# Fractional-Fourier-transform calculation through the fast-Fourier-transform algorithm

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A method for the calculation of the fractional Fourier transform (FRT) by means of the fast Fourier transform (FFT) algorithm is presented. The process involves mainly two FFT's in cascade; thus the process has the same complexity as this algorithm. The method is valid for fractional orders varying from −1 to 1. Scaling factors for the FRT and Fresnel diffraction when calculated through the FFT are discussed. © 1996 Optical Society of America

*Key words:* Fractional Fourier transform, fast Fourier transform, Fresnel integral.

## 1. Introduction

Most of optics is based on performing operations on an image or on manipulating its Fourier transform. The gap between both domains (spatial and frequency domains) is very seldom used. It is interesting to obtain an analytical expression of the amplitude distribution in these intermediate domains. This may be accomplished in several ways. Fresnel diffraction allows the calculation of the field distribution in cases in which the transformation from the spatial domain to the Fourier domain is obtained by free-space propagation (like in a conventional Fourier transformer).

Recently, Mendlovic and Ozaktas[1,2] introduced a new tool for image analysis in optics, coined the fractional Fourier transform (FRT). This transformation was defined mathematically by McBride and Kerr[3] based on research by Namias.[4] An operational definition of FRT in optics was stated by the use of propagation in a gradient-index (GRIN) medium.[1,2] Such a medium provides, by combining self-focusing and propagation, the Fourier transform of an input plane at a given distance, which depends on the fiber. The FRT's of different orders are defined as the amplitude field distributions as they propagate along the medium between the input and the Fourier planes.

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Lohmann[5] gave a different definition of the FRT that is based on the Wigner distribution functions (WDF's). In that paper he proposed two optical setups for FRT implementation on the basis of bulk optics (Fig. 1). The equivalence between the GRIN medium and the bulk-optics formulations was proved in Ref. 6. Many contributions have shown the usefulness of the FRT as a tool for optical information processing.[7–9] Problems like optical diffraction theory and optical systems have been treated with the FRT as well.[10–12]

Using either operation, the FRT or Fresnel diffraction, permits a continuous transition from the object domain to the Fourier domain to be accomplished. In a more general context, ABCD-matrix formalism provides a general framework for optical transformations including, as particular cases, the Fourier, FRT, and Fresnel transforms (see, for instance, Ref. 13). The final integral expressions obtained with these transformations provides, in principle, a tool for calculating any diffraction pattern in an optical system. Nevertheless, in the general case there are no analytical solutions, so numerical approximations are needed. Unfortunately, the direct calculation of the integrals is inefficient from the computational point of view.

The availability of fast algorithms for digital calculation of the Fourier transform has greatly contributed to the spread of Fourier processing and analyzing methods. Even when optics can provide any ABCD transform in real time, computer simulations of optical setups are often used as preliminary stages for the final design to provide fast and accurate preliminary tests. Therefore, it is interesting to design an efficient algorithm for fast FRT calculation

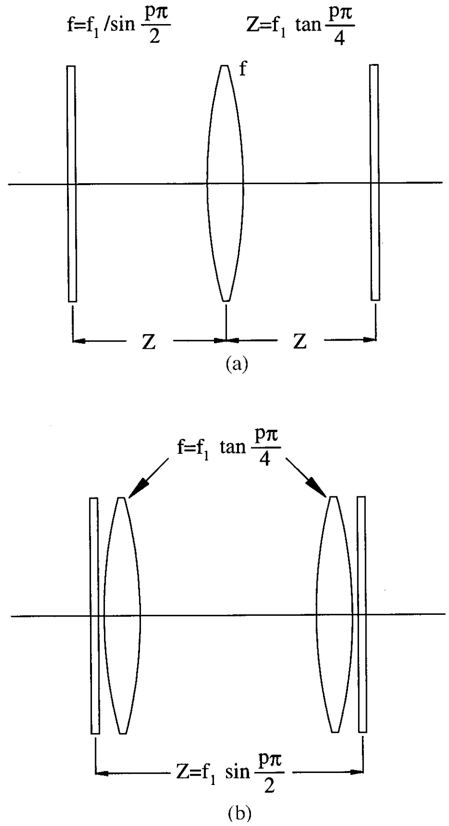


Fig. 1. Schematic diagrams of the optical setups for obtaining an FRT: (a) a type I module and (b) a type II module.

with applications in optics and in digital signal processing.

