Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **5** (2011), 110–121. doi:10.2298/AADM110131002F

FRACTIONAL *h*-DIFFERENCE EQUATIONS ARISING FROM THE CALCULUS OF VARIATIONS

Rui A. C. Ferreira, Delfim F. M. Torres

The recent theory of fractional h-difference equations introduced in [**9**], is enriched with useful tools for the explicit solution of discrete equations involving left and right fractional difference operators. New results for the right fractional h sum are proved. Illustrative examples show the effectiveness of the obtained results in solving fractional discrete Euler–Lagrange equations.

1. INTRODUCTION

The fractional calculus is a generalization of (integer order) differential calculus, allowing to define derivatives (and integrals) of real or complex order [**25, 30**]. It is a mathematical subject that has proved to be very useful in applied fields such as economics, engineering, and physics [**3, 21, 22, 29**]. Several definitions of fractional derivatives, including Riemann–Liouville, Caputo, Riesz, Riesz-Caputo, Weyl, Grunwald–Letnikov, Hadamard, and Chen derivatives, are available in the literature [**2, 14, 17, 26**]. The most common used fractional derivative is the Riemann–Liouville [**1, 13, 16, 27**]. Analogously, one can define a discrete fractional derivative in different ways. In 1989, Miller and Ross introduced the discrete analogue of the Riemann–Liouville fractional derivative and proved some properties of the fractional difference operator [**24**]. More results on the theory introduced by Miller and Ross are given in the works of Atici and Eloe [**5, 6**]. See also [**10, 11, 18**], and [**7**] for applications to the Gompertz fractional difference equation and tumor growth models. Regarding other fractional discrete definitions, we refer the reader to [**4, 8, 19, 28**] and references therein. Here we follow [**9**], i.e., we adopt a more general fractional h-difference Riemann–Liouville operator. The presence of the h parameter is particularly interesting from the numerical point of view,

2010 Mathematics Subject Classification. Primary. 39A12. Secondary 49J05, 49K05.
Keywords and Phrases. Fractional discrete calculus, fractional difference calculus of variations, Euler–Lagrange equations, explicit solutions.

because when h tends to zero the solutions of the fractional difference equations can be seen as approximations to the solutions of corresponding Riemann–Liouville fractional differential equations [**9, 15**] (cf. Proposition 3).

In the recent work of Bastos et al. [**9**], necessary optimality conditions of first and second order are proved for the fractional h-difference variational problem

$$\mathcal{L}(y) = \sum_{t=a/h}^{b/h-1} L\big(th, y(\sigma_h(th)), ({}_a\Delta_h^\alpha y)(th), ({}_h\Delta_b^\alpha y)(th)\big)h \longrightarrow \min\,, \qquad (1)$$
$$(y(a) = A),\ (y(b) = B),$$

as well as transversality conditions when the boundary conditions $y(a) = A$ or $y(b) = B$ are not given (see Section 2 for definitions and notations). The main result of [**9**] gives an Euler–Lagrange type equation for problem (1), but no clues are devised for the solution of such fractional h-difference equations. Instead, some examples are solved numerically [**8, 9**]. Here we develop further the subject of the calculus of variations within the fractional discrete setting, by obtaining explicit solutions to the fractional difference Euler–Lagrange equations [**8, 9**].

Our results are given in Section 2, where we prove some new formulas for the fractional h-difference operator. The obtained results are then used in Section 3 to solve two illustrative examples of (1), for which the global minimizers are explicitly found in exact form. This is in contrast with [**8, 9**], where all the solutions are obtained via approximated numerical computations.

2. MAIN RESULTS

Before stating and proving our results, we introduce some definitions and notations. Let $h > 0$ and put $\mathbb{T} = \{a, a + h, a + 2h, \ldots, b\}$ with $a \in \mathbb{R}$ and $b = a + kh$ for $k \in \{2, 3, \ldots\}$. Let us denote by $\mathcal{F}_\mathbb{T}$ the set of real valued functions defined on $\mathbb{T}$, $\sigma_h(t) = t + h$, and $\rho_h(t) = t - h$.

Definition 1. *For a function $f \in \mathcal{F}_\mathbb{T}$ the forward h-difference operator is defined as*

$$(\Delta_h f)(t) = \frac{f(\sigma_h(t)) - f(t)}{h}\,, \quad t \in \{a, a + h, a + 2h, \ldots, \rho_h(b)\},$$

while the h-difference sum is given by

$$({}_a\Delta_h^{-1} f)(t) = \sum_{k=a/h}^{t/h-1} f(kh)h\,, \quad t \in \{a, a + h, a + 2h, \ldots, \sigma_h(b)\}.$$

Definition 2. *For arbitrary $x, y \in \mathbb{R}$ the h-factorial function is defined by*

$$x_h^{(y)} = h^y \frac{\Gamma\left(\frac{x}{h} + 1\right)}{\Gamma\left(\frac{x}{h} + 1 - y\right)}\,,$$

where Γ is the Euler gamma function. We use the convention that division at a pole yields zero.

In [**9**] it is remarked that in the case $h = 1$, then $x_h^{(y)}$ coincides with the falling factorial power. One also expects to see that $x_h^{(y)}$ converges to x^y when h tends to zero. Since this is not addressed in [**9**], we prove it here.

