# Fractional generalization of gradient and Hamiltonian systems

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**Abstract**
We consider a fractional generalization of Hamiltonian and gradient systems.
We use differential forms and exterior derivatives of fractional orders. We
derive fractional generalization of Helmholtz conditions for phase space.
Examples of fractional gradient and Hamiltonian systems are considered. The
stationary states for these systems are derived.

PACS numbers: 45.20.−d, 05.45.−a

## 1. Introduction

Derivatives and integrals of fractional order [1, 2] have found many applications in recent
studies in physics. The interest in fractional analysis has been growing continually during
the past few years. Fractional analysis has numerous applications: kinetic theories [3, 4, 9],
statistical mechanics [10–12], dynamics in complex media [13–17] and many others [5–8].

The theory of derivatives of non-integer order goes back to Leibniz, Liouville, Grunwald,
Letnikov and Riemann. In the past few decades, many authors have pointed out that fractional-
order models are more appropriate than integer-order models for various real materials.
Fractional derivatives provide an excellent instrument for the description of memory and
hereditary properties of various materials and processes. This is the main advantage of
fractional derivatives in comparison with classical integer-order models in which such effects
are, in fact, neglected. The advantages of fractional derivatives become apparent in modelling
mechanical and electrical properties of real materials, as well as in the description of rheological
properties of rocks, and in many other fields.

In this paper, we use a fractional generalization of exterior calculus that was suggested in
[18, 19]. Fractional generalizations of differential forms and exterior derivatives were defined
in [18]. It allows us to consider the fractional generalization of Hamiltonian and gradient
dynamical systems [20, 21]. The suggested class of fractional gradient and Hamiltonian
systems is wider than the usual class of gradient and Hamiltonian dynamical systems. The

gradient and Hamiltonian systems can be considered as a special case of fractional gradient and Hamiltonian systems.

In section 2, a brief review of gradient systems and exterior calculus is considered to fix notation and provide a convenient reference. In section 3, a brief review of fractional (exterior) calculus will be given to fix notations and provide a convenient reference. In section 4, a definition of fractional generalization of gradient systems is suggested. In section 5, we consider a fractional gradient system that cannot be considered as a gradient system. In section 6, we prove that a dynamical system that is defined by the well-known Lorenz equations [23, 24] can be considered as a fractional gradient system. In section 7, a brief review of Hamiltonian systems is considered to fix notations and provide a convenient reference. In section 8, we consider the fractional generalization of Hamiltonian systems and Helmholtz conditions. In section 9, the simple example of fractional Hamiltonian systems is discussed. Finally, a short conclusion is given in section 10.

## 2. Gradient systems

In this section, a brief review of gradient systems and exterior calculus [21] is considered to fix notations and provide a convenient reference.

Gradient systems arise in dynamical systems theory [20–22]. They are described by the equation $d\mathbf{x}/dt = -\text{grad}\, V(x)$, where $\mathbf{x} \in R^n$. In Cartesian coordinates, the gradient is given by $\text{grad}\, V = \mathbf{e}_i \partial V/\partial x_i$, where $\mathbf{x} = \mathbf{e}_i x_i$. Here and later, we mean the sum on the repeated indices $i$ and $j$ from 1 to $n$.

**Definition 1.** *A dynamical system that is described by the equations*

$$\frac{dx_i}{dt} = F_i(x) \qquad (i = 1, \ldots, n) \tag{1}$$

*is called a gradient system in $R^n$ if the differential 1-form*

$$\omega = F_i(x)\, dx_i \tag{2}$$

*is an exact form $\omega = -dV$, where $V = V(x)$ is a continuously differentiable function (0-form).*

Here, d is the exterior derivative [21]. Let $V = V(x)$ be a real, continuously differentiable function on $R^n$. The exterior derivative of the function $V$ is the 1-form $dV = dx_i \partial V/\partial x_i$ written in a coordinate chart $(x_1, \ldots, x_n)$.

In mathematics [21], the concepts of closed form and exact form are defined for differential forms by the equation $d\omega = 0$ for a given form $\omega$ to be a closed form and $\omega = dh$ for an exact form. It is known that to be exact is a sufficient condition to be closed. In abstract terms, the question of whether this is also a necessary condition is a way of detecting topological information, by differential conditions.

Let us consider the 1-form (2). The formula for the exterior derivative d of differential form (2) is

$$d\omega = \frac{1}{2}\left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}\right) dx_j \wedge dx_i,$$

where $\wedge$ is the wedge product. Therefore, the condition for $\omega$ to be closed is

$$\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i} = 0.$$

In this case, if $V(x)$ is a potential function then $dV = dx_i \partial V / \partial x_i$. The implication from 'exact' to 'closed' is then a consequence of the symmetry of the second derivatives:

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i}. \tag{3}$$

If the function $V = V(x)$ is a smooth function, then the second derivative commutes, and equation (3) holds.

The fundamental topological result here is the Poincare lemma. It states that for a contractible open subset $X$ of $R^n$, any smooth $p$-form $\beta$ defined on $X$ that is closed, is also exact, for any integer $p > 0$ (this has content only when $p$ is at most $n$). This is not true for an open annulus in the plane, for some 1-forms $\omega$ that fail to extend smoothly to the whole disc, so that some topological condition is necessary. A space $X$ is contractible if the identity map on $X$ is homotopic to a constant map. Every contractible space is simply connected. A space is simply connected if it is path connected and every loop is homotopic to a constant map.

