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Fractional Fokker–Planck equation for nonlinear stochastic differential equations driven by non-Gaussian Lévy stable noises

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The Fokker–Planck equation has been very useful for studying dynamic behavior of stochastic differential equations driven by Gaussian noises. However, there are both theoretical and empirical reasons to consider similar equations driven by strongly non-Gaussian noises. In particular, they yield strongly non-Gaussian anomalous diffusion which seems to be relevant in different domains of Physics. In this paper, we therefore derive a fractional Fokker–Planck equation for the probability distribution of particles whose motion is governed by a *nonlinear* Langevin-type equation, which is driven by a Lévy stable noise rather than a Gaussian. We obtain in fact a general result for a Markovian forcing. We also discuss the existence and uniqueness of the solution of the fractional Fokker–Planck equation.
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I. INTRODUCTION AND MOTIVATION

The Fokker–Planck equation is one of the most celebrated equations in Physics, since it has been very useful for studying[1] the dynamic behavior of stochastic differential equations driven by Gaussian noises. However, it turns out that many physical phenomena are outside of this framework. For instance, it has been argued that diffusion by geophysical turbulence[2–7] corresponds, loosely speaking, to a series of sticking (pauses), when the particle is trapped by a coherent structure, and (fast) flights, when the particle moves in the jet flow. A similar phenomenology is observed for zoo plankton grazing.[8,9]

Although there have been some attempts[6] to analyze and quantify this behavior with the help of the classical Fokker–Planck equation, i.e., assuming finite moments of all orders, some laboratory experiments[3–5] or numerical simulations of geostrophic turbulence[10] show that this phenomenology could be rather a consequence of the presence of heavy tails (i.e., power law falloff) for the probability distribution and a strong anisotropy with a clearly preferred direction of diffusion. One can conclude[11] that if the processes are additive, the corresponding walks are Lévy motions.

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Let us recall that indeed stable Lévy motions $L(t)$ generalize the Brownian motion $B(t)$ in the sense that first they are also motions (e.g., Refs. 12 and 13) whose increments $\Delta L(t,\Delta t)=L(t+\Delta t)-L(t)$ are stationary (therefore ΔL has no statistical dependence on t) and independent for any nonoverlapping time lags Δt. Therefore, $L(t)$ corresponds to the sum of independent, identically distributed Lévy stable variables.[14–18] The second common property is that these increments satisfy a ''stability property:'' up to a rescaling and recentring, the sum of different steps has the same probability distribution as one of the steps. Lévy stable variables are precisely defined by this property. The stability property implies in both cases a property of attraction: under rather general conditions a renormalized sum of independent identically distributed variables converge to a stable law. Furthermore, there are no other attractive laws. This explains why the stable property is so important. The attraction property corresponds to a broad generalization of the central limit theorem, with the important difference that whereas the classical theorem (Gaussian case) is satisfied with the condition that the variance is finite, the convergence towards a Lévy law is obtained with the condition that *not only* the variance of the summands X_i is infinite, but also that all their moments of order q equal to or larger than a critical order α $(0<\alpha<2)$ are infinite. This critical order α is called the Lévy stability index and corresponds to the exponent of the power law of probability distribution tails:

$$\text{any } \; s \gg 1: \; \Pr(|\Delta L|>s)\approx s^{-\alpha} \Leftrightarrow \text{any } \; q\geqslant\alpha: \; E(|X|^q)=\infty, \tag{1}$$

where Pr denotes the probability, $E(\,)$ is the mathematical expectation, and s is a given (large) non-negative threshold. This statistical divergence of a Lévy motion is due to jumps, whereas a Brownian motion is almost surely continuous.

This index is the most important of the four parameters defining a Lévy stable law. The second one is the skewness β $(-1\leqslant\beta\leqslant 1)$ which defines the degree of asymmetry of the law, which is maximal for $\beta=-1$ or $\beta=+1$, and the law is symmetric when $\beta=0$. In spite of its name and some common properties, β nevertheless does not correspond to the classical skewness of a quasi-Gaussian law. The latter is indeed undefined for a stable Lévy law due to the above-mentioned statistical divergences. The center γ corresponds to the statistical mean when defined (i.e., $\alpha>1$) and/or to the median when symmetric (i.e., $\beta=0$). The scale parameter $D(D\geqslant 0)$ corresponds to a generalization of the variance of the Gaussian case. More precisely, as discussed below, it corresponds to the intensity scale of the cumulant of (possibly noninteger) order α. It yields an anomalous[19] generalization of the classical Einstein relation: $\mathrm{Var}[X(t)-X(t_0)]=2D(t-t_0)$, where Var( ) denotes the variance. Finally, let us emphasize that the Gaussian case corresponds to the limit case $\alpha=2$, which also implies $\beta=0$, i.e., no asymmetry.

Further comments are now in order on the relevance of Lévy motions in Physics. On the one hand, claims in favor of the relevance of Lévy motions have been made on many physical phenomena ranging from subrecoil laser cooling[20,21] to diffusion by flows in porous media,[22,23] including finance fluctuations,[24,25] see Refs. 26 and 27 for other examples. Many systems indeed display a phenomenology rather similar to that we reported above on geostrophic turbulence.