Proposition 3. *For $x \geq 0$ and $y \in \mathbb{R}$,*

$$\lim_{h\to 0} x_h^{(y)} = x^y. \tag{2}$$

Proof. Equality (2) is a straightforward consequence of the following well-known asymptotic formula for the Gamma function:

$$\lim_{x\to+\infty} \frac{\Gamma(x+\beta)}{\Gamma(x)} = (x+\beta-1)^\beta, \quad \beta \in \mathbb{R}$$

(see, e.g., inequality (33) and Corollary 3 in [**23**]). Indeed, starting from the definition of $x_h^{(y)}$ and introducing the new variable $t = x/h + 1 - y$, we have

$$x_h^{(y)} = \frac{x^y}{(t+y-1)^y} \cdot \frac{\Gamma(t+y)}{\Gamma(t)}$$

for any $y \in \mathbb{R}$. We obtain (2) taking the limit $t \to +\infty$ or, equivalently, the limit $h \to 0^+$. $\square$

The motivation for the next definition can be found in [**9**].

Definition 4 ([**9**])**.** *Let $f \in \mathcal{F}_\mathbb{T}$. The* left and right fractional h-sum *of order $\nu > 0$ are, respectively, the operators ${}_a\Delta_h^{-\nu} : \mathcal{F}_\mathbb{T} \to \mathcal{F}_{\widetilde{\mathbb{T}}_\nu^+}$ and ${}_h\Delta_b^{-\nu} : \mathcal{F}_\mathbb{T} \to \mathcal{F}_{\widetilde{\mathbb{T}}_\nu^-}$, $\widetilde{\mathbb{T}}_\nu^\pm = \{a \pm \nu h, a \pm \nu h \pm h, \ldots, b \pm \nu h\}$, given by*

$$({}_a\Delta_h^{-\nu} f)(t) = \frac{1}{\Gamma(\nu)} \sum_{k=a/h}^{t/h-\nu} (t - \sigma_h(kh))_h^{(\nu-1)} f(kh)h,$$

$$({}_h\Delta_b^{-\nu} f)(t) = \frac{1}{\Gamma(\nu)} \sum_{k=t/h+\nu}^{b/h} (kh - \sigma_h(t))_h^{(\nu-1)} f(kh)h.$$

We define $({}_a\Delta_h^0 f)(t) = f(t)$ and $({}_h\Delta_b^0 f)(t) = f(t)$.

Definition 5. *Let $0 < \alpha \leq 1$ and set $\gamma = 1 - \alpha$. The* left fractional h-difference *${}_a\Delta_h^\alpha f$ and the* right fractional h-difference *${}_h\Delta_b^\alpha f$ of order α of a function $f \in \mathcal{F}_\mathbb{T}$ are defined, respectively, by*

$$({}_a\Delta_h^\alpha f)(t) = (\Delta_h\, {}_a\Delta_h^{-\gamma} f)(t), \quad t \in \{a + \gamma h, a + \gamma h + h, \ldots, \rho_h(b) + \gamma h\},$$
$$({}_h\Delta_b^\alpha f)(t) = -(\Delta_h\, {}_h\Delta_b^{-\gamma} f)(t), \quad t \in \{a - \gamma h, a - \gamma h - h, \ldots, \rho_h(b) - \gamma h\}.$$

Remark 6. We define fractional sums/differences for functions on a bounded domain. This is done so, because of the problems of the calculus of variations we consider here. Nevertheless, one can use our definitions for functions with unbounded domains: an unbounded domain from above for the left fractional sum/difference, an unbounded domain from below for the right fractional sum/difference.

Let us now recall a result that will be used later in finding solutions to the boundary value problems originated from the fractional h-difference calculus of variations.

Theorem 7 (Theorem 2.10 of [**9**])**.** *Let $f \in \mathcal{F}_{\mathbb{T}}$ and $\nu \geq 0$. Then,*

$$({}_a\Delta_h^{-\nu}\Delta_h f)(t) = (\Delta_{h\,a}\Delta_h^{-\nu} f)(t) - \frac{\nu}{\Gamma(\nu+1)}(t-a)_h^{(\nu-1)}f(a)$$

for all $t \in \{a+\nu h, a+\nu h+h, \ldots, \rho_h(b)+\nu h\}$.

The next lemma permits to shorten the proofs of our main results. Essentially, it allow us to borrow information from the formulas obtained in [**5**].

