[4] O. L. Davies (editor), The design and analysis of industrial experiments (Hafner Publ. Co., New York, N.Y., 1954)

[5] O. Kempthorne, The design and analysis of experiments (John Wiley & Sons, Inc., New York, N.Y., 1952)

[6] W. J. Youden and P. W. Zimmerman, Field trials with fibre pots, Contributions from Boyce Thompson Institute **8**, 317–331 (1936).

8. Designs

Design $2^4 3^1$

There are four factors at 2 levels and one factor at 3 levels. 21 effects are estimated from 36 treatment combinations. This is a ¾ fraction.

Experimental Plan

S_1S' S_2S' S_3S'

Analysis

The matrix $\frac{1}{32}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ is used to estimate $3\begin{bmatrix} \mu \\ A_3A_4 \end{bmatrix}$, $2\begin{bmatrix} A_3B_1 \\ A_4B_1 \end{bmatrix}$ $6\begin{bmatrix} A_3B_1^2 \\ A_4B_1^2 \end{bmatrix}$,

and the matrix $\frac{1}{48}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_3 \\ A_1A_2 \\ A_4 \end{bmatrix}$, $\begin{bmatrix} A_1 \\ A_2A_3 \\ A_2A_4 \end{bmatrix}$, $\begin{bmatrix} A_2 \\ A_1A_3 \\ A_1A_4 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^4

Set	S_1	S_2	S_3
$x_1+x_2+x_3=0$		0	1
$x_3+x_4=0$		1	0

Treatment Combinations

S_1	S_2	S_3
0000	0001	0011
1100	0110	0100
0111	1010	1000
1011	1101	1111

There is only one set S′ of treatment combinations from the 3^1, viz, the full replicate.

Design $2^5 3^1$

There are five factors at 2 levels and one factor at 3 levels. 28 effects are estimated from 48 treatment combinations. This is a ½ fraction.

Experimental Plan

S S′

Analysis

Completely orthogonal

Construction

Sets of Treatment Combinations from the 2^5

Set	S
$x_1+x_2+x_3+x_4+x_5=0$	

Treatment Combinations

S	
00000	01100
00011	10100
00101	11000
01001	01111
10001	10111
00110	11011
01010	11101
10010	11110

There is only one set S′ of treatment combinations from the 3^1, viz, the full replicate.

Design $2^6 3^1$

There are six factors at 2 levels and one factor at 3 levels. 36 effects are estimated from 48 treatment combinations. This is a ¼ fraction.

Experimental Plan

S_1S_1' S_2S_2' S_3S_3'

Analysis

The matrix $\frac{1}{128}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_1A_3 \\ A_2A_5 \end{bmatrix}$, $\begin{bmatrix} A_1A_4 \\ A_2A_6 \end{bmatrix}$, $\begin{bmatrix} A_1A_5 \\ A_2A_3 \end{bmatrix}$, $\begin{bmatrix} A_1A_6 \\ A_2A_4 \end{bmatrix}$;

the matrix $\frac{1}{128}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_3A_4 \\ A_5A_6 \end{bmatrix}$, $\begin{bmatrix} A_3A_6 \\ A_4A_5 \end{bmatrix}$;

the matrix $\frac{1}{64}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_1A_2 \\ A_3A_5 \\ A_4A_6 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^6

Set S_1	S_2	S_3
$x_1+x_2+x_3+x_5=0$	0	1
$x_1+x_2+x_4+x_6=0$	1	0

Treatment combinations

S_1	S_2	S_3
000000	000100	100001
000101	000001	100100
110000	110001	010001
110101	110100	010100
011001	011000	001000
011100	011101	001101
001010	001011	000010
001111	001110	000111
100011	100010	111000
100110	100111	111101
101001	101000	110010
101100	101101	110111
111010	111011	101011
111111	111110	101110
010011	010010	011011
010110	010111	011110

Sets of Treatment Combinations from the 3^1

Set S_1'	S_2'	S_3'
$z_1=0$	1	2

Treatment Combinations

S_1'	S_2'	S_3'
0	1	2

Design $2^7 3^1$

There are seven factors at 2 levels and one factor at 3 levels. 45 effects are estimated from 96 treatment combinations. This is a ¼ fraction.

Experimental Plan

S_1S_1' S_2S_2' S_3S_3'

Analysis

The matrix $\frac{1}{256}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_1A_2 \\ A_3A_4 \end{bmatrix}$, $\begin{bmatrix} A_1A_3 \\ A_2A_4 \end{bmatrix}$, $\begin{bmatrix} A_1A_4 \\ A_2A_3 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^7

Set	S_1	S_2	S_3
$x_1+x_2+x_3+x_4=0$		0	1
$x_3+x_4+x_5+x_6+x_7=0$		1	0

Treatment Combinations

S_1	S_2	S_3
0000000	0000001	1000000
1100000	1100001	0100000
0011000	0011001	1011000
1111000	1111001	0111000
1001100	1001101	0001100
0101100	0101101	1101100
1010100	1010101	0010100
0110100	0110101	1110100
1001010	1001011	0001010
0101010	0101011	1101010
1010010	1010011	0010010
0110010	0110011	1110010
0000110	0000111	1000110
1100110	1100111	0100110
0011110	0011111	1011110
1111110	1111111	0111110
1001001	1001000	0001001
0101001	0101000	1101001
1010001	1010000	0010001
0110001	0110000	1110001
0000101	0000100	1000101
1100101	1100100	0100101
0011101	0011100	1011101
1111101	1111100	0111101
0000011	0000010	1000011
1100011	1100010	0100011
0011011	0011010	1011011
1111011	1111010	0111011
1001111	1001110	0001111
0101111	0101110	1101111
1010111	1010110	0010111
0110111	0110110	1110111

Sets of Treatment Combinations from the 3^1

Set	S_1'	S_2'	S_3'
$z_1=0$	1	2	

Treatment Combinations

S_1'	S_2'	S_3'
0	1	2

Design $2^8 3^1$

There are eight factors at 2 levels and one factor at 3 levels. 55 effects are estimated from 96 treatment combinations. This is a ⅛ fraction.

Experimental Plan

S_1S_1' S_2S_2' S_3S_3'

Analysis

The matrix $\frac{1}{128}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_1A_3 \\ A_2A_4 \\ A_6A_7 \end{bmatrix}$;

the matrix $\frac{1}{256}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_2A_6 \\ A_4A_7 \end{bmatrix}$, $\begin{bmatrix} A_2A_7 \\ A_4A_6 \end{bmatrix}$;

the matrix $\frac{1}{256}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_1A_2 \\ A_3A_4 \end{bmatrix}$, $\begin{bmatrix} A_1A_4 \\ A_2A_3 \end{bmatrix}$, $\begin{bmatrix} A_1A_6 \\ A_3A_7 \end{bmatrix}$, $\begin{bmatrix} A_1A_7 \\ A_3A_6 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^8

Set	S_1	S_2	S_3
$x_1+x_2+x_5+x_7+x_8=$	0	0	0
$x_1+x_3+x_6+x_7=$	0	0	1
$x_1+x_2+x_3+x_4=$	0	1	0

Treatment Combinations

S_1	S_2	S_3
00000000	10000010	00000011
00001001	10001011	00001010
00000111	10000101	00000100
00001110	10001100	00001101
00110011	01000001	00110000
00111010	01001000	00111001
00110100	01000110	00110111
00111101	01001111	00111110
01010001	00100011	01010010
01011000	00101010	01011011
01010110	00100100	01010101
01011111	00101101	01011100
10010010	00010000	10010001
10011011	00011001	10011000
10010101	00010111	10010110
10011100	00011110	10011111
01100010	11100000	01100001
01101011	11101001	01101000
01100101	11100111	01100110
01101100	11101110	01101111
10100001	11010011	10100010
10101000	11011010	10101011
10100110	11010100	10100101
10101111	11011101	10101100
11000011	10110001	11000000
11001010	10111000	11001001
11000100	10110110	11000111
11001101	10111111	11001110
11110000	01110010	11110011
11111001	01111011	11111010
11110111	01110101	11110100
11111110	01111100	11111101

Sets of Treatment Combinations from the 3^1

Set	S_1'	S_2'	S_3'
$z_1=$	0	1	2

Treatment Combinations

S_1'	S_2'	S_3'
0	1	2

Design $2^9 3^1$

There are nine factors at 2 levels and one factor at 3 levels. 66 effects are estimated from 128 treatment combinations. This is a 1/12 fraction.

Experimental Plan

S_1S_1' S_2S_2' S_3S_3' S_4S_2'

Analysis

The matrix $\frac{1}{64}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_1B_1 \\ A_4A_6 \end{bmatrix}$, $\begin{bmatrix} A_4B_1 \\ A_1A_6 \end{bmatrix}$, $\begin{bmatrix} A_6B_1 \\ A_1A_4 \end{bmatrix}$

and the matrix $\frac{1}{576}\begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$ is used to estimate $\begin{bmatrix} \mu \\ B_1^2 \end{bmatrix}$, $\begin{bmatrix} A_1 \\ A_1B_1^2 \end{bmatrix}$, $\begin{bmatrix} A_2 \\ A_2B_1^2 \end{bmatrix}$, $\begin{bmatrix} A_3 \\ A_3B_1^2 \end{bmatrix}$, $\begin{bmatrix} A_4 \\ A_4B_1^2 \end{bmatrix}$, $\begin{bmatrix} A_5 \\ A_5B_1^2 \end{bmatrix}$, $\begin{bmatrix} A_6 \\ A_6B_1^2 \end{bmatrix}$, $\begin{bmatrix} A_7 \\ A_7B_1^2 \end{bmatrix}$, $\begin{bmatrix} A_8 \\ A_8B_1^2 \end{bmatrix}$, $\begin{bmatrix} A_9 \\ A_9B_1^2 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^9

Set	S_1	S_2	S_3	S_4
$x_1+x_2+x_3+x_4+x_9=$	1	0	0	1
$x_1+x_2+x_5+x_6+x_8=$	0	1	0	1
$x_2+x_3+x_5+x_7=$	0	0	1	1
$x_1+x_2+x_3+x_4+x_5+x_6+x_7=$	1	1	1	1

Treatment Combinations

S_1	S_2	S_3	S_4
000100000	101000100	010000011	101100000
001011000	100111100	011111011	100011000
110111000	011011100	100011011	011111000
100010100	001110000	110110111	001010100
011110100	110010000	001010111	110110100
010001100	111101000	000101111	111001100
101101100	000001000	111001111	000101100
100000010	001100110	110100001	001000010
011100010	110000110	001000001	110100010
010011010	111111110	000111001	111011010
101111010	000011110	111011001	000111010
111010110	010110010	101110101	010010110
000110110	101010010	010010101	101110110
001001110	100101010	011101101	100001110
110101110	011001010	100001101	011101110
101010001	000110101	111110010	000010001
010110001	111010101	000010010	111110001
011001001	110101101	001101010	110001001
100101001	001001101	110001010	001101001
110000101	011100001	100100110	011000101
001100101	100000001	011000110	100100101
000011101	101111001	010111110	101011101
111111101	010011001	101011110	010111101
110010011	011110111	100110000	011010011
001110011	100010111	011010000	100110011
000001011	101101111	010101000	101001011
111101011	010001111	101001000	010101011
101000111	000100011	111100100	000000111
010100111	111000011	000000100	111100111
011011111	110111011	001111100	110011111
100111111	001011011	110011100	001111111
111000000	010100100	101100011	010000000

Sets of Treatment Combinations from the 3^1

Set	S_1'	S_2'	S_3'
$z_1=$	0	1	2

Treatment Combinations

S_1'	S_2'	S_3'
0	1	2

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Design $2^3 3^2$

There are three factors at 2 levels and two factors at 3 levels. 27 effects are estimated from 36 treatment combinations. This is a ½ fraction.

Experimental Plan

$S_1S'_1 \quad S_2S'_2 \quad S_2S'_3$

Analysis

The matrix $\frac{1}{96}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_3 \\ A_1A_2 \end{bmatrix}, \begin{bmatrix} A_2 \\ A_1A_3 \end{bmatrix}, \begin{bmatrix} A_1 \\ A_2A_3 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^3

Set	S_1	S_2
$x_1+x_2+x_3=0$		1

Treatment Combinations

S_1	S_2
000	111
110	100
101	010
011	001

Sets of Treatment Combinations from the 3^2

Set	S'_1	S'_2	S'_3
$z_1+z_2=0$		1	2

Treatment Combinations

S'_1	S'_2	S'_3
00	10	20
12	01	02
21	22	11

Design $2^4 3^2$

There are four factors at 2 levels and two factors at 3 levels. 35 effects are estimated from 72 treatment combinations. This is a ½ fraction.

Experimental Plan

$S_1S'_1 \quad S_2S'_2 \quad S_2S'_3$

Analysis

The matrix $\frac{1}{192}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_1A_2 \\ A_3A_4 \end{bmatrix}, \begin{bmatrix} A_1A_3 \\ A_2A_4 \end{bmatrix}, \begin{bmatrix} A_1A_4 \\ A_2A_3 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^4

Set	S_1	S_2
$x_1+x_2+x_3+x_4=0$		1

Treatment Combinations

S_1	S_2
0000	1000
1111	0100
1100	0010
1010	0001
1001	1110
0110	1101
0101	1011
0011	0111

Sets of Treatment Combinations from the 3^2

Set	S'_1	S'_2	S'_3
$z_1+z_2=0$		1	2

Treatment Combinations

S'_1	S'_2	S'_3
00	10	20
12	01	02
21	22	11

Design $2^5 3^2$

There are five factors at 2 levels and two factors at 3 levels. 44 effects are estimated from 72 treatment combinations. This is a ¼ fraction.

Experimental Plan

S_1S_1' S_2S_2' S_3S_3'

Analysis

The matrix $\frac{1}{96}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_3 \\ A_1A_2 \\ A_4A_5 \end{bmatrix}$ and

the matrix $\frac{1}{192}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_1 \\ A_2A_3 \end{bmatrix}$, $\begin{bmatrix} A_2 \\ A_1A_3 \end{bmatrix}$, $\begin{bmatrix} A_4 \\ A_3A_5 \end{bmatrix}$, $\begin{bmatrix} A_5 \\ A_3A_4 \end{bmatrix}$, $\begin{bmatrix} A_1A_4 \\ A_2A_5 \end{bmatrix}$, $\begin{bmatrix} A_1A_5 \\ A_2A_4 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^5

Set	S_1	S_2	S_3
$x_1+x_2+x_3=0$	0	0	1
$x_3+x_4+x_5=0$	0	1	0

Treatment Combinations

S_1	S_2	S_3
00000	00001	10000
00011	00010	10011
11000	11001	01000
11011	11010	01011
10101	10100	00101
10110	10111	00110
01101	01100	11101
01110	01111	11110

Sets of Treatment Combinations from the 3^2

Set	S_1'	S_2'	S_3'
$z_1+z_2=$	0	1	2

Treatment Combinations

S_1'	S_2'	S_3'
00	10	20
21	01	02
12	22	11

Design $2^6 3^2$

There are six factors at 2 levels and two factors at 3 levels. 54 effects are estimated from 96 treatment combinations. This is a ⅙ fraction.