On the other hand, important questions have been raised. In particular, Ref. 28 questioned the resulting infinite variance of the advecting field for porous media. Indeed, it turns out that recent estimates[29] of the power law of the probability distributions of the hydraulic conductivity yields an exponent $\alpha\approx 3.5$. The question of finite variance might apply to other examples, in particular for atmospheric turbulence where different studies[30] yield a critical exponent $\alpha\approx 7$ for the wind field. Therefore, in spite of their clear phenomenological interest, the relevance of pure Lévy motions could be questioned.

The main goal of this paper is to clarify and define a framework adequate for handling motions more general than pure Lévy motions and which are nevertheless generated by the latter. We will do it by building upon a series of rather recent works[31–37,19,38] which show that the probability density of particles moving with a Lévy motion satisfies a generalized Fokker–Planck equation involving fractional orders of differentiation. Indeed, it could be first argued in a ''very formal and phenomenological'' manner[31] that a fractional power of the Laplacian yields an anomalous scaling for the corresponding diffusion.

A fractional Fokker–Planck equation was obtained in a less formal manner by Refs. 32 and 35 in the framework of the continuous time random walks (CTRWs) model of anomalous diffusion.[33] However, this method does not involve directly a stable Lévy process, but a walk sharing some behavior common with the latter, without being equivalent to it. A different fractional Fokker–Planck equation was introduced[37] with the help of a phenomenological and interesting transformation of the classical Fick law into a fractional Fick law. However, it is not clear that its solution corresponds to a (non-negative) probability distribution. A rather distinct approach was followed by Refs. 34 and 19 since it starts with a *linear* Langevin-type equation with random forces which are *exact* stable Lévy processes, which can be symmetric as well as asymmetric, and with no limitation on the possible values of the Lévy index α. The fundamental mathematical tool which is used is the second characteristic (or cumulant generating) function of the motion defined by this Langevin-type equation. The particular case of symmetric processes correspond to what was previously inferred by Refs. 31, 32, 35, and 37. However, it was shown that in the more general case of asymmetric processes, a new nontrivial advective–diffusive term appears. This is confirmed with the help of a reinterpretation of the characteristic function of a Lévy motion.[38]

We already discussed that theoretically and empirically the nonfiniteness of the variance could be questioned. There are two more general questions: the inhomogeneities of the medium, which are first emphasized for the introduction of the Lévy motions, are finally reduced to a (homogeneous) distribution of times when the particle is strongly kicked. As soon as this representation is granted, the medium (and its properties) does not intervene any longer. This is very restrictive and for instance incompatible with the multifractality of the medium[39,8] (or of the diffusion) when observed. The second reason is that the underlying processes are thought to be strongly nonlinear, whereas the transport is modeled with the help of a (stochastic) linear equation.

Both the successes and limitations of the previous results plead in favor of investigating a local and nonlinear modeling with the help of Lévy motions. This is the reason that we investigate the properties of *nonlinear* Langevin-type equation forced by a Lévy stable motion.

II. STATEMENT OF THE PROBLEM

Further to our above discussion, we consider the following *nonlinear* Langevin-type equation for a stochastic (real) quantity $X(t)$ (e.g., location of a particle):

$$dX(t)=m(X(t),t)dt+\sigma(X(t),t)dL, \tag{2}$$

where the driving source is a Lévy stable motion $L(t)$ instead of Brownian motion $B(t)$. The latter case corresponds to the basis of stochastic calculus (e.g., Ref. 40) and the corresponding differential equation is often called the Ito–Skorokhod equation. The extension to Lévy stable motion $L(t)$ is rather natural and straightforward (e.g., Ref. 41) due to the common properties of $L(t)$ and $B(t)$ that we discussed in Sec. I, i.e., their infinitesimal increments are independent identically distributed and furthermore stable.

More precisely the Ito stochastic calculus corresponds to consider that the dL is, similarly to dB, a forward increment in time [it should be understood as $dL(t,dt)=L(t+dt)-L(t)$]. This means that the value of X at time t is determined by events prior to the application of the stochastic force $dL(t)$, which acts only from time t to $t+dt$.

The Eq. (2) can also be understood under its integral form

$$X(t)=X(t_0)+\int m(X(t),t)dt+\int \sigma(X(t),t)dL, \tag{3}$$

where the last term corresponds to a stochastic integration of a stochastic process. The integration of a stochastic process $\Phi(t)$ [in the case of Eq. (2): $\Phi(t)=\sigma(X(t),t)$] with respect to the Lévy motion L, is rather straightforward in the case of step processes:[42]

$$\Phi(t)=\Phi_n, \quad \text{for } t\in(t_n,t_{n+1}), \quad n=0,1,...,N-1; \int \Phi(t)dL=\sum_{n=0}^{N-1} \Phi_n(L(t_{n+1})-L(t_n)) \tag{4}$$

and this rather suggestive definition is naturally extended to functional spaces in which the step processes are dense.

In order to establish local properties, for instance the time evolution of the probability of the particles, we will use the differential form [Eq. (2)], whereas Refs. 34 and 19 rather used the integral form [Eq. (4)] which becomes cumbersome in the nonlinear case and is in fact useful only to establish global properties (Sec. IX).