Lemma 8. *Let $f \in \mathcal{F}_{\mathbb{T}}$ and $\nu \geq 0$. Then,*

$$({}_a\Delta_h^{-\nu} f)(t) = h^\nu ({}_{a/h}\Delta_1^{-\nu} s \mapsto f(sh))\left(\frac{t}{h}\right), \quad t \in \{a+\nu h, a+\nu h+h, \ldots, b+\nu h\}.$$

Proof. We have

$$\begin{aligned}
&({}_a\Delta_h^{-\nu} f)(t)\\
&\quad= \frac{1}{\Gamma(\nu)}\sum_{k=a/h}^{t/h-\nu}(t-\sigma_h(kh))_h^{(\nu-1)}f(kh)h = \frac{1}{\Gamma(\nu)}\sum_{k=a/h}^{t'-\nu}(t'h-\sigma_h(kh))_h^{(\nu-1)}f(kh)h\\
&\quad= \frac{1}{\Gamma(\nu)}\sum_{k=a/h}^{t'-\nu}h^{\nu-1}(t'-\sigma_1(k))_1^{(\nu-1)}f(kh)h = h^\nu\left({}_{a/h}\Delta_1^{-\nu} s \mapsto f(sh)\right)\left(\frac{t}{h}\right),
\end{aligned}$$

and therefore the proof is done. □

Lemma 9 (Lemma 2.3 of [**5**])**.** *Let μ, ν be two real numbers such that $\mu, \mu+\nu \in \mathbb{R}\backslash\{\ldots,-2,-1\}$. Then,*

$$\left({}_\mu\Delta_1^{-\nu} s \mapsto s_1^{(\mu)}\right)(t) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}t_1^{(\mu+\nu)}, \quad t = \mu+\nu, \mu+\nu+1, \ldots$$

Corollary 10. *Suppose that $\frac{\mu}{h}, \frac{\mu}{h}+\nu \in \mathbb{R}\backslash\{\ldots,-2,-1\}$. Then,*

$$\left({}_a\Delta_h^{-\nu} s \mapsto (s-a+\mu)_h^{(\mu/h)}\right)(t) = \frac{\Gamma\left(\frac{\mu}{h}+1\right)}{\Gamma\left(\frac{\mu}{h}+\nu+1\right)}(t-a+\mu)_h^{(\mu/h+\nu)}$$

for all $t \in \{a+\nu h, a+\nu h+h, \ldots\}$.

Proof. The result is a simple consequence of Lemma 8 and Lemma 9. □

Corollary 11. *Suppose that* $-\frac{\mu}{h}, -\frac{\mu}{h}+\nu \in \mathbb{R}\backslash\{\ldots,-2,-1\}$. *Then,*

$$\left({}_h\Delta_b^{-\nu} s \mapsto (b-\mu-s)_h^{(-\mu/h)}\right)(t) = \frac{\Gamma\left(-\frac{\mu}{h}+1\right)}{\Gamma\left(-\frac{\mu}{h}+\nu+1\right)}(b-\mu-t)_h^{(-\mu/h+\nu)}$$

for all $t \in \{b-\nu h, b-\nu h-h, \ldots\}$.

Proof. Using Corollary 10, we get

$$\begin{aligned}
&\left({}_h\Delta_b^{-\nu} s \mapsto (b-\mu-s)_h^{(-\mu/h)}\right)(t)\\
&\quad = \frac{1}{\Gamma(\nu)}\sum_{k=t/h+\nu}^{b/h}(kh-\sigma_h(t))_h^{(\nu-1)}(b-\mu-kh)_h^{(-\mu/h)}h\\
&\quad = \frac{1}{\Gamma(\nu)}\sum_{k=(t-b+\mu)/h+\nu}^{\mu/h}(kh+b-\mu-\sigma_h(t))_h^{(\nu-1)}(-kh)_h^{(-\mu/h)}h\\
&\quad = \frac{1}{\Gamma(\nu)}\sum_{k=-\mu/h}^{(b-\mu-t)/h-\nu}(b-\mu-t-\sigma_h(kh))_h^{(\nu-1)}(kh)_h^{(-\mu/h)}h\\
&\quad = \left({}_{-\mu}\Delta_h^{-\nu} s \mapsto s_h^{(-\mu/h)}\right)(b-\mu-t) = \frac{\Gamma\left(-\frac{\mu}{h}+1\right)}{\Gamma\left(-\frac{\mu}{h}+\nu+1\right)}(b-\mu-t)_h^{(-\mu/h+\nu)}
\end{aligned}$$

and the proof is complete. □

We now state and prove the law of exponents for the fractional h-difference sums.

Theorem 12. *Let* $f \in \mathcal{F}_{\mathbb{T}}$ *and* $\mu, \nu \geq 0$. *Then,*

$$\left({}_{a+\nu h}\Delta_h^{-\mu}\, {}_a\Delta_h^{-\nu} f\right)(t) = \left({}_a\Delta_h^{-(\mu+\nu)} f\right)(t), \tag{3}$$

where $t \in \{a+(\mu+\nu)h, a+(\mu+\nu)h+h, \ldots, b+(\mu+\nu)h\}$; *and*

$$\left({}_h\Delta_{b-\nu h}^{-\mu}\ {}_h\Delta_b^{-\nu} f\right)(t) = \left({}_h\Delta_b^{-(\mu+\nu)} f\right)(t), \tag{4}$$

where $t \in \{b-(\mu+\nu)h, b-(\mu+\nu)h-h, \ldots, a-(\mu+\nu)h\}$.