Experimental Plan

S_1S_1' S_2S_2' S_3S_3' S_4S_3'

Analysis

The matrix $\frac{1}{240}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_1B_1 \\ A_1B_2 \end{bmatrix}$, $\begin{bmatrix} A_2B_1 \\ A_2B_2 \end{bmatrix}$, $\begin{bmatrix} A_3B_1 \\ A_3B_2 \end{bmatrix}$, $\begin{bmatrix} A_4B_1 \\ A_4B_2 \end{bmatrix}$, $\begin{bmatrix} A_5B_1 \\ A_5B_2 \end{bmatrix}$, $\begin{bmatrix} A_6B_1 \\ A_6B_2 \end{bmatrix}$;

the matrix $\frac{1}{720}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_1B_1^2 \\ A_1B_2^2 \end{bmatrix}$, $\begin{bmatrix} A_2B_1^2 \\ A_2B_2^2 \end{bmatrix}$, $\begin{bmatrix} A_3B_1^2 \\ A_3B_2^2 \end{bmatrix}$, $\begin{bmatrix} A_4B_1^2 \\ A_4B_2^2 \end{bmatrix}$, $\begin{bmatrix} A_5B_1^2 \\ A_5B_2^2 \end{bmatrix}$, $\begin{bmatrix} A_6B_1^2 \\ A_6B_2^2 \end{bmatrix}$;

the matrix $\frac{1}{192}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_1 \\ A_2A_5 \\ A_4A_6 \end{bmatrix}$, $\begin{bmatrix} A_4 \\ A_3A_5 \\ A_1A_6 \end{bmatrix}$, $\begin{bmatrix} A_5 \\ A_3A_4 \\ A_1A_2 \end{bmatrix}$;

the matrix $\frac{1}{192}\begin{bmatrix} 4 & 2 & -2 \\ 2 & 3 & -1 \\ -2 & -1 & 3 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_2 \\ A_1A_5 \\ A_3A_6 \end{bmatrix}$, $\begin{bmatrix} A_3 \\ A_4A_5 \\ A_2A_6 \end{bmatrix}$, $\begin{bmatrix} A_6 \\ A_1A_4 \\ A_2A_3 \end{bmatrix}$;

the matrix $\frac{1}{58{,}752}\begin{bmatrix} 680 & 204 & -68 & 0 & 0 \\ 204 & 1377 & 51 & 102 & 102 \\ -68 & 51 & 153 & 34 & 34 \\ 0 & 102 & 34 & 340 & -68 \\ 0 & 102 & 34 & -68 & 340 \end{bmatrix}$ is used to estimate $\begin{bmatrix} \mu \\ B_1B_2 \\ B_1^2B_2^2 \\ B_1^2 \\ B_2^2 \end{bmatrix}$

and the matrix $\frac{1}{1{,}152}\begin{bmatrix} 20 & 4 & -2 & 2 \\ 4 & 20 & 2 & -2 \\ -2 & 2 & 11 & 1 \\ 2 & -2 & 1 & 11 \end{bmatrix}$ is used to estimate $\begin{bmatrix} B_1 \\ B_2 \\ B_1B_2^2 \\ B_1^2B_2 \end{bmatrix}$

Construction

Sets of Treatment Combinations from the 2^6

Set	S_1	S_2	S_3	S_4
$x_1+x_2+x_3+x_4=0$	0	1	1	
$x_3+x_4+x_5=0$	0	1	0	
$x_2+x_4+x_5+x_6=0$	1	0	1	

Treatment Combinations

S_1	S_2	S_3	S_4
000000	001100	000101	100001
111100	110000	111001	011101
011010	010110	011111	111011
100110	101010	100011	000111
001101	000001	001000	101100
110001	111101	110100	010000
010111	011011	010010	110110
101011	100111	101110	001010

Sets of Treatment Combinations from the 3^2

Set	S_1'	S_2'	S_3'
$z_1+z_2=0$	1	2	

Treatment Combinations

S_1'	S_2'	S_3'
00	10	20
12	01	02
21	22	11

Design $2^7 3^2$

There are seven factors at 2 levels and two factors at 3 levels. 65 effects are estimated from 144 treatment combinations. This is a ⅛ fraction.

Experimental Plan

S_1S_1' S_2S_2' S_3S_3'

Analysis

The matrix $\frac{1}{192}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_7 \\ A_4A_6 \\ A_3A_5 \end{bmatrix}, \begin{bmatrix} A_1A_2 \\ A_5A_6 \\ A_3A_4 \end{bmatrix}$;

the matrix $\frac{1}{384}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_4 \\ A_6A_7 \end{bmatrix}, \begin{bmatrix} A_6 \\ A_4A_7 \end{bmatrix}, \begin{bmatrix} A_1A_5 \\ A_2A_6 \end{bmatrix}, \begin{bmatrix} A_1A_6 \\ A_2A_5 \end{bmatrix}$;

the matrix $\frac{1}{384}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_3 \\ A_5A_7 \end{bmatrix}, \begin{bmatrix} A_5 \\ A_3A_7 \end{bmatrix}, \begin{bmatrix} A_1A_3 \\ A_2A_4 \end{bmatrix}, \begin{bmatrix} A_1A_4 \\ A_2A_3 \end{bmatrix}, \begin{bmatrix} A_3A_6 \\ A_4A_5 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^7

Set	S_1	S_2	S_3
$x_1+x_2+x_3+x_4=0$	1	0	
$x_3+x_4+x_5+x_6=0$	0	1	
$x_1+x_2+x_3+x_6+x_7=0$	0	0	

Treatment Combinations

S_1	S_2	S_3
0000000	1000001	0000100
1100000	0100001	1100100
0110100	1110101	0110000
1010100	0010101	1010000
0101010	1101011	0101110
1001010	0001011	1001110
0011110	1011111	0011010
1111110	0111111	1111010
0011001	1011000	0011101
1111001	0111000	1111101
0101101	1101100	0101001
1001101	0001100	1001001
0110011	1110010	0110111
1010011	0010010	1010111
0000111	1000110	0000011
1100111	0100110	1100011

Sets of treatment Combinations from the 3^2

Set	S_1'	S_2'	S_3'
$z_1+z_2=0$	1	2	

Treatment Combinations

S_1'	S_2'	S_3'
00	10	20
12	01	02
21	22	11

Design $2^8 3^2$

There are eight factors at 2 levels and two factors at 3 levels. 77 effects are estimated from 144 treatment combinations. This is a 1/16 fraction.

Experimental Plan

S_1S_1' S_2S_2' S_3S_3' S_4S_4' S_5S_5' S_6S_6' S_7S_7' S_8S_8' S_8S_9'

Analysis

The matrix $\frac{1}{1536}\begin{bmatrix} 11 & -1 & -1 & 1 \\ -1 & 11 & -1 & 1 \\ -1 & -1 & 11 & 1 \\ 1 & 1 & 1 & 11 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_1A_2 \\ A_3A_4 \\ A_5A_6 \\ A_7A_8 \end{bmatrix}$, $\begin{bmatrix} A_1A_3 \\ A_2A_4 \\ A_6A_7 \\ A_5A_8 \end{bmatrix}$, $\begin{bmatrix} A_1A_4 \\ A_2A_3 \\ A_5A_7 \\ A_6A_8 \end{bmatrix}$, $\begin{bmatrix} A_1A_5 \\ A_2A_6 \\ A_4A_7 \\ A_3A_8 \end{bmatrix}$, $\begin{bmatrix} A_1A_6 \\ A_2A_5 \\ A_3A_7 \\ A_4A_8 \end{bmatrix}$, $\begin{bmatrix} A_1A_7 \\ A_4A_5 \\ A_3A_6 \\ A_2A_8 \end{bmatrix}$, $\begin{bmatrix} A_2A_7 \\ A_4A_6 \\ A_3A_5 \\ A_1A_8 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^8

Set	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
$x_1+x_2+x_3+x_4=1$	1	1	1	0	0	0	0	
$x_1+x_2+x_5+x_6=1$	1	0	0	1	1	0	0	
$x_2+x_3+x_5+x_7=0$	1	1	0	1	0	1	0	
$x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8=1$	1	1	1	1	1	1	1	

Treatment Combinations

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
01110000	01000000	00100000	00010000	00001000	00111000	00000010	00000001
01001100	10110000	11010000	11100000	11111000	11001000	11110010	11110001
10000000	10001100	11101100	11011100	11000100	11110100	11001110	11001101
10111100	01111100	00011100	00101100	00110100	00000100	00111110	00111101
11101010	00101010	01001010	01111010	01100010	01010010	01101000	01101011
00011010	11011010	10111010	10001010	10010010	10100010	10011000	10011011
00100110	11100110	10000110	10110110	10101110	10011110	10100100	10100111
11010110	00010110	01110110	01000110	01011110	01101110	01010100	01010111
00101001	11101001	10001001	10111001	10100001	10010001	10101011	10101000
11011001	00011001	01111001	01001001	01010001	01100001	01011011	01011000
11100101	00100101	01000101	01110101	01101101	01011101	01100111	01100100
01000011	10000011	11100011	10000101	10011101	10101101	10010111	10010100
10110011	01110011	00010011	11010011	11001011	11111011	11000001	11000010
10001111	01001111	00101111	00100011	00111011	00001011	00110001	00110010
01111111	10111111	11011111	11101111	00000111	00110111	00001101	00001110
00010101	11010101	10110101	00011111	11110111	11000111	11111101	11111110

Sets of Treatment Combinations from the 3^2

Set	S_1'	S_2'	S_3'	S_4'	S_5'	S_6'	S_7'	S_8'	S_9'
$z_1+z_2=0$	0	0	1	1	1	2	2	2	

Treatment Combinations

S_1'	S_2'	S_3'	S_4'	S_5'	S_6'	S_7'	S_8'	S_9'
00	12	21	10	01	22	02	20	11

Design $2^2 3^3$

There are two factors at 2 levels and three factors at 3 levels. 34 effects are estimated from **54** treatment combinations. This is a ½ fraction.

Experimental Plan

S_1S_1' S_2S_2' S_2S_3'

Analysis

The matrix $\frac{1}{48}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ is used to estimate

$$3\begin{bmatrix} A_1 \\ A_2 \end{bmatrix},\ 3\begin{bmatrix} \mu \\ A_1A_2 \end{bmatrix},\ 2\begin{bmatrix} A_1B_1 \\ A_2B_1 \end{bmatrix},\ 2\begin{bmatrix} A_1B_2 \\ A_2B_2 \end{bmatrix},\ 2\begin{bmatrix} A_1B_3 \\ A_2B_3 \end{bmatrix},\ 6\begin{bmatrix} A_1B_1^2 \\ A_2B_1^2 \end{bmatrix},\ 6\begin{bmatrix} A_1B_2^2 \\ A_2B_2^2 \end{bmatrix},\ 6\begin{bmatrix} A_1B_3^2 \\ A_2B_3^2 \end{bmatrix}.$$

Construction

Sets of Treatment Combinations from the 2^2

Set	S_1	S_2
$x_1+x_2=0$	1	

Treatment Combinations

S_1	S_2
00	10
11	01

Sets of Treatment Combinations from the 3^3

Set	S_1'	S_2'	S_3'
$z_1+z_2+z_3=0$	1	2	

Treatment Combinations

S_1'	S_2'	S_3'
000	001	020
222	010	002
120	022	011
102	100	110
012	121	101
210	112	122
201	220	200
021	202	212
111	211	221

Design $2^4 3^3$

There are four factors at 2 levels and three factors at 3 levels. 53 effects are estimated from 108 treatment combinations. This is a ¼ fraction.

Experimental Plan

S_1S_1' S_2S_2' S_3S_3'

Analysis

The matrix $\frac{1}{48}\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$ is used to estimate $6\begin{bmatrix}\mu\\ A_3A_4\end{bmatrix}$, $4\begin{bmatrix}A_3B_1\\ A_4B_1\end{bmatrix}$, $4\begin{bmatrix}A_3B_2\\ A_4B_2\end{bmatrix}$, $4\begin{bmatrix}A_3B_3\\ A_4B_3\end{bmatrix}$, $12\begin{bmatrix}A_3B_1^2\\ A_4B_1^2\end{bmatrix}$, $12\begin{bmatrix}A_3B_2^2\\ A_4B_2^2\end{bmatrix}$, $12\begin{bmatrix}A_3B_3^2\\ A_4B_3^2\end{bmatrix}$,

and the matrix $\frac{1}{144}\begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix}$ is used to estimate $\begin{bmatrix}A_1\\ A_2A_3\\ A_2A_4\end{bmatrix}$, $\begin{bmatrix}A_2\\ A_1A_3\\ A_1A_4\end{bmatrix}$, $\begin{bmatrix}A_3\\ A_4\\ A_1A_2\end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^4

Set	S_1	S_2	S_3
$x_1+x_2+x_3=0$		0	1
$x_1+x_2+x_4=0$		1	0

Treatment Combinations

S_1	S_2	S_3
0000	0001	1110
1100	1101	1001
0111	0110	0101
1011	1010	0010

Sets of Treatment Combinations from the 3^3

Set	S_1'	S_2'	S_3'
$z_1+z_2+z_3=0$		1	2

Treatment Combinations

S_1'	S_2'	S_3'
000	001	020
222	010	002
120	022	011
102	100	110
012	121	101
210	112	122
201	220	200
021	202	212
111	211	221

Design $2^7 3^3$

There are seven factors at 2 levels and three factors at 3 levels. 89 effects are estimated from 432 treatment combinations. This is a ⅛ fraction.

Experimental Plan

S_1S_1' S_2S_2' S_3S_3'

Analysis

The matrix $\frac{1}{576}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_1A_2 \\ A_5A_6 \\ A_3A_4 \end{bmatrix}$, $\begin{bmatrix} A_7 \\ A_1A_4 \\ A_2A_3 \end{bmatrix}$;

the matrix $\frac{1}{1152}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_1A_5 \\ A_2A_6 \end{bmatrix}$, $\begin{bmatrix} A_1A_6 \\ A_2A_5 \end{bmatrix}$, $\begin{bmatrix} A_1 \\ A_4A_7 \end{bmatrix}$, $\begin{bmatrix} A_4 \\ A_1A_7 \end{bmatrix}$;

the matrix $\frac{1}{1152}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_3 \\ A_2A_7 \end{bmatrix}$, $\begin{bmatrix} A_1A_3 \\ A_2A_4 \end{bmatrix}$, $\begin{bmatrix} A_3A_5 \\ A_4A_6 \end{bmatrix}$, $\begin{bmatrix} A_3A_6 \\ A_4A_5 \end{bmatrix}$, $\begin{bmatrix} A_2 \\ A_3A_7 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^7

Set	S_1	S_2	S_3
$x_1+x_2+x_3+x_4=0$		1	0
$x_3+x_4+x_5+x_6=0$		0	1
$x_1+x_3+x_5+x_6+x_7=0$		0	0

Treatment Combinations

S_1	S_2	S_3
0000000	0100000	0011100
1111000	1011000	1100100
0110100	0010100	0101000
1001100	1101100	1010000
0110010	0010010	0101110
1001010	1101010	1010110
0000110	0100110	0011010
1111110	1011110	1100010
1100001	1000001	1111101
0011001	0111001	0000101
1010101	1110101	1001001
0101101	0001101	0110001
1010011	1110011	1001111
0101011	0001011	0110111
1100111	1000111	1111011
0011111	0111111	0000011

Sets of Treatment Combinations from the 3^3

Set	S_1'	S_2'	S_3'
$z_1+z_2+z_3=0$		1	2

Treatment Combinations

S_1'	S_2'	S_3'
000	001	020
222	010	002
120	022	011
102	100	110
012	121	101
210	112	122
201	220	200
021	202	212
111	211	221

Design $2^1 3^4$

There is one factor at 2 levels and there are four factors at 3 levels. 42 effects are estimated from 81 treatment combinations. This is a ½ fraction.

Experimental Plan

S_1S_1' S_2S_2' S_2S_3'

Analysis

The matrix $\frac{1}{72}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ is used to estimate $3\begin{bmatrix} \mu \\ A_1 \end{bmatrix}$, $2\begin{bmatrix} B_1 \\ A_1B_1 \end{bmatrix}$, $2\begin{bmatrix} B_2 \\ A_1B_2 \end{bmatrix}$, $2\begin{bmatrix} B_3 \\ A_1B_3 \end{bmatrix}$, $2\begin{bmatrix} B_4 \\ A_1B_4 \end{bmatrix}$, $6\begin{bmatrix} B_1^2 \\ A_1B_1^2 \end{bmatrix}$, $6\begin{bmatrix} B_2^2 \\ A_1B_2^2 \end{bmatrix}$, $6\begin{bmatrix} B_3^2 \\ A_1B_3^2 \end{bmatrix}$, $6\begin{bmatrix} B_4^2 \\ A_1B_4^2 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^1

Set	S_1	S_2
$x_1 =$	0	1

Treatment Combinations

S_1	S_2
0	1

Sets of Treatment Combinations from the 3^4

Set	S_1'	S_2'	S_3'
$z_1+z_2+z_3+z_4=$	0	1	2

Treatment Combinations

S_1'	S_2'	S_3'
0000	1000	2000
1110	2110	0110
2220	0220	1220
2001	0001	1001
0111	1111	2111
1221	2221	0221
1002	2002	0002
2112	0112	1112
0222	1222	2222
0120	1120	2120
1200	2200	0200
2010	0010	1010
2121	0121	1121
0201	1201	2201
1011	2011	0011
1122	2122	0122
2202	0202	1202
0012	1012	2012
0210	1210	2210
1020	2020	0020
2100	0100	1100
2211	0211	1211
0021	1021	2021
1101	2101	0101
1212	2212	0212
2022	0022	1022
0102	1102	2102

Design $2^2 3^4$

There are two factors at 2 levels and four factors at 3 levels. 52 effects are estimated from 162 treatment combinations. This is a ½ fraction.