After having emphasized the similarities between $L(t)$ and $B(t)$, it is important to underline the nontrivial consequences due to the fact, contrary to the Gaussian case which has all its moments finite, Lévy motions have a finite critical order of divergence of statistical moments $(0<\alpha<2)$. These include the fact that the mathematical techniques which could be used can be rather distinct. For instance, our derivation will rely on the use of the second characteristic function of the increments, Sec. III, instead of probabilities of the increments as done usually for the derivation of the classical Fokker–Planck equation. An obvious reason is that the former are relatively simple (see Sec. VII), while the latter are not, with the only exception of the three following cases: $\alpha=2$, $\beta=0$; $\alpha=1$, $\beta=0$; $\alpha=1/2$, $\beta=1$. The fundamental reason is that both the stability property and the divergence of moments are related to the presence of a cumulant of noninteger order α. In relation to this problem, the convenient L^2 Hilbert structure of Gaussian processes is reduced to a L^α Banach structure for stable Lévy processes. This is particularly important for the integral equation (3), when defining functional spaces where step processes are dense.

The linear case, which is the hitherto studied case, corresponds to

$$m(X(t),t)\equiv m=\text{const}; \quad \sigma(X(t),t)\equiv\sigma=\text{const}. \tag{5}$$

$X(t)-X(t_0)$ is also a Lévy motion which has the same Lévy stability index α as its increments, but with a different center or trend and scale or amplitude.

In the nonlinear case, $m(X(t),t)$ and $\sigma(X(t),t)$ are (possibly nonlinear) functions of $X(t)$ and t, which satisfy certain regularity constraints to be discussed later (Sec. IX). They correspond to inhomogeneities of the medium, which were ignored in the linear case. As a possibly important, but simple example, let us mention the Lévy extension of the so-called geometric Brownian motion, which is rather ubiquitous and for instance is at the core of the Black–Scholes model for option pricing: $m(X(t),t)=mX(t)$ and $\sigma(X(t),t)=\sigma X(t)$, where σ is the votality constant of the price $X(t)$ of a given stock share.

We will demonstrate the following proposition:

Proposition 1: The transition probability density:

$$\forall t\geqslant t_0: \ \ p(x,t|x_0,t_0)=\Pr(X(t)=x|X(t_0)=x_0) \tag{6}$$

corresponding to the nonlinear stochastic differential equation (2), with a Lévy forcing of parameters $\alpha\neq 1$ *or* $\beta=0$, γ, $D\geqslant 0$, *is solution of the following fractional Fokker–Planck equation*:

$$\begin{aligned}\frac{\partial}{\partial t}p(x,t|x_0,t_0)=&-\frac{\partial}{\partial x}(\gamma\sigma(x,t)+m(x,t))p(x,t|x_0,t_0)\\&-D\Big[(-\Delta)^{\alpha/2}(|\sigma(x,t)|^{\alpha}p(x,t|x_0,t_0))\\&+\beta\omega(\alpha)\frac{\partial}{\partial x}(-\Delta)^{(\alpha-1)/2}(|\sigma(x,t)|^{\alpha-1}\sigma(x,t)p(x,t|x_0,t_0))\Big]\end{aligned} \tag{7}$$

with the initial condition

$$p(x,t_0|x_0,t_0)=\delta(x-x_0), \tag{8}$$

where $\delta(x-x_0)$ *is the degenerate Dirac measure in* x_0 *and* $\omega(\alpha)$ *is defined by*

$$\alpha \neq 1: \quad \omega(\alpha) = \tan \frac{\pi\alpha}{2} \tag{9}$$

and where the fractional powers of the Laplacian Δ will be discussed in Sec. VI. Proposition 1 and Eq. (7) are for scalar processes (i.e., $\Delta \equiv \partial^2/\partial x^2$) and their extension to vector processes will be discussed and presented in Sec. VIII. One may note that the fractional diffusive isotropic operator $-(-\Delta)^{\alpha/2}$ applies via a fractional diffusivity $|\sigma(x,t)|^{\alpha}$, whereas the advective–diffusive term corresponds to a conjugate action of a fractional diffusive term $-(-\Delta)^{(\alpha-1)/2}|\sigma(x,t)|^{\alpha-1}$ and a convective term $(\partial/\partial x)\sigma(x,t)$ on the transition probability.

This fractional Fokker–Planck equation will be established with the help of the much more general proposition.

Proposition 2: The inverse Fourier transform of the second characteristic function or cumulant generating function of the increments of a Markov process $X(t)$ generates by convolution the Fokker–Planck equation of evolution of its transition probability $p(x,t|x_0,t_0)$.

We will demonstrate this proposition in a straightforward, yet rigorous way. More precisely, we will establish the following:

$$\frac{\partial p}{\partial t}(x,t|x_0,t_0) = \int dy \, \frac{\partial \tilde{K}}{\partial t}(x-y|y,t)p(y,t|x_0,t_0), \tag{10}$$

where $\tilde{K}$ is the inverse Fourier transform of the cumulant generating function of the increments. The $\tilde{K}$ arguments will become explicit in Sec. III.

Equation (10) not only holds for processes with stationary and independent increments, as in the linear case [Eq. (5)] but also for any Markov process, including those defined by the nonlinear Langevin-type equation [Eq. (2) with $m \neq \text{const}$, $\sigma \neq \text{const}$]. As a consequence of Eq. (10), we will demonstrate the following.