Proof. We prove (3) only, (4) being accomplished analogously. First, note that if $\nu = 0$ or $\mu = \nu = 0$, then the equality is valid by definition. Therefore, assume that $\nu - 1, \nu - 1 + \mu \in \mathbb{R}\backslash\{\ldots,-2,-1\}$. Then,

$$
\begin{aligned}
&\left({}_{a+\nu h}\Delta_h^{-\mu}\, {}_a\Delta_h^{-\nu} f\right)(t)\\
&= \frac{h^2}{\Gamma(\nu)\Gamma(\mu)} \sum_{s=a/h+\nu}^{t/h-\mu} (t-\sigma_h(sh))_h^{(\mu-1)} \sum_{r=a/h}^{s-\nu} (sh-\sigma_h(rh))_h^{(\nu-1)} f(rh)\\
&= \frac{h^2}{\Gamma(\nu)\Gamma(\mu)} \sum_{r=a/h}^{t/h-(\mu+\nu)} \sum_{s=r+\nu}^{t/h-\mu} (t-\sigma_h(sh))_h^{(\mu-1)} (sh-\sigma_h(rh))_h^{(\nu-1)} f(rh)\\
&= \frac{h^2}{\Gamma(\nu)\Gamma(\mu)} \sum_{r=a/h}^{t/h-(\mu+\nu)} \sum_{s=\nu-1}^{t/h-r-1-\mu} (t-\sigma_h[(s+r+1)h])_h^{(\mu-1)} (sh)_h^{(\nu-1)} f(rh)\\
&= \frac{h^2}{\Gamma(\nu)\Gamma(\mu)} \sum_{r=a/h}^{t/h-(\mu+\nu)} \sum_{s=\nu-1}^{(t-\sigma_h(rh))/h-\mu} (t-\sigma_h(rh)-\sigma_h(sh))_h^{(\mu-1)} (sh)_h^{(\nu-1)} f(rh)\\
&= \frac{h}{\Gamma(\nu)} \sum_{r=a/h}^{t/h-(\mu+\nu)} \left({}_{(\nu-1)h}\Delta_h^{-\mu} s \mapsto s_h^{(\nu-1)}\right)(t-\sigma_h(rh)) f(rh)\\
&= \frac{h}{\Gamma(\nu)} \sum_{r=a/h}^{t/h-(\mu+\nu)} \frac{\Gamma(\nu)}{\Gamma(\nu+\mu)} (t-\sigma_h(rh))_h^{(\nu+\mu-1)} f(rh) = \left({}_a\Delta_h^{-(\mu+\nu)} f\right)(t),
\end{aligned}
$$

which shows the intended equality. □

The next theorem is crucial in order to solve some fractional h-difference Euler–Lagrange equations (see the examples in Section 3).

Theorem 13. *Let $0 < \alpha \leq 1$ and $f \in \mathcal{F}_{\mathbb{T}}$. Then,*

(5) $\quad ({}_h\Delta_b^{\alpha} f)(t) = 0, \quad t \in \{a-(1-\alpha)h, a-(1-\alpha)h+h, \ldots, \rho_h(b)-(1-\alpha)h\},$

if and only if

(6) $$f(t) = \frac{c}{\Gamma(\alpha)} (b-(1-\alpha)h-t)_h^{(\alpha-1)}, \quad t \in \{a, a+h, \ldots, b\},$$

where c is an arbitrary constant.

Proof. If f is given as in (6), we immediately get (5) using Corollary 11. Suppose now that equality in (5) holds in the mentioned domain. Then, by definition of fractional difference (for $c \in \mathbb{R}$),

(7) $\quad \left({}_h\Delta_b^{-(1-\alpha)} f\right)(t) = c, \quad t \in \{a-(1-\alpha)h, a+h-(1-\alpha)h, \ldots, b-(1-\alpha)h\}.$

Applying the operator ${}_h\Delta^{-\alpha}_{b-(1-\alpha)h}$ to both sides of equality in (7), and using (4) of Theorem 12, we get $({}_h\Delta_b^{-1} f)(t) = c\left({}_h\Delta^{-\alpha}_{b-(1-\alpha)h} 1\right)(t)$, $t \in \{a-h, a, \ldots, b-h\}$. Corollary 11 now implies that

(8) $$({}_h\Delta_b^{-1} f)(t) = \frac{c}{\Gamma(\alpha+1)} (b-(1-\alpha)h-t)_h^{(\alpha)}, \quad t \in \{a-h, a, \ldots, b-h\}.$$

An application of the operator Δ_h to both sides of the equality in (8) gives

$$f(\sigma_h(t)) = c\frac{\alpha}{\Gamma(\alpha+1)}(b-(1-\alpha)h-\sigma_h(t))_h^{(\alpha-1)}, \quad t \in \{a-h, a, \ldots, b-2h\},$$

or

$$f(t) = \frac{c}{\Gamma(\alpha)}(b-(1-\alpha)h-t)_h^{(\alpha-1)}, \quad t \in \{a, a+h, \ldots, b-h\}. \tag{9}$$

Setting $t = b-h$ in (8) we get $f(b)h = ch^\alpha$, i.e., equality in (9) is also valid when $t = b$. □