The first method proposed for FRT digital calculation was based on decomposing the input distribution into the eigenmodes of the FRT operator—the Hermite–Gaussian functions. A phase-propagation factor transforms each of these functions independently. Addition of the propagated modes constructs the final FRT.[1,2] This method is time consuming, especially for large image sizes, because the algorithm is quadratic with the input size, $O(N^2)$. In Ref. 14 two efficient algorithms for discrete FRT calculation were proposed that work in $O(N \log N)$ time. Nevertheless, this method is not appropriate for low fractional orders as a result of subsampling fast-oscillating functions. This results in a low accuracy for small orders and, consequently, in an important loss of information.

Here we propose an alternative method that is conceptually simpler than the above-mentioned one. It consists of a direct translation of the type II optical setup [see Fig. 1(b)] into digital calculations. The full correspondence between the mathematical calculations and the optical system permits a simple analysis of each step of the process.

In Section 2 we report preliminaries about the discrete Fourier transform (DFT). The relation between the DFT and an optically obtained Fourier transform is discussed as well. Section 3 contains a brief introduction of the Fresnel-diffraction integral calculation. The requirements for Fresnel integral discretization is also discussed. In Section 4 we recall the optical FRT definition. In Section 5, the algorithm for the FRT calculation is presented. Finally, in Section 6 we outline the conclusions.

## 2. Discrete Fourier Transform

When an input transparency $u(x)$ is placed in the front focal plane of a lens with focal length $f$, then in the back focal plane an exact, scaled version of the Fourier transform $\tilde{u}(\tilde{x})$ is obtained:

$$\tilde{u}(\tilde{x}) = \frac{\exp(jkf)}{j\lambda f} \int_{-\infty}^{+\infty} u(x) \exp\left(-j \frac{2\pi}{\lambda f} x\tilde{x}\right) \mathrm{d}x, \quad (1)$$

where $k = 2\pi/\lambda$. The product $\lambda f$ acts as the scaling factor between the Fourier transform and the actual distribution obtained. This definition permits writing the final amplitude of the optically obtained Fourier transform. However, except for few functions, the result of the Fourier integral cannot be expressed in terms of analytical functions, and numerical computations are needed. For doing so, one must sample the input and output signals at positions $x = m\Delta x$ and $\tilde{x} = \tilde{m}\Delta\tilde{x}$, respectively, with $m$ and $\tilde{m}$ being integer variables. The proper sampling interval will avoid the loss of information while it minimizes the number of operations.

Given a discrete distribution $u_m$, its DFT $\tilde{u}_{\tilde{m}}$ is defined as

$$\tilde{u}_{\tilde{m}} = \sum_{m=-N/2}^{N/2-1} u_m \exp\left(-j \frac{2\pi}{N} m\tilde{m}\right), \quad (2)$$

where $N$ is the length of both the input and output vectors. Note that, with some additional constraints, expression (2) can be shown to be the sampled Fourier transform of the discretized input function. If one assumes the sampling intervals are $\Delta x$ and $\Delta\tilde{x}$ for the input and output matrices, respectively, Eqs. (1) and (2) are matched if the following scaling relation is fulfilled:

$$N\Delta x\Delta\tilde{x} = \lambda f, \qquad L_x L_{\tilde{x}} = \lambda f N, \quad (3)$$

where $L_x = N\Delta x$ and $L_{\tilde{x}} = N\Delta\tilde{x}$ are the global sizes of the input and output transparencies, respectively.

The expression shown in Eq. (2) can efficiently be calculated when the samples are ordered conveniently. It provides a fast algorithm for calculation that is known as the fast Fourier transform (FFT), which was introduced by Cooley and Tukey.[15]

## 3. Fresnel-Diffraction Calculation

The Fresnel integral can be expressed in terms of the Fourier transform, permitting, in principle, a direct calculation by the use of the FFT. Nevertheless the scaling factor impedes the unique calculation for the full range of distances from the object plane to infinity. Thus a separate treatment of two different cases is needed.

### A. From Middle Range to Fourier Transform

The Fresnel pattern $u'(x')$ at a distance $z$ of a distribution $u(x)$, illuminated by a plane wave, is given by convolution between the input signal and the free-space propagation kernel:

$$u'(x') = \frac{\exp(jkz)}{j\lambda z}\left[u(x') * \exp\left(j\frac{\pi}{\lambda z}x'^2\right)\right], \quad (4)$$

where the asterisk denotes the convolution operation. This can be written in a more explicit form as

$$u'(x') = \frac{\exp(jkz)}{j\lambda z}\exp\left(j\frac{\pi}{\lambda z}x'^2\right) \times \int_{-\infty}^{+\infty}\left[u(x)\exp\left(j\frac{\pi}{\lambda z}x^2\right)\right]\exp\left(-j\frac{2\pi}{\lambda z}xx'\right)\mathrm{d}x. \quad (5)$$

Aside from the factor dependent on $z$ in the first line of expression (5) it consists of a Fourier transform of the product of the input and a quadratic phase factor. The whole result is also multiplied by an additional quadratic phase.