Proposition 3: When the increment's cumulant generating function of a Markov process $X(t)$ is defined by its expansion in cumulants C_n, its Fokker–Planck equation is

$$\frac{\partial p}{\partial t}(x,t|x_0,t_0) = \sum_{n \in J} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n}[C_n(x,t)p(x,t|x_0,t_0)]. \tag{11}$$

An obviously sufficient condition of convergence is obtained when the set J of the orders of differentiation n is finite. This is true in particular for Gaussian forcing: $J = \{1,2\}$. It corresponds to the classical Fokker–Planck equation. On the other hand, $J = \mathbf{N}$ would correspond to an analytic expansion of cumulants. In spite of its interest, we will not discuss the latter case in this paper, nor its relationship to the classical Kramers–Moyal expansion (e.g., Ref. 43).

Below, we concentrate on the case of a finite, but nonanalytic expansion: $J = \{1,\alpha\}$ (noninteger α, $0 < \alpha < 2$), since it corresponds to the Lévy extension (Sec. VII and yields Proposition 1 with the help of fractional derivatives, as discussed in Sec. VI.

III. THE CUMULANT GENERATING FUNCTION OF THE INCREMENTS

The first and second (conditional) characteristic functions are, respectively, the moment generating function $Z_X(k,t-t_0|x_0,t_0)$ and the cumulant generating function $K_X(k,t-t_0|x_0,t_0)$, associated with the transition probability $p(x,t|x_0,t_0)$ of a process $X(t)$. These are defined by the Fourier transform of the latter, with k being the conjugate variable of $x-x_0$:

$$F[p(x,t|x_0,t_0)] \equiv Z_X(k,t-t_0|x_0,t_0) \tag{12}$$

$$\equiv \exp(K_X(k,t-t_0|x_0,t_0)) \tag{13}$$

$$\equiv E[\exp(ik(X(t)-X(t_0)))|X(t_0)=x_0], \tag{14}$$

where $E[\cdot|\cdot]$ denote the conditional mathematical expectation, F and F^{-1}, respectively, the Fourier transform and its inverse:

$$F[f]=\hat{f}(k)=\int_{-\infty}^{\infty} dx \exp(ikx)f(x), \tag{15}$$

$$F^{-1}[\hat{f}]=f(x)=\int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp(-ikx)\hat{f}(k). \tag{16}$$

The corresponding quantities for increments $\delta X(\delta t)=X(t+\delta t)-X(t)$, corresponding to a given time lag $\delta t>0$, are defined in a similar way:

$$F[p(x+\delta x,t+\delta t|x,t)]=\delta Z_X(k,\delta t|x,t) \tag{17}$$

$$\equiv \exp(\delta K_X(k,\delta t|x,t)) \tag{18}$$

$$=E[\exp(ik(X(t+\delta t)-X(t))|X(t)=x], \tag{19}$$

where k is the conjugate variable of δx. The cumulants of the increments C_n are the coefficients of the Taylor expansion of δK_X:

$$\delta K_X(k,\delta t|x,t)=\delta t \sum_{n\in J} \frac{(ik)^n}{n!} C_n(x,t)+o(\delta t). \tag{20}$$

As already mentioned, the classical case corresponds to an analytic expansion of δK_X, i.e., $J\subseteq \mathbf{N}$, whereas we will be interested by a finite but nonanalytic expansion $J=\{1,\alpha\}$ (noninteger α, $0<\alpha<2$).

IV. PROCESSES WITH STATIONARY AND INDEPENDENT INCREMENTS

Let us first consider the simple subcase of a process with stationary and independent increments. It corresponds to $C_n(x,t)\equiv C_n=\text{const}$ in Eqs. (11) and (20) and as already discussed in Sec. I, it includes the linear case [Eq. (5)] of the Langevin-type equation (2).

However, we believe that the following derivation is not only somewhat pedagogical on the role of the characteristic functions for the nonlinear case, but also terser than derivations previously presented for the linear case.

The stationarity of the increments implies that the transition probability depends only on the time and space lags, i.e.,

$$p(x,t|x_0,t_0)=p(x-x_0,t-t_0) \tag{21}$$

and similarly, the characteristic functions of the increments are no longer conditioned, for instance,

$$Z_X(k,t-t_0|x_0,t_0)\equiv Z_X(k,t-t_0), \tag{22}$$

$$K_X(k,t-t_0|x_0,t_0)\equiv K_X(k,t-t_0). \tag{23}$$

On the other hand, the independence of the increments implies that the transition probabilities satisfy a convolution (over any possible intermediate position y) for any given time lag δt:

$$\forall \delta t>0: \; p(x-x_0,t+\delta t-t_0)=\int dy\, p(x-y,\delta t)(y-x_0,t-t_0) \tag{24}$$

and the corresponding characteristic functions merely factor (respectively, add). Therefore, we have,

$$Z_X(k,t+\delta t-t_0)-Z_X(k,t-t_0)=Z_X(k,t-t_0)(\delta Z_X(k,\delta t)-1). \tag{25}$$

This in turn leads to

$$Z_X(k,t+\delta t-t_0)-Z_X(k,t-t_0)=Z_X(k,t-t_0)\,\delta K_X(k,\delta t)+o(\delta t). \tag{26}$$

Its inverse Fourier transform yields

$$p(x,t+\delta t|x_0,t_0)-p(x,t|x_0,t_0)=\int dy\, F^{-1}[\delta K_X(k,\delta t)]p(y-x_0,t-t_0)+o(\delta t). \tag{27}$$

This demonstrates (in the limit $\delta t\to 0$) Proposition 2 and Eq. (10), as well as Proposition 3, since Eq. (27) corresponds, with the help of Eq. (20), to

$$p(x,t+\delta t|x_0,t_0)-p(x,t|x_0,t_0)=\delta t\sum_{n\in J}\frac{(-1)^n}{n!}\left[C_n\int dy\;\delta^{(n)}_{x-y}p(y,t|x_0,t_0)\right]+o(\delta t), \tag{28}$$

where δ^n_x denotes the nth derivative of the Dirac function. Therefore, we obtain

$$\frac{\partial}{\partial t}p(x,t|x_0,t_0)=\sum_{n\in J}\frac{(-1)^n}{n!}C_n\frac{\partial^n}{\partial x^n}p(x,t|x_0,t_0) \tag{29}$$

which corresponds to the linear case of Eq. (11).