Experimental Plan

S_1S_1' S_2S_2' S_1S_3'

Analysis

The matrix $\frac{1}{3}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ is used to estimate $96\begin{bmatrix} A_1B_1 \\ A_2B_1 \end{bmatrix}$, $96\begin{bmatrix} A_1B_2 \\ A_2B_2 \end{bmatrix}$, $96\begin{bmatrix} A_1B_3 \\ A_2B_3 \end{bmatrix}$, $96\begin{bmatrix} A_1B_4 \\ A_2B_4 \end{bmatrix}$, $288\begin{bmatrix} A_1B_1^2 \\ A_2B_1^2 \end{bmatrix}$, $288\begin{bmatrix} A_1B_2^2 \\ A_2B_2^2 \end{bmatrix}$, $288\begin{bmatrix} A_1B_3^2 \\ A_2B_3^2 \end{bmatrix}$, $288\begin{bmatrix} A_1B_4^2 \\ A_2B_4^2 \end{bmatrix}$, $144\begin{bmatrix} \mu \\ A_1A_2 \end{bmatrix}$, $144\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^2

Set	S_1	S_2
$x_1+x_2=$	0	1

Treatment Combinations

S_1	S_2
00	01
11	10

Sets of Treatment Combinations from the 3^4

Set	S_1'	S_2'	S_3'
$z_1+z_2+z_3+z_4=$	0	1	2

Treatment Combinations

S_1'		S_2'		S_3'	
0000	1011	1000	2011	2000	0011
1110	1122	2110	2122	0110	0122
2220	2202	0220	0202	1220	1202
2001	0012	0001	1012	1001	2012
0111	0210	1111	1210	2111	2210
1221	1020	2221	2020	0221	0020
1002	2100	2002	0100	0002	1100
2112	2211	0112	0211	1112	1211
0222	0021	1222	1021	2222	2021
0120	1101	1120	2101	2120	0101
1200	1212	2200	2212	0200	0212
2010	2022	0010	0022	1010	1022
2121	0102	0121	1102	1121	2102
0201		1201		2201	

Design $2^3 3^4$

There are three factors at 2 levels and four factors at 3 levels. 63 effects are estimated from 162 treatment combinations. This is a ¼ fraction.

Experimental Plan

S_1S_1' S_2S_2' S_3S_3' S_4S_4' S_1S_5' S_2S_6' S_3S_7' S_4S_8' S_1S_9'

Analysis

The matrix $\frac{1}{1728}\begin{bmatrix} 11 & -1 & -1 & -1 \\ -1 & 11 & -1 & -1 \\ -1 & -1 & 11 & -1 \\ -1 & -1 & -1 & 11 \end{bmatrix}$ is used to estimate $\begin{bmatrix} \mu \\ A_1A_2 \\ A_1A_3 \\ A_2A_3 \end{bmatrix}$

and the matrix $\frac{1}{88}\begin{bmatrix} 10 & -1 & -1 \\ -1 & 10 & -1 \\ -1 & -1 & 10 \end{bmatrix}$ is used to estimate $18\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$, $12\begin{bmatrix} A_1B_1 \\ A_2B_1 \\ A_3B_1 \end{bmatrix}$, $12\begin{bmatrix} A_1B_2 \\ A_2B_2 \\ A_3B_2 \end{bmatrix}$,

$12\begin{bmatrix} A_1B_3 \\ A_2B_3 \\ A_3B_3 \end{bmatrix}$, $12\begin{bmatrix} A_1B_4 \\ A_2B_4 \\ A_3B_4 \end{bmatrix}$, $36\begin{bmatrix} A_1B_1^2 \\ A_2B_1^2 \\ A_3B_1^2 \end{bmatrix}$, $36\begin{bmatrix} A_1B_2^2 \\ A_2B_2^2 \\ A_3B_2^2 \end{bmatrix}$, $36\begin{bmatrix} A_1B_3^2 \\ A_2B_3^2 \\ A_3B_3^2 \end{bmatrix}$, $36\begin{bmatrix} A_1B_4^2 \\ A_2B_4^2 \\ A_3B_4^2 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^3

Set	S_1	S_2	S_3	S_4
$x_1+x_2=0$	0	1	1	
$x_1+x_3=0$	1	0	1	

Treatment Combinations

S_1	S_2	S_3	S_4
000	001	010	100
111	110	101	011

Sets of Treatment Combinations from the 3^4

Set	S_1'	S_2'	S_3'	S_4'	S_5'	S_6'	S_7'	S_8'	S_9'
$z_1+z_2+z_3=0$	0	0	1	1	1	2	2	2	
$z_2+2z_3+z_4=0$	1	2	0	1	2	0	1	2	

Treatment Combinations

S_1'	S_2'	S_3'	S_4'	S_5'	S_6'	S_7'	S_8'	S_9'
0000	0001	0002	1000	1001	1002	2000	2001	2002
1110	1111	1112	2110	2111	2112	0110	0111	0112
2220	2221	2222	0220	0221	0222	1220	1221	1222
1201	1202	1200	2201	2202	2200	0201	0202	0200
2011	2012	2010	0011	0012	0010	1011	1012	1010
0121	0122	0120	1121	1122	1120	2121	2122	2120
2102	2100	2101	0102	0100	0101	1102	1100	1101
0212	0210	0211	1212	1210	1211	2212	2210	2211
1022	1020	1021	2022	2020	2021	0022	0020	0021

Design $2^4 3^4$

There are four factors at 2 levels and four factors at 3 levels. 75 effects are estimated from 162 treatment combinations. This is a ⅛ fraction.

Experimental Plan

$S_1S'_1 \quad S_2S'_2 \quad S_3S'_3 \quad S_4S'_4 \quad S_5S'_5 \quad S_6S'_6 \quad S_7S'_7 \quad S_8S'_8 \quad S_1S'_9$

Analysis

$$\text{The matrix } \frac{1}{2160}\begin{bmatrix} 14 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 14 & 1 & -1 & -1 & 1 & -1 \\ -1 & 1 & 14 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 14 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 14 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 & 14 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & 14 \end{bmatrix} \text{ is used to estimate } \begin{bmatrix} \mu \\ A_1A_2 \\ A_1A_3 \\ A_1A_4 \\ A_2A_3 \\ A_2A_4 \\ A_3A_4 \end{bmatrix}$$

$$\text{and the matrix } \frac{1}{96}\begin{bmatrix} 11 & 1 & -1 & 1 \\ 1 & 11 & 1 & -1 \\ -1 & 1 & 11 & 1 \\ 1 & -1 & 1 & 11 \end{bmatrix} \text{ is used to estimate } 18\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix},\ 12\begin{bmatrix} A_1B_1 \\ A_2B_1 \\ A_3B_1 \\ A_4B_1 \end{bmatrix},\ 12\begin{bmatrix} A_1B_2 \\ A_2B_2 \\ A_3B_2 \\ A_4B_2 \end{bmatrix},$$

$$12\begin{bmatrix} A_1B_3 \\ A_2B_3 \\ A_3B_3 \\ A_4B_3 \end{bmatrix},\ 12\begin{bmatrix} A_1B_4 \\ A_2B_4 \\ A_3B_4 \\ A_4B_4 \end{bmatrix},\ 36\begin{bmatrix} A_1B_1^2 \\ A_2B_1^2 \\ A_3B_1^2 \\ A_4B_1^2 \end{bmatrix},\ 36\begin{bmatrix} A_1B_2^2 \\ A_2B_2^2 \\ A_3B_2^2 \\ A_4B_2^2 \end{bmatrix},\ 36\begin{bmatrix} A_1B_3^2 \\ A_2B_3^2 \\ A_3B_3^2 \\ A_4B_3^2 \end{bmatrix},\ 36\begin{bmatrix} A_1B_4^2 \\ A_2B_4^2 \\ A_3B_4^2 \\ A_4B_4^2 \end{bmatrix}.$$

Construction

Sets of Treatment Combinations from the 2^4

Set	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
$x_1+x_2=$	1	1	1	1	0	0	0	0
$x_3+x_4=$	1	1	0	0	1	0	1	0
$x_2+x_3=$	1	0	1	0	0	1	1	0

Treatment Combinations

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
1010	1001	0100	1000	0001	1100	0010	0000
0101	0110	1011	0111	1110	0011	1101	1111

Sets of Treatment Combinations from the 3^4

Set	S'_1	S'_2	S'_3	S'_4	S'_5	S'_6	S'_7	S'_8	S'_9
$z_1+z_2+z_3=$	0	0	0	1	1	1	2	2	2
$z_2+2z_3+z_4=$	0	1	2	0	1	2	0	1	2

Treatment Combinations

S'_1	S'_2	S'_3	S'_4	S'_5	S'_6	S'_7	S'_8	S'_9
0000	0001	0002	1000	1001	1002	2000	2001	2002
1110	1111	1112	2110	2111	2112	0110	0111	0112
2220	2221	2222	0220	0221	0222	1220	1221	1222
1201	1202	1200	2201	2202	2200	0201	0202	0200
2011	2012	2010	0011	0012	0010	1011	1012	1010
0121	0122	0120	1121	1122	1120	2121	2122	2120
2102	2100	2101	0102	0100	0101	1102	1100	1101
0212	0210	0211	1212	1210	1211	2212	2210	2211
1022	1020	1021	2022	2020	2021	0022	0020	0021

Design $2^6 3^4$

There are six factors at 2 levels and four factors at 3 levels. 102 effects are estimated from 324 treatment combinations. This is a $\frac{1}{16}$ fraction.

Experimental Plan

S_1S_1' S_2S_2' S_3S_3' S_4S_4' S_5S_5' S_6S_6' S_7S_7' S_8S_8' S_8S_9'

Analysis

The matrix $\frac{1}{3456}\begin{bmatrix} 11 & -1 & -1 & 1 \\ -1 & 11 & -1 & 1 \\ -1 & -1 & 11 & 1 \\ 1 & 1 & 1 & 11 \end{bmatrix}$ is used to estimate $\begin{bmatrix} \mu \\ A_1A_5 \\ A_2A_4 \\ A_3A_6 \end{bmatrix}$;

the matrix $\frac{1}{4032}\begin{bmatrix} 13 & 1 & 1 & 1 & 1 & 1 \\ 1 & 13 & -1 & -1 & -1 & -1 \\ 1 & -1 & 13 & -1 & -1 & -1 \\ 1 & -1 & -1 & 13 & -1 & -1 \\ 1 & -1 & -1 & -1 & 13 & -1 \\ 1 & -1 & -1 & -1 & -1 & 13 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_3 \\ A_6 \\ A_1A_2 \\ A_1A_4 \\ A_2A_5 \\ A_4A_5 \end{bmatrix}$;

the matrix $\frac{1}{4032}\begin{bmatrix} 13 & -1 & 1 & -1 & 1 & -1 \\ -1 & 13 & 1 & -1 & 1 & -1 \\ 1 & 1 & 13 & 1 & -1 & 1 \\ -1 & -1 & 1 & 13 & 1 & -1 \\ 1 & 1 & -1 & 1 & 13 & 1 \\ -1 & -1 & 1 & -1 & 1 & 13 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_2 \\ A_4 \\ A_1A_3 \\ A_1A_6 \\ A_3A_5 \\ A_5A_6 \end{bmatrix}$, $\begin{bmatrix} A_1 \\ A_5 \\ A_2A_3 \\ A_2A_6 \\ A_3A_4 \\ A_4A_6 \end{bmatrix}$;

the matrix $\frac{1}{80}\begin{bmatrix} 9 & -1 \\ -1 & 9 \end{bmatrix}$ is used to estimate $24\begin{bmatrix} A_1B_1 \\ A_5B_1 \end{bmatrix}$, $24\begin{bmatrix} A_1B_2 \\ A_5B_2 \end{bmatrix}$, $24\begin{bmatrix} A_1B_3 \\ A_5B_3 \end{bmatrix}$, $24\begin{bmatrix} A_1B_4 \\ A_5B_4 \end{bmatrix}$, $24\begin{bmatrix} A_2B_1 \\ A_4B_1 \end{bmatrix}$, $24\begin{bmatrix} A_2B_2 \\ A_4B_2 \end{bmatrix}$, $24\begin{bmatrix} A_2B_3 \\ A_4B_3 \end{bmatrix}$, $24\begin{bmatrix} A_2B_4 \\ A_4B_4 \end{bmatrix}$, $72\begin{bmatrix} A_1B_1^2 \\ A_5B_1^2 \end{bmatrix}$, $72\begin{bmatrix} A_1B_2^2 \\ A_5B_2^2 \end{bmatrix}$, $72\begin{bmatrix} A_1B_3^2 \\ A_5B_3^2 \end{bmatrix}$, $72\begin{bmatrix} A_1B_4^2 \\ A_5B_4^2 \end{bmatrix}$, $72\begin{bmatrix} A_2B_1^2 \\ A_4B_1^2 \end{bmatrix}$, $72\begin{bmatrix} A_2B_2^2 \\ A_4B_2^2 \end{bmatrix}$, $72\begin{bmatrix} A_2B_3^2 \\ A_4B_3^2 \end{bmatrix}$, $72\begin{bmatrix} A_2B_4^2 \\ A_4B_4^2 \end{bmatrix}$;

the matrix $\frac{1}{80}\begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix}$ is used to estimate $24\begin{bmatrix} A_3B_1 \\ A_6B_1 \end{bmatrix}$, $24\begin{bmatrix} A_3B_2 \\ A_6B_2 \end{bmatrix}$, $24\begin{bmatrix} A_3B_3 \\ A_6B_3 \end{bmatrix}$, $24\begin{bmatrix} A_3B_4 \\ A_6B_4 \end{bmatrix}$, $72\begin{bmatrix} A_3B_1^2 \\ A_6B_1^2 \end{bmatrix}$, $72\begin{bmatrix} A_3B_2^2 \\ A_6B_2^2 \end{bmatrix}$, $72\begin{bmatrix} A_3B_3^2 \\ A_6B_3^2 \end{bmatrix}$, $72\begin{bmatrix} A_3B_4^2 \\ A_6B_4^2 \end{bmatrix}$.

Sets of Treatment Combinations from the 2^6

Set	S_1	S_2	S_3	S_4	S_5	S_5	S_7	S_8
$x_1+x_2+x_3=$	1	1	1	1	0	0	0	0
$x_3+x_4+x_5=$	1	1	0	0	1	1	0	0
$x_1+x_3+x_5+x_6=$	0	1	0	1	0	1	0	1
$x_1+x_2+x_3+x_4+x_5+x_6=$	1	1	1	1	1	1	1	1

Treatment Combinations

S_1	S_2	S_3	S_4
111000	001000	010000	100000
010011	100011	111011	001011
100101	010101	001101	111101
001110	111110	100110	010110

S_5	S_6	S_7	S_8
000100	000010	000111	000001
101111	101001	101100	101010
011001	011111	011010	011100
110010	110100	110001	110111

Sets of Treatment Combinations from the 3^4

Set	S_1'	S_2'	S_3'	S_4'	S_5'	S_6'	S_7'	S_8'	S_9'
$z_1+z_2+z_3=$	0	0	0	1	1	1	2	2	2
$z_2+2z_3+z_4=$	0	1	2	0	1	2	0	1	2

Treatment Combinations

S_1'	S_2'	S_3'	S_4'	S_5'	S_6'	S_7'	S_8'	S_9'
0000	0001	0002	1000	1001	1002	2000	2001	2002
1110	1111	1112	2110	2111	2112	0110	0111	0112
2220	2221	2222	0220	0221	0222	1220	1221	1222
1201	1202	1200	2201	2202	2200	0201	0202	0200
2011	2012	2010	0011	0012	0010	1011	1012	1010
0121	0122	0120	1121	1122	1120	2121	2122	2120
2102	2100	2101	0102	0100	0101	1102	1100	1101
0212	0210	0211	1212	1210	1211	2212	2210	2211
1022	1020	1021	2022	2020	2021	0022	0020	0021

Design $2^1 3^5$

There is one factor at 2 levels and there are five factors at 3 levels. 62 effects are estimated from 162 treatment combinations. This is a ⅓ fraction.