**Proposition 1.** *If a smooth vector field $\mathbf{F} = \mathbf{e}_i F_i(x)$ of system (1) satisfies the relations*

$$\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i} = 0 \tag{4}$$

*on a contractible open subset $X$ of $R^n$, then the dynamical system (1) is the gradient system such that*

$$\frac{dx_i}{dt} = -\frac{\partial V(x)}{\partial x_i}. \tag{5}$$

This proposition is a corollary of the Poincaré lemma. The Poincaré lemma states that for a contractible open subset $X$ of $R^n$, any smooth 1-form (2) defined on $X$ that is closed, is also exact.

The equations of motion for the gradient system on a contractible open subset $X$ of $R^n$ can be represented in the form (5). Therefore, the gradient systems can be defined by the potential function $V = V(x)$.

If the exact differential 1-form $\omega$ is equal to zero ($dV = 0$), then we get the equation

$$V(x) - C = 0, \tag{6}$$

which defines the stationary states of the gradient dynamical system (5). Here, $C$ is a constant.

## 3. Fractional differential forms

If the partial derivatives in the definition of the exterior derivative $d = dx_i \partial / \partial x_i$ are allowed to assume fractional order, a fractional exterior derivative can be defined [18] by the equation

$$d^\alpha = (dx_i)^\alpha \mathbf{D}_{x_i}^\alpha. \tag{7}$$

Here, we use the fractional derivative $\mathbf{D}_x^\alpha$ in the Riemann–Liouville form [1] that is defined by the equation

$$\mathbf{D}_x^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial x^m} \int_0^x \frac{f(y)\,dy}{(x-y)^{\alpha-m+1}}, \tag{8}$$

where $m$ is the first whole number greater than or equal to $\alpha$. The initial point of the fractional derivative [1] is set to zero. The derivative of powers $k$ of $x$ is

$$\mathbf{D}_x^\alpha x^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} x^{k-\alpha}, \tag{9}$$

where $k \geqslant 1$ and $\alpha \geqslant 0$. The derivative of a constant $C$ need not be zero

$$\mathbf{D}_x^\alpha C = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} C. \tag{10}$$

For example, the fractional exterior derivative of order $\alpha$ of $x_1^k$, with the initial point taken to be zero and $n = 2$, is given by

$$\mathrm{d}^\alpha x_1^k = (\mathrm{d}x_1)^\alpha \mathbf{D}_{x_1}^\alpha x_1^k + (\mathrm{d}x_2)^\alpha \mathbf{D}_{x_2}^\alpha x_1^k. \tag{11}$$

Using equation (9), we get the following relation for the fractional exterior derivative of $x_1^k$:

$$\mathrm{d}^\alpha x_1^k = (\mathrm{d}x_1)^\alpha \frac{\Gamma(k+1)x_1^{k-\alpha}}{\Gamma(k+1-\alpha)} + (\mathrm{d}x_2)^\alpha \frac{x_1^k x_2^{-\alpha}}{\Gamma(1-\alpha)}.$$

## 4. Fractional gradient systems

A fractional generalization of exterior calculus was suggested in [18, 19]. A fractional exterior derivative and the fractional differential forms were defined [18]. It allows us to consider the fractional generalization of gradient systems.

Let us consider a dynamical system that is defined by the equation $\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{F}$, on a subset $X$ of $R^n$. In Cartesian coordinates, we can use the following equation:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = F_i(x), \tag{12}$$

where $i = 1, \ldots, n$, $\mathbf{x} = \mathbf{e}_i x_i$ and $\mathbf{F} = \mathbf{e}_i F_i(x)$. The fractional analogue of definition 1 has the form

**Definition 2.** *A dynamical system (12) is called a fractional gradient system if the fractional differential 1-form*

$$\omega_\alpha = F_i(x)(\mathrm{d}x_i)^\alpha \tag{13}$$

*is an exact fractional form $\omega_\alpha = -\mathrm{d}^\alpha V$, where $V = V(x)$ is a continuously differentiable function.*

Using the definition of the fractional exterior derivative, equation (13) can be represented as

$$\omega_\alpha = -\mathrm{d}^\alpha V = -(\mathrm{d}x_i)^\alpha \mathbf{D}_{x_i}^\alpha V.$$

Therefore, we have $F_i(x) = -\mathbf{D}_{x_i}^\alpha V$.

Note that equation (13) is a fractional generalization of equation (2). If $\alpha = 1$, then equation (13) leads us to equation (2). Obviously, a fractional 1-form $\omega_\alpha$ can be closed when the 1-form $\omega = \omega_1$ is not closed. The fractional generalization of the Poincaré lemma is considered in [19]. Therefore, we have the following proposition.

**Proposition 2.** *If a smooth vector field $\mathbf{F} = \mathbf{e}_i F_i(x)$ on a contractible open subset $X$ of $R^n$ satisfies the relations*

$$\mathbf{D}_{x_j}^\alpha F_i - \mathbf{D}_{x_i}^\alpha F_j = 0, \tag{14}$$

*then the dynamical system (12) is a fractional gradient system such that*

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -\mathbf{D}_{x_i}^\alpha V(x), \tag{15}$$

*where $V(x)$ is a continuous differentiable function and $\mathbf{D}_{x_i}^\alpha V = -F_i$.*

**Proof.**   This proposition is a corollary of the fractional generalization of Poincaré lemma [19]. The Poincare lemma is shown [18, 19] to be true for the exterior fractional derivative.