Remark 14. Similar steps as those done in the proof of Theorem 13 permit us to prove the following equivalence: for $0 < \alpha \leq 1$, $c \in \mathbb{R}$, and $f \in \mathcal{F}_\mathbb{T}$, we have

$$({}_h\Delta_b^\alpha f)(t) = c, \quad t \in \{a-(1-\alpha)h, a+h-(1-\alpha)h, \ldots, b-h-(1-\alpha)h\}$$

if and only if

$$f(t) = \frac{c(b+\alpha h - b\alpha - \alpha^2 h - t) + d\alpha}{\Gamma(\alpha+1)}(b-(1-\alpha)h-t)_h^{(\alpha-1)}, \quad t \in \{a, a+h, \ldots, b\},$$

where d is an arbitrary constant. Indeed, from Corollary 11 and Theorem 12, we have

$$\begin{aligned}
&\big({}_h\Delta_b^\alpha f\big)(t) = c \Leftrightarrow \big({}_h\Delta_b^{-(1-\alpha)} f\big)(t) = -ct + d\\
&\Leftrightarrow \big({}_h\Delta_b^{-(1-\alpha)} f\big)(t) = c(b+\alpha h - t - b - \alpha h) + d\\
&\Leftrightarrow \big({}_h\Delta_b^{-(1-\alpha)} f\big)(t) = c(b+\alpha h - t) - c(b+\alpha h) + d\\
&\Leftrightarrow \big({}_h\Delta_{b-(1-\alpha)h}^{-\alpha}\, {}_h\Delta_b^{-(1-\alpha)} f\big)(t) = c\big({}_h\Delta_{b-(1-\alpha)h}^{-\alpha} s \mapsto b+\alpha h - s\big)(t)\\
&\qquad - [c(b+\alpha h)-d]\big({}_h\Delta_{b-(1-\alpha)h}^{-\alpha} 1\big)(t)\\
&\Leftrightarrow ({}_h\Delta_b^{-1} f)(t) = \frac{c}{\Gamma(\alpha+2)}(b+\alpha h - t)_h^{(\alpha+1)} - \frac{c(b+\alpha h)-d}{\Gamma(\alpha+1)}(b-(1-\alpha)h-t)_h^{(\alpha)}\\
&\Leftrightarrow f(t) = \frac{c}{\Gamma(\alpha+1)}(b+\alpha h - t)_h^{(\alpha)} - \frac{c(b+\alpha h)-d}{\Gamma(\alpha)}(b-(1-\alpha)h-t)_h^{(\alpha-1)}\\
&\Leftrightarrow f(t) = \frac{c}{\Gamma(\alpha+1)} h^\alpha \frac{\Gamma\Big(\frac{b+\alpha h - t}{h}+1\Big)}{\Gamma\Big(\frac{b+\alpha h - t}{h}+1-\alpha\Big)} - \frac{c(b+\alpha h)-d}{\Gamma(\alpha)}(b-(1-\alpha)h-t)_h^{(\alpha-1)}\\
&\Leftrightarrow f(t) = \frac{c}{\Gamma(\alpha+1)}(b+\alpha h - t)h^{\alpha-1} \frac{\Gamma\Big(\frac{b-(1-\alpha)h-t}{h}+1\Big)}{\Gamma\Big(\frac{b-(1-\alpha)h-t}{h}+1-(\alpha-1)\Big)}\\
&\qquad - \frac{c(b+\alpha h)-d}{\Gamma(\alpha)}(b-(1-\alpha)h-t)_h^{(\alpha-1)}\\
&\Leftrightarrow f(t) = \frac{c(b+\alpha h - b\alpha - \alpha^2 h - t) + d\alpha}{\Gamma(\alpha+1)}(b-(1-\alpha)h-t)_h^{(\alpha-1)}.
\end{aligned}$$

We end this section enunciating the analogue of Theorem 13 for the left fractional h-difference.

Theorem 15. *Let $0 < \alpha \le 1$ and $f \in \mathcal{F}_{\mathbb{T}}$. Then,*

$$({}_a\Delta_h^\alpha f)(t) = 0, \quad t \in \{a + (1-\alpha)h, a + (1-\alpha)h + h, \ldots, \rho_h(b) + (1-\alpha)h\},$$

if and only if $f(t) = \frac{c}{\Gamma(\alpha)}(t - (1-\alpha)h - a)_h^{(\alpha-1)}$, $\quad t \in \{a, a+h, \ldots, b\}$, *where c is an arbitrary constant.*

Proof. The proof is analogous to the one of Theorem 13. □

3. APPLICATIONS TO THE CALCULUS OF VARIATIONS

We now give two examples of application of our results. The main achievement is to obtain explicit solutions for some problems of the calculus of variations. In this section we omit the subscript h in σ_h and ρ_h. For convenience of notation we write $y^\sigma(t) = y(\sigma(t))$.