V. MORE GENERAL MARKOV PROCESSES

In the case of a Markov process which does not have stationary and independent increments, there is no longer a simple convolution equation [Eq. (24)] of the transition probabilities, nor a simple factorization of characteristic functions [Eq. (25)]. However, the former satisfies a generalized convolution equation which corresponds to the Chapman–Kolmogorov identity[17] valid for any Markov process $X(t)$:

$$\forall\,\delta t>0:\;\; p(x,t+\delta t|x_0,t_0))=\int dy\; p(x,t+\delta t|y,t)p(y,t|x_0,t_0) \tag{30}$$

which indeed reduces to a mere convolution [Eq. (24)] in the case of processes with stationary and independent increments. This identity can be written under the equivalent form:

$$p(x,t+\delta t|x_0,t_0)=\int dy\int\frac{dk}{2\pi}e^{-iky+\delta K_X(k,\delta t|y,t)}p(y,t|x_0,t_0). \tag{31}$$

Noting that we have

$$p(x,t|x_0,t_0)=\int dy\; p(y,t|x_0,t_0)\int\frac{dk}{2\pi}e^{-iky}, \tag{32}$$

we obtain

$$p(x,t+\delta t|x_0,t_0)-p(x,t|x_0,t_0)=\delta t\int dy\, F^{-1}[\delta K_X(k,\delta t|y,t)]p(y,t|x_0,t_0)+o(\delta t). \tag{33}$$

In the limit $\delta t\to 0$, this corresponds to Proposition 2 and Eq. (10). When $J\subseteq\mathbf{N}$, it yields with the help of Eq. (20):

$$\delta p(x,t|x_0,t_0)=\delta t\sum_{n\in J}\int dy\ \delta_{x-y}^{(n)}\left[\frac{(-1)^n}{n!}C_n(y,t)p(y,t|x_0,t_0)\right]+o(\delta t). \tag{34}$$

The limit $\delta t\to 0$ corresponds to Eq. (11) and demonstrates Proposition 3 for a Markow process.

VI. EXTENSION TO FRACTIONAL ORDERS

In the two preceding sections (Secs. IV and V), the fact that the indices $n\in J$ should be integers intervened at best only in the correspondence between (integer order) differentiation $\partial^n/\partial x^n$ [in Eq. (11)] and powers of the conjugate variable k^n [in Eq. (20)]. However, by the very definition of fractional differentiation (e.g., Ref. 44), this correspondence holds also for noninteger orders. However, there is not a unique definition of fractional differentiation and therefore, as discussed in some details in Ref. 19, we cannot expect to have a unique expression of the fractional Fokker–Planck equation.

Since in the following it will be sufficient to consider an expansion of the characteristic function involving fractional powers of only the wave number $|k|$, it is interesting to consider Riesz's definition of a fractional differentiation. Indeed, the latter corresponds to consider fractional powers of the Laplacian:

$$-(-\Delta)^{\alpha/2}f(x)=F^{-1}[|k|^{\alpha}\hat{f}(k)] \tag{35}$$

which has furthermore the advantage of being valid for the vector cases. However, we will see in Sec. VIII that in general it does not apply in a straightforward manner for vector stable Lévy motions. Indeed the latter introduces rather (one-dimensional) directional Laplacians, i.e., (one-dimensional) Laplacians along a given direction $\underline{u}(|\underline{u}|=1)$:

$$-(-\Delta_{\underline{u}})^{\alpha/2}f(x)=F^{-1}[|(\underline{k},\underline{u})|^{\alpha}\hat{f}(k)], \tag{36}$$

where (.,.) denotes the scalar product. On the other hand, it will be useful to consider the fractional power of the contraction of the Laplacian tensor $\underline{\underline{\Delta}}$:

$$\Delta_{i,j}=\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} \tag{37}$$

by a tensor $\underline{\underline{\sigma}}$ ($\underline{\underline{\sigma}}^*$ denotes its transpose), with the following definition:

$$-(-\underline{\underline{\Delta}}:\underline{\underline{\sigma}}.\underline{\underline{\sigma}}^*)^{\alpha/2}\equiv F^{-1}[|(\underline{k},\underline{\underline{\sigma}}.\underline{\underline{\sigma}}^*.\underline{k}|^{\alpha/2}]=F^{-1}[|\underline{\underline{\sigma}}^*.\underline{k}|^{\alpha}]. \tag{38}$$