Experimental Plan

$S_1 S'_1 \quad S_2 S'_2$

Analysis

Completely Orthogonal

Construction

Sets of Treatment Combinations from the 2^1

Set S_1	S_2
$x_1=0$	1

Treatment Combinations

S_1	S_2
0	1

Sets of Treatment Combinations from the 3^5

Set	S'_1	S'_2
$z_1+z_2+z_3+z_4+z_5=$	0	1

Treatment Combinations

S'_1			S'_2		
00000	00111	00222	10000	10111	10222
11100	11211	11022	21100	21211	21022
22200	22011	22122	02200	02011	02122
20010	20121	20202	00010	00121	00202
01110	01221	01002	11110	11221	11002
12210	12021	12102	22210	22021	22102
10020	10101	10212	20020	20101	20212
21120	21201	21012	01120	01201	01012
02220	02001	02112	12220	12001	12112
01200	01011	01122	11200	11011	11122
12000	12111	12222	22000	22111	22222
20100	20211	20022	00100	00211	00022
21210	21021	21102	01210	01021	01102
02010	02121	02202	12010	12121	12202
10110	10221	10002	20110	20221	20002
11220	11001	11112	21220	21001	21112
22020	22101	22212	02020	02101	02212
00120	00201	00012	10120	10201	10012
02100	02211	02022	12100	12211	12022
10200	10011	10122	20200	20011	20122
21000	21111	21222	01000	01111	01222
22110	22221	22002	02110	02221	02002
00210	00021	00102	10210	10021	10102
11010	11121	11202	21010	21121	21202
12120	12201	12012	22120	22201	22012
20220	20001	20112	00220	00001	00112
01020	01101	01212	11020	11101	11212

Design $2^2 3^5$

There are two factors at 2 levels and five factors at 3 levels. 74 effects are estimated from 162 treatment combinations. This is a ⅙ fraction.

Experimental Plan

$S_1S'_1 \quad S_2S'_2 \quad S_2S'_3$

Analysis

The matrix $\frac{1}{3}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ is used to estimate $144\begin{bmatrix} \mu \\ A_1A_2 \end{bmatrix}$, $144\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, $96\begin{bmatrix} A_1B_1 \\ A_2B_1 \end{bmatrix}$, $96\begin{bmatrix} A_1B_2 \\ A_2B_2 \end{bmatrix}$, $96\begin{bmatrix} A_1B_3 \\ A_2B_3 \end{bmatrix}$, $96\begin{bmatrix} A_1B_4 \\ A_2B_4 \end{bmatrix}$, $96\begin{bmatrix} A_1B_5 \\ A_2B_5 \end{bmatrix}$, $288\begin{bmatrix} A_1B_1^2 \\ A_2B_1^2 \end{bmatrix}$, $288\begin{bmatrix} A_1B_2^2 \\ A_2B_2^2 \end{bmatrix}$, $288\begin{bmatrix} A_1B_3^2 \\ A_2B_3^2 \end{bmatrix}$, $288\begin{bmatrix} A_1B_4^2 \\ A_2B_4^2 \end{bmatrix}$, $288\begin{bmatrix} A_1B_5^2 \\ A_2B_5^2 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^2

Set	S_1	S_2
$x_1+x_2=0$		1

Treatment Combinations

S_1	S_2
00	10
11	01

Sets of Treatment Combinations from the 3^5

Set	S'_1	S'_2	S'_3
$z_1+z_2+z_3+z_4+z_5=0$		0	0
$z_1+z_2+2z_3=0$		1	2

Treatment Combinations

S'_1		S'_2		S'_3	
00000	22101	02100	21201	01200	20001
21000	20211	20100	22011	22200	21111
12000	11211	11100	10011	10200	12111
10110	02211	12210	01011	11010	00111
01110	00012	00210	02112	02010	01212
22110	21012	21210	20112	20010	22212
20220	12012	22020	11112	21120	10212
11220	10122	10020	12222	12120	11022
02220	01122	01020	00222	00120	02022
00021	22122	02121	21222	01221	20022
21021	20202	20121	22002	22221	21102
12021	11202	11121	10002	10221	12102
10101	02202	12201	01002	11001	00102
01101		00201		02001	

Design $2^4 3^5$

There are four factors at 2 levels and five factors at 3 levels. 101 effects are estimated from 324 treatment combinations. This is a 1/12 fraction.

Experimental Plan

S_1S_1' S_2S_2' S_3S_3'

Analysis

The matrix $\frac{1}{432}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_3 \\ A_4 \\ A_1A_2 \end{bmatrix}$, $\begin{bmatrix} A_1 \\ A_2A_3 \\ A_2A_4 \end{bmatrix}$, $\begin{bmatrix} A_2 \\ A_1A_3 \\ A_1A_4 \end{bmatrix}$, and

the matrix $\frac{1}{3}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ is used to estimate $288\begin{bmatrix} \mu \\ A_3A_4 \end{bmatrix}$, $192\begin{bmatrix} A_3B_1 \\ A_4B_1 \end{bmatrix}$, $192\begin{bmatrix} A_3B_2 \\ A_4B_2 \end{bmatrix}$, $192\begin{bmatrix} A_3B_3 \\ A_4B_3 \end{bmatrix}$,

$192\begin{bmatrix} A_3B_4 \\ A_4B_4 \end{bmatrix}$, $192\begin{bmatrix} A_3B_5 \\ A_4B_5 \end{bmatrix}$, $576\begin{bmatrix} A_3B_1^2 \\ A_4B_1^2 \end{bmatrix}$, $576\begin{bmatrix} A_3B_2^2 \\ A_4B_2^2 \end{bmatrix}$, $576\begin{bmatrix} A_3B_3^2 \\ A_4B_3^2 \end{bmatrix}$, $576\begin{bmatrix} A_3B_4^2 \\ A_4B_4^2 \end{bmatrix}$, $576\begin{bmatrix} A_3B_5^2 \\ A_4B_5^2 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^4

Set	S_1	S_2	S_3
$x_1+x_2+x_3=0$		1	0
$x_1+x_2+x_4=0$		0	1

Sets of Treatment Combinations from the 3^5

Set	S_1'	S_2'	S_3'
$z_1+z_2+z_3+z_4+z_5=0$		0	0
$z_1+z_2+2z_3=0$		1	2

Treatment Combinations

S_1	S_2	S_3
0000	0010	0001
1100	1110	1101
1011	1001	1010
0111	0101	0110

Treatment Combinations

S_1'	S_2'	S_3'
00000	02100	01200
21000	20100	22200
12000	11100	10200
10110	12210	11010
01110	00210	02010
22110	21210	20010
20220	22020	21120
11220	10020	12120
02220	01020	00120
00021	02121	01221
21021	20121	22221
12021	11121	10221
10101	12201	11001
01101	00201	02001
22101	12012	20001
20211	20211	21111
11211	10011	12111
02211	01011	00111
00012	02112	01212
21012	20112	22212
12012	11112	10212
10122	12222	11022
01122	00222	02022
22122	21222	20022
20202	22002	21102
11202	10002	12102
02202	01002	00102

Design $2^1 3^6$

There is one factor at 2 levels and there are six factors at 3 levels. 86 effects are estimated from 243 treatment combinations. This is a 1/6 fraction.

Experimental Plan

S_1S_1' S_2S_2' S_2S_3'

Analysis

The matrix $\frac{1}{3}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ is used to estimate $216\begin{bmatrix} \mu \\ A_1 \end{bmatrix}$, $144\begin{bmatrix} B_1 \\ A_1B_1 \end{bmatrix}$, $144\begin{bmatrix} B_2 \\ A_1B_2 \end{bmatrix}$, $144\begin{bmatrix} B_3 \\ A_1B_3 \end{bmatrix}$, $144\begin{bmatrix} B_4 \\ A_1B_4 \end{bmatrix}$, $144\begin{bmatrix} B_5 \\ A_1B_5 \end{bmatrix}$, $144\begin{bmatrix} B_6 \\ A_1B_6 \end{bmatrix}$, $432\begin{bmatrix} B_1^2 \\ A_1B_1^2 \end{bmatrix}$, $432\begin{bmatrix} B_2^2 \\ A_1B_2^2 \end{bmatrix}$, $432\begin{bmatrix} B_3^2 \\ A_1B_3^2 \end{bmatrix}$, $432\begin{bmatrix} B_4^2 \\ A_1B_4^2 \end{bmatrix}$, $432\begin{bmatrix} B_5^2 \\ A_1B_5^2 \end{bmatrix}$, $432\begin{bmatrix} B_6^2 \\ A_1B_6^2 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^1

Set	S_1	S_2
$x_1=0$	1	

Treatment Combinations

S_1	S_2
0	1

Sets of Treatment Combinations from the 3^6

Set	S_1'	S_2'	S_3'
$z_1+z_3+z_4+z_5+2z_6=0$		0	0
$z_2+2z_3+z_5+2z_6=0$		1	2

Treatment Combinations

S_1'			S_2'			S_3'		
000000	000011	000022	010000	010011	010022	020000	020011	020022
110020	110001	110012	120020	120001	120012	100020	100001	100012
220010	220021	220002	200010	200021	200002	210010	210021	210002
101010	101021	101002	111010	111021	111002	121010	121021	121002
211000	211011	211022	221000	221011	221022	201000	201011	201022
021020	021001	021012	001020	001001	001012	011020	011001	011012
202020	202001	202012	212020	212001	212012	222020	222001	222012
012010	012021	012002	022010	022021	022002	002010	002021	002002
122000	122011	122022	102000	102011	102022	112000	112011	112022
200100	200111	200122	210100	210111	210122	220100	220111	220122
010120	010101	010112	020120	020101	020112	000120	000101	000112
120110	120121	120102	100110	100121	100102	110110	110121	110102
001110	001121	001102	011110	011121	011102	021110	021121	021102
111100	111111	111122	121100	121111	121122	101100	101111	101122
221120	221101	221112	201120	201101	201112	211120	211101	211112
102120	102101	102112	112120	112101	112112	122120	122101	122112
212110	212121	212102	222110	222121	222102	202110	202121	202102
022100	022111	022122	002100	002111	002122	012100	012111	012122
100200	100211	100222	110200	110211	110222	120200	120211	120222
210220	210201	210212	220220	220201	220212	200220	200201	200212
020210	020221	020202	000210	000221	000202	010210	010221	010202
201210	201221	201202	211210	211221	211202	221210	221221	221202
011200	011211	011222	021200	021211	021222	001200	001211	001222
121220	121201	121212	101220	101201	101212	111220	111201	111212
002220	002201	002212	012220	012201	012212	022220	022201	022212
112210	112221	112202	122210	122221	122202	102210	102221	102202
222200	222211	222222	202200	202211	202222	212200	212211	212222

Design $2^2 3^6$

There are two factors at 2 levels and six factors at 3 levels. 100 effects are estimated from 486 treatment combinations. This is a 1/6 fraction.

Experimental Plan

$S_1S_1' \quad S_2S_2' \quad S_1S_3'$

Analysis

The matrix $\frac{1}{3}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ is used to estimate $432\begin{bmatrix} \mu \\ A_1A_2 \end{bmatrix}$, $432\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, $288\begin{bmatrix} A_1B_1 \\ A_2B_1 \end{bmatrix}$,

$288\begin{bmatrix} A_1B_2 \\ A_2B_2 \end{bmatrix}$, $288\begin{bmatrix} A_1B_3 \\ A_2B_3 \end{bmatrix}$, $288\begin{bmatrix} A_1B_4 \\ A_2B_4 \end{bmatrix}$, $288\begin{bmatrix} A_1B_5 \\ A_2B_5 \end{bmatrix}$, $288\begin{bmatrix} A_1B_6 \\ A_2B_6 \end{bmatrix}$, $864\begin{bmatrix} A_1B_1^2 \\ A_2B_1^2 \end{bmatrix}$, $864\begin{bmatrix} A_1B_2^2 \\ A_2B_2^2 \end{bmatrix}$,

$864\begin{bmatrix} A_1B_3^2 \\ A_2B_3^2 \end{bmatrix}$, $864\begin{bmatrix} A_1B_4^2 \\ A_2B_4^2 \end{bmatrix}$, $864\begin{bmatrix} A_1B_5^2 \\ A_2B_5^2 \end{bmatrix}$, $864\begin{bmatrix} A_1B_6^2 \\ A_2B_6^2 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^2

Set	S_1	S_2
$x_1+x_2=$	0	1

Treatment Combinations

S_1	S_2
00	01
11	10

Sets of Treatment Combinations from the 3^6

Set	S_1'	S_2'	S_3'
$z_1+z_3+z_4+z_5+2z_6=$	0	0	0
$z_2+2z_3+z_5+2z_6=$	0	1	2

Treatment Combinations

S_1'

000000	022100	202001	011211	120102
110020	100200	012021	121201	001102
220010	210220	122011	002201	111122
101010	020210	200111	112221	221112
211000	201210	010101	222211	102112
021020	011200	120121	000022	212102
202020	121220	001121	110012	022122
012010	002220	111111	220002	100222
122000	112210	221101	101002	210212
200100	222200	102101	211022	020202
010120	000011	212121	021012	201202
120110	110001	022111	202012	011222
001110	220021	100211	012002	121212
111100	101021	210201	122022	002212
221120	211011	020221	200122	112202
102120	021001	201221	010112	222222
212110				

S_2'

010000	002100	212001	021211	100102
120020	110200	022021	101201	011102
200010	220220	102011	012201	121122
111010	000210	210111	122221	201112
221000	211210	020101	202211	112112
001020	021200	100121	010022	222102
212020	101220	011121	120012	002122
022010	012220	121111	200002	110222
102000	122210	201101	111002	220212
210100	202200	112101	221022	000202
020120	010011	222121	001012	211202
100110	120001	002111	212012	021222
011110	200021	110211	022002	101212
121100	111021	220201	102022	012212
201120	221011	000221	210122	122202
112120	001001	211221	020112	202222
222110				

S_3'

020000	012100	222001	001211	110102
100020	120200	002021	111201	021102
210010	200220	112011	022201	101122
121010	010210	220111	102221	211112
201000	221210	000101	212211	122112
011020	001200	110121	020022	202102
222020	111220	021121	100012	012122
002010	022220	101111	210002	120222
112000	102210	211101	121002	200212
220100	212200	122101	201022	010202
000120	020011	202121	011012	221202
110110	100001	012111	222012	011222
021110	210021	120211	002002	111212
101100	121021	200201	112022	022212
211120	201011	010221	220122	102202
122120	011001	221221	000112	212222
202110				

Design $2^3 3^6$

There are three factors at 2 levels and six factors at 3 levels. 115 effects are estimated from 486 treatment combinations. This is a $\frac{1}{12}$ fraction.