EXAMPLE 16. Let us consider the following data: let $a \in \mathbb{R}$, $h > 0$, $b = a + kh$ with $k \in \{2, 3, \ldots\}$, and $0 < \alpha \le 1$. Moreover, let A and B be to given real numbers. We want to find a function $y \in \mathcal{F}_{\mathbb{T}}$ that solves the problem

$$\mathcal{L}(y) = \sum_{t=a/h}^{b/h-1} ({}_a\Delta_h^\alpha y)^2(th)h \longrightarrow \min, \quad y(a) = A, \quad y(b) = B. \tag{10}$$

By [**9**, Theorem 3.5] we have that if $\hat{y}$ is a minimizer of $\mathcal{L}$ given in (10), then

$$({}_h\Delta_{\rho(b)}^\alpha \, {}_a\Delta_h{}^\alpha \hat{y})(t) = 0, \quad t \in \{a, a+h, \ldots, b-2h\}. \tag{11}$$

REMARK 17. At a first glance the sum in (10) and the equation in (11) seem to be meaningless due to the possible values of the variable t. However, they aren't by the fact that the authors in [**9**] used the following notation for the difference operators:

$$({}_a\Delta_h^\alpha f)(t) = ({}_a\Delta_h^\alpha f)(t + (1-\alpha)h), \quad t \in \{a, a+h, \ldots, b-h\},$$
$$({}_h\Delta_b^\alpha f)(t) = ({}_h\Delta_b^\alpha f)(t - (1-\alpha)h), \quad t \in \{a, a+h, \ldots, b-h\}.$$

An application of our Theorem 13 to the equality in (11) gives

$$({}_a\Delta_h^\alpha \hat{y})(t) = \frac{c}{\Gamma(\alpha)}(\rho(b) - (1-\alpha)h - t)_h^{(\alpha-1)}, \quad t \in \{a, a+h, \ldots, \rho(b)\},$$

with $c \in \mathbb{R}$ or

$$(\Delta_h {}_a\Delta_h^{1-\alpha} \hat{y})(t) = \frac{c}{\Gamma(\alpha)}(\rho(b) - (t + (1-\alpha)h))_h^{(\alpha-1)}, \quad t \in \{a, a+h, \ldots, \rho(b)\}.$$

We now remember Remark 17 and apply the operator ${}_{a+(1-\alpha)h}\Delta_h^{-\alpha}$ to both sides of this equality. From Theorems 7 and 12 it follows that

$$({}_{a+(1-\alpha)h}\Delta_h^{-\alpha} \Delta_h {}_a\Delta_h^{1-\alpha} \hat{y})(t) = \frac{c}{\Gamma(\alpha)}\big({}_{a+(1-\alpha)h}\Delta_h^{-\alpha} s \mapsto (\rho(b) - s)_h^{(\alpha-1)}\big)(t) \tag{12}$$

$$\Leftrightarrow \hat{y}(t) = \frac{c}{\Gamma(\alpha)}\big({}_{a+(1-\alpha)h}\Delta_h^{-\alpha} s \mapsto (\rho(b) - s)_h^{(\alpha-1)}\big)(t) + \frac{1}{\Gamma(\alpha)}(t - (a + (1-\alpha)h))_h^{(\alpha-1)} \hat{y}(a),$$

with $t \in \{a+h, a+2h, \ldots, b\}$. The constant c is determined by the end condition $\hat{y}(b) = B$.

Remark 18. We point out that if $\alpha = 1$ we get the "straight line" connecting the points (a, A) and (b, B) as the solution of the Euler–Lagrange equation (12), i.e., $\hat{y}(t) = \frac{B-A}{b-a}(t-a) + A$. This result can be found, e.g., in [**12, 20**].

We now show that the function $\hat{y}$ given by (12) furnishes in fact a global minimum to the problem (10). To do that, we recall the fractional h-summation by parts formula obtained by the authors in [**9**] (we continue to use here the notation mentioned in Remark 17).

Theorem 19 (Theorem 3.2 of [**9**]). *Let f and g be real valued functions defined on $\{a, a+h, \ldots, b-h\}$ and $\{a, a+h, \ldots, b\}$, respectively. Fix $0 < \alpha \leq 1$ and put $\gamma = 1 - \alpha$. Then,*

$$\sum_{t=a/h}^{b/h-1} f(th)({}_a\Delta_h^\alpha g)(th)h = h^\gamma f(\rho(b))g(b) - h^\gamma f(a)g(a) + \sum_{t=a/h}^{b/h-2} \left({}_h\Delta_{\rho(b)}^\alpha f\right)(th)g^\sigma(th)h$$
$$+\frac{\gamma\, g(a)}{\Gamma(\gamma+1)}\left(\sum_{t=a/h}^{b/h-1}(th+\gamma h-a)_h^{(\gamma-1)}f(th)h - \sum_{t=\sigma(a)/h}^{b/h-1}(th+\gamma h-\sigma(a))_h^{(\gamma-1)}f(th)h\right).$$

Before proceeding, we need the following definition:

Definition 20. *We say that a Lagrangian $L(t,u,v) : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ is jointly convex in (u, v) if*

$$L(t,u,v) - L(t,u',v') \geq (u-u')L_u(t,u',v') + (v-v')L_v(t,u',v'), \quad u,u',v,v' \in \mathbb{R},$$

provided the partial derivatives L_u and L_v exist.

We are now able to prove the following theorem.