To calculate this integral by the FFT and bearing in mind the relation between Eqs. (1) and (2), one must sample the above expression with the scaling factor $N\Delta x\Delta x' = \lambda z$. Then Eq. (5) converts into

$$u'_{m'} = \frac{\exp(jkz)}{j\lambda z}\exp\left(j\pi\frac{\lambda z}{L_x^2}m'^2\right) \times \sum_{m=-N/2}^{N/2-1}\left\{\left[u_m\exp\left(j\frac{\pi}{\lambda z}\frac{L_x^2}{N^2}m^2\right)\right] \times \exp\left(-j\frac{2\pi}{N}mm'\right)\right\}. \quad (6)$$

The Fresnel pattern is calculated by means of the FFT of the term in square brackets in Eq. (6). The above-written sampling condition implies that the $N$ elements of the input and output vectors fulfill relation $L_xL_{x'} = \lambda zN$; thus for large distances the output matrix contains the sampling of a larger area of the output plane.

Unfortunately, this method cannot be applied for an arbitrary value of the distance $z$. For small distances the quadratic phase factor multiplying the input has a too rapid variation with the spatial coordinate, impeding an accurate sampling for numerical calculations. Let us estimate the range for a good sampling area. For a given distribution of the field, the phase factor increases its frequency linearly with the distance to center of the vector. If we assume that in the border of the signal one period of the quadratic phase factor is smaller than the sampling interval, it is easy to obtain the approximate condition

$$z \geq z_1 \equiv \frac{L_x^2}{\lambda N}. \quad (7)$$

The above assumption may look too weak, as this yields to a sampling in the border of the image just within the Nyquist limit. Nevertheless, in real situations the image is usually padded with zeros in the border, with a good sampling being obtained for the region of interest of the image. The main reason for padding with zeros is to avoid, after Fresnel diffraction, the external part of the image going out of the support matrix. Moreover, if an FFT algorithm is used, because of aliasing the portion that runs out of the image will enter by the opposite side of the matrix. As an example of the validity of the above-described method, real values can be introduced into expression (7). For parameter values of $\lambda = 632.8$ nm, $N = 256$, and $L_x = 10$ mm, the method is valid for distances greater than 300 mm.

### B. From Object Plane to Middle Range

The Fresnel pattern in the range from the object plane to $z_1$ cannot be calculated in the same way described in Subsection 3.A. Bearing in mind that the difficulty comes from the high frequencies of a quadratic phase factor we find that it is more convenient to operate in the Fourier domain. Physically this procedure equals propagating the Fourier transform of the signal instead of the signal itself. In this domain the convolution in Eq. (4) is transformed into the product

$$\tilde{u}'(\nu) = \frac{\exp(jkz)}{j\lambda z}\tilde{u}(\nu)\exp(-j\pi\lambda z\nu^2). \quad (8)$$

The relation of the spatial frequency $\nu$ and the spatial coordinate in the Fourier plane $\tilde{x}$ is given by $\tilde{x} = \lambda z\nu$, where $\lambda z$ acts as the scaling factor. Discretization of the quadratic exponential term in Eq. (8) produces

$$\exp[-j\pi\lambda z(\Delta\nu)^2\tilde{m}^2] = \exp\left[-j\pi\lambda z\left(\frac{\Delta\tilde{x}}{\lambda z}\right)^2\tilde{m}^2\right] = \exp\left(-j\pi\frac{\lambda z}{L_x^2}\tilde{m}^2\right). \quad (9)$$

Now the frequencies of the quadratic phase factor are low for small $z$ values, just contrary to the previous method. This result permits the calculation of the Fresnel pattern for short distances from the object plane (provided that the Fresnel approximation is still valid). The method consists of performing a Fourier transform of the input signal, multiplying by the phase factor in Eq. (9), and performing an inverse Fourier transform on the result to obtain the Fresnel pattern. In the discrete case, and by the use of the

DFT, i.e.,

$$u'_{m'} = \frac{\exp[j(2\pi/\lambda)z]}{j\lambda z} \times \sum_{\tilde{m}=-N/2}^{N/2-1} \left\{ \left[ \sum_{m=-N/2}^{N/2-1} u_m \exp\left(-j\frac{2\pi}{N} m\tilde{m}\right) \right] \times \exp\left(-j\pi \frac{\lambda z}{{L_x}^2} \tilde{m}^2\right) \exp\left(j\frac{2\pi}{N} \tilde{m}m'\right) \right\}. \quad (10)$$

The main difference between this method and the one described in Subsection 3.A is that the use of two Fourier transforms (one direct and one inverse) cancels the scale factor between the input and output. The actual sizes of both the input and output matrices are identical ($L_x = L_{x'}$).