Experimental Plan

$S_1S_1' \quad S_2S_2' \quad S_3S_3' \quad S_4S_4' \quad S_1S_5' \quad S_2S_6' \quad S_3S_7' \quad S_4S_8' \quad S_1S_9'$

Analysis

$$\text{The matrix } \frac{1}{5184}\begin{bmatrix} 11 & -1 & -1 & -1 \\ -1 & 11 & -1 & -1 \\ -1 & -1 & 11 & -1 \\ -1 & -1 & -1 & 11 \end{bmatrix} \text{ is used to estimate } \begin{bmatrix} \mu \\ A_1A_2 \\ A_1A_3 \\ A_2A_3 \end{bmatrix}$$

$$\text{and the matrix } \frac{1}{88}\begin{bmatrix} 10 & -1 & -1 \\ -1 & 10 & -1 \\ -1 & -1 & 10 \end{bmatrix} \text{ is used to estimate } 54\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}, \quad 36\begin{bmatrix} A_1B_1 \\ A_2B_1 \\ A_3B_1 \end{bmatrix}, \quad 36\begin{bmatrix} A_1B_2 \\ A_2B_2 \\ A_3B_2 \end{bmatrix},$$

$$36\begin{bmatrix} A_1B_3 \\ A_2B_3 \\ A_3B_3 \end{bmatrix}, \quad 36\begin{bmatrix} A_1B_4 \\ A_2B_4 \\ A_3B_4 \end{bmatrix}, \quad 36\begin{bmatrix} A_1B_5 \\ A_2B_5 \\ A_3B_5 \end{bmatrix}, \quad 36\begin{bmatrix} A_1B_6 \\ A_2B_5 \\ A_3B_6 \end{bmatrix}, \quad 108\begin{bmatrix} A_1B_1^2 \\ A_2B_1^2 \\ A_3B_1^2 \end{bmatrix}, \quad 108\begin{bmatrix} A_1B_2^2 \\ A_2B_2^2 \\ A_3B_2^2 \end{bmatrix}, \quad 108\begin{bmatrix} A_1B_3^2 \\ A_2B_3^2 \\ A_3B_3^2 \end{bmatrix},$$

$$108\begin{bmatrix} A_1B_4^2 \\ A_2B_4^2 \\ A_3B_4^2 \end{bmatrix}, \quad 108\begin{bmatrix} A_1B_5^2 \\ A_2B_5^2 \\ A_3B_5^2 \end{bmatrix}, \quad 108\begin{bmatrix} A_1B_6^2 \\ A_2B_6^2 \\ A_3B_6^2 \end{bmatrix}.$$

Sets of Treatment Combinations from the 2^3

Set	S_1	S_2	S_3	S_4
$x_1+x_2=0$	0	1	1	
$x_1+x_3=0$	1	0	1	

Treatment Combinations

S_1	S_2	S_3	S_4
000	001	010	100
111	110	101	011

Sets of Treatment Combinations from the 3^6

Set	S_1'	S_2'	S_3'	S_4'	S_5'	S_6'	S_7'	S_8'	S_9'
$z_1+z_3+z_4+z_5+2z_6=0$	0	0	0	0	0	0	0	0	
$z_2+2z_3+z_5+2z_6=0$	0	0	1	1	1	2	2	2	
$z_1+z_2+z_3+z_5=0$	1	2	0	1	2	0	1	2	

Treatment Combinations

S_1'	S_2'	S_3'	S_4'	S_5'	S_6'	S_7'	S_8'	S_9'
000000	211000	122000	200010	111010	022010	100020	011020	222020
010120	221120	102120	210100	121100	002100	110110	021110	202110
020210	201210	112210	220220	101220	012220	120200	001200	212200
101010	012010	220010	001020	212020	120020	201000	112000	020000
111100	022100	200100	011110	222110	100110	211120	122120	000120
121220	002220	210220	021200	202200	110200	221210	102210	010210
202020	110020	021020	102000	010000	221000	002010	210010	121010
212110	120110	001110	112120	020120	201120	012100	220100	101100
222200	100200	011200	122210	000210	211210	022220	200220	111220
200111	111111	022111	100121	011121	222121	000101	211101	122101
210201	121201	002201	110211	021211	202211	010221	221221	102221
220021	101021	012021	120001	001001	212001	020011	201011	112011
001121	212121	120121	201101	112101	020101	101111	012111	220111
011211	222211	100211	211221	122221	000221	111201	022201	200201
021001	202001	110001	221011	102011	010011	121021	002021	210021
102101	010101	221101	002111	210111	121111	202121	110121	021121
112221	020221	201221	012201	220201	101201	212211	120211	001211
122011	000011	211011	022021	200021	111021	222001	100001	011001
100222	011222	222222	000202	211202	122202	200212	111212	022212
110012	021012	202012	010022	221022	102022	210002	121002	002002
120102	001102	212102	020112	201112	112112	220122	101122	012122
201202	112202	020202	101212	012212	220212	001222	212222	120222
211022	122022	000022	111002	022002	200002	011012	222012	100012
221112	102112	010112	121122	002122	210122	021102	202102	110102
002212	210212	121212	202222	110222	021222	102202	010202	221202
012002	220002	101002	212012	120012	001012	112022	020022	201022
022122	200122	111122	222102	100102	011102	122112	000112	211112

Design $2^4 3^6$

There are four factors at 2 levels and six factors at 3 levels. 131 effects are estimated from 486 treatment combinations. This is a $\frac{1}{24}$ fraction.

Experimental Plan

S_1S_1' S_2S_2' S_3S_3' S_4S_4' S_5S_5' S_6S_6' S_7S_7' S_8S_8' S_8S_9'

Analysis

$$\text{The matrix } \frac{1}{6480}\begin{bmatrix} 14 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 14 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 14 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 14 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 14 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 14 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 14 \end{bmatrix} \text{ is used to estimate } \begin{bmatrix} \mu \\ A_1A_2 \\ A_1A_3 \\ A_1A_4 \\ A_2A_3 \\ A_2A_4 \\ A_3A_4 \end{bmatrix}$$

$$\text{and the matrix } \frac{1}{96}\begin{bmatrix} 11 & 1 & 1 & -1 \\ 1 & 11 & -1 & 1 \\ 1 & -1 & 11 & 1 \\ -1 & 1 & 1 & 11 \end{bmatrix} \text{ is used to estimate } 54\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}, \quad 36\begin{bmatrix} A_1B_1 \\ A_2B_1 \\ A_3B_1 \\ A_4B_1 \end{bmatrix}, \quad 36\begin{bmatrix} A_1B_2 \\ A_2B_2 \\ A_3B_2 \\ A_4B_2 \end{bmatrix},$$

$$36\begin{bmatrix} A_1B_3 \\ A_2B_3 \\ A_3B_3 \\ A_4B_3 \end{bmatrix}, \quad 36\begin{bmatrix} A_1B_4 \\ A_2B_4 \\ A_3B_4 \\ A_4B_4 \end{bmatrix}, \quad 36\begin{bmatrix} A_1B_5 \\ A_2B_5 \\ A_3B_5 \\ A_4B_5 \end{bmatrix}, \quad 36\begin{bmatrix} A_1B_6 \\ A_2B_6 \\ A_3B_6 \\ A_4B_6 \end{bmatrix}, \quad 108\begin{bmatrix} A_1B_1^2 \\ A_2B_1^2 \\ A_3B_1^2 \\ A_4B_1^2 \end{bmatrix}, \quad 108\begin{bmatrix} A_1B_2^2 \\ A_2B_2^2 \\ A_3B_2^2 \\ A_4B_2^2 \end{bmatrix}, \quad 108\begin{bmatrix} A_1B_3^2 \\ A_2B_3^2 \\ A_3B_3^2 \\ A_4B_3^2 \end{bmatrix},$$

$$108\begin{bmatrix} A_1B_4^2 \\ A_2B_4^2 \\ A_3B_4^2 \\ A_4B_4^2 \end{bmatrix}, \quad 108\begin{bmatrix} A_1B_5^2 \\ A_2B_5^2 \\ A_3B_5^2 \\ A_4B_5^2 \end{bmatrix}, \quad 108\begin{bmatrix} A_1B_6^2 \\ A_2B_6^2 \\ A_3B_6^2 \\ A_4B_6^2 \end{bmatrix}.$$

Sets of Treatment Combinations from the 2^4

Set	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
$x_1+x_2=0$	0	1	1	0	0	1	1	
$x_1+x_3=0$	1	0	1	0	1	0	1	
$x_1+x_4=1$	1	1	1	0	0	0	0	

Treatment Combinations

S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
0001	0011	0101	1000	0000	0010	0100	1001
1110	1100	1010	0111	1111	1101	1011	0110

Sets of Treatment Combinations from the 3^6

Set	S_1'	S_2'	S_3'	S_4'	S_5'	S_6'	S_7'	S_8'	S_9'
$z_1+z_3+z_4+z_5+2z_6=0$	0	0	0	0	0	0	0	0	
$z_2+2z_3+z_5+2z_6=0$	0	0	1	1	1	2	2	2	
$z_1+z_2+z_3+z_5=0$	1	2	0	1	2	0	1	2	

Treatment Combinations

S_1'	S_2'	S_3'	S_4'	S_5'	S_6'	S_7'	S_8'	S_9'
000000	211000	122000	200010	111010	022010	100020	011020	222020
010120	221120	102120	210100	121100	002100	110110	021110	202110
020210	201210	112210	220220	101220	012220	120200	001200	212200
101010	012010	220010	001020	212020	120020	201000	112000	020000
111100	022100	200100	011110	222110	100110	211120	122120	000120
121220	002220	210220	021200	202200	110200	221210	102210	010210
202020	110020	021020	102000	010000	221000	002010	210010	121010
212110	120110	001110	112120	020120	201120	012100	220100	101100
222200	100200	011200	122210	000210	211210	022220	200220	111220
200111	111111	022111	100121	011121	222121	000101	211101	122101
210201	121201	002201	110211	021211	202211	010221	221221	102221
220021	101021	012021	120001	001001	212001	020011	201011	112011
001121	212121	120121	201101	112101	020101	101111	012111	220111
011211	222211	100211	211221	122221	000221	111201	022201	200201
021001	202001	110001	221011	102011	010011	121021	002021	210021
102101	010101	221101	002111	210111	121111	202121	110121	021121
112221	020221	201221	012201	220201	101201	212211	120211	001211
122011	000011	211011	022021	200021	111021	222001	100001	011001
100222	011222	222222	000202	211202	122202	200212	111212	022212
110012	021012	202012	010022	221022	102022	210002	121002	002002
120102	001102	212102	020112	201112	112112	220122	101122	012122
201202	112202	020202	101212	012212	220212	001222	212222	120222
211022	122022	000022	111002	022002	220002	011012	222012	100012
221112	102112	010112	121122	002122	210122	021102	202102	110102
002212	210212	121212	202222	110222	021222	102202	010202	221202
012002	220002	101002	212012	120012	001012	112022	020022	201022
022122	200122	111122	222102	100102	011102	122112	000112	211112

Design $2^1 3^7$

There is one factor at 2 levels and there are seven factors at 3 levels. 114 effects are estimated from 243 treatment combinations. This is a 1/18 fraction.

Experimental Plan

$S_1 S_1' \quad S_2 S_2' \quad S_2 S_3'$

Analysis

The matrix $\frac{1}{3}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ is used to estimate $216\begin{bmatrix} \mu \\ A_1 \end{bmatrix}$, $144\begin{bmatrix} B_1 \\ A_1B_1 \end{bmatrix}$, $144\begin{bmatrix} B_2 \\ A_1B_2 \end{bmatrix}$,

$144\begin{bmatrix} B_3 \\ A_1B_3 \end{bmatrix}$, $144\begin{bmatrix} B_4 \\ A_1B_4 \end{bmatrix}$, $144\begin{bmatrix} B_5 \\ A_1B_5 \end{bmatrix}$, $144\begin{bmatrix} B_6 \\ A_1B_6 \end{bmatrix}$, $144\begin{bmatrix} B_7 \\ A_1B_7 \end{bmatrix}$, $432\begin{bmatrix} B_1^2 \\ A_1B_1^2 \end{bmatrix}$, $432\begin{bmatrix} B_2^2 \\ A_1B_2^2 \end{bmatrix}$,

$432\begin{bmatrix} B_3^2 \\ A_1B_3^2 \end{bmatrix}$, $432\begin{bmatrix} B_4^2 \\ A_1B_4^2 \end{bmatrix}$, $432\begin{bmatrix} B_5^2 \\ A_1B_5^2 \end{bmatrix}$, $432\begin{bmatrix} B_6^2 \\ A_1B_6^2 \end{bmatrix}$, $432\begin{bmatrix} B_7^2 \\ A_1B_7^2 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^1

Set	S_1	S_2
$x_1=$	0	1

Treatment Combinations

S_1	S_2
0	1

Sets of Treatment Combinations from the 3^7

Set	S_1'	S_2'	S_3'
$z_1+z_3+z_4+z_5+2z_6$	$=0$	0	0
$z_2+2z_3+z_5+2z_6+z_7$	$=1$	1	1
$z_1+z_2+z_3+z_5+2z_7$	$=0$	1	2

Treatment Combinations

S_1'

2000100	2120120	1111111	0102102
0010200	0201120	2121211	1112202
1020000	1211220	0202211	2122002
2101000	2221020	1212011	2100212
0111100	0002020	2222111	0110012
1121200	1012120	2200021	1120112
2202200	2022220	0210121	2201112
0212000	1100201	1220221	0211212
1222100	2110001	2001221	1221012
1200010	0120101	0011021	2002012
2210110	1201101	1021121	0012112
0220210	2211201	2102121	1022212
1001210	0221001	0112221	1000122
2011010	1002001	1122021	2010222
0021110	2012101	0200002	0020022
1102110	0022201	1210102	1101022
2112210	0000111	2220202	2111122
0122010	1010211	0001202	0121222
0100220	2020011	1011002	1202222
1110020	0101011	2021102	2212022
			0222122

S_2'

2000210	2120200	1111221	0102212
0010010	0201200	2121021	1112012
1020110	1211000	0202021	2122112
2101110	2221100	1212121	2100022
0111210	0002100	2222221	0110122
1121010	1012200	2200101	1120222
2202010	2022000	0210201	2201222
0212110	1100011	1220001	0211022
1222210	2110111	2001001	1221122
1200120	0120211	0011101	2002122
2210220	1201211	1021201	0012222
0220020	2211011	2102201	1022022
1001020	0221111	0112001	1000202
2011120	1002111	1122101	2010002
0021220	2012211	0200112	0020102
1102220	0022011	1210212	1101102
2112020	0000221	2220012	2111202
0122120	1010021	0001212	0121002
0100000	2020121	1011112	1202002
1110100	0101121	2021212	2212102
			0222202

S_3'

2000020	2120010	1111001	0102022
0010120	0201010	2121101	1112122
1020220	1211110	0202101	2122222
2101220	2221210	1212201	2100102
0111020	0002210	2222001	0110202
1121120	1012010	2200211	1120002
2202120	2022110	0210011	2201002
0212220	1100121	1220111	0211102
1222020	2110221	2001111	1221202
1200200	0120021	0011211	2002202
2210000	1201021	1021011	0012002
0220100	2211121	2102011	1022102
1001100	0221221	0112111	1000012
2011200	1002221	1122211	2010112
0021000	2012021	0200222	0020212
1102000	0022121	1210022	1101212
2112100	0000001	2220122	2111012
0122200	1010101	0001122	0121112
0100110	2020201	1011222	1202112
1110210	0101201	2021022	2212212
			0222012

Design $2^2 3^7$

There are two factors at 2 levels and seven factors at 3 levels. 130 effects are estimated from 486 treatment combinations. This is a 1/18 fraction.

