J. Indian Inst. Sci., Jan.–Feb. 2005, **85**, 11–26

Fractional Fourier transform: A novel tool for signal processing

RAJIV SAXENA* AND KULBIR SINGH†
*Principal, Rustamji Institute of Technology, BSF Academy, Tekanpur 475 005, India.
†Department of Electronics and Communication Engineering, Thapar Institute of Engineering and Technology, Patiala 147 004, India.
emails: ksingh@tiet.ac.in; rskulbir@yahoo.com; Phones: 0175-2393482 (R); 0175-2393334(O); Mob: 098720-93482; Fax: 0175-2393005.

Received on April 27, 2004; Revised on November 10, 2004.

Abstract

The fractional Fourier transform (FRFT) is the generalization of the classical Fourier transform. It depends on a parameter a (= $ap/2$) and can be interpreted as a rotation by an angle a in the time-frequency plane or decomposition of the signal in terms of chirps. This paper discusses discrete FRFT (DFRFT), time-frequency distributions related to FRFT, optimal filter and beamformer in FRFT domain, filtering using window functions and other fractional transforms along with simulation results.

Keywords: Fractional Fourier transform, signal processing and analysis.

1. Introduction

The Fourier transform (FT) is undoubtedly one of the most valuable and frequently used tools in signal processing and analysis [1]. A generalization of Fourier transform, the fractional Fourier transform (commonly referred to as FRFT in the literature), was first introduced by Victor Namias in 1980 [2]. He was apparently unaware of the previous works of N. Wiener in 1929, H. Weyl in 1930, E. U. Condon in 1937, H. Kober in 1939, A. P. Guinand in 1956, A. L. Patterson in 1959, V. Bargmann in 1961, De Bruijn in 1973 and R. S. Khare in 1974 [3–4], and of others. Though the idea was the same, these authors discussed the FRFT in a broader context and not by the same name. Mustard [5] in 1987 did considerable work considering Condon and Bargmann as his base without citing Namias' work. Moreover, as FRFT is a special case of linear canonical transform (LCT), all the work previously done on LCT covers FRFT in some sense. In some cases, FRFT is not given any special attention but in other cases the authors have commented on it as one-parameter subclass with the FT as a special case.

In 1980, Victor Namias established that the other transforms could also be fractionalized [6]. The refinement and mathematical description was given by McBride and Keer in 1987 [7]. FRFT has established itself as a powerful tool for the analysis of time-varying signals

†Author for correspondence.
*On lein from M.I.T.S., Gwalior 474 005, India.

in a very short span of time [8]. Furthermore, a general definition of FRFT for all classes of signals (one- and multidimensional, continuous and discrete, and periodic and nonperiodic) was given by Cariolario *et al.* [9]. With the advent of computers and enhanced computational capabilities, the discrete Fourier transform (DFT) came into existence in the evaluation of FT for real-time processing. Further, these capabilities are enhanced by the introduction of DSP processors and fast Fourier transform (FFT) algorithms. On similar lines, there arises a need for discretization of FRFT. DFT has only one basic definition and nearly 200 algorithms are available for fast computation of DFT. But when FRFT is analyzed in discrete domain, there are many definitions of discrete fractional Fourier transform (DFRFT) [10–13]. It is also established that none of these definitions satisfies all the properties of continuous FRFT [14]. Santhanam and McClellan [11] first reported the work on DFRFT in 1995. Thereafter, within a short span of time, a lot many definitions of DFRFT came into existence. These are classified by Pei and Ding [15] in 2000 according to the methodology of its calculation.

The FRFT has been found to have several applications in the areas of optics [8, 16] and signal processing [17–19]. It also leads to generalization of notion of space (or time) and frequency domains which are central concepts of signal processing. It has many applications in solution of differential equations, optical beam propagation and spherical mirror resonators, optical diffraction theory, quantum mechanics, statistical optics, optical system design and optical signal processing, signal detectors, correlation and pattern recognition, space or time-variant filtering, multiplexing [14], signal and image recovery, restoration and enhancement [20, 21], study of space or time-frequency distributions (TFDs) [22], etc. The fractional Fourier transform is likely to have something to offer in every area in which FT and related concepts are used. Therefore, applications of the transform have been studied mostly in the areas of optics and wave propagation, and signal analysis and processing.

2. Fractional operations

Going from the whole of an entity to its fractions represents a relatively major conceptual leap. The fourth power of 3 may be defined as $3^4 = 3 \times 3 \times 3 \times 3$, but it is not obvious from this definition how $3^{3.5}$ might be defined. It must have taken sometime before the common definition $3^{3.5} = 3^{7/2} = \sqrt{3^7}$ emerged. The first and the second derivatives of the function $f(x)$ are commonly denoted by: $df(x)/dx$ and

$$\frac{d^2 f(x)}{dx^2} = \frac{d}{dx}\left[\frac{df(x)}{dx}\right] = \frac{d[df(x)/dx]}{dx} = \left(\frac{d}{dx}\right)^2 f(x),$$