Theorem 21. *Consider the set $S = \{f \in \mathcal{F}_\mathbb{T} : f(a) = A, f(b) = B\}$. Suppose that the Lagrangian $L(t,u,v) : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ of the minimization problem*

$$\mathcal{L}(y) = \sum_{t=a/h}^{b/h-1} L(th, y^\sigma(th), \left({}_a\Delta_h^\alpha y\right)(th))h \longrightarrow \min\,, \quad y(a) = A\,, \quad y(b) = B,$$

is jointly convex in (u, v). Assume that the function $\hat{y} \in S$ satisfies the Euler–Lagrange equation for this problem, i.e.,

$$L_u[\hat{y}](t) + \left({}_h\Delta_{\rho(b)}^\alpha L_v[\hat{y}]\right)(t) = 0, \quad t \in \{a, a+h, \ldots, b-2h\}, \tag{13}$$

where $[y](s) = (s, y^\sigma(s), ({}_a\Delta_h^\alpha y)(s)))$. Then, $\hat{y}$ furnishes a global minimum to $\mathcal{L}$ in the set S.

Proof. Let $y \in S$ be an arbitrary function. Suppose that $\hat{y} \in S$ satisfies equation in (13). Since $L(t,u,v)$ is jointly convex in (u,v), we get, with the use of Theorem 19,

$$\begin{aligned}\sum_{t=a/h}^{b/h-1}\{L[y]-L[\hat{y}]\}h &\geq \sum_{t=a/h}^{b/h-1}\{(y^\sigma-\hat{y}^\sigma)L_u[\hat{y}]+({}_a\Delta_h^\alpha y-{}_a\Delta_h^\alpha \hat{y})L_v[\hat{y}]\}h\\ &=\sum_{t=a/h}^{b/h-2}\Big\{(y^\sigma-\hat{y}^\sigma)\big(L_u[\hat{y}]+{}_h\Delta_{\rho(b)}^\alpha L_v[\hat{y}]\big)\Big\}h=0.\end{aligned}$$

The theorem is proved. □

It is clear that the Lagrangian $L(t,u,v)=v^2$ in (10) is jointly convex in (u,v). Therefore, the function $\hat{y}$ defined in (12) furnishes a global minimum to (10).

We end solving another fractional difference problem of the calculus of variations.

Example 22. Let $a \in \mathbb{R}$, $h>0$, $b=a+kh$ with $k \in \{2,3,\ldots\}$, $0<\alpha\leq 1$, and A and B be two given real numbers. We consider the following variational problem:

$$\mathcal{L}(y)=\sum_{t=a/h}^{b/h-1}\left[\frac{1}{2}({}_a\Delta_h^\alpha y)^2(th)-y^\sigma(th)\right]h \longrightarrow \min\,,\quad y(a)=A\,,\quad y(b)=B\,. \tag{14}$$

The Euler–Lagrange equation for problem (14) is

$$\Big({}_h\Delta_{\rho(b)}^\alpha\,{}_a\Delta_h^\alpha y\Big)(t)=1,\quad t\in\{a,a+h,\ldots,b-2h\}. \tag{15}$$

In view of Remark 14, we get from equality in (15) that

$$\big({}_a\Delta_h^\alpha y\big)(t)=\frac{\rho(b)+\alpha h-\rho(b)\alpha-\alpha^2 h-t+d\alpha}{\Gamma(\alpha+1)}(\rho(b)-(1-\alpha)h-t)_h^{(\alpha-1)},$$

for a constant $d\in\mathbb{R}$ to be determined. Following the same steps as those done for Example 16, we get

$$\begin{aligned}\hat{y}(t)=&\left({}_{a+(1-\alpha)h}\Delta_h^{-\alpha}\, s\mapsto \frac{b-\rho(b)\alpha-\alpha^2h-s+d\alpha}{\Gamma(\alpha+1)}(\rho(b)-s)_h^{(\alpha-1)}\right)(t)\\ &+\frac{1}{\Gamma(\alpha)}(t-(a+(1-\alpha)h))_h^{(\alpha-1)}y(a),\end{aligned} \tag{16}$$

for $t\in\{a+h,a+2h,\ldots,b\}$. Finally, we show that the Lagrangian $L(t,u,v)=\frac{1}{2}v^2-u$ is jointly convex in (u,v). Indeed, for $u,v,u',v'\in\mathbb{R}$ we have

$$\frac{1}{2}v^2-u-\left(\frac{1}{2}{v'}^2-u'\right)\geq -(u-u')+(v-v')v' \Leftrightarrow \frac{1}{2}(v-v')^2\geq 0.$$

We conclude that $\hat{y}$ given by (16) is the global minimizer of (14).

Acknowledgment. The authors were supported by the *Portuguese Foundation for Science and Technology* (FCT) through the *Center for Research and Development in Mathematics and Applications* (CIDMA). They are very grateful to a referee for valuable remarks and comments, which significantly contributed to the quality of the paper.