Experimental Plan

$S_1S_1' \quad S_2S_2' \quad S_1S_3'$

Analysis

The matrix $\frac{1}{3}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ is used to estimate $432\begin{bmatrix} \mu \\ A_1A_2 \end{bmatrix}$, $432\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, $288\begin{bmatrix} A_1B_1 \\ A_2B_1 \end{bmatrix}$,

$288\begin{bmatrix} A_1B_2 \\ A_2B_2 \end{bmatrix}$, $288\begin{bmatrix} A_1B_3 \\ A_2B_3 \end{bmatrix}$, $288\begin{bmatrix} A_1B_4 \\ A_2B_4 \end{bmatrix}$, $288\begin{bmatrix} A_1B_5 \\ A_2B_5 \end{bmatrix}$, $288\begin{bmatrix} A_1B_6 \\ A_2B_6 \end{bmatrix}$, $288\begin{bmatrix} A_1B_7 \\ A_2B_7 \end{bmatrix}$, $864\begin{bmatrix} A_1B_1^2 \\ A_2B_1^2 \end{bmatrix}$,

$864\begin{bmatrix} A_1B_2^2 \\ A_2B_2^2 \end{bmatrix}$, $864\begin{bmatrix} A_1B_3^2 \\ A_2B_3^2 \end{bmatrix}$, $864\begin{bmatrix} A_1B_4^2 \\ A_2B_4^2 \end{bmatrix}$, $864\begin{bmatrix} A_1B_5^2 \\ A_2B_5^2 \end{bmatrix}$, $864\begin{bmatrix} A_1B_6^2 \\ A_2B_6^2 \end{bmatrix}$, $864\begin{bmatrix} A_1B_7^2 \\ A_2B_7^2 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^2

Set	S_1	S_2
$x_1+x_2=$	0	1

Treatment Combinations

S_1	S_2
00	01
11	10

Sets of Treatment Combinations from the 3^7

Set	S_1'	S_2'	S_3'
$z_1+z_3+z_4+z_5+2z_6=$	0	0	0
$z_2+2z_3+z_5+2z_6+z_7=$	1	1	1
$z_1+z_2+z_3+z_5+2z_7=$	0	1	2

Treatment Combinations

S_1

2000100	2112210	0221001	2001221	0110012
0010200	0122010	1002001	0011021	1120112
1020000	0100220	2012101	1021121	2201112
2101000	1110020	0022201	2102121	0211212
0111100	2120120	0000111	0112221	1221012
1121200	0201120	1010211	1122021	2002012
2202200	1211220	2020011	0200002	0012112
0212000	2221020	0101011	1210102	1022212
1222100	0002020	1111111	2220202	1000122
1200010	1012120	2121211	0001202	2010222
2210110	2022220	0202211	1011002	0020022
0220210	1100201	1212011	2021102	1101022
1001210	2110001	2222111	0102102	2111122
2011010	0120101	2200021	1112202	0121222
0021110	1201101	0210121	2122002	1202222
1102110	2211201	1220221	2100212	2212022
				0222122

S_2'

2000210	2112020	0221111	2001001	0110122
0010010	0122120	1002111	0011101	1120222
1020110	0100000	2012211	1021201	2201222
2101110	1110100	0022011	2102201	0211022
0111210	2120200	0000221	0112001	1221122
1121010	0201200	1010021	1122101	2002122
2202010	1211000	2020121	0200112	0012222
0212110	2221100	0101121	1210212	1022022
1222210	0002100	1111221	2220012	1000202
1200120	1012200	2121021	0001012	2010002
2210220	2022000	0202021	1011112	0020102
0220020	1100011	1212121	2021212	1101102
1001020	2110111	2222221	0102212	2111202
2011120	0120211	2200101	1112012	0121002
0021220	1201211	0210201	2122112	1202002
1102220	2211011	1220001	2100022	2212102
				0222202

S_3'

2000020	2112100	0221221	2001111	0110202
0010120	0122200	1002221	0011211	1120002
1020220	0100110	2012021	1021011	2201002
2101220	1110210	0022121	2102011	0211102
0111020	2120010	0000001	0112111	1221202
1121120	0201010	1010101	1122211	2002202
2202120	1211110	2020201	0200222	0012002
0212220	2221210	0101201	1210022	1022102
1222020	0002210	1111001	2220122	1000012
1200200	1012010	2121101	0001122	2010112
2210000	2022110	0202101	1011222	0020212
0220100	1100121	1212201	2021022	1101212
1001100	2110221	2222001	0102022	2111012
2011200	0120021	2200211	1112122	0121112
0021000	1201021	0210011	2122222	1202112
1102000	2211121	1220111	2100102	2212212
				0222012

Design $2^3 3^7$

There are three factors at 2 levels and seven factors at 3 levels. 147 effects are estimated from 486 treatment combinations. This is a $1/36$ fraction.

Experimental Plan

S_1S_1' S_2S_2' S_3S_3' S_4S_4' S_1S_5' S_2S_6' S_3S_7' S_4S_8' S_4S_9'

Analysis

The matrix $\frac{1}{5184}\begin{bmatrix} 11 & 1 & 1 & -1 \\ 1 & 11 & -1 & 1 \\ 1 & -1 & 11 & 1 \\ -1 & 1 & 1 & 11 \end{bmatrix}$ is used to estimate $\begin{bmatrix} \mu \\ A_1A_2 \\ A_1A_3 \\ A_2A_3 \end{bmatrix}$

and the matrix $\frac{1}{88}\begin{bmatrix} 10 & 1 & 1 \\ 1 & 10 & -1 \\ 1 & -1 & 10 \end{bmatrix}$ is used to estimate 54 $\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$, 36 $\begin{bmatrix} A_1B_1 \\ A_2B_1 \\ A_3B_1 \end{bmatrix}$, 36 $\begin{bmatrix} A_1B_2 \\ A_2B_2 \\ A_3B_2 \end{bmatrix}$, 36 $\begin{bmatrix} A_1B_3 \\ A_2B_3 \\ A_3B_3 \end{bmatrix}$,

36 $\begin{bmatrix} A_1B_4 \\ A_2B_4 \\ A_3B_4 \end{bmatrix}$, 36 $\begin{bmatrix} A_1B_5 \\ A_2B_5 \\ A_3B_5 \end{bmatrix}$, 36 $\begin{bmatrix} A_1B_6 \\ A_2B_6 \\ A_3B_6 \end{bmatrix}$, 36 $\begin{bmatrix} A_1B_7 \\ A_2B_7 \\ A_3B_7 \end{bmatrix}$, 108 $\begin{bmatrix} A_1B_1^2 \\ A_2B_1^2 \\ A_3B_1^2 \end{bmatrix}$, 108 $\begin{bmatrix} A_1B_2^2 \\ A_2B_2^2 \\ A_3B_2^2 \end{bmatrix}$, 108 $\begin{bmatrix} A_1B_3^2 \\ A_2B_3^2 \\ A_3B_3^2 \end{bmatrix}$,

108 $\begin{bmatrix} A_1B_4^2 \\ A_2B_4^2 \\ A_3B_4^2 \end{bmatrix}$, 108 $\begin{bmatrix} A_1B_5^2 \\ A_2B_5^2 \\ A_3B_5^2 \end{bmatrix}$, 108 $\begin{bmatrix} A_1B_6^2 \\ A_2B_6^2 \\ A_3B_6^2 \end{bmatrix}$, 108 $\begin{bmatrix} A_1B_7^2 \\ A_2B_7^2 \\ A_3B_7^2 \end{bmatrix}$.

Sets of Treatment Combinations from the 2^3

Set	S_1	S_2	S_3	S_4
$x_1+x_2=0$	0	1	1	0
$x_1+x_3=0$	1	0	1	0

Treatment Combinations

S_1	S_2	S_3	S_4
000	001	010	100
111	110	101	011

Sets of Treatment Combinations from the 3^7

Set	S'_1	S'_2	S'_3	S'_4	S'_5	S'_6	S'_7	S'_8	S'_9
$z_1+z_3+z_4+z_5+2z_6=0$	0	0	0	0	0	0	0	0	
$z_2+2z_3+z_5+2z_6+z_7=1$	1	1	1	1	1	1	1	1	
$z_1+z_2+z_3+z_5+2z_7=0$	0	0	1	1	1	2	2	2	
$z_1+2z_2+z_3+2z_4+2z_5+2z_6+2z_7=0$	1	2	0	1	2	0	1	2	

Treatment Combinations

S'_1	S'_2	S'_3	S'_4	S'_5	S'_6	S'_7	S'_8	S'_9
0000111	1010211	2020011	0120211	1100011	2110111	0210011	1220111	2200211
1111111	2121211	0101011	1201211	2211011	0221111	1021011	2001111	0011211
2222111	0202211	1212011	2012211	0022011	1002111	2102011	0112111	1122211
0011021	1021121	2001221	0101121	1111221	2121021	0221221	1201021	2211121
1122021	2102121	0112221	1212121	2222221	0202021	1002221	2012021	0022121
2200021	0210121	1220221	2020121	0000221	1010021	2110221	0120021	1100121
0022201	1002001	2012101	0112001	1122101	2102201	0202101	1212201	2222001
1100201	2110001	0120101	1220001	2200101	0210201	1010101	2020201	0000001
2211201	0221001	1201101	2001001	0011101	1021201	2121101	0101201	1111001
2002012	0012112	1022212	2122112	0102212	1112012	2212212	0222012	1202112
0110012	1120112	2100212	0200112	1210212	2220012	0020212	1000012	2010112
1221012	2201112	0211212	1011112	2021212	0001012	1101212	2111012	0121112
2010222	0020022	1000122	2100022	0110122	1120222	2220122	0200222	1210022
0121222	1101022	2111122	0211022	1221122	2201222	0001122	1011222	2021022
1202222	2212022	0222122	1022022	2002122	0012222	1112122	2122222	0102022
2021102	0001202	1011002	2111202	0121002	1101102	2201002	0211102	1221202
0102102	1112202	2122002	0222202	1202002	2212102	0012002	1022102	2002202
1210102	2220202	0200002	1000202	2010002	0020102	1120002	2100102	0110202
1001210	2011010	0021110	1121010	2101110	0111210	1211110	2221210	0201010
2112210	0122010	1102110	2202010	0212110	1222210	2022110	0002210	1012010
0220210	1200010	2210110	0010010	1020110	2000210	0100110	1110210	2120010
1012120	2022220	0002020	1102220	2112020	0122120	1222020	2202120	0212220
2120120	0100220	1110020	2210220	0220020	1200120	2000020	0010120	1020220
0201120	1211220	2221020	0021220	1001020	2011120	0111020	1121120	2101220
1020000	2000100	0010200	1110100	2120200	0100000	1200200	2210000	0220100
2101000	0111100	1121200	2221100	0201200	1211000	2011200	0021000	1001100
0212000	1222100	2202200	0002100	1012200	2022000	0122200	1102000	2112100

Design $2^1 3^8$

There is one factor at 2 levels and there are eight factors at 3 levels. 146 effects are estimated from 243 treatment combinations. This is a 1/64 fraction.

Experimental Plan

$S_1S_1' \quad S_2S_2' \quad S_2S_3'$

Analysis

The matrix $\frac{1}{3}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ is used to estimate $216\begin{bmatrix} \mu \\ A_1 \end{bmatrix}$, $144\begin{bmatrix} B_1 \\ A_1B_1 \end{bmatrix}$, $144\begin{bmatrix} B_2 \\ A_1B_2 \end{bmatrix}$,

$144\begin{bmatrix} B_3 \\ A_1B_3 \end{bmatrix}$, $144\begin{bmatrix} B_4 \\ A_1B_4 \end{bmatrix}$, $144\begin{bmatrix} B_5 \\ A_1B_5 \end{bmatrix}$, $144\begin{bmatrix} B_6 \\ A_1B_6 \end{bmatrix}$, $144\begin{bmatrix} B_7 \\ A_1B_7 \end{bmatrix}$, $144\begin{bmatrix} B_8 \\ A_1B_8 \end{bmatrix}$, $432\begin{bmatrix} B_1^2 \\ A_1B_1^2 \end{bmatrix}$,

$432\begin{bmatrix} B_2^2 \\ A_1B_2^2 \end{bmatrix}$, $432\begin{bmatrix} B_3^2 \\ A_1B_3^2 \end{bmatrix}$, $432\begin{bmatrix} B_4^2 \\ A_1B_4^2 \end{bmatrix}$, $432\begin{bmatrix} B_5^2 \\ A_1B_5^2 \end{bmatrix}$, $432\begin{bmatrix} B_6^2 \\ A_1B_6^2 \end{bmatrix}$, $432\begin{bmatrix} B_7^2 \\ A_1B_7^2 \end{bmatrix}$, $432\begin{bmatrix} B_8^2 \\ A_1B_8^2 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^1

Set	S_1	S_2
$x_1 =$	0	1

Treatment Combinations

S_1	S_2
0	1

Sets of Treatment Combinations from the 3^8

Set	S_1'	S_2'	S_3'
$z_2+z_3+z_4+z_5+z_6+z_7=$	0	0	0
$z_1+z_3+z_4+2z_5+2z_6+z_8=$	0	0	0
$z_1+2z_3+2z_5+z_6+z_7=$	0	0	0
$z_2+2z_3+2z_6+z_7=$	0	1	2

Treatment Combinations

S_1'

00000000	21021020	11102111	01210202
11110000	02010120	22212111	12020202
22220000	10120120	00201211	20100202
00212100	21200120	11011211	02021012
11022100	02222220	22121211	10101012
22102100	10002220	01012021	21211012
00121200	21112220	12122021	02200112
11201200	02211001	20202021	10010112
22011200	10021001	01221121	21120112
01202010	21101001	12001121	02112212
12012010	02120101	20111121	10222212
20122010	10200101	01100221	21002212
01111110	21010101	12210221	00220022
12221110	02002201	20020221	11000022
20001110	10112201	01122002	22110022
01020210	21222201	12202002	00102122
12100210	00110011	20012002	11212122
20210210	11220011	01001102	22022122
02101020	22000011	12111102	00011222
10211020	00022111	20221102	11121222
			22201222

S_2'

00000121	21021111	11102202	01210020
11110121	02010211	22212202	12020020
22220121	10120211	00201002	20100020
00212221	21200211	11011002	02021100
11022221	02222011	22121002	10101100
22102221	10002011	01012112	21211100
00121021	21112011	12122112	02200200
11201021	02211122	20202112	10010200
22011021	10021122	01221212	21120200
01202101	21101122	12001212	02112000
12012101	02120222	20111212	10222000
20122101	10200222	01100012	21002000
01111201	21010222	12210012	00220110
12221201	02002022	20020012	11000110
20001201	10112022	01122120	22110110
01020001	21222022	12202120	00102210
12100001	00110102	20012120	11212210
20210001	11220102	01001220	22022210
02101111	22000102	12111220	00011010
10211111	00022202	20221220	11121010
			22201010

S_3'

00000212	21021202	11102020	01210111
11110212	02010002	22212020	12020111
22220212	10120002	00201120	20100111
00212012	21200002	11011120	02021221
11022012	02222102	22121120	10101221
22102012	10002102	01012200	21211221
00121112	21112102	12122200	02200021
11201112	02211210	20202200	10010021
22011112	10021210	01221000	21120021
01202222	21101210	12001000	02112121
12012222	02120010	20111000	10222121
20122222	10200010	01100100	21002121
01111022	21010010	12210100	00220201
12221022	02002110	20020100	11000201
20001022	10112110	01122211	22110201
01020122	21222110	12202211	00102001
12100122	00110220	20012211	11212001
20210122	11220220	01001011	22022001
02101202	22000220	12111011	00011101
10211202	00022020	20221011	11121101
			22201101

Design $2^2 3^8$

There are two factors at 2 levels and eight factors at 3 levels. 164 effects are estimated from 486 treatment combinations. This is a $\frac{1}{64}$ fraction.