Relations (14) are the fractional generalization of relations (4). Note that the fractional derivative of a constant need not be zero (10). Therefore, we see that constants $C$ in the equation $V(x) = C$ cannot define a stationary state of the gradient system (15). It is easy to see that

$$\mathbf{D}^{\alpha}_{x_i} V(x) = \mathbf{D}^{\alpha}_{x_i} C = \frac{x_i^{-\alpha}}{\Gamma(1-\alpha)} C \neq 0.$$

In order to define stationary states of fractional gradient systems, we consider the solutions of the system of the equations $\mathbf{D}^{\alpha}_{x_i} V(x) = 0$. □

**Proposition 3.** *The stationary states of gradient system (15) are defined by the equation*

$$V(x) - \left| \prod_{i=1}^{n} x_i \right|^{\alpha-m} \sum_{k_1=0}^{m-1} \cdots \sum_{k_n=0}^{m-1} C_{k_1,\ldots,k_n} \left( \prod_{i=1}^{n} (x_i)^{k_i} \right) = 0. \tag{16}$$

*The $C_{k_1,\ldots,k_n}$ are constants and m is the first whole number greater than or equal to $\alpha$.*

**Proof.**   In order to define the stationary states of a fractional gradient system, we consider the solution of the equation

$$\mathbf{D}^{\alpha}_{x_i} V(x) = 0. \tag{17}$$

This equation can be solved by using equation (8). Let $m$ be the first whole number greater than or equal to $\alpha$; then we have the solution [1, 2] of equation (17) in the form

$$V(x) = |x_i|^{\alpha} \sum_{k=0}^{m-1} a_k(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)(x_i)^k, \tag{18}$$

where $a_k$ are functions of the other coordinates. Using equation (18) for $i = 1, \ldots, n$, we get the solution of the system of equation (17) in the form (16).

If we consider $n = 2$ such that $x = x_1$ and $y = x_2$, we have the equations of motion for the fractional gradient system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\mathbf{D}^{\alpha}_x V(x, y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\mathbf{D}^{\alpha}_y V(x, y). \tag{19}$$

The stationary states of this system are defined by the equation

$$V(x, y) - |xy|^{\alpha-1} \sum_{k=0}^{m-1} \sum_{l=0}^{m-1} C_{kl} x^k y^l = 0.$$

The $C_{kl}$ are constants and $m$ is the first whole number greater than or equal to $\alpha$. □

## 5. Examples of fractional gradient system

In this section, we consider a fractional gradient systems that cannot be considered as a gradient system. We prove that the class of fractional gradient systems is wider than the usual class of gradient dynamical systems. The gradient systems can be considered as a special case of fractional gradient systems.

**Example 1.**   Let us consider the dynamical system that is defined by the equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F_x, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = F_y, \tag{20}$$

where the right-hand sides have the form

$$F_x = acx^{1-k} + bx^{-k}, \qquad F_y = (ax + b)y^{-k}, \tag{21}$$

where $a \neq 0$. This system cannot be considered as a gradient dynamical system. Using

$$\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} = ay^{-k} \neq 0,$$

we get that $\omega = F_x \, dx + F_y \, dy$ is not closed form

$$d\omega = -ay^{-k} dx \wedge dy.$$

Note that relation (14) in the form

$$\mathbf{D}_y^\alpha F_x - \mathbf{D}_x^\alpha F_y = 0$$

is satisfied for the system (21), if $\alpha = k$ and the constant $c$ is defined by

$$c = \frac{\Gamma(1 - \alpha)}{\Gamma(2 - \alpha)}.$$

Therefore, this system can be considered as a fractional gradient system with the linear potential function

$$V(x, y) = \Gamma(1 - \alpha)(ax + b),$$

where $\alpha = k$.

**Example 2.** Let us consider the dynamical system that is defined by equation (20) with

$$F_x = an(n - 1)x^{n-2} + ck(k - 1)x^{k-2}y^l, \tag{22}$$

$$F_y = bm(m - 1)y^{m-2} + cl(l - 1)x^k y^{l-2}, \tag{23}$$

where $k \neq 1$ and $l \neq 1$. It is easy to derive that

$$\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} = cklx^{k-2}y^{l-2}[(k - 1)y - (l - 1)x] \neq 0,$$

and the differential form $\omega = F_x \, dx + F_y \, dy$ is not closed $d\omega \neq 0$. Therefore, this system is not a gradient dynamical system. Using conditions (14) in the form

$$\mathbf{D}_y^2 F_x - \mathbf{D}_x^2 F_y = \frac{\partial^2 F_x}{\partial y^2} - \frac{\partial^2 F_y}{\partial x^2} = 0,$$

we get $d^\alpha \omega = 0$ for $\alpha = 2$. As a result, we have that this system can be considered as a fractional gradient system with the potential function

$$V(x, y) = ax^n + by^m + cx^k y^l.$$

In the general case, the fractional gradient system cannot be considered as a gradient system. The gradient systems can be considered as a special case of fractional gradient systems such that $\alpha = 1$.

## 6. Lorenz system as a fractional gradient system

In this section, we prove that dynamical systems that are defined by the well-known Lorenz equations [23, 24] are the fractional gradient system.

The well-known Lorenz equations [23, 24] are defined by

$$\frac{dx}{dt} = F_x, \qquad \frac{dy}{dt} = F_y, \qquad \frac{dz}{dt} = F_z,$$

where the right-hand sides $F_x$, $F_y$ and $F_z$ have the forms

$$F_x = \sigma(y - x), \qquad F_y = (r - z)x - y, \qquad F_z = xy - bz.$$

The parameters $\sigma$, $r$ and $b$ can be equal to the following values:

$$\sigma = 10, \qquad b = 8/3, \qquad r = 470/19 \simeq 24.74.$$

The dynamical system which is defined by the Lorenz equations cannot be considered as a gradient dynamical system. It is easy to see that

$$\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} = z + \sigma - r, \qquad \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = -y, \qquad \frac{\partial F_y}{\partial z} - \frac{\partial F_z}{\partial y} = -2x.$$

Therefore, $\omega = F_x\, dx + F_y\, dy + F_z\, dz$ is not a closed 1-form and we have

$$d\omega = -(z + \sigma - r)\, dx \wedge dy + y\, dx \wedge dz + 2x\, dy \wedge dz.$$