As the distance $z$ is increased the Fresnel pattern will widen, producing aliasing as the object information goes across the border of the matrix. This fact is more marked in the case of small values of $z$ because the scale is the same for both the input and output. Aside from this effect, there is also the need for fair sampling in the quadratic phase factor. As in the above-described case an estimation of the validity range can be made, yielding the approximate relation

$$z \le z_2 \equiv \frac{{L_x}^2}{\lambda N}. \quad (11)$$

Under the conditions exposed in Subsection 3.A, both ranges overlap, permitting a calculation in the full range from the object plane to the Fraunhofer diffraction region, provided that the proper method is selected according to the distance.

## 4. Fractional Fourier Transform

The FRT of the order $0 < |p| < 2$ of an input function $u(x)$ can be expressed as[10]

$$u^p(a') = C_p \exp\left\{ j\frac{\pi}{\tan[p(\pi/2)]} a'^2 \right\} \times \int_{-\infty}^{+\infty} u(a) \exp\left\{ -j\frac{\pi}{\tan[p(\pi/2)]} a^2 \right\} \times \exp\left\{ -j\frac{2\pi}{\sin[p(\pi/2)]} aa' \right\} \mathrm{d}a, \quad (12)$$

where $C_p$ is the constant factor

$$C_p = \frac{\exp(-j\pi\, \mathrm{sgn}\{\sin[p(\pi/2)]\}/4 + jp(\pi/4)}{|\sin[p(\pi/2)]|^{1/2}}. \quad (13)$$

For values of $p = 0$ and $p = \pm 2$ the FRT is defined separately:

$$u^0(a') = \int_{-\infty}^{+\infty} u(a)\, \delta(a - a')\mathrm{d}a,$$

$$u^{\pm 2}(a') = \int_{-\infty}^{+\infty} u(a)\, \delta(a + a')\mathrm{d}a, \quad (14)$$

respectively.

Two important properties can be deduced from expressions (14)[3]:

- The Fourier property

$$(\mathscr{F}^1 u)(a') = \mathscr{F}[u(a)](a') = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u(a)\exp(-j2\pi a'a)\mathrm{d}x.$$

- The semigroup property

$$\mathscr{F}^a[\mathscr{F}^b f] = \mathscr{F}^{a+b} f. \quad (15)$$

Note that, from these properties, the orders $P = 4k$, with $k$ being an integer number, correspond to an exact image of the input. The FRT of order two can be interpreted as imaging with spatial inversion. Thus, the FRT of an arbitrary order of an input vector can be calculated by the performance of the FRT of the nearest integer order (which reduces to a trivial case) and the computation of the FRT of the result by use of the remainder order. Therefore, the operational-orders range reduces to [0, 1].

The above mathematical expressions [Eq. (12)] can be physically interpreted and rewritten in a more suitable form. Mendlovic and Ozaktas[1,2] proposed the FRT calculation of an input function by means of its propagation through a GRIN medium. The kernel of the FRT obtained is based on the Hermite–Gaussian functions and leads to the same result as in Eq. (12). Lohmann[5] showed, through the WDF properties, that the FRT can also be optically obtained by interleaving free-space propagation and lenses.

Two possible FRT setups are shown in Figs. 1(a) and 1(b). It is interesting to think about these setups in terms of operations in the Wigner domain. The $p$th-order FRT of an input vector is connected by a rotation of $p\pi/2$ of its WDF. This rotation may be decomposed into three shearing operations of the WDF—one in the $x$ direction, one in the $y$ direction ($\nu$ shearing), and finally another in the $x$ direction. A $\nu - x - \nu$ shearing will provide the same result. These operations can be performed optically by means of free-space propagation and lenses. After free-space propagation the WDF of the resulting signal is an $x$-sheared version of the WDF of the original signal. Analogously, a $\nu$ shearing of the WDF is connected with the action of a lens. The setups shown in Fig. 1 can be interpreted in these terms.