respectively. Similarly, higher-order derivatives are defined. Now what is the 2.5$^{\text{th}}$ derivative of a function? It may not be clear from the above definition. Let $F(m)$ denote the FT of $f(x)$. The FT of the nth derivative of $f(x)$, [i.e. $(d^n f(x)/dx^n)$] is known to be given by $(i2pm)^n F(m)$, for any positive integer n. Now, let us generalize this property by replacing n with the real-order a and take it as the a^{th} derivative of $f(x)$. Thus to find $d^a f(x)/dx^a$, the a^{th} derivative of $f(x)$, find the inverse Fourier transform of $(i2pm)^a F(m)$. Both of these examples deal with the fractions of an operation performed on an entity, rather than fractions of the entity itself. $4^{0.5}$ is the square root of the integer 4. The function $[f(x)]^{0.5}$ is the square root of the function $f(x)$. But $d^{0.5}f(x)/dx^{0.5}$ is the 0.5^{th} derivative of $f(x)$, $(df(x)/dx)^{0.5}$ being the square root of the derivative operator d/dx. The process of going from the whole of an entity to its fractions underlies several of the more important conceptual developments, e.g. fuzzy logic, where the binary 1 and 0 are replaced by continuous values representing certainty or uncertainty of a proposition [23].

3. Fractional Fourier transform (FRFT)

FRFT is a generalization of FT. It is not only richer in theory and more flexible in application, but is also not expensive in implementation. It is a powerful tool for the analysis of time-varying signals. With the advent of FRFT and related concepts, it is seen that the properties and applications of the conventional FT are special cases of those of the FRFT. However, in every area where FT and frequency domain concepts are used, there exists the potential for generalization and implementation by using FRFT. In this section, the basic concept of FRFT and generalization of FT is described.

FT of a function can be considered as a linear differential operator acting on that function. The FRFT generalizes this differential operator by letting it depend on a continuous parameter a. Mathematically, a^{th} order FRFT is the a^{th} power of FT operator.

3.1. *Definition*

The FRFT of a function $s\,(x_1)$ can be given as:

$$F^{a}[s(x_1)] = S(x) = \frac{\exp i(\frac{p}{4}-\frac{p}{2})}{\sqrt{2p\sin\boldsymbol{a}}}\exp\left(-\frac{i}{2}x^2\cot\boldsymbol{a}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{i}{2}x_1^2\cot\boldsymbol{a}-\frac{ix_1x}{\sin\boldsymbol{a}}\right)s(x_1)dx_1, \quad (1)$$

and the inverse FRFT can be given as

$$F^{-a}[s(x_1)] = \frac{\exp - i(\frac{p}{4}-\frac{p}{2})}{\sqrt{2p\sin\boldsymbol{a}}}\exp\left(+\frac{i}{2}x^2\cot\boldsymbol{a}\right)\int_{-\infty}^{\infty}\exp\left(+\frac{i}{2}x_1^2\cot\boldsymbol{a}-\frac{ix_1x}{\sin\boldsymbol{a}}\right)s(x_1)dx_1, \quad (2)$$

where $\boldsymbol{a} = ap/2$.

Different cases are discussed in the following section.

i) When $\boldsymbol{a} = p/2$, i.e. a = 1

$$F^{a}[s(x_1)] = F^{1}[s(x_1)] = \frac{1}{\sqrt{2p}}\int_{-\infty}^{\infty}s(x_1)\exp(-ixx_1)dx_1, \quad (3)$$

is the ordinary Fourier transform.

ii) When $\boldsymbol{a} = 0$, i.e. $a = 0$, the transform kernel reduces to identity operation. When $\boldsymbol{a}$ approaches 0, $\sin\boldsymbol{a}$ approaches $\boldsymbol{a}$, $\cot\boldsymbol{a}$ approaches $1/\boldsymbol{a}$ and using the fact in the sense of generalized functions

$$\lim_{e\to 0}\frac{1}{\sqrt{ipe}}\exp\left(-\frac{x^2}{ie}\right) = d\,(x), \quad (4)$$

so that we have,

Table 1
Various kernels available with FRFT

Value of parameter a	$a = ap/2$	Kernel	Fractional operator	Operation on signal
0 or 4	0 or $2p$	$d(x - x_1)$	$F^0 = F^4 = I$	Identity operator
1	$p/2$	$\exp(ixx_1)$	$F^1 = F$	Fourier operator
2	p	$d(x + x_1)$	$F^2 = FF = I$	Reflection operator
3	$3p/2$	$\exp(-ixx_1)$	$F^3 = FF^2 = F^{-1}$	Inverse Fourier operator

$$F^0[s(x_1)] = \int_{-\infty}^{\infty} d(x - x_1)s(x_1)dx_1 = s(x_1). \tag{5}$$

A similar procedure can be applied to the case.

iii) When $a = \pi$, i.e. $a = 2$ and the result turns out to be

$$F^2[s(x_1)] = \int_{-\infty}^{\infty} d(x + x_1)s(x_1)dx_1 = s(-x_1). \tag{6}$$

So, for an angle from 0 to $2p$, we have the values of a from 0 to 4. It can be shown that the transform kernel is periodic with a period 4. Table I gives the various kernels of FRFT for variation of a from 0 to 4.

Many FRFT definitions are found in the literature, which converge to the original definition. Among them the most commonly used is:

$$F^a[s(t)] = \sqrt{1 - i\cot a}\int_{-\infty}^{\infty} \exp[ip(f^2 \cot a - 2ft\csc a + t^2 \cot a)]s(t)dt \tag{7}$$

$$= \sqrt[4]{-\csc a \exp(ip\,a)}\,\exp(ip\, f^2 \cot a)\int_{-\infty}^{\infty} \exp(t^2 \cot a - 2ft\csc a)s(t)dt, \tag{7a}$$

where $a = ap/2$ and $\sqrt[4]{.}$ denotes the complex fourth root z, with $-p/4 \le \arg\, z \le p/4$.