For the Lorenz equations, conditions (14) can be satisfied in the form

$$\mathbf{D}_y^2 F_x - \mathbf{D}_x^2 F_y = 0, \qquad \mathbf{D}_z^2 F_x - \mathbf{D}_x^2 F_z = 0, \qquad \mathbf{D}_z^2 F_y - \mathbf{D}_y^2 F_z = 0.$$

As a result, we get that the Lorenz system can be considered as a fractional gradient dynamical system with the potential function

$$V(x, y, z) = \frac{1}{6}\sigma x^3 - \frac{1}{2}\sigma y x^2 + \frac{1}{2}(z - r)xy^2 + \frac{1}{6}y^3 - \frac{1}{2}xyz^2 + \frac{b}{6}z^3. \qquad (24)$$

The potential (24) uniquely defines the Lorenz system. Using equation (16), we can get that the stationary states of the Lorenz system are defined by the equation

$$V(x, y, z) + C_{00} + C_x x + C_y y + C_z z + C_{xy}xy + C_{xz}xz + C_{yz}yz = 0, \qquad (25)$$

where $C_{00}$, $C_x$, $C_y$, $C_z C_{xy}$, $C_{xz}$ and $C_{yz}$ are the constants and $\alpha = m = 2$. The plot of stationary states of the Lorenz system with the constants $C_{00} = 1$, $C_x = C_y = C_z = C_{xy} = C_{xz} = C_{yz} = 0$ and parameters $\sigma = 10$, $b = 3$ and $r = 25$ is shown in figures 1 and 2.

Note that the Rossler system [25], which is defined by the equations

$$\frac{dx}{dt} = -(y + z), \qquad \frac{dx}{dt} = x + 0.2y, \qquad \frac{dz}{dt} = 0.2 + (x - c)z,$$

can be considered as a fractional gradient system with the potential function

$$V(x, y, z) = \frac{1}{2}(y + z)x^2 - \frac{1}{2}xy^2 - \frac{1}{30}y^3 - \frac{1}{10}z^2 - \frac{1}{6}(x - c)z^3. \qquad (26)$$

This potential uniquely defines the Rossler system. The stationary states of the Rossler system are defined by equation (25), where the potential function is defined by (26). The plot of stationary states of the Rossler system for the constants $C_{00} = 1$, $C_x = C_y = C_z = C_{xy} = C_{xz} = C_{yz} = 0$ and parameter $c = 1$ is shown in figures 3 and 4.

Let us note the interesting qualitative property of surfaces (25) which is difficult to see from the figures. The surfaces of the stationary states of the Lorenz and Rossler systems separate the three-dimensional Euclidean space into some number of areas. We have eight areas for the Lorenz system and four areas for the Rossler system. This separation has the interesting property for some values of parameters. All regions are connected with each other. Beginning movement from one of the areas, it is possible to appear in any other area, not crossing a surface. Any two points from different areas can be connected by a curve which does not cross a surface. It is difficult to see this property from figures 1–4.
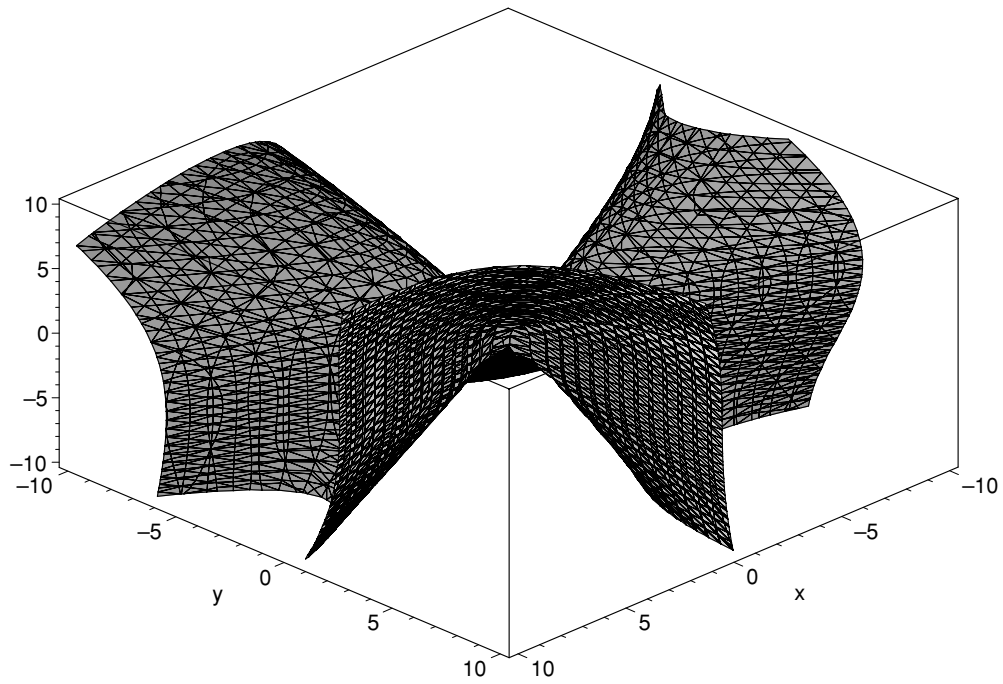
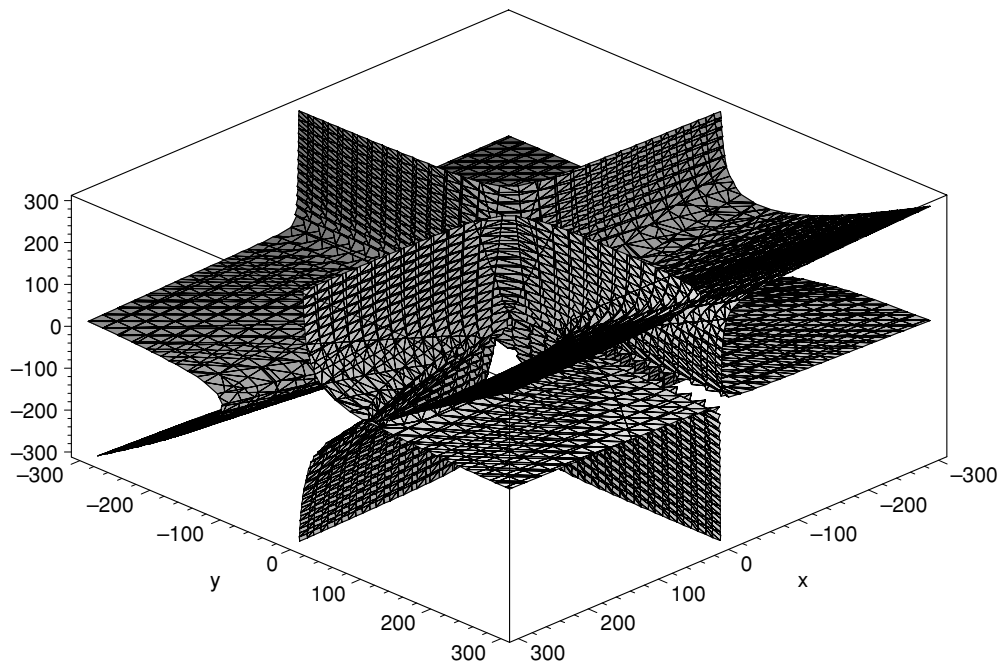**Figure 1.** Stationary states of the Lorenz system.