VII. LÉVY CASE

The second characteristic function of the increments δL of the (scalar) Lévy forcing is the following:

$$\delta K_L(k,\delta t)=\delta t\left[ik\gamma-D|k|^{\alpha}\left(1-i\beta\frac{k}{|k|}\right)\omega(k,\alpha)\right]+o(\delta t), \tag{39}$$

where $\omega(k,\alpha)$ is defined by

$$\alpha\neq 1:\quad \omega(k,\alpha)\equiv\omega(\alpha)=\tan\frac{\pi\alpha}{2},\quad \alpha=1:\quad \omega(k,\alpha)=\frac{\pi}{2}\log|k|. \tag{40}$$

Considering an Ito-type forward integration of Eq. (2), the increments δL generates the following (first) characteristic function for the increments δX of the motion $X(t)$:

$$\delta Z_X(k,\delta t|x-\delta x,t)=E(e^{ikm(X,t)})\,\delta Z_{\sigma L}(k,\delta t|x,t)+o(\delta t) \tag{41}$$

which yields the following elementary cumulant generating function δK_X:

$$\delta K_X(k,\delta t|x,t)=\delta t\left[ikm(x,t)+ik\gamma\sigma(x,t)-D|k|^{\alpha}|\sigma(x,t)|^{\alpha}\left(1-i\beta\frac{k\sigma(x,t)}{|k||\sigma(x,t)|}\,\omega(k,\alpha)\right)\right]+o(\delta t) \tag{42}$$

and which is of the same type as Eq. (20), with $J=\{1,\alpha\}$. Therefore, as discussed in Sec. VI, we have fractional differentiations in the corresponding Eq. (11), which will precisely correspond to Eq. (7), and therefore establishes Proposition 1.

VIII. EXTENSION TO VECTOR PROCESSES

With but one important exception, the extension of the previous results to higher dimensions is rather straightforward. The starting point of this extension is the following nonlinear stochastic equation ($\underline{X}(t)\in R^d$):

$$d\underline{X}(t)=\underline{m}(\underline{X}(t),t)dt+\underline{\underline{\sigma}}(\underline{X}(t),t).d\underline{L}, \tag{43}$$

where $\underline{m}$ is the natural d-dimensional vector extension of the deterministic-like trend, $\underline{\underline{\sigma}}$ is the $d\times d'$-dimensional tensor extension of the modulation of the random driving force, and $\underline{L}$ is a d'-dimensional Lévy stable motion. As discussed below, the expression of the characteristic function of the latter corresponds to the source of the difficulty in extending the scalar results to higher dimensions. On the contrary, it is straightforward to check that Propositions 2 and 3 are valid in the vector case, with the following extensions ($\underline{x}\in R^d$) for Eq. (10):

$$\frac{\partial p}{\partial t}(\underline{x},t|\underline{x}_0,t_0)=\int d\underline{y}\frac{\partial\tilde{K}}{\partial t}(\underline{x}-\underline{y}|\underline{y},t)p(\underline{y},t|\underline{x}_0,t_0) \tag{44}$$

and for Eq. (11) ($\underline{n}\in J\subseteq\mathbf{N}^d,|\underline{n}|=\Sigma_{i=1}^{d}n_i$):

$$\frac{\partial p}{\partial t}(\underline{x},t|\underline{x}_0,t_0)=\sum_{\underline{n}\in J}\frac{(-1)^{|\underline{n}|}}{(n_1)!(n_2)!..(n_d)!}\frac{\partial^{|\underline{n}|}}{\partial x_1^{n_1}\partial x_2^{n_2}..\partial x_d^{n_d}}[C_{\underline{n}}(\underline{x},t)p(\underline{x},t|\underline{x}_0,t_0)]. \tag{45}$$

On the other hand, Eq. (43) yields the following extension to Eq. (41):

$$\delta Z_{\underline{X}}(k,\delta t|\underline{x},t)=e^{i\,\underline{k}.\underline{m}(\underline{x},t)}\delta Z_{\underline{\underline{\sigma}}.\underline{L}}(\underline{k},\delta t|\underline{x},t) \tag{46}$$

and therefore we have

$$\delta K_{\underline{X}}(\underline{k},\delta t|\underline{x},t)=i\,\underline{k}.\underline{m}(\underline{x},t)+\delta K_{\underline{L}}(\underline{\underline{\sigma}}^*.\underline{k},\delta t|\underline{x},t)+o(\delta t). \tag{47}$$

Let us recall that a stable Lévy vector in the classical sense[14,45,46] (see Ref. 47 for a discussion on a rather straightforward generalization, or Refs. 48, 49, and 50 for a more abstract generalization) corresponds to the limit of a sum of jumps, with a power-law distribution, along random directions $\underline{u}\in\partial B_1$, B_1 being the unit ball, distributed according to a (positive) measure $d\Sigma(\underline{u})$. The latter, which generalizes the scale parameter D of the scalar case, is the source of the difficulty since in general the probability distribution of a stable Lévy vector depends on this measure, and therefore is a nonparametric distribution. However, as discussed below, there is at least a trivial exception: the case of isotropic stable Lévy vectors.