At this point we have three different means of obtaining the FRT of the order $p$ of a distribution: GRIN media, type I systems, and type II systems. To reach the purpose of this paper, one must choose the appropriate way, which is the one to provide the shortest calculation time when discretization is performed. GRIN-media-based algorithms present a calculation time that depends on the squared number of samples. The calculation time is reduced down to the $N(\log N)$ order when algorithms based on bulk-optics systems are considered. Type II systems need only one Fresnel transformation, so the final algorithm will be based on that configuration.

In Ref. 5 the final output obtained through a type II system [Fig. 1(b)] is expressed as

$$u^p(x') = \frac{\exp(jkz)}{j\lambda z} \int \left[ u(x) \exp\left( -j \frac{\pi}{\lambda f} x^2 \right) \right] \exp\left[ j \frac{\pi}{\lambda z} \times (x^2 + x'^2 - 2xx') \right] \exp\left( -j \frac{\pi}{\lambda f} x'^2 \right) \mathrm{d}x. \tag{16}$$

Writing the physical parameters of the system as $f = f_1/\tan(p\pi/4)$ and $z = f_1 \sin(p\pi/2)$, with $f_1$ being an arbitrary fixed length, Eq. (16) converts, aside from constant factors, into Eq. (12). The scaling factor of the transformation is determined by the product $\lambda f_1$.

Note that the phase factor $\exp(jkz)/j\lambda z$ coming from the Fresnel-diffraction integral has no corresponding term in Eq. (12). One can notice that writing $z$ as $f_1 \sin(p\pi/2)$ in this factor yields a phase that depends on $f_1/\lambda$; thus there are different combinations of $\lambda$ and $f$ that provide the FRT with the same scaling factor except for a different global phase factor. On the other hand, with GRIN media the phase factor in Eq. (13) depends on only the fractional order, without explicit dependence of the physical characteristics of the fiber.

Because of the arbitrary phase factor that is introduced when the FRT is performed through bulk-optics systems, the FRT obtained by GRIN media has to be the reference definition; Eqs. (12)–(14) have been adopted here for the final calculations.

## 5. Digital Fractional Fourier Transform

A direct calculation of expression (16) by the FFT is complicated because the same problem that arises for Fresnel diffraction appears here. With type II systems the free-space propagation distance can be chosen to be small, and we can apply the same method that was used for the short-distance Fresnel-diffraction calculation (see Section 3.B).

The whole algorithm consists of writing the sampled versions of the transformations performed in the optical system. Note that here $L_x = L_{\tilde{x}} = L_{x'}$, so the sampling intervals are $\Delta x = \Delta \tilde{x} = \Delta x' = \sqrt{\lambda f_1 / N}$, in the object, Fourier, and fractional domains, respectively.

The different steps are as follows:

1. Action of the first lens: This consists of the multiplication of the input matrix by the quadratic phase factor, given by

$$\exp\left[ -j \frac{\pi}{N} \tan\left( p \frac{\pi}{4} \right) m^2 \right].$$

2. Free-space propagation of the result of step 1: According to Section 3.B this is accomplished by
   (a) the FFT,
   (b) multiplication of the output matrix by the quadratic phase factor:

$$\exp\left[ -j \frac{\pi}{N} \sin\left( p \frac{\pi}{2} \right) \tilde{m}^2 \right],$$

   (c) the inverse FFT.

3. Action of the second lens: this again means the multiplication of the matrix by the quadratic phase factor:

$$\exp\left[ -j \frac{\pi}{N} \tan\left( p \frac{\pi}{4} \right) m'^2 \right].$$

Note that there are no axial distances in the process. The fractional order or the rotation angle of the WDF, which are not dependent on the actual setup, is used instead.

The conditions for a fair sampling will be

$$\left| \sin\left( p \frac{\pi}{2} \right) \right| < 1, \tag{17a}$$

$$\left| \tan\left( p \frac{\pi}{4} \right) \right| < 1. \tag{17b}$$

Condition (17a) is always fulfilled, whereas condition (17b) will hold for only $|p| < 1$. The properties of the FRT permit extension for an arbitrary range.