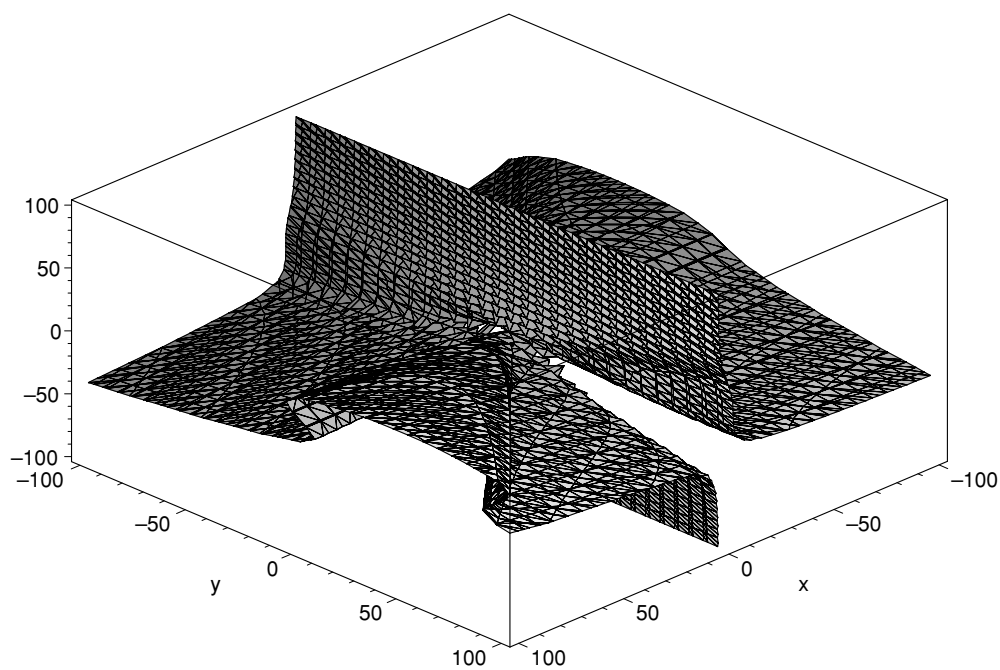**Figure 2.** Stationary states of the Lorenz system.

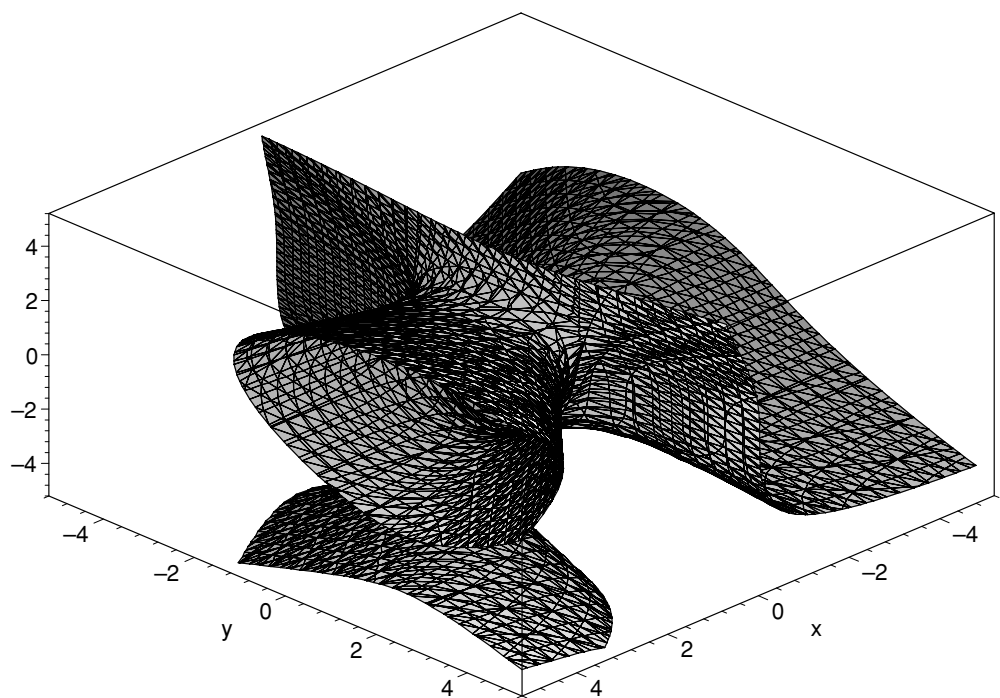**Figure 3.** Stationary states of the Rossler system.