Experimental Plan

$S_1S_1' \quad S_2S_2' \quad S_1S_3'$

Analysis

The matrix $\frac{1}{3}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ is used to estimate $432\begin{bmatrix} \mu \\ A_1A_2 \end{bmatrix}$, $432\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, $288\begin{bmatrix} A_1B_1 \\ A_2B_1 \end{bmatrix}$, $288\begin{bmatrix} A_1B_2 \\ A_2B_2 \end{bmatrix}$, $288\begin{bmatrix} A_1B_3 \\ A_2B_3 \end{bmatrix}$, $288\begin{bmatrix} A_1B_4 \\ A_2B_4 \end{bmatrix}$, $288\begin{bmatrix} A_1B_5 \\ A_2B_5 \end{bmatrix}$, $288\begin{bmatrix} A_1B_6 \\ A_2B_6 \end{bmatrix}$, $288\begin{bmatrix} A_1B_7 \\ A_2B_7 \end{bmatrix}$, $288\begin{bmatrix} A_1B_8 \\ A_2B_8 \end{bmatrix}$, $864\begin{bmatrix} A_1B_1^2 \\ A_2B_1^2 \end{bmatrix}$, $864\begin{bmatrix} A_1B_2^2 \\ A_2B_2^2 \end{bmatrix}$, $864\begin{bmatrix} A_1B_3^2 \\ A_2B_3^2 \end{bmatrix}$, $864\begin{bmatrix} A_1B_4^2 \\ A_2B_4^2 \end{bmatrix}$, $864\begin{bmatrix} A_1B_5^2 \\ A_2B_5^2 \end{bmatrix}$, $864\begin{bmatrix} A_1B_6^2 \\ A_2B_6^2 \end{bmatrix}$, $864\begin{bmatrix} A_1B_7^2 \\ A_2B_7^2 \end{bmatrix}$, $864\begin{bmatrix} A_1B_8^2 \\ A_2B_8^2 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^2

Set	S_1	S_2
$x_1+x_2=$	0	1

Treatment Combinations

S_1	S_2
00	01
11	10

Sets of Treatment Combinations from the 3^8

Set	S_1'	S_2'	S_3'
$z_2+z_3+z_4+z_5+z_6+z_7=$	0	0	0
$z_1+z_3+z_4+2z_5+2z_6+z_8=$	0	0	0
$z_1+2z_3+2z_5+z_6+z_7=$	0	0	0
$z_2+2z_3+2z_6+z_7=$	0	1	2

Treatment Combinations

S_1'

00000000	21021020	11102111	01210202
11110000	02010120	22212111	12020202
22220000	10120120	00201211	20100202
00212100	21200120	11011211	02021012
11022100	02222220	22121211	10101012
22102100	10002220	01012021	21211012
00121200	21112220	12122021	02200112
11201200	02211001	20202021	10010112
22011200	10021001	01221121	21120112
01202010	21101001	12001121	02112212
12012010	02120101	20111121	10222212
20122010	10200101	01100221	21002212
01111110	21010101	12210221	00220022
12221110	02002201	20020221	11000022
20001110	10112201	01122002	22110022
01020210	21222201	12202002	00102122
12100210	00110011	20012002	11212122
20210210	11220011	01001102	22022122
02101020	22000011	12111102	00011222
10211020	00022111	20221102	11121222
			22201222

S_2'

00000121	21021111	11102202	01210020
11110121	02010211	22212202	12020020
22220121	10120211	00201002	20100020
00212221	21200211	11011002	02021100
11022221	02222011	22121002	10101100
22102221	10002011	01012112	21211100
00121021	21112011	12122112	02200200
11201021	02211122	20202112	10010200
22011021	10021122	01221212	21120200
01202101	21101122	12001212	02112000
12012101	02120222	20111212	10222000
20122101	10200222	01100012	21002000
01111201	21010222	12210012	00220110
12221201	02002022	20020012	11000110
20001201	10112022	01122120	22110110
01020001	21222022	12202120	00102210
12100001	00110102	20012120	11212210
20210001	11220102	01001220	22022210
02101111	22000102	12111220	00011010
10211111	00022202	20221220	11121010
			22201010

S_3'

00000212	21021202	11102020	01210111
11110212	02010002	22212020	12020111
22220212	10120002	00201120	20100111
00212012	21200002	11011120	02021221
11022012	02222102	22121120	10101221
22102012	10002102	01012200	21211221
00121112	21112102	12122200	02200021
11201112	02211210	20202200	10010021
22011112	10021210	01221000	21120021
01202222	21101210	12001000	02112121
12012222	02120010	20111000	10222121
20122222	10200010	01100100	21002121
01111022	21010010	12210100	00220201
12221022	02002110	20020100	11000201
20001022	10112110	01122211	22110201
01020122	21222110	12202211	00102001
12100122	00110220	20012211	11212001
20210122	11220220	01001011	22022001
02101202	22000220	12111011	00011101
10211202	00022020	20221011	11121101
			22201101

Design $2^1 3^9$

There is one factor at 2 levels and there are nine factors at 3 levels. 182 effects are estimated from 243 treatment combinations. This is a $1/162$ fraction.

Experimental Plan

S_1S_1' S_2S_2' S_2S_3'

Analysis

The matrix $\frac{1}{3}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ is used to estimate $216\begin{bmatrix} \mu \\ A_1 \end{bmatrix}$, $144\begin{bmatrix} B_1 \\ A_1B_1 \end{bmatrix}$, $144\begin{bmatrix} B_2 \\ A_1B_2 \end{bmatrix}$, $144\begin{bmatrix} B_3 \\ A_1B_3 \end{bmatrix}$, $144\begin{bmatrix} B_4 \\ A_1B_4 \end{bmatrix}$, $144\begin{bmatrix} B_5 \\ A_1B_5 \end{bmatrix}$, $144\begin{bmatrix} B_6 \\ A_1B_6 \end{bmatrix}$, $144\begin{bmatrix} B_7 \\ A_1B_7 \end{bmatrix}$, $144\begin{bmatrix} B_8 \\ A_1B_8 \end{bmatrix}$, $144\begin{bmatrix} B_9 \\ A_1B_9 \end{bmatrix}$, $432\begin{bmatrix} B_1^2 \\ A_1B_1^2 \end{bmatrix}$, $432\begin{bmatrix} B_2^2 \\ A_1B_2^2 \end{bmatrix}$, $432\begin{bmatrix} B_3^2 \\ A_1B_3^2 \end{bmatrix}$, $432\begin{bmatrix} B_4^2 \\ A_1B_4^2 \end{bmatrix}$, $432\begin{bmatrix} B_5^2 \\ A_1B_5^2 \end{bmatrix}$, $432\begin{bmatrix} B_6^2 \\ A_1B_6^2 \end{bmatrix}$, $432\begin{bmatrix} B_7^2 \\ A_1B_7^2 \end{bmatrix}$, $432\begin{bmatrix} B_8^2 \\ A_1B_8^2 \end{bmatrix}$, $432\begin{bmatrix} B_9^2 \\ A_1B_9^2 \end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^1

Set	S_1	S_2
$x_1=$	0	1

Treatment Combinations

S_1	S_2
0	1

Sets of Treatment Combinations from the 3^9

Set	S'_1	S'_2	S'_3
$z_2+z_3+z_4+z_5+z_6+z_7=$	0	0	0
$z_1+z_3+z_4+2z_5+2z_6+z_8=$	0	0	0
$z_1+z_2+2z_4+2z_5+z_6+z_9=$	0	0	0
$z_1+z_2+2z_3+z_5+2z_6=$	0	0	0
$z_1+2z_2+2z_3+z_4+z_6=$	0	1	2

Treatment Combinations

S'_1

000000000	202211020	101211111	000211202
121210000	001202120	222121111	121121202
212120000	122112120	021112211	212001202
011111100	210022120	112022211	021201012
102021100	012010220	200202211	112111012
220201100	100220220	012102021	200021012
022222200	221100220	100012021	002012112
110102200	022011001	221222021	120222112
201012200	110221001	020210121	211102112
010212010	201101001	111120121	010120212
101122010	000122101	202000121	101000212
222002010	121002101	001021221	222210212
021020110	212212101	122201221	001110022
112200110	011200201	210111221	122020022
200110110	102110201	011022002	210200022
002101210	220020201	102202002	012221122
120011210	002220011	220112002	100101122
211221210	120100011	022100102	221011122
020121020	211010011	110010102	020002222
111001020	010001111	201220102	111212222
			202122222

S'_2

001210210	200121200	102121021	001121112
122120210	002112000	220001021	122001112
210000210	120022000	022022121	210211112
012021010	211202000	110202121	022111222
100201010	010220100	201112121	110021222
221111010	101100100	010012201	201201222
020102110	222010100	101222201	000222022
111012110	020221211	222102201	121102022
202222110	111101211	021120001	212012022
011122220	202011211	112000001	011000122
102002220	001002011	200210001	102210122
220212220	122212011	002201101	220120122
022200020	210122011	120111101	002020202
110110020	012110111	211021101	120200202
201020020	100020111	012202212	211110202
000011120	221200111	100112212	010101002
121221120	000100221	221022212	101011002
212101120	121010221	020010012	222221002
021001200	212220221	111220012	021212102
112211200	011211021	202100012	112122102
			200002102

S'_3

002120120	201001110	100001201	002001022
120000120	000022210	221211201	120211022
211210120	121202210	020202001	211121022
010201220	212112210	111112001	020021102
101111220	011100010	202022001	111201102
222021220	102010010	011222111	202111102
021012020	220220010	102102111	001102202
112222020	021101121	220012111	122012202
200102020	112011121	022000211	210222202
012002100	200221121	110210211	012210002
100212100	002212221	201120211	100120002
221122100	120122221	000111011	221000002
020110200	211002221	121021011	000200112
111020200	010020021	212201011	121110112
202200200	101200021	010112122	212020112
001221000	222110021	101022122	011011212
122101000	001010101	222202122	102221212
210011000	122220101	021220222	220101212
022211110	210100101	112100222	022122012
110121110	012121201	200010222	110002012
			201212012

Design $2^3 3^3$

There are three factors at 2 levels and three factors at 3 levels. 43 effects are estimated from 72 treatment combinations. This is a ⅛ fraction.

Experimental Plan

S_1S_1' S_2S_2' S_3S_3' S_4S_3'

Analysis

The matrix $\frac{1}{180}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ is used to estimate $\begin{bmatrix} L(B_1) \\ L(B_2B_3) \end{bmatrix}, \begin{bmatrix} L(B_2) \\ L(B_1B_3) \end{bmatrix}, \begin{bmatrix} L(B_3) \\ L(B_1B_2) \end{bmatrix},$

and the matrix $\frac{1}{540}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$ is used to estimate $\begin{bmatrix} Q(B_1) \\ Q(B_2B_3) \end{bmatrix}, \begin{bmatrix} Q(B_2) \\ Q(B_1B_3) \end{bmatrix}, \begin{bmatrix} Q(B_3) \\ Q(B_1B_2) \end{bmatrix},$

and the matrix $\frac{1}{1296}\begin{bmatrix} 30 & 0 & -3 & -3 & 3 & -3 \\ 0 & 10 & -3 & 1 & 3 & 1 \\ -3 & -3 & 30 & 0 & 6 & 0 \\ -3 & 1 & 0 & 10 & 0 & -2 \\ 3 & 3 & 6 & 0 & 30 & 0 \\ -3 & 1 & 0 & -2 & 0 & 10 \end{bmatrix}$ is used to estimate $\begin{bmatrix} L(B_1B_2^2) \\ Q(B_1B_2^2) \\ L(B_1B_3^2) \\ Q(B_1B_3^2) \\ L(B_2B_3^2) \\ Q(B_2B_3^2) \end{bmatrix}.$

Construction

Sets of Treatment Combinations from the 2^3

Set	S_1	S_2	S_3	S_4
$x_1+x_2=0$	0	1	1	
$x_1+x_3=0$	1	0	1	

Treatment Combinations

S_1	S_2	S_3	S_4
000	001	010	100
111	110	101	011

Sets of Treatment Combinations from the 3^3

Set	S_1'	S_2'	S_3'
$z_1+z_2+z_3=0$	1	2	

Treatment Combinations

S_1'	S_2'	S_3'
000	001	020
222	010	002
120	022	011
102	100	110
012	121	101
210	112	122
201	220	200
021	202	212
111	211	221

Design $2^5 3^3$

There are five factors at 2 levels and three factors at 3 levels. 64 effects are estimated from 144 treatment combinations. This is a ⅙ fraction.

Experimental Plan

S_1S_1' S_2S_2' S_3S_3' S_4S_3'

Analysis

The matrix $\frac{1}{360}\begin{bmatrix}4 & 1\\ 1 & 4\end{bmatrix}$ is used to estimate $\begin{bmatrix}L(B_1)\\ L(B_2B_3)\end{bmatrix}$, $\begin{bmatrix}L(B_2)\\ L(B_1B_3)\end{bmatrix}$, $\begin{bmatrix}L(B_3)\\ L(B_1B_2)\end{bmatrix}$,

and the matrix $\frac{1}{1080}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}$ is used to estimate $\begin{bmatrix}Q(B_1)\\ Q(B_2B_3)\end{bmatrix}$, $\begin{bmatrix}Q(B_2)\\ Q(B_1B_3)\end{bmatrix}$, $\begin{bmatrix}Q(B_3)\\ Q(B_1B_2)\end{bmatrix}$,

and the matrix $\frac{1}{2592}\begin{bmatrix}30 & 0 & -3 & -3 & 3 & -3\\ 0 & 10 & -3 & 1 & 3 & 1\\ -3 & -3 & 30 & 0 & 6 & 0\\ -3 & 1 & 0 & 10 & 0 & -2\\ 3 & 3 & 6 & 0 & 30 & 0\\ -3 & 1 & 0 & -2 & 0 & 10\end{bmatrix}$ is used to estimate $\begin{bmatrix}L(B_1B_2^2)\\ Q(B_1B_2^2)\\ L(B_1B_3^2)\\ Q(B_1B_3^2)\\ L(B_2B_3^2)\\ Q(B_2B_3^2)\end{bmatrix}$.

Construction

Sets of Treatment Combinations from the 2^5

Set	S_1	S_2	S_3	S_4
$x_1+x_2+x_3+x_4+x_5=1$		1	1	1
$x_1+x_2+x_3=1$		0	1	0
$x_3+x_4=1$		1	0	0

Treatment Combinations

S_1	S_2	S_3	S_4
00100	00010	10000	00001
11100	11010	01000	11001
01011	01101	11111	01110
10011	10101	00111	10110

Sets of Treatment Combinations from the 3^3

Set	S_1'	S_2'	S_3'
$z_1+z_2+z_3=0$		1	2

Treatment Combinations

S_1'	S_2'	S_3'
000	001	020
222	010	002
120	022	011
102	100	110
012	121	101
210	112	122
201	220	200
021	202	212
111	211	221

Design $2^6 3^3$

There are six factors at 2 levels and three factors at 3 levels. 76 effects are estimated from 288 treatment combinations. This is a ⅙ fraction.

Experimental Plan

$S_1S_1' \quad S_2S_2' \quad S_3S_3' \quad S_4S_3'$

Analysis

The matrix $\frac{1}{720}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ is used to estimate $\begin{bmatrix} L(B_1) \\ L(B_2B_3) \end{bmatrix}$, $\begin{bmatrix} L(B_2) \\ L(B_1B_3) \end{bmatrix}$, $\begin{bmatrix} L(B_3) \\ L(B_1B_2) \end{bmatrix}$,

and the matrix $\frac{1}{2160}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$ is used to estimate $\begin{bmatrix} Q(B_1) \\ Q(B_2B_3) \end{bmatrix}$, $\begin{bmatrix} Q(B_2) \\ Q(B_1B_3) \end{bmatrix}$, $\begin{bmatrix} Q(B_3) \\ Q(B_2B_3) \end{bmatrix}$,

and the matrix $\frac{1}{5184}\begin{bmatrix} 30 & 0 & -3 & -3 & 3 & -3 \\ 0 & 10 & -3 & 1 & 3 & 1 \\ -3 & -3 & 30 & 0 & 6 & 0 \\ -3 & 1 & 0 & 10 & 0 & -2 \\ 3 & 3 & 6 & 0 & 30 & 0 \\ -3 & 1 & 0 & -2 & 0 & 10 \end{bmatrix}$ is used to estimate $\begin{bmatrix} L(B_1B_2^2) \\ Q(B_1B_2^2) \\ L(B_1B_3^2) \\ Q(B_1B_3^2) \\ L(B_2B_3^2) \\ Q(B_2B_3^2) \end{bmatrix}$.

Sets of Treatment Combinations from the 2^6

Set	S_1	S_2	S_3	S_4
$x_2+x_4+x_5=1$	1	0	0	
$x_1+x_4+x_5=0$	1	0	1	
$x_3+x_4+x_5+x_6=0$	1	1	0	

Treatment Combinations

S_1	S_2	S_3	S_4
010000	000010	000001	100000
100101	110111	110100	010101
011001	001011	001000	101001
101100	111110	111101	011100
010110	000100	000111	100110
100011	110001	110010	010011
011111	001101	001110	101111
101010	111000	111011	011010

Sets of Treatment Combinations from the 3^3

Set	S_1'	S_2'	S_3'
$z_1+z_2+z_3=0$	1	2	

Treatment Combinations

S_1'	S_2'	S_3'
000	001	020
222	010	002
120	022	011
102	100	110
012	121	101
210	112	122
201	220	200
021	202	212
111	211	221

Design $2^5 3^4$

There are five factors at 2 levels and four factors at 3 levels. 88 effects are estimated from 216 treatment combinations. This is a 1/12 fraction.