The FRFT of a function is equivalent to a four-step process:

1. Multiplying the function with a chirp,
2. Taking its Fourier transform,
3. Again multiplying with a chirp, and
4. Then multiplication with an amplitude factor.

The above-described type of FRFT is also known as Chirp FRFT (CFRFT). A version of weighted FRFT (WFRFT) is also available in the literature. It gives inferior results compared to CFRFT and hence is not popular and is not in common use [9].

The FRFT kernel can be written as $j_a(f, t)$:

$$F^a[s(t)] = \int_{-\infty}^{\infty} j_a(f, t)s(t)dt = S_a(f). \tag{8}$$

Let F^a be an operator generating an FRFT, for any given fraction; a belongs to R, maps a signal $s(t)$, t belongs to R. F^a will be an FRFT operator if it is:

i) linear

ii) verifies FT condition $F^1 = F$.

iii) has additive property $F^{a+b} = F^a F^b$ for every choice of a and b.

It can be seen from here that the order parameter or fraction a can be freely manipulated, as if it denotes a power of FT operator.

Given the widespread use of conventional FT in science and engineering, it is important to recognize this integral transform as the fractional power of FT. Indeed, it has been this recognition, which has inspired most of the recent applications replacing the ordinary FT with FRFT (which is more general and includes FT as special case) adding an additional degree of freedom to problem, represented by the fraction or order parameter a. This, in turn may allow either a more general formulation of the problem or improvement based on possibility of optimization over a (as in optimal Wiener filter resulting in smaller mean square error at practically no additional cost).

4. Generalized operations

In this section, some of the important properties of FRFT are discussed. The properties of FRFT are useful not only in deriving the direct and inverse transform of many time-varying functions but also in obtaining several valuable results in signal processing.

As in the case of the conventional Fourier and Laplace transforms, an operational calculus exists in FRFT also. McBride and Keer developed the necessary framework for the idea of operational calculus of Victor Namias in a function-space setting [7]. The critical analysis of Namias approach of fractionalization was done in order to clarify some ambiguities and modifications.

Recalling some of the well-established properties of Fourier transform, the operational calculus for FRFT is described in Table II. The fractional Fourier transform of some simple functions is given in Table III.

When two functions multiply (or convolve) in time domain ($a = 0$) they get convolved (or multiplied) in frequency domain ($a = 1$). More generally, multiplication (or convolution) in the a^{th} domain is convolution (or multiplication) in $(a \pm 1)^{\text{th}}$ domain (which is orthogonal to a^{th} domain). In $(a \pm 2)^{\text{th}}$ domain (which is sign-flipped version of the a^{th} domain) convolution (or multiplication) operation in the a^{th} domain remains the same. The concept of fractional convolution and correlation has been developed differently by various authors [14, 18, 24].

Now the question is how to evaluate FRFT? The answer obviously comes with the help of computers because of its computational complexity. So this leads to the requirement of the discrete version of FRFT so that it can be evaluated with the help of a computer. In contrast to the case of DFT where it has one basic definition and a lot of algorithms are available for its fast computation, FRFT has many definitions in discrete domain. The basic

Table II
Important properties of FRFT

Sl no	Properties	Calculus
1.	Multiplication rule	$F_a(gf) = g\left(x\cos a + \frac{1}{i}\sin a\,\frac{d}{dx}\right)F_a(f).$
2.	The division rule	$F_a(f/x) = (i/\sin a)\exp\left(-\frac{ix^2}{2}\cot a\right)\int_{-\infty}^{x}\exp\left(+\frac{ix^2}{2}\cot a\right)F_a(f)dx.$
3.	Mixed product rule	$F_a\left(x\frac{d}{dx}\right) = -(\sin a + ix^2\cos a)\sin a\,.F_a(f) + x\cos 2a\,\frac{d}{dx}F_a(f) - \frac{1}{2}\sin 2a\,\frac{d^2}{dx^2}F_a(f).$
4.	Differentiation rule	$F_a\left(\frac{df}{dx}\right) = \left(-ix\sin a + \cos a\,\frac{d}{dx}\right)F_a(f).$
		$F_a\,\frac{d^m}{dx^m} = \left(-ix\sin a + \cos a\,\frac{d}{dx}\right)^m F_a.$
5.	Integration rule	$F_a\int_a^x f(x)dx = \sec x\exp\left(-\frac{ix^2}{2}\tan a\right)\int_a^x\exp\left(+\frac{ix^2}{2}\tan a\right)F_a(f)dx.$
6.	Shift rule	$F_a\ f(x+k) = \exp\left[-ik\sin a\left(x+\frac{k}{2}\cos a\right)\right]F_a(f)_{[x+k\cos a]}.$
7.	Similarity rule	$F_a f(-x) = F_{a-p}\ f(x).$
8.	Convolution rule	$f *^a g = \exp(-ibt^2)\int_{-\infty}^{\infty} f(t)e^{ibt^2}g(t-t)e^{ib(t-t)^2}dt$, where $b = 0.5\cot(0.5pa)$

problem for the signal-processing community is the exact definition of DFRFT. Which one can be used for signal-processing applications with the least possible error and which one has the fastest evaluation?