**Figure 4.** Stationary states of the Rossler system.

## 7. Hamiltonian systems

In this section, a brief review of Hamiltonian systems is considered to fix notations and provide a convenient reference.

Let us consider the canonical coordinates $(q^1, \ldots, q^n, p^1, \ldots, p^n)$ in the phase space $R^{2n}$. We consider a dynamical system that is defined by the equations

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = G^i(q, p), \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = F^i(q, p). \tag{27}$$

The definition of Hamiltonian systems can be realized in the following form [27, 28].

**Definition 3.** *A dynamical system (27) on the phase space $R^{2n}$ is called a Hamiltonian system if the differential 1-form*

$$\beta = G^i \, \mathrm{d}p_i - F^i \, \mathrm{d}q_i \tag{28}$$

*is a closed form $\mathrm{d}\beta = 0$, where $\mathrm{d}$ is the exterior derivative. A dynamical system is called a non-Hamiltonian system if the differential 1-form $\beta$ is nonclosed $\mathrm{d}\beta \neq 0$.*

The exterior derivative for the phase space is defined as

$$\mathrm{d} = \mathrm{d}q_i \frac{\partial}{\partial q_i} + \mathrm{d}p_i \frac{\partial}{\partial p_i}. \tag{29}$$

Here and later, we mean the sum on the repeated indices $i$ and $j$ from 1 to $n$.

**Proposition 4.** *If the right-hand sides of equations (27) satisfy the Helmholtz conditions [26–28] for the phase space, which have the following forms:*

$$\frac{\partial G^i}{\partial p_j} - \frac{\partial G^j}{\partial p_i} = 0, \tag{30}$$

$$\frac{\partial G^j}{\partial q_i} + \frac{\partial F^i}{\partial p_j} = 0, \tag{31}$$

$$\frac{\partial F^i}{\partial q_j} - \frac{\partial F^j}{\partial q_i} = 0, \tag{32}$$

*then the dynamical system (27) is a Hamiltonian system.*

**Proof.** In the canonical coordinates $(q, p)$, the vector fields that define the system have the components $(G^i, F^i)$, which are used in equation (27). Let us consider the 1-form that is defined by the equation

$$\beta = G^i \, \mathrm{d}p_i - F^i \, \mathrm{d}q_i.$$

The exterior derivative for this form can be written by the relation

$$\mathrm{d}\beta = \mathrm{d}(G^i \, \mathrm{d}p_i) - \mathrm{d}(F^i \, \mathrm{d}q_i).$$

It now follows that

$$\mathrm{d}\beta = \frac{\partial G^i}{\partial q_j} \, \mathrm{d}q_j \wedge \mathrm{d}p_i + \frac{\partial G^i}{\partial p_j} \, \mathrm{d}p_j \wedge \mathrm{d}p_i - \frac{\partial F^i}{\partial q_j} \, \mathrm{d}q_j \wedge \mathrm{d}q_i - \frac{\partial F^i}{\partial p_j} \, \mathrm{d}p_j \wedge \mathrm{d}q_i.$$

Here, $\wedge$ is the wedge product. This equation can be rewritten in an equivalent form as

$$\mathrm{d}\beta = \left( \frac{\partial G^j}{\partial q_i} + \frac{\partial F^i}{\partial p_j} \right) \mathrm{d}q_i \wedge \mathrm{d}p_j + \frac{1}{2} \left( \frac{\partial G^j}{\partial p_i} - \frac{\partial G^i}{\partial p_j} \right) \mathrm{d}p_i \wedge \mathrm{d}p_j + \frac{1}{2} \left( \frac{\partial F^i}{\partial q_j} - \frac{\partial F^j}{\partial q_i} \right) \mathrm{d}q_i \wedge \mathrm{d}q_j.$$

Here, we use the skew symmetry of $dq_i \wedge dq_j$ and $dp_i \wedge dp_j$ with respect to the indices $i$ and $j$. It is obvious that conditions (30)–(32) lead to the equation $d\beta = 0$.                    □

Some of Hamiltonian systems can be defined by the unique function.

**Proposition 5.** *A dynamical system (27) on the phase space $R^{2n}$ is a Hamiltonian system that is defined by the Hamiltonian $H = H(q, p)$ if the differential 1-form*

$$\beta = G^i \, dp_i - F^i \, dq_i$$

*is an exact form $\beta = dH$, where d is the exterior derivative and $H = H(q, p)$ is a continuous differentiable unique function on the phase space.*

**Proof.** Suppose that the differential 1-form $\beta$, which is defined by equation (28), has the form

$$\beta = dH = \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i.$$

In this case, vector fields $(G^i, F^i)$ can be represented in the form

$$G^i(q, p) = \frac{\partial H}{\partial p_i}, \qquad F^i(q, p) = -\frac{\partial H}{\partial q_i}.$$

If $H = H(q, p)$ is a continuous differentiable function, then conditions (30)–(32) are satisfied. Using proposition 4, we get that this system is a Hamiltonian system. The equations of motion for the Hamiltonian system (27) can be written in the form

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \tag{33}$$

which is uniquely defined by the Hamiltonian $H$.