As was explained in Section 4, there is not complete equivalence between Eq. (12) and the result obtained with our algorithm. A constant factor of $\exp(-i\pi/4)/|\sin(p\pi/2)|^{1/2}$ is introduced instead of the constant $C_p$ [Eq. (13)] when the FRT is calculated with our algorithm. So, if a completely equivalent result is desired, a matching constant $M_p$,

$$M_p = \exp\left\{ -i\pi \operatorname{sgn}\left[ \sin\left( p \frac{\pi}{2} \right) \right] \Big/ 4 + ip \frac{\pi}{4} + \pi/4 \right\}, \tag{18}$$

has to be introduced. This phase is not important for experimental implementations because it cannot be measured. Nevertheless, it is important for theoretical calculations.

As an example of the application of our method, we have applied it to a rectangle function [see Fig. 2(a)] for which an analytic solution is possible. The results of the direct calculation are presented in Fig. 2(b). In Fig. 2(c) the calculation has been done with an algorithm based on the propagation of a GRIN fiber.[1,2] Finally, in Fig. 2(d) the FRT obtained through the algorithm introduced here is presented. The methods that yielded the results shown in both

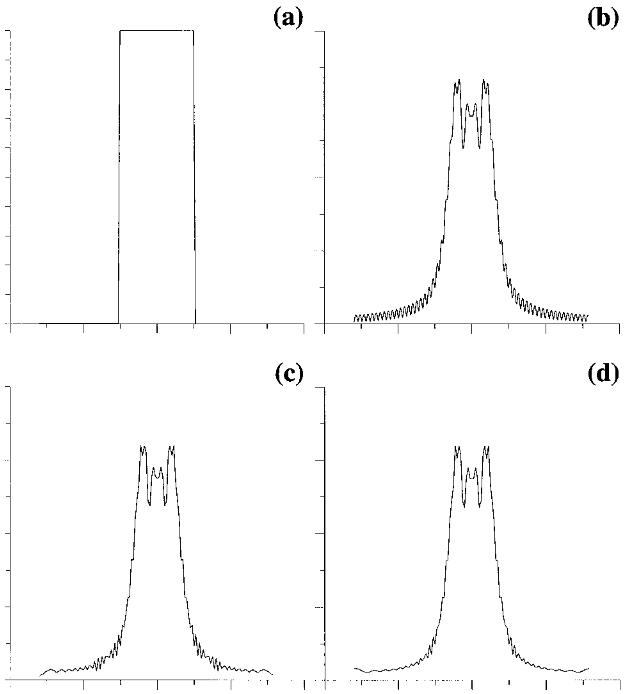


Fig. 2. (a) Input vector for the calculation of the FRT. (b) Modulus of the FRT of the order 0.5 of the vector in (a), calculated through direct integration. (c) Same as (b) but calculated through Hermite–Gaussian functions. (d) Same as (b) but calculated through the algorithm introduced in this paper.

Figs. 2(c) and 2(d) provide quite similar results, but the calculation time for our algorithm is significantly faster than the one based on Hermite–Gaussian functions. Using MATLAB routines on a PC computer for the implementation of these algorithms for an input vector of 128 elements shows that the time ratio is 15. As the complexity of the GRIN-media-based algorithm is of the order of $N^2$, whereas that of ours is $O(N \log N)$, the time ratio will dramatically increase with the size of the input.

Comparing these results with the exact calculation means that two different sources of error must be considered. First is the sampling error, which produces a loss of fine details. This effect appears when one discretizes the quadratic phase factors (or the Hermite–Gaussian function for the first method) and when one samples the output. The second is due to aliasing, which is especially noticeable for $x$ coordinates in the border of the image. To avoid these effects, oversampling the input matrix is advisable.[14] In this case, to obtain the original size of the matrix, decimation at the end of the algorithm is necessary.

## 6. Conclusions

The FFT has been used to obtain a numerical calculation of the FRT. The algorithm is based on the performance of the FRT by the application of a lens, free-space propagation, and a second lens. Free-space propagation is accomplished in the Fourier domain to avoid undesirable scaling. Two FFT's and multiplication by three quadratic phase factors are needed, the process being of the order of $O[N \log(N)]$. Because of the incomplete equivalence between the FRT's calculated through GRIN media and bulk optics, the introduction of a matching phase factor is discussed. The same principles can be used to calculate the Fresnel diffraction in the near and far fields, in this case with different algorithms.

Although we have designed the algorithm for one-dimensional signals, the results can easily be extended to two-dimensional cases, provided that all the operations performed are separable. Thus, performing the FRT of a bidimensional input matrix reduces to the calculation of the FRT in the $x$ and $y$ directions in cascade. The calculation time in this case is increased up to $O(N^2 \log N)$, in contrast with a complexity of $O(N^3)$ for a GRIN-media-based algorithm.

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