5. Types of DFRFT

There are a lot of definitions of DFRFT in the literature but none obeys all the properties of continuous FRFT. So in 2000, Pei and Ding [15] classified these definitions according to the methodologies used for calculations. This is given as under.

5.1. *Direct form of DFRFT*

The simplest way to derive the DFRFT is sampling the continuous FRFT and computing it directly, but this method of evaluation loses the properties of unitary, additivity, reversibility and closed-form properties. Its domain is therefore confined.

5.2. *Improved sampling-type DFRFT*

In this class of DFRFT, the continuous FRFT is properly sampled and it is observed that the resultant DFRFT has fast algorithm. It gives results similar to those of continuous FRFT. The major constraint in this class is that it is nonorthogonal, nonadditive and is applicable to only a set of signals [10].

Table III
Fractional Fourier transforms of simple functions

Function $f(x)$	Fractional Fourier transform $F_a f(x)$
$\exp(-x^2/2)$	$\exp(-x^2/2)$
$H_n(x)\exp(-x^2/2)$	$H_n(x)\ \exp(-x^2/2)$
$\exp(-x^2/2 + ax)$	$\exp\left(-\frac{x^2}{2} - \frac{ia^2}{2} e^{ia} \sin a + axe^{ia}\right)$
$d(x)$	$\frac{\exp(ip/4 - ia/2)}{\sqrt{2p \sin a}} \exp\left(\left[-\frac{ix^2}{2} \cot a\right]\right)$
$d(x - a)$	$\frac{\exp(ip/4 - ia/2)}{\sqrt{2p \sin a}} \exp\left[-\frac{i}{2} \cot a\,(x^2 + a^2) + iax \operatorname{cosec} a\right]$
1	$\frac{e^{-ia/2}}{\sqrt{\cos a}} \exp\left[+\frac{ix^2}{2} \tan a\right]$
$\exp(ikx)$	$\frac{e^{-ia/2}}{\sqrt{\cos a}} \exp\left[\frac{i}{2} \tan a\,(k^2 + x^2) + ikx \sec a\right]$

5.3. *Linear combination-type DFRFT*

This DFRFT is derived by using the linear combination of identity operator, DFT, time inverse operation and IDFT. The results do not match with the continuous FRFT. The transform matrix is orthogonal, additive and reversible [9, 11, 25].

5.4. *Eigenvector decomposition-type DFRFT*

Pei and Ding derived another type of DFRFT by searching the eigenvectors and eigenvalues of the DFT matrix and computed the fractional power of the DFT matrix. This type of DFRFT will work similar to the continuous FRFT and will also satisfy the properties of orthogonality, additivity and reversibility. The eigenvectors cannot be expressed in closed form and they also lack the fast computational algorithms [12].

5.5. *Group theory-type DFRFT*

The concept of group theory is used in deriving this definition of DFRFT as the multiplication of DFT and periodic chirps. It satisfies the rotational property of Wigner distribution, the additivity property and the reversibility property of FRFT but can be derived only when the fractional order equals some specified angles [26].

5.6. *Impulse train-type DFRFT*

This DFRFT can be viewed as a special case of continuous FRFT in which the input function is periodic and equally spaced impulse train. In this, if in a period Δ_o, the number of impulses are N, then N should be equal to Δ_o^2. It satisfies many properties of the FRFT and has fast computational algorithms but is not defined for all fractions [13].

The problem of nonavailability of perfect and proper DFRFT expression in closed form still persists. Researchers have started the use of available DFRFTs for convolution, filtering and multiplexing in the fractional Fourier domain [14].

Subsequently, FRFT can be generalized into special affine Fourier transforms (AFT). Work has started to find out discrete affine Fourier transform (DAFT) by sampling the AFT. The DAFT appears good in concept by sacrificing the additivity property. It is suitable for practical applications due to simpler and closed-form expression of discrete fractional convolution and correlation. It is being used for computing FRFT, discrete filter design and pattern recognition [15].

6. TFDs similar to FRFT

Time or space/frequency distributions are functions of time (or space) and temporal (or spatial) frequency content of signals for different times (or locations). It is tempting to view time–frequency representations of a signal, just as the time domain and frequency domain representations. This is justified by the fact that they often contain the same (or almost the same) information as these other representations. Time-frequency distribution functions similar to FRFT are included [27].

6.1. *Wigner distribution function (WDF)*

The WDF is a time-frequency representation that maps an one-dimensional (1D) time (or space in optics)-varying signal into a two-dimensional (2D) signal representation of both time and frequency. The WDF can be interpreted as a joint time-frequency power spectrum distribution function under the restriction of the uncertainty principle. A more general definition is known as the cross-WDF. The cross-WDF can be interpreted as a joint time-frequency cross spectrum distribution function and is defined as

$$W_{u_0,v_0}(x,v) = \int_0 u\left(x + \frac{x'}{2}\right) v_0^*\left(x - \frac{x'}{2}\right) e^{-2ip\,x'v}\,dx'. \tag{9}$$

The cross-WDF satisfies a large number of desirable mathematical properties. Among these are the marginal properties

$$\int W_{u_0,v_0}(x,v)\,dv = u_0(x)v_0^*(x), \tag{10}$$

$$\int W_{u_0,v_0}(x,v)\,dx = u_1(v)v_1^*(v), \tag{11}$$

where we use the FT notation as $u_1(v) = FT\{u_0(x)\}$.