If the exact differential 1-form $\beta$ is equal to zero ($dH = 0$), then the equation

$$H(q, p) - C = 0 \tag{34}$$

defines the stationary states of the Hamiltonian system (27). Here, $C$ is a constant.                    □

## 8. Fractional Hamiltonian systems

Fractional generalization of the differential form (28), which is used in the definition of the Hamiltonian system, can be defined in the following form:

$$\beta_\alpha = G^i (dp_i)^\alpha - F^i (dq_i)^\alpha.$$

Let us consider the canonical coordinates $(x^1, \ldots, x^n, x^{n+1}, \ldots, x^{2n}) = (q^1, \ldots, q^n, p^1, \ldots, p^n)$ in the phase space $R^{2n}$ and a dynamical system that is defined by the equations

$$\frac{dq_i}{dt} = G^i(q, p), \qquad \frac{dp_i}{dt} = F^i(q, p). \tag{35}$$

The fractional generalization of Hamiltonian systems can be defined by using fractional generalization of differential forms [18].

**Definition 4.** *A dynamical system (35) on the phase space $R^{2n}$ is called a fractional Hamiltonian system if the fractional differential 1-form*

$$\beta_\alpha = G^i (dp_i)^\alpha - F^i (dq_i)^\alpha$$

*is a closed fractional form*

$$d^\alpha \beta_\alpha = 0, \tag{36}$$

*where* $d^\alpha$ *is the fractional exterior derivative. A dynamical system is called a fractional non-Hamiltonian system if the fractional differential 1-form $\beta_\alpha$ is a nonclosed fractional form* $d^\alpha \beta_\alpha \neq 0$.

The fractional exterior derivative for the phase space $R^{2n}$ is defined as

$$d^\alpha = (dq_i)^\alpha \mathbf{D}^\alpha_{q_i} + (dp_i)^\alpha \mathbf{D}^\alpha_{p_i}. \tag{37}$$

For example, the fractional exterior derivative of order $\alpha$ of $q^k$, with the initial point taken to be zero and $n = 2$, is given by

$$d^\alpha q^k = (dq)^\alpha \mathbf{D}^\alpha_q q^k + (dp)^\alpha \mathbf{D}^\alpha_p q^k. \tag{38}$$

Using equations (9) and (10), we have the following relation for the fractional exterior derivative (37):

$$d^\alpha q^k = (dq)^\alpha \frac{\Gamma(k+1)q^{k-\alpha}}{\Gamma(k+1-\alpha)} + (dp)^\alpha \frac{q^k p^{-\alpha}}{\Gamma(1-\alpha)}.$$

Let us consider a fractional generalization of the Helmholtz conditions.

**Proposition 6.** *If the right-hand sides of equations (35) satisfy the fractional generalization of the Helmholtz conditions in the following form:*

$$\mathbf{D}^\alpha_{p_j} G^i - \mathbf{D}^\alpha_{p_i} G^j = 0, \tag{39}$$

$$\mathbf{D}^\alpha_{q_i} G^j + \mathbf{D}^\alpha_{p_j} F^i = 0, \tag{40}$$

$$\mathbf{D}^\alpha_{q_j} F^i - \mathbf{D}^\alpha_{q_i} F^j = 0, \tag{41}$$

*then dynamical system (35) is a fractional Hamiltonian system.*

**Proof.**   In the canonical coordinates $(q, p)$, the vector fields that define the system have the components $(G^i, F^i)$, which are used in equation (27). The 1-form $\beta_\alpha$ is defined by the equation

$$\beta_\alpha = G^i (dp_i)^\alpha - F^i (dq_i)^\alpha. \tag{42}$$

The exterior derivative for this form can now be given by the relation

$$d^\alpha \beta_\alpha = d^\alpha (G^i (dp_i)^\alpha) - d^\alpha (F^i (dq_i)^\alpha).$$

Using the rule

$$\mathbf{D}^\alpha_x (fg) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \left( \mathbf{D}^{\alpha-k}_x f \right) \frac{\partial^k g}{\partial x^k}$$

and the relation

$$\frac{\partial^k}{\partial x^k} ((dx_i)^\alpha) = 0 \quad (k \geqslant 1),$$

we get that

$$d^\alpha (A^i (dx_i)^\alpha) = \sum_{k=0}^{\infty} (dx_j)^\alpha \wedge \binom{\alpha}{k} \left( \mathbf{D}^{\alpha-k}_{x_j} A^i \right) \frac{\partial^k}{\partial x_j^k} (dx_i)^\alpha = (dx_j)^\alpha \wedge (dx_i)^\alpha \binom{\alpha}{0} \left( \mathbf{D}^\alpha_{x_j} A^i \right).$$

Here, we use

$$\binom{\alpha}{k} = \frac{(-1)^{k-1} \alpha \Gamma(k-\alpha)}{\Gamma(1-\alpha) \Gamma(k+1)}.$$

Therefore, we have

$$\mathrm{d}^\alpha \beta_\alpha = \mathbf{D}^\alpha_{q_j} G^i (\mathrm{d}q_j)^\alpha \wedge (\mathrm{d}p_i)^\alpha + \mathbf{D}^\alpha_{p_j} G^i (\mathrm{d}p_j)^\alpha \wedge (\mathrm{d}p_i)^\alpha$$
$$- \mathbf{D}^\alpha_{q_j} F^i (\mathrm{d}q_j)^\alpha \wedge (\mathrm{d}q_i)^\alpha - \mathbf{D}^\alpha_{p_j} F^i (\mathrm{d}p_j)^\alpha \wedge (\mathrm{d}q_i)^\alpha.$$