Experimental Plan

S_1S_1' S_2S_2' S_3S_3' S_4S_4' S_5S_5' S_6S_6' S_7S_7' S_8S_8' S_9S_9'

$S_{10}S_1'$ $S_{11}S_2'$ $S_{12}S_3'$

Analysis

The matrix $\frac{1}{288}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ is used to estimate $\begin{bmatrix} A_1A_2 \\ A_3A_5 \\ A_3A_4 \end{bmatrix}$, $\begin{bmatrix} A_2A_3 \\ A_1A_4 \\ A_1A_5 \end{bmatrix}$, $\begin{bmatrix} A_1A_3 \\ A_2A_5 \\ A_2A_4 \end{bmatrix}$,

and the matrix $\frac{1}{3}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ is used to estimate $192\begin{bmatrix} \mu \\ A_4A_5 \end{bmatrix}$, $192\begin{bmatrix} A_4 \\ A_5 \end{bmatrix}$, $128\begin{bmatrix} A_4B_1 \\ A_5B_1 \end{bmatrix}$,

$128\begin{bmatrix} A_4B_2 \\ A_5B_2 \end{bmatrix}$, $128\begin{bmatrix} A_4B_3 \\ A_5B_3 \end{bmatrix}$, $128\begin{bmatrix} A_4B_4 \\ A_5B_4 \end{bmatrix}$, $384\begin{bmatrix} A_4B_1^2 \\ A_5B_1^2 \end{bmatrix}$, $384\begin{bmatrix} A_4B_2^2 \\ A_5B_2^2 \end{bmatrix}$, $384\begin{bmatrix} A_4B_3^2 \\ A_5B_3^2 \end{bmatrix}$, $384\begin{bmatrix} A_4B_4^2 \\ A_5B_4^2 \end{bmatrix}$,

and the matrix $\frac{1}{3240}\begin{bmatrix} 24 & 0 & -3 & 3 \\ 0 & 8 & 3 & 1 \\ -3 & 3 & 24 & 0 \\ 3 & 1 & 0 & 8 \end{bmatrix}$ is used to estimate $\begin{bmatrix} L(B_1) \\ Q(B_1) \\ L(B_2B_3) \\ Q(B_2B_3) \end{bmatrix}$, $\begin{bmatrix} L(B_2) \\ Q(B_2) \\ L(B_1B_3) \\ Q(B_1B_3) \end{bmatrix}$,

$\begin{bmatrix} L(B_3) \\ Q(B_3) \\ L(B_1B_2) \\ Q(B_1B_2) \end{bmatrix}$,

and the matrix $\frac{1}{3888}\begin{bmatrix} 30 & 0 & -6 & 0 & -3 & 3 \\ 0 & 10 & 0 & -2 & 3 & 1 \\ -6 & 0 & 30 & 0 & -3 & 3 \\ 0 & -2 & 0 & 10 & 3 & 1 \\ -3 & 3 & -3 & 3 & 30 & 0 \\ 3 & 1 & 3 & 1 & 0 & 10 \end{bmatrix}$ is used to estimate $\begin{bmatrix} L(B_2B_3^2) \\ Q(B_2B_3^2) \\ L(B_1B_2^2) \\ Q(B_1B_2^2) \\ L(B_1B_3^2) \\ Q(B_1B_3^2) \end{bmatrix}$.

Sets of Treatment Combinations from the 2^5

Set	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}
$x_1+x_2=0$	0	0	0	1	1	1	1	0	0	1	1	
$x_1+x_3=0$	1	0	1	0	1	0	1	0	1	0	1	
$x_1+x_2+x_3+x_4=0$	0	0	0	0	0	0	0	1	1	1	1	
$x_1+x_2+x_3+x_5=0$	0	1	1	0	0	1	1	0	0	0	0	

Treatment Combinations

S_1	S_2	S_3	S_4	S_5	S_6
00000	00111	00001	11001	10100	10011
11111	11000	11110	00110	01011	01100

S_7	S_8	S_9	S_{10}	S_{11}	S_{12}
10101	10010	00010	00101	10110	10001
01010	01101	11101	11010	01001	01110

Sets of Treatment Combinations from the 3^4

Set	S_1'	S_2'	S_3'	S_4'	S_5'	S_6'	S_7'	S_8'	S_9'
$z_1+z_2+z_3=0$	0	0	1	1	1	2	2	2	
$z_2+2z_3+z_4=0$	1	2	0	1	2	0	1	2	

Treatment Combinations

S_1'	S_2'	S_3'	S_4'	S_5'	S_6'	S_7'	S_8'	S_9'
0000	0001	0002	1000	1001	1002	2000	2001	2002
1110	1111	1112	2110	2111	2112	0110	0111	0112
2220	2221	2222	0220	0221	0222	1220	1221	1222
1201	1202	1200	2201	2202	2200	0201	0202	0200
2011	2012	2010	0011	0012	0010	1011	1012	1010
0121	0122	0120	1121	1122	1120	2121	2122	2120
2102	2100	2101	0102	0100	0101	1102	1100	1101
0212	0210	0211	1212	1210	1211	2212	2210	2211
1022	1020	1021	2022	2020	2021	0022	0020	0021

Design $2^3 3^5$

There are three factors at 2 levels and five factors at 3 levels. 87 effects are estimated from 216 treatment combinations. This is a ⅛ fraction.

Experimental Plan

$S_1S_1' \quad S_2S_2' \quad S_3S_3' \quad S_4S_4'$

Analysis

The matrix $\frac{1}{540}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ is used to estimate $\begin{bmatrix} L(B_1B_4^2) \\ L(B_2B_5^2) \end{bmatrix}$, $\begin{bmatrix} L(B_1B_5^2) \\ L(B_2B_4^2) \end{bmatrix}$,

and the matrix $\frac{1}{1620}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$ is used to estimate $\begin{bmatrix} Q(B_1B_4^2) \\ Q(B_2B_5^2) \end{bmatrix}$, $\begin{bmatrix} Q(B_1B_5^2) \\ Q(B_2B_4^2) \end{bmatrix}$,

and the matrix

$$\frac{1}{3240}\begin{bmatrix} 24 & 0 & -3 & 3 \\ 0 & 8 & 3 & 1 \\ -3 & 3 & 24 & 0 \\ 3 & 1 & 0 & 8 \end{bmatrix}$$

is used to estimate

$$\begin{bmatrix} L(B_1) \\ Q(B_1) \\ L(B_2B_3^2) \\ Q(B_2B_3^2) \end{bmatrix}, \begin{bmatrix} L(B_2) \\ Q(B_2) \\ L(B_1B_3^2) \\ Q(B_1B_3^2) \end{bmatrix},$$

and the matrix

$$\frac{1}{3240}\begin{bmatrix} 24 & 0 & 3 & 3 \\ 0 & 8 & -3 & 1 \\ 3 & -3 & 24 & 0 \\ 3 & 1 & 8 & 0 \end{bmatrix}$$

is used to estimate

$$\begin{bmatrix} L(B_4) \\ Q(B_4) \\ L(B_3B_5^2) \\ Q(B_3B_5^2) \end{bmatrix}, \begin{bmatrix} L(B_5) \\ Q(B_5) \\ L(B_3B_4^2) \\ Q(B_3B_4^2) \end{bmatrix}$$

and the matrix

$$\frac{1}{3888}\begin{bmatrix} 30 & 0 & -6 & 0 & -3 & 3 \\ 0 & 10 & 0 & -2 & 3 & 1 \\ -6 & 0 & 30 & 0 & -3 & 3 \\ 0 & -2 & 0 & 10 & 3 & 1 \\ -3 & 3 & -3 & 3 & 30 & 0 \\ 3 & 1 & 3 & 1 & 0 & 10 \end{bmatrix}$$

is used to estimate

$$\begin{bmatrix} L(B_1B_2^2) \\ Q(B_1B_2^2) \\ L(B_2B_3) \\ Q(B_2B_3) \\ L(B_1B_3) \\ Q(B_1B_3) \end{bmatrix},$$

and the matrix

$$\frac{1}{3888}\begin{bmatrix} 30 & 0 & 6 & 0 & -3 & -3 \\ 0 & 10 & 0 & -2 & -3 & 1 \\ 6 & 0 & 30 & 0 & 3 & 3 \\ 0 & -2 & 0 & 10 & -3 & 1 \\ -3 & -3 & 3 & -3 & 30 & 0 \\ -3 & 1 & 3 & 1 & 0 & 10 \end{bmatrix}$$

is used to estimate

$$\begin{bmatrix} L(B_3B_4) \\ Q(B_3B_4) \\ L(B_4B_5^2) \\ Q(B_4B_5^2) \\ L(B_3B_5) \\ Q(B_3B_5) \end{bmatrix},$$

and the matrix

$$\frac{1}{3888}\begin{bmatrix} 30 & 0 & -6 & 0 & 3 & -3 \\ 0 & 10 & 0 & -2 & 3 & 1 \\ -6 & 0 & 30 & 0 & 3 & -3 \\ 0 & -2 & 0 & 10 & 3 & 1 \\ 3 & 3 & 3 & 3 & 30 & 0 \\ -3 & 1 & -3 & 1 & 0 & 10 \end{bmatrix}$$

is used to estimate

$$\begin{bmatrix} L(B_3) \\ Q(B_3) \\ L(B_1B_2) \\ Q(B_1B_2) \\ L(B_4B_5) \\ Q(B_4B_5) \end{bmatrix}.$$

Sets of Treatment Combinations from the 2^3

Set	S_1	S_2	S_3	S_4
$x_1+x_2=0$	0	1	1	
$x_1+x_3=0$	1	0	1	

Treatment Combinations

S_1	S_2	S_3	S_4
000	001	010	100
111	110	101	011

Sets of Treatment Combinations from the 3^5

Set	S'_1	S'_2	S'_3	S'_4
$z_1+z_2+z_3+z_4+z_5=0$	0	0	1	
$z_1+z_2+2z_3=0$	1	2	0	

Treatment Combinations

S'_1	S'_2	S'_3	S'_4
00000	02100	01200	00001
21000	20100	22200	21001
12000	11100	10200	12001
10110	12210	11010	10111
01110	00210	02010	01111
22110	21210	20010	22111
20220	22020	21120	20221
11220	10020	12120	11221
02220	01020	00120	02221
00021	02121	01221	00022
21021	20121	22221	21022
12021	11121	10221	12022
10101	12201	11001	10102
01101	00201	02001	01102
22101	21201	20001	22102
20211	22011	21111	20212
11211	10011	12111	11212
02211	01011	00111	02212
00012	02112	01212	00010
21012	20112	22212	21010
12012	11112	10212	12010
10122	12222	11022	10120
01122	00222	02022	01120
22122	21222	20022	22120
20202	22002	21102	20200
11202	10002	12102	11200
02202	01002	00102	02200

Design $2^5 3^5$

There are five factors at 2 levels and five factors at 3 levels. 116 effects are estimated from 432 treatment combinations. This is a 1/18 fraction.

Experimental Plan

$S_1S_1' \quad S_2S_2' \quad S_3S_3' \quad S_4S_4'$

Analysis

The matrix $\frac{1}{1080}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ is used to estimate $\begin{bmatrix} L(B_1B_4^2) \\ L(B_2B_5^2) \end{bmatrix}$, $\begin{bmatrix} L(B_1B_5^2) \\ L(B_2B_4^2) \end{bmatrix}$,

and the matrix $\frac{1}{3240}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$ is used to estimate $\begin{bmatrix} Q(B_1B_4^2) \\ Q(B_2B_5^2) \end{bmatrix}$, $\begin{bmatrix} Q(B_1B_5^2) \\ Q(B_2B_4^2) \end{bmatrix}$,

and the matrix
$$\frac{1}{6480}\begin{bmatrix} 24 & 0 & -3 & 3 \\ 0 & 8 & 3 & 1 \\ -3 & 3 & 24 & 0 \\ 3 & 1 & 0 & 8 \end{bmatrix}$$
is used to estimate
$$\begin{bmatrix} L(B_1) \\ Q(B_1) \\ L(B_2B_3^2) \\ Q(B_2B_3^2) \end{bmatrix}, \begin{bmatrix} L(B_2) \\ Q(B_2) \\ L(B_1B_3^2) \\ Q(B_1B_3^2) \end{bmatrix},$$

and the matrix
$$\frac{1}{6480}\begin{bmatrix} 24 & 0 & 3 & 3 \\ 0 & 8 & -3 & 1 \\ 3 & -3 & 24 & 0 \\ 3 & 1 & 0 & 8 \end{bmatrix}$$
is used to estimate
$$\begin{bmatrix} L(B_4) \\ Q(B_4) \\ L(B_3B_5^2) \\ Q(B_3B_5^2) \end{bmatrix}, \begin{bmatrix} L(B_5) \\ Q(B_5) \\ L(B_3B_4^2) \\ Q(B_3B_4^2) \end{bmatrix},$$

and the matrix
$$\frac{1}{7776}\begin{bmatrix} 30 & 0 & -6 & 0 & -3 & 3 \\ 0 & 10 & 0 & -2 & 3 & 1 \\ -6 & 0 & 30 & 0 & -3 & 3 \\ 0 & -2 & 0 & 10 & 3 & 1 \\ -3 & 3 & -3 & 3 & 30 & 0 \\ 3 & 1 & 3 & 1 & 0 & 10 \end{bmatrix}$$
is used to estimate
$$\begin{bmatrix} L(B_1B_2^2) \\ Q(B_1B_2^2) \\ L(B_2B_3) \\ Q(B_2B_3) \\ L(B_1B_3) \\ Q(B_1B_3) \end{bmatrix},$$

and the matrix
$$\frac{1}{7776}\begin{bmatrix} 30 & 0 & 6 & 0 & -3 & -3 \\ 0 & 10 & 0 & -2 & -3 & 1 \\ 6 & 0 & 30 & 0 & 3 & 3 \\ 0 & -2 & 0 & 10 & -3 & 1 \\ -3 & -3 & 3 & -3 & 30 & 0 \\ -3 & 1 & 3 & 1 & 0 & 10 \end{bmatrix}$$
is used to estimate
$$\begin{bmatrix} L(B_3B_4) \\ Q(B_3B_4) \\ L(B_4B_5^2) \\ Q(B_4B_5^2) \\ L(B_3B_5) \\ Q(B_3B_5) \end{bmatrix},$$

and the matrix
$$\frac{1}{7776}\begin{bmatrix} 30 & 0 & -6 & 0 & 3 & -3 \\ 0 & 10 & 0 & -2 & 3 & 1 \\ -6 & 0 & 30 & 0 & 3 & -3 \\ 0 & -2 & 0 & 10 & 3 & 1 \\ 3 & 3 & 3 & 3 & 30 & 0 \\ -3 & 1 & -3 & 1 & 0 & 10 \end{bmatrix}$$
is used to estimate
$$\begin{bmatrix} L(B_3) \\ Q(B_3) \\ L(B_1B_2) \\ Q(B_1B_2) \\ L(B_4B_5) \\ Q(B_4B_5) \end{bmatrix}.$$

Sets of Treatment Combinations from the 2^5

Set	S_1	S_2	S_3	S_4
$x_1+x_2+x_3+x_4+x_5=1$	1	1	1	
$x_1+x_2+x_3=1$	0	1	0	
$x_3+x_4=1$	1	0	0	

Treatment Combinations

S_1	S_2	S_3	S_4
00100	00010	10000	00001
11100	11010	01000	11001
01011	01101	11111	01110
10011	10101	00111	10110

Sets of Treatment Combinations from the 3^5

Set	S_1'	S_2'	S_3'	S_4'
$z_1+z_2+z_3+z_4+z_5=0$	0	0	1	
$z_1+z_2+2z_3=0$	1	2	0	

Treatment Combinations

S_1'	S_2'	S_3'	S_4'
00000	02100	01200	00001
21000	20100	22200	21001
12000	11100	10200	12001
10110	12210	11010	10111
01110	00210	02010	01111
22110	21210	20010	22111
20220	22020	21120	20221
11220	10020	12120	11221
02220	01020	00120	02221
00021	02121	01221	00022
21021	20121	22221	21022
12021	11121	10221	12022
10101	12201	11001	10102
01101	00201	02001	01102
22101	21201	20001	22102
20211	22011	21111	20212
11211	10011	12111	11212
02211	01011	00111	02212
00012	02112	01212	00010
21012	20112	22212	21010
12012	11112	10212	12010
10122	12222	11022	10120
01122	00222	02022	01120
22122	21222	20022	22120
20202	22002	21102	20200
11202	10002	12102	11200
02202	01002	00102	02200

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