6.2. *Ambiguity function (AF)*

The AF can be interpreted as a joint time–frequency autocorrelation function. Once again, a more general definition known as the cross-AF is seldom used. The cross-AF can be interpreted as a joint time-frequency cross-correlation function and is defined as

$$A_{u_0,v_0}(h,y) = \int u_0\left(h' + \frac{h}{2}\right) v_0^*\left(h' - \frac{h}{2}\right) e^{-2p\,ih'y}\,dh'. \tag{12}$$

The cross-WDF and the cross-AF are related by a double Fourier connection [27]

$$A_{u_0,v_0}(h,y) = \iint W_{u_0,v_0}(x,v) e^{-2pi(hx+yv)} dxdv. \tag{13}$$

6.3. *Radon–Wigner transform*

The Radon transform of the WDF (RW) has already been investigated and associated with the FRFT. The RW with an angle $f = P(p/2)$ is equal to

$$R\{W_{U_0}(x,x)\}|_{f = p(p/2)} = |\,\mathrm{FRFT}_p\,[u_0(x)]\,|^2 = |\,u_p(x)\,|^2 . \tag{14}$$

The 2D convolution of two auto-WDFs is related to an 1D radial convolution of modulo square FRFTs [23].

7. Applications in signal processing

The intimate relationship of the FRFT to time-frequency representations as well as the central importance of FT suggests that the FRFT should have many applications in optics, signal analysis and processing, especially for wave and beam propagation, wave field reconstruction, phase retrieval and phase–space tomography, study of time- or space-frequency distributions. It is now also being used in biometrics for iris verification.

In signal-processing application, it is basically used for filtering, signal recovery, signal reconstruction, signal synthesis, beamforming, signal detectors, correlators, image recovery, restoration and enhancement, pattern recognition, optimal Wiener filtering and matched filtering. It can also be used for multistage and multichannel filtering, multiplexing in fractional Fourier domains, fractional joint transform correlators, adaptive windowed FRFT and applications with different orders in the two dimensions. Some of the above applications are discussed in detail from the point of view of signal processing.

7.1. *Filtering using FRFT*

In signal recovery typically the received signal is related to the transmitted signal through some system and it is desired to estimate the transmitted signal. The received signal can be observed with some finite accuracy determined by noise or other errors. Signal restoration problems are signal recovery problems where the received signal is a distorted, noisy or otherwise degraded version of the transmitted signal. Signal reconstruction problems are signal recovery problems where the received signal is some, perhaps quite complicated, mapping of the transmitted signal to another space such that the received signal has no direct resemblance to the transmitted signal. In signal synthesis a desired output signal is specified and input to the system is to be chosen so that the required signal is observed at the output. All these inverse problems are mathematically similar. In each case the problem is to estimate the input from knowledge of output, also using any available prior knowledge regarding the nature of the input and/or the nature and statistics of the measurement error or noise or the specified tolerance [14, 17].

In most of the signal-processing applications, the signal which is to be recovered is degraded by known distortion, blur and/or noise and the problem is to reduce or eliminate these degradations. The concept of filtering in fractional Fourier domain is to realize, flexibly and efficiently, general shift-variant linear filters for a variety of applications. Figure 1 shows Wigner distribution of a desired signal and noise superimposed in single plot. It is clear from here that signals with significant overlap in both time and frequency domains may have little or no overlap in a fractional Fourier domain. The solution to such problems depends on the observation model and the objectives, as well as the prior knowledge available about the desired signal, degradation process and noise. The problem is then to find the optimal estimation operators with respect to some design criteria that removes or minimizes these degradations. The most commonly used observation model is: $y(t) = h(t) * x(t) + n(t)$ where $h(t)$ is a system that degrades the desired signal $x(t)$, and $n(t)$ is additive noise, possibly nonstationary noise and $y(t)$ is the signal at the input of the proposed filter. A frequently used criterion for optimal filtering is mean square error (MSE). For an arbitrary degradation model (nonstationary processes), the resulting optimal recovery will not be time invariant and thus cannot be expressed as a convolution and cannot be realized by filtering in conventional Fourier domain (multiplying the Fourier transform of a function with its filter function in that domain). The optimal filtering can be obtained depending upon the criteria of optimization. The main criteria of optimization are minimum mean squared error (MMSE), maximum signal-to-noise ratio (SNR) and minimum variance. Each criterion has its own advantages and disadvantages. The problem considered in this paper is to minimize the MSE for arbitrary degradation model and nonstationary processes by filtering in fractional Fourier domains. The MMSE method has been used to obtain the optimum weights. The objective is to recover the desired signal free from noise and fading in the received signal, in stationary and moving source problems. Let the filter input be $y(t)$ and the reference signal be $x(t)$. The weights of the filter can be chosen in order to minimize the MSE between the output and the reference signal.

$$J(w) = E\{\| y(t) - x(t) \|^2\}, \tag{15}$$

where $\|\bullet\|$ is the L_2 norm given by $\| y(t) \|^2 = \int_{-\infty}^{\infty} y(t) y^*(t) dt.$ The optimum weights can be found by setting the derivative of $J(w)$ to $w*$ equal to zero. They are given as

$$w_{opt} = R_y^{-1} r_{yx}, \tag{16}$$

where R_y is the covariance matrix of the received signal and r_{yx} is the cross-covariance between the input of the filter and the desired signal.