Corresponding to our previous remarks, a (classical) stable Lévy vector has the following (Fourier) cumulant generating function:

$$K_{\underline{L}}(\underline{k}) = \delta t \left[i(\underline{k},\underline{\gamma}) - \int_{\underline{u}\in\partial B_1} (i\underline{k},\underline{u})^{\alpha}\, d\Sigma(\underline{u}) \right] + o(\delta t) \tag{48}$$

which yields with the help of Eq. (47):

$$\frac{\partial}{\partial t}\tilde{K}_{\underline{X}}(\underline{k}) = -\operatorname{div}(\underline{m} + \underline{\underline{\sigma}}.\underline{\gamma}) - F^{-1}\left[\int_{\underline{u}\in\partial B_1} (i\underline{\underline{\sigma}}^*(\underline{x},t).\underline{k},\underline{u})^{\alpha}\, d\Sigma(\underline{u}) \right]. \tag{49}$$

The scalar case [Eq. (39)] corresponds to

$$0 \leqslant p \leqslant 1:\ \ \beta = 2p-1,\ \ d\Sigma(u) = D\cos\left(\frac{\pi\alpha}{2}\right)[p\,\delta_{(u-1)} + (1-p)\,\delta_{(u+1)}]. \tag{50}$$

For any dimension d, the second term on the right-hand side of Eq. (49) corresponds to a fractional differentiation operator of order α. This operator can be slightly rearranged. With the help of the odd $d\Sigma^{-}(\underline{u})$ and even $d\Sigma^{+}(\underline{u})$ parts of the measure $d\Sigma(\underline{u})$:

$$2\,d\Sigma^{+}(\underline{u}) = d\Sigma(\underline{u}) + d\Sigma(-\underline{u}), \quad 2\,d\Sigma^{-}(\underline{u}) = d\Sigma(\underline{u}) - d\Sigma(-\underline{u}) \tag{51}$$

and the identity (θ being the Heaviside function):

$$(ik)^{\alpha} = |k|^{\alpha}[\theta(k)e^{i(\alpha\pi/2)} + \theta(-k)e^{-i(\alpha\pi/2)}] \tag{52}$$

one can write the extension of Eq. (7) under the following form:

$$\begin{aligned} \frac{\partial}{\partial t}p(\underline{x},t|\underline{x}_0,t_0) = {} & -\operatorname{div}[\underline{m}(\underline{x},t) + \underline{\underline{\sigma}}(\underline{x},t).\underline{\gamma})]p(\underline{x},t|\underline{x}_0,t_0) \\ & -[\langle(-\underline{\underline{\Delta}}:\underline{\underline{\sigma}}.\underline{\underline{\sigma}}^*)^{\alpha/2}\rangle_{\Sigma^+} - \langle(\underline{\nabla}.\underline{\underline{\sigma}}^*).(-\underline{\underline{\Delta}}:\underline{\underline{\sigma}}.\underline{\underline{\sigma}}^*)^{\alpha-1/2}\rangle_{\Sigma^-}] \\ & \times p(\underline{x},t|\underline{x}_0,t_0), \end{aligned} \tag{53}$$

where the symmetric fractional diffusive and, respectively, the antisymmetric advective–diffusive terms are defined, similarly to Eq. (38), in the following manner:

$$-\langle(-\underline{\underline{\Delta}}:\underline{\underline{\sigma}}.\underline{\underline{\sigma}}^*)^{\alpha/2}\rangle_{\Sigma^+} = \int_{\underline{u}\in\partial B_1} d\Sigma^{+}(\underline{u})F^{-1}[|(\underline{\underline{\sigma}}^*(\underline{x},t).\underline{k},\underline{u})|^{\alpha}] \tag{54}$$

$$-\langle(\boldsymbol{\underline{\nabla}}.\underline{\underline{\sigma}}^*).(-\underline{\underline{\Delta}}:\underline{\underline{\sigma}}.\underline{\underline{\sigma}}^*)^{\alpha-1/2}\rangle_{\Sigma^-} = \int_{\underline{u}\in\partial B_1} d\Sigma^{-}(\underline{u})F^{-1}[(-i\underline{\underline{\sigma}}^*(\underline{x},t).\underline{k},\underline{u})|(\underline{\underline{\sigma}}^*(\underline{x},t).\underline{k},\overline{\underline{u}})|^{\alpha-1}]. \tag{55}$$

In general, each term corresponds to a rather complex integration (which is indicated by the symbol $\langle . \rangle_{\Sigma}$) by the measure $d\Sigma$ of directional fractional Laplacians [Eq. (36)]. However, the symmetric term becomes simpler as soon as the even part $d\Sigma^{+}$ of the measure $d\Sigma$ is isotropic. Indeed, the integration over directions yields only a prefactor D:

$$\langle -(\underline{\underline{\Delta}}:\underline{\underline{\sigma}}.\underline{\underline{\sigma}}^*)^{\alpha/2}\rangle_{\Sigma^+} = D(-\underline{\underline{\Delta}}:\underline{\underline{\sigma}}.\underline{\underline{\sigma}}^*)^{\alpha/2}D = \int_{\underline{u}\in\partial B_1} d\Sigma^{+}(\underline{u})|(\underline{u}_1,\underline{u})|^{\alpha} \tag{56}$$

and for $\alpha=2$ this corresponds to the classical term $(\underline{\underline{\Delta}}:\underline{\underline{\sigma}}.\underline{\underline{\sigma}}^*)$ of the standard d-dimensional Fokker–Planck equation. If $d\Sigma$ itself is rotation invariant, then the asymmetric operator vanishes, since $d\Sigma^{-}=0$. If furthermore, $\underline{\underline{\sigma}}$ is scalar, i.e., $\underline{\underline{\sigma}} = \sigma\underline{\underline{1}}$, then one obtains the following Fractional Fokker–Planck equation:

$$\frac{\partial}{\partial t} p(\underline{x},t|\underline{x}_0,t_0) = -\operatorname{div}[\underline{\underline{\sigma}}.\underline{\gamma}(\underline{x},t) + \underline{m}(\underline{x},t)] p(\underline{x},t|\underline{x}_0,t_0) \tag{57}$$

$$-D[(-\Delta)^{\alpha/2}] |\sigma(x,t)|^{\alpha} p(\underline{x},t|\underline{x}_0,t_0). \tag{58}$$