This equation can be rewritten in an equivalent form

$$\mathrm{d}^\alpha \beta_\alpha = \left(\mathbf{D}^\alpha_{q_i} G^j + \mathbf{D}^\alpha_{p_j} F^i\right)(\mathrm{d}q_i)^\alpha \wedge (\mathrm{d}p_j)^\alpha + \tfrac{1}{2}\left(\mathbf{D}^\alpha_{p_i} G^j - \mathbf{D}^\alpha_{p_j} G^i\right)(\mathrm{d}p_i)^\alpha \wedge (\mathrm{d}p_j)^\alpha$$
$$+ \tfrac{1}{2}\left(\mathbf{D}^\alpha_{q_j} F^i - \mathbf{D}^\alpha_{q_i} F^j\right)(\mathrm{d}q_i)^\alpha \wedge (\mathrm{d}q_j)^\alpha.$$

Here, we use the skew symmetry of $\wedge$. It is obvious that conditions (39)–(41) lead to the equation $\mathrm{d}^\alpha \beta_\alpha = 0$, i.e., $\beta_\alpha$ is a closed fractional form. $\qquad\square$

Let us define the Hamiltonian for the fractional Hamiltonian systems.

**Proposition 7.** *A dynamical system (35) on the phase space $R^{2n}$ is a fractional Hamiltonian system that is defined by the Hamiltonian $H = H(q, p)$ if the fractional differential 1-form*

$$\beta_\alpha = G^i (\mathrm{d}p_i)^\alpha - F^i (\mathrm{d}q_i)^\alpha$$

*is an exact fractional form*

$$\beta_\alpha = \mathrm{d}^\alpha H, \tag{43}$$

*where $\mathrm{d}^\alpha$ is the fractional exterior derivative and $H = H(q, p)$ is a continuous differentiable function on the phase space.*

**Proof.** Suppose that the fractional differential 1-form $\beta_\alpha$, which is defined by equation (42), has the form

$$\beta_\alpha = \mathrm{d}^\alpha H = (\mathrm{d}p_i)^\alpha \mathbf{D}^\alpha_{p_i} H + (\mathrm{d}q_i)^\alpha \mathbf{D}^\alpha_{q_i} H.$$

In this case, vector fields $(G^i, F^i)$ can be represented in the form

$$G^i(q, p) = \mathbf{D}^\alpha_{p_i} H, \qquad F^i(q, p) = -\mathbf{D}^\alpha_{q_i} H.$$

Therefore, the equations of motion for fractional Hamiltonian systems can be written in the form

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \mathbf{D}^\alpha_{p_i} H, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\mathbf{D}^\alpha_{q_i} H. \tag{44}$$

The fractional differential 1-form $\beta_\alpha$ for the fractional Hamiltonian system with Hamiltonian $H$ can be written in the form $\beta_\alpha = \mathrm{d}^\alpha H$. If the exact fractional differential 1-form $\beta_\alpha$ is equal to zero ($\mathrm{d}^\alpha H = 0$), then we can get the equation that defines the stationary states of the Hamiltonian system. $\qquad\square$

**Proposition 8.** *The stationary states of the fractional Hamiltonian system (44) are defined by the equation*

$$H(q, p) - \left| \prod_{i=1}^n q_i p_i \right|^{\alpha - m} \sum_{k_1=0, l_1=0}^{m-1} \cdots \sum_{k_n=0, l_n=0}^{m-1} C_{k_1,\ldots,k_n,l_1,\ldots,l_n} \prod_{i=1}^n (q_i)^{k_i} (p_i)^{l_i} = 0, \tag{45}$$

*where $C_{k_1,\ldots,k_n,l_1,\ldots,l_n}$ are constants and $m$ is the first whole number greater than or equal to $\alpha$.*

**Proof.** This proposition is a corollary of proposition 3. $\qquad\square$

## 9. Example of fractional Hamiltonian system

Let us consider a dynamical system in the phase space $R^2$ ($n = 1$) that is defined by the equation

$$\frac{dq}{dt} = \mathbf{D}_p^\alpha H, \qquad \frac{dp}{dt} = -\mathbf{D}_q^\alpha H, \tag{46}$$

where the fractional order $0 < \alpha \leqslant 1$ and the Hamiltonian $H(q, p)$ has the form

$$H(q, p) = ap^2 + bq^2. \tag{47}$$

If $\alpha = 1$, then equation (46) describes the linear harmonic oscillator.

If the exact fractional differential 1-form

$$\beta_\alpha = d^\alpha H = (dp)^\alpha \mathbf{D}_p^\alpha H + (dq)^\alpha \mathbf{D}_q^\alpha H$$

is equal to zero ($d^\alpha H = 0$), then the equation

$$H(q, p) - C|qp|^{\alpha-1} = 0$$

defines the stationary states of the system (46). Here, $C$ is a constant. If $\alpha = 1$, we get the usual stationary-state equation (34).

Using equation (47), we get the following equation for stationary states:

$$|qp|^{1-\alpha}(ap^2 + bq^2) = C. \tag{48}$$

If $\alpha = 1$, then we get the equation $ap^2 + bq^2 = C$, which describes the ellipse.

## 10. Conclusion

Fractional derivatives and integrals [1, 2] have found many applications in recent studies in physics. The interest in fractional analysis has been growing continually during the past few years [3–17]. Using the fractional derivatives and fractional differential forms, we consider the fractional generalization of gradient and Hamiltonian systems. In the general case, the fractional gradient and Hamiltonian systems cannot be considered as gradient and Hamiltonian systems. The class of fractional gradient and Hamiltonian systems is wider than the usual class of gradient and Hamiltonian dynamical systems. The gradient and Hamiltonian systems can be considered as a special case of fractional gradient and Hamiltonian systems. Therefore, it is possible to generalize the application of catastrophe and bifurcation theory from gradient to a wider class of fractional gradient dynamical systems. Note that quantization of the fractional Hamiltonian systems can be realized by the method suggested in [29–32].

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