Figure 2 shows that the MSE is less in the case of $a = -0.3$ domain (optimum FRFT domain) as compared to $a = 0$ (time domain) and $a = 1$(frequency domain). So filtering is to be done in optimum FRFT domain for least MSE.

7.2. *Filtering using window functions*

In the case of FT, the limits to evaluate the integral are from –ve infinity to infinity and for calculation with computer (discrete Fourier transform) the truncation is done up to N points. This gives oscillations at the discontinuity (Gibbs phenomenon) when the signal is reconstructed back. In order to suppress the Gibbs phenomenon, window functions are used.

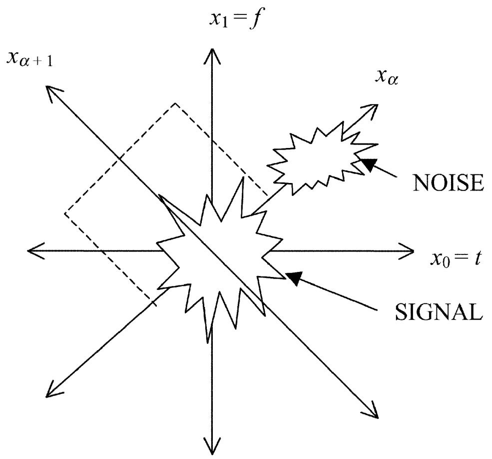


FIG. 1. Filtering in fractional Fourier domain as observed in time-frequency or time-space plane; a = 0.5 is drawn.

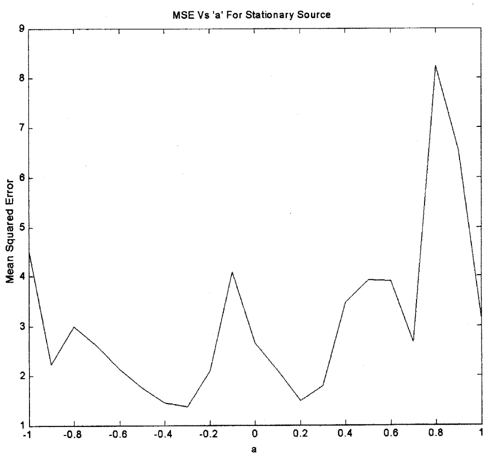


FIG. 2. Plot for variation of mean squared error for various FRFT domains.

Similarly, for computation of FRFT with computer (i.e. DFRFT), the window functions are to be analyzed. This section describes the behavior of window functions in various fraction domains.

Window functions have been successfully used in various areas of signal processing and communications such as spectrum estimation, digital filter design, speech processing, and in other fields. These functions are frequently used to produce realizable systems in diverse engineering disciplines. A complete review of many window functions and their properties was presented by Harris [28]. Discussions on this topic have been going on quite intensely [29]. A quick check of the literature reveals that there are no fewer than 46 different common window functions. All windowing functions are designed to reduce the side lobes of the spectral output of FFT routines. Whilst applying the window function reduces the side lobe leakage, it causes the main lobe to broaden reducing the resolution. This is a trade-off that has to be made, one should choose the weighing function, which best suites the application.

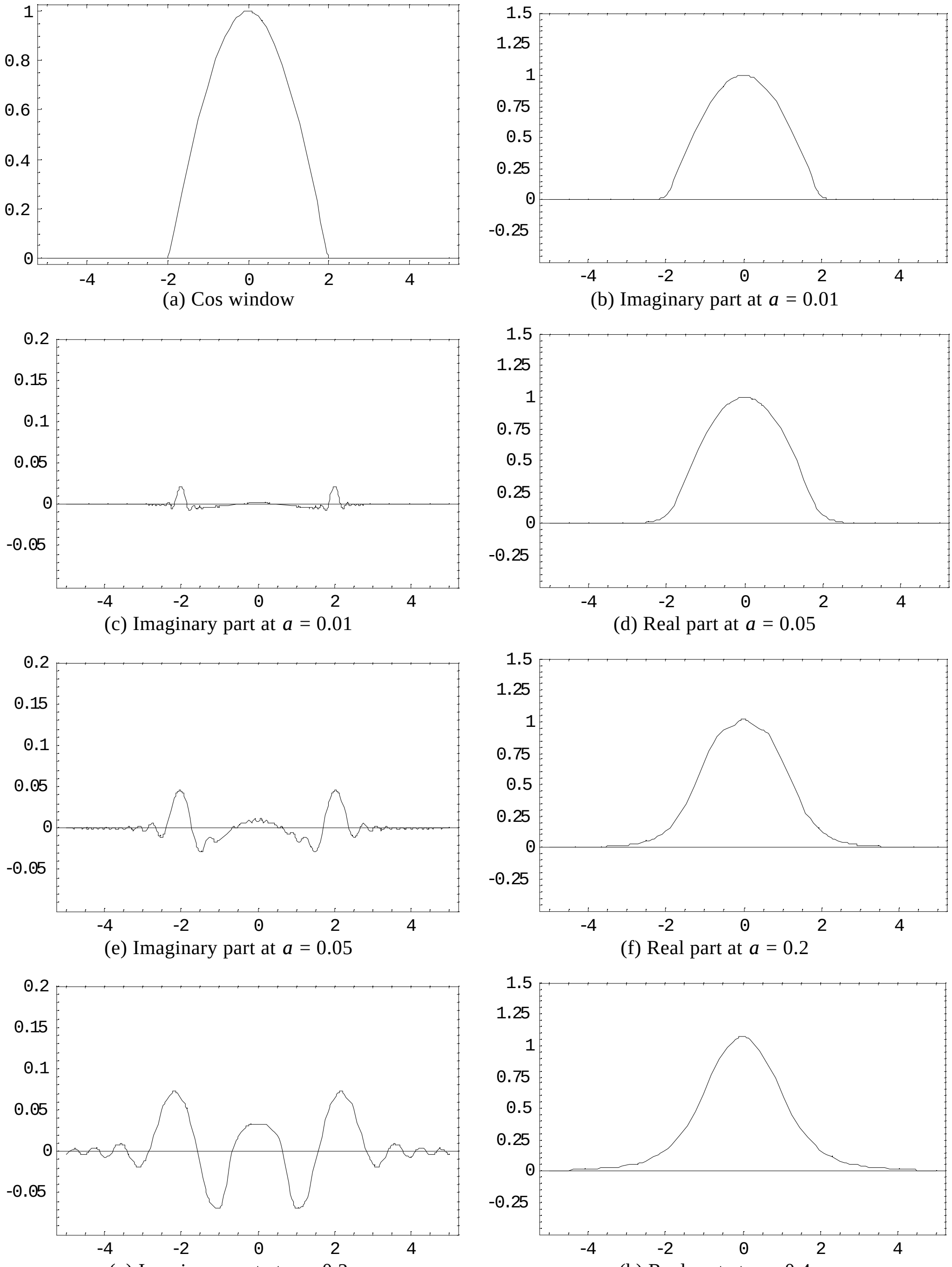
Recently, the FRFT has been invented by a number of researchers, and being used in almost all applications where Fourier transforms were used. The windows can also be analyzed using FRFT. An attempt is made to evaluate the FRFT of the cos window.