Therefore, as one might expect the rotation symmetries yield a rather trivial extension of the standard Gaussian case: a fractional power α of the d-dimensional Laplacian, as in the pure scalar case [Eq. (7)]. Obviously, the integration performed in Eq. (53) is also greatly simplified as soon as $d\Sigma(\underline{u})$ is discrete, i.e. its support corresponds to a discrete set of directions $\underline{u}_i$.

On the other hand, let us note that the framework of generalized stable Lévy vectors,[47–50] allows one to introduce a much stronger anisotropy than the measure $d\Sigma$ allows for classical stable Lévy vectors. This therefore diminishes the importance of the asymmetry of the latter. Indeed, the components of a generalized stable Lévy vector do not necessarily have the same Lévy stability index, the latter being generalized into a second rank tensor. Similarly, the differential operators involved in the corresponding fractional Fokker–Planck equation no longer have a unique order of differentiation. This is rather easy to check in case of a discrete measure $d\Sigma(\underline{u})$ and we will explore the general case elsewhere.

IX. EXISTENCE AND UNIQUENESS OF THE SOLUTION

The preceding sections established a generalization of the Fokker–Planck equation for the evolution of the probability distribution of nonlinear stochastic differential equations driven by Lévy stable noises. This is the main goal of this paper. Naturally, one would also like to have if possible a theorem of existence and uniqueness of the solution of this equation. Due to its origin, such a theorem will also imply that the solution will remain positive and normalized, as required for a transition probability. In this section we argue that the general results obtained[51] in the classical Gaussian case ($\alpha=2$) are also relevant for the Lévy extension, whereas up until now existence and uniqueness conditions of partial fractional differential equations have been scarcely explored (see however Refs. 52 and 53) and therefore we cannot rely on general results.

The classical Fokker–Planck equation belongs to the well-explored domain of parabolic equations. Existence and uniqueness of the solution fundamentally result[54] from the fact that the linear operator $A=-\Delta$ is a (self-adjoint) positive generator of a semigroup of contraction operators $T(t)=e^{-tA}$, $t\geqslant 0$. In the case of constant coefficients (linear Langevin equation), the solution is directly obtained with the help of $T(t)$ and this ensures its existence, uniqueness, and positiveness. Note that in our case, the semigroup action corresponds to the equation of convolution [Eq. (24)].

Similar results hold for a Lipschitz variation of the coefficients, i.e.,

$$|m(x,t)-m(y,t)| + |\sigma(x,t)-\sigma(y,t)| \leqslant D|x-y| \tag{59}$$

as well as a condition of slow growth in time of the coefficients $m(X(t),t)$ and $\sigma(X(t),t)$, e.g.,

$$|m(x,t)| + |\sigma(x,t)| \leqslant C|1+x|, \tag{60}$$

where D and C are given positive constants.

These conditions have been extensively used for the classical Fokker–Planck equation with non constant coefficients (e.g., Ref. 43). Considering now the fractional generalization, it is important to note that the fractional power of the Laplacian $-(-\Delta)^{\alpha/2}$ remains positive, since its definition Eq. (35) corresponds to replacing the eigenvalues k^2 by eigenvalues having as real part $|k|^{\alpha}$. Therefore, we remain inside of the previous framework of contraction semigroup and the previous results should hold.

This could also be seen from the integral form of the differential equation. Indeed, in the classical case, the Lipschitz condition is classical for the Brownian forcing,[55,41] as well as for the more general case of martingale and semimartingale forcing.[56–58] The latest case is relevant for the stable Lévy forcing. The Lipschitz condition can be rather understood as a condition of convergence of the Picard iteration method towards a fixed point:

$$X^{n+1}(t)=X(t_0)+\int m(X^n(t),t)dt+\int \sigma(X^n(t),t)dL;X^0(t)=X(t_0). \tag{61}$$

On the other hand, the condition of slow growth (60) in time prevents a finite explosion time, i.e., $X(t)$ remains finite for any given finite time t: this condition is rather general, since it is already required by the deterministic part of the Langevin-type equation.

X. CONCLUSION

We have derived a fractional Fokker–Planck equation, i.e., a kinetic equation which involves fractional derivatives, for the evolution of the probability distribution of nonlinear stochastic differential equations driven by non-Gaussian Lévy stable noises. We first established this equation in the scalar case, where it has a rather compact expression with the help of fractional powers of the Laplacian, and then discussed and presented its extension to the vector case. This fractional Fokker–Planck equation generalizes broadly previous results obtained for a linear Langevin-type equation with a Lévy forcing, as well as the standard Fokker–Planck equation for a nonlinear Langevin equation with a Gaussian forcing. As suggested in Ref. 36 and in the comments of Ref. 19 on Refs. 32 and 35, we will show elsewhere, that the present results could be extended to include fractional time derivative in the Langevin equation and in the corresponding Fokker–Planck equation. This is particularly important for multifractal modeling, since the generators of dynamic universal multifractals[30] are defined by this type of equations.

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