REFERENCES

1. R. Almeida, D. F. M. Torres: *Calculus of variations with fractional derivatives and fractional integrals.* Appl. Math. Lett., **22** (12) (2009), 1816–1820.
2. R. Almeida, D. F. M. Torres: *Leitmann's direct method for fractional optimization problems.* Appl. Math. Comput., **217** (3) (2010), 956–962.
3. R. Almeida, D. F. M. Torres: *Necessary and sufficient conditions for the fractional calculus of variations with Caputo derivatives.* Commun. Nonlinear Sci. Numer. Simul., **16** (3) (2011), 1490–1500.
4. G. A. Anastassiou: *Nabla discrete fractional calculus and nabla inequalities.* Math. Comput. Modelling, **51** (5–6) (2010), 562–571.
5. F. M. Atici, P. W. Eloe: *A transform method in discrete fractional calculus.* Int. J. Difference Equ., **2** (2) (2007), 165–176.
6. F. M. Atici, P. W. Eloe: *Initial value problems in discrete fractional calculus.* Proc. Amer. Math. Soc., **137** (3) (2009), 981–989.
7. F. M. Atici, S. Şengül: *Modeling with fractional difference equations.* J. Math. Anal. Appl., **369** (1) (2010), 1–9.
8. N. R. O. Bastos, R. A. C. Ferreira, D. F. M. Torres: *Necessary optimality conditions for fractional difference problems of the calculus of variations.* Discrete Contin. Dyn. Syst., **29** (2) (2011), 417–437.
9. N. R. O. Bastos, R. A. C. Ferreira, D. F. M. Torres: *Discrete-time fractional variational problems.* Signal Process., **91** (3) (2011), 513–524.
10. N. R. O. Bastos, D. Mozyrska, D. F. M. Torres: *Fractional derivatives and integrals on time scales via the inverse generalized Laplace transform.* Int. J. Math. Comput., **11** (J11) (2011), 1–9.
11. N. R. O. Bastos, D. F. M. Torres: *Combined delta-nabla sum operator in discrete fractional calculus.* Commun. Frac. Calc., **1** (1) (2010), 41–47.
12. M. Bohner: *Calculus of variations on time scales.* Dynam. Systems Appl., **13** (3–4) (2004), 339–349.
13. R. A. El-Nabulsi, D. F. M. Torres: *Necessary optimality conditions for fractional action-like integrals of variational calculus with Riemann-Liouville derivatives of order* (α, β). Math. Methods Appl. Sci., **30** (15) (2007), 1931–1939.
14. R. A. El-Nabulsi, D. F. M. Torres: *Fractional actionlike variational problems.* J. Math. Phys., **49** (5) (2008), 053521, 7 pp.
15. G. S. F. Frederico, D. F. M. Torres: *A formulation of Noether's theorem for fractional problems of the calculus of variations.* J. Math. Anal. Appl., **334** (2) (2007), 834–846.

16. G. S. F. Frederico, D. F. M. Torres: *Fractional conservation laws in optimal control theory.* Nonlinear Dynam., **53** (3) (2008), 215–222.

17. G. S. F. Frederico, D. F. M. Torres: *Fractional Noether's theorem in the Riesz-Caputo sense.* Appl. Math. Comput., **217** (3) (2010), 1023–1033.

18. C. S. Goodrich: *Existence of a positive solution to a system of discrete fractional boundary value problems.* Appl. Math. Comput., **217** (9) (2011), 4740–4753.

19. H. L. Gray, N. F. Zhang: *On a new definition of the fractional difference.* Math. Comp., **50** (182) (1988), 513–529.

20. G. Sh. Guseinov: *Discrete calculus of variations.* In "Global analysis and applied mathematics", 170–176, Amer. Inst. Phys., Melville, NY, 2004.

21. R. L. Magin: *Fractional calculus in bioengineering.* Begell House, 2006.

22. A. B. Malinowska, D. F. M. Torres: *Generalized natural boundary conditions for fractional variational problems in terms of the Caputo derivative.* Comput. Math. Appl. **59** (9) (2010), 3110–3116.

23. M. Merkle: *Representations of error terms in Jensen's and some related inequalities with applications.* J. Math. Anal. Appl., **231** (1) (1999), 76–90.

24. K. S. Miller, B. Ross: *Fractional difference calculus.* In "Univalent functions, fractional calculus, and their applications (Kōriyama, 1988)", 139–152, Horwood, Chichester, 1989.

25. K. S. Miller, B. Ross: *An introduction to the fractional calculus and fractional differential equations.* Wiley, New York, 1993.

26. D. Mozyrska, D. F. M. Torres: *Minimal modified energy control for fractional linear control systems with the Caputo derivative.* Carpathian J. Math., **26** (2) (2010), 210–221.

27. D. Mozyrska, D. F. M. Torres: *Modified optimal energy and initial memory of fractional continuous-time linear systems.* Signal Process., **91** (3) (2011), 379–385.

28. M. D. Ortigueira: *Fractional central differences and derivatives.* J. Vib. Control, **14** (9–10) (2008), 1255–1266.

29. J. Sabatier, O. P. Agrawal, J. A. Tenreiro Machado: *Advances in fractional calculus.* Springer, Dordrecht, 2007.

30. S. G. Samko, A. A. Kilbas, O. I. Marichev: *Fractional integrals and derivatives.* Translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.

Department of Mathematics,
Faculty of Engineering and Natural Sciences,
Lusophone University of Humanities and Technologies,
1749-024 Lisbon, Portugal
E-mail: ruiacferreira@ua.pt

(Received August 15, 2010)
(Revised January 31, 2011)

Department of Mathematics,
University of Aveiro,
3810-193 Aveiro, Portugal
E-mail: delfim@ua.pt