The window functions are selected in the duration from $x_1 = -2$ to 2 and the maximum magnitude of the window is taken as unity. The expression used to evaluate the FRFT of cos window functions is given by eqn (1).

7.2.1. *FRFT of cos window*

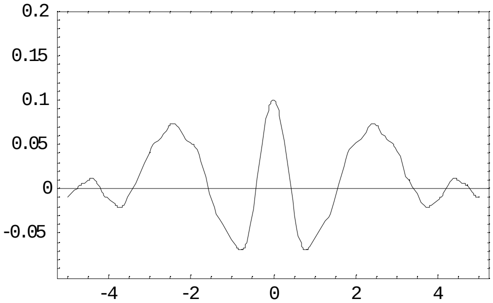
The cos window function is defined by the following expression:

$$s(x_1) = \cos\left[\frac{1}{4}px_1\right] \quad |x_1| \leq 2. \tag{17}$$

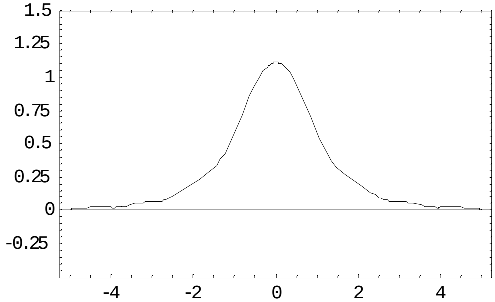


(a) Cos window
(b) Imaginary part at $a = 0.01$
(c) Imaginary part at $a = 0.01$
(d) Real part at $a = 0.05$
(e) Imaginary part at $a = 0.05$
(f) Real part at $a = 0.2$
(g) Imaginary part at $a = 0.2$
(h) Real part at $a = 0.4$

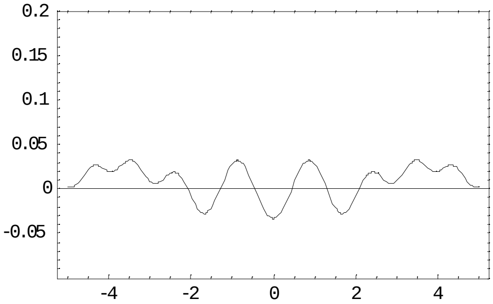
FIG. 3. (Contd.)



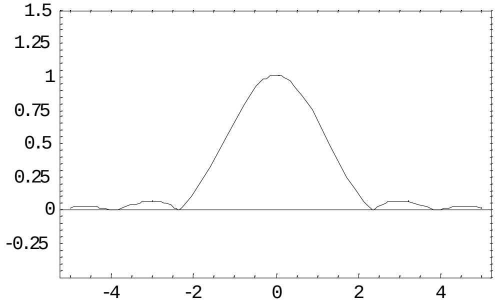
(i) Imaginary part at a = 0.4



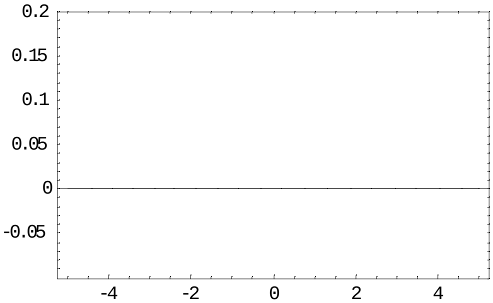
(j) Real part at a = Pi/4



(k) Imaginary part at a = Pi/4



(l) Real part at a = Pi/2



(m) Imaginary part at a = Pi/2

FIG. 3. FRFT of cos window computed at various angles.

The FRFT of this window at different values of a is plotted in Fig. 3.

7.2.2. *Observations of window functions in FRFT*

In general, it was observed that maximum number of oscillations were present in rectangular window and the number decreases in both the real as well as imaginary parts as we moved from a = 0.01 to a = Pi/2. In cos window the number of oscillations first increased up to a = 0.05 and then decreased.

In the FRFT analysis of window functions, it can be concluded that maximum number of harmonics are present in the rectangular window at every value of alpha as compared to other windows. This can be attributed to the fact that when the rectangular window is used to truncate the infinite function, the Gibbs phenomenon occurs and usually an undesirable approximation results. At a discontinuity, the approximation has a fixed percentage overshoot with ripples before and after the discontinuity. If the number of terms increases, the ripples do not decrease, but are squeezed into narrower interval about the discontinuity. Moreover, it is impossible to obtain an infinite slope using only a finite number of terms. For this reason, the rectangular window is not of much practical use. To overcome the presence of large oscillations in both the pass and the stop bands, we should choose a window function that contains a taper and decays towards zero gradually instead of abruptly.

In all, it can be concluded that the side lobe amplitudes of cos are considerably smaller than for the rectangular window and thus eliminate the ringing effect. However, for the same size of window, the width of the main lobe is also wider for this window compared to the rectangular window. Consequently, these window functions provide more smoothing as they are tangentially terminating with very less harmonics. The common feature observed in all the windows is that the maximum energy contents are centered in the main lobe. The study of different window parameters for different values of a reveals that some of the window parameters showed variation with change in the value of a.

7.2.3. *Beamforming*

Beamforming is a very useful tool in sensor array signal processing and is used widely for direction of arrival estimation, interference suppression and signal enhancement. The FT-based method of beamforming is out of use these days because of its inherent shortcoming to handle time-varying signals. In the active radar problem where the chirp signals are transmitted or an accelerating source reflects the sinusoidal signal as a chirp signal, FRFT can be applied because of the ability of FRFT to handle chirp signals better than the FT. Therefore, as discussed earlier, the replacement of FT with FRFT should improve the performance considerably. As FRFT gives rotational effect to time-frequency plane, a chirp signal that is an oblique line in time-frequency plane transforms into a harmonic which is vertical line in this plane. The WD gives an idea about the energy distribution of a signal in time-frequency plane and FRFT rotates the WD in clockwise direction by an angle a in the time-frequency plane. This way, the chirp signal (which is not compact in either spatial or time domain) is converted to harmonic signal as there exists an order for which the signal is compact. After this, the MSE is calculated for various FRFT domains from $a = -1$ to $a = 1$. The optimal domain is searched for minimum MSE. Filtering in this optimal domain is seen to be significantly better than in conventional Fourier domain [19]. From Fig. 4, it is clear that there is a significant improvement in the performance of beamformer in optimum FRFT domain as compared to space and frequency domains. The proposed method of obtaining optimum a is based on frame-by-frame basis. In practice, the optimum a that gives minimum MSE requires an efficient online procedure for its computation.

8. Transforms of fractional orders

In 1980, Victor Namias established that other transforms can also be fractionalized. This has given a lot of motivation to fractional signal processing. Research in every field of science and engineering is trying to generalize various transforms on lines similar to FRFT. Literature is available for fractional Hartley transform, fractional sine transform, fractional

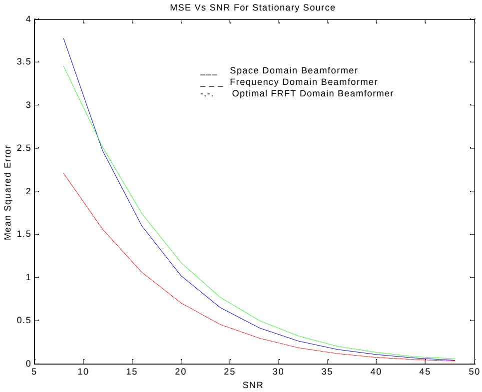


FIG. 4. Plot for comparison of mean squared error with varying SNR for different beamformers.

cosine transform, fractional wave packet transform, fractional spline wavelet transform, fractional Hilbert transform and fractal approach for image processing, etc. It is obvious from this that an era has been opened up for a generalization of the problems to get better results in every area of engineering by using fractional domains of a transform [30].

9. Conclusions

This paper shows that the fractional Fourier transforms lead to a generalization of time (or space) and frequency domains, which are the central concepts in signal analysis and processing as well as other areas. The intimate relationship of FRFT to time-frequency representations, as well as central importance of the Fourier transforms suggests that FRFT should have many applications in signal analysis and processing. The merits of FRFT are that it is not only richer in theory and more flexible in application but the cost of implementation is also low. The most important aspect of the FRFT is its use in time-varying signals for which the FT fails to work. The potential for generalization and implementation of FRFT is still there in all the areas where FT was used earlier. The FRFT provides additional degree of freedom to the problem as parameter a gives multidirectional applications in various areas of optics and signal processing in particular and physics and mathematics in general. The discrete version of FRFT discussed is still not well established and a closed-form expression does not exist. But the available definitions of DFRFT are being used for various applications in signal processing. The demerits of FT are known in the field of mobile communication where it was discarded way back, but the evolution of FRFT has given a ray of hope for its usage in this field.

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