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FRACTIONAL HERMITE-HADAMARD-TYPE INEQUALITIES FOR INTERVAL-VALUED FUNCTIONS

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Abstract. In this paper, we define interval-valued right-sided Riemann-Liouville fractional integrals. Later, we handle Hermite-Hadamard inequality and Hermite-Hadamard-type inequalities via interval-valued Riemann-Liouville fractional integrals.

1. Introduction

The Hermite-Hadamard inequality discovered by C. Hermite and J. Hadamard (see, e.g., [12], [37, p. 137]) is one of the most well established inequalities in the theory of convex functions with a geometrical interpretation and many applications. These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that the Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. The Hermite-Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied (see, for example, [2], [6]-[8], [13]-[15], [22], [34], [36], [44]-[46]).

On the other hand, interval analysis is a particular case of set-valued analysis which is the study of sets in the spirit of mathematical analysis and general topology. It was introduced as an attempt to handle interval uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena. An old example of interval enclosure is Archimedes' method which is related to computing the circumference of a circle. In 1966, the first book related to interval analysis was written by Moore who is known as the first user of intervals in computational mathematics [29]. After his book, several scientists started to investigate the theory and application of interval arithmetic. Nowadays, because of its applications, interval analysis is a useful tool in various areas which are interested intensely in uncertain data. You can see applications in computer graphics,

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experimental and computational physics, error analysis, robotics, and many others [20], [26], [43].

What's more, several important inequalities (Hermite-Hadamard, Ostrowski, etc.) have been studied for the interval-valued functions in recent years. In [4],[5] Chalco-Cano et al. obtained Ostrowski-type inequalities for interval-valued functions by using Hukuhara derivative for interval-valued functions. In [16], Román Flores et al. established Minkowski and Beckenbach's inequalities for interval-valued functions. For the others, please see [9], [10], [16]–[18], [48]. However, inequalities were studied for more general set-valued maps. For example, Sadowska gave the Hermite-Hadamard inequality in [41]. For the other studies, you can see [28], [32], [35].

The purpose of this paper is to complete the Riemann-Liouville fractional integrals for interval-valued functions and to obtain Hermite-Hadamard inequality via these integrals. Furthermore, Hermite-Hadamard-type inequalities will be proved using these integrals.

2. Interval calculus

A real-valued interval X is a bounded, closed subset of $\mathbb{R}$ defined by

$$X=\left[\underline{X},\overline{X}\right]=\left\{t\in\mathbb{R}:\underline{X}\ \le t\le\overline{X}\right\},$$

where $\underline{X}$, $\overline{X}\in\mathbb{R}$ and $\underline{X}\le\overline{X}$. The numbers $\underline{X}$ and $\overline{X}$ are called the left and the right endpoints of interval X, respectively. When $\overline{X}=\underline{X}=a$, the interval X is said to be degenerate and we use the form $X=a=[a,a]$. Also, we call X positive if $\underline{X}>0$ or negative if $\overline{X}<0$. The set of all closed intervals of $\mathbb{R}$, the sets of all closed positive intervals of $\mathbb{R}$, and closed negative intervals of $\mathbb{R}$ are denoted by $\mathbb{R}_{\mathcal{I}}$, $\mathbb{R}_{\mathcal{I}}^{+}$, and $\mathbb{R}_{\mathcal{I}}^{-}$, respectively. The Hausdorff-Pompeiu distance between the intervals X and Y is defined by

$$d\left(X,Y\right)=d\left(\left[\underline{X},\overline{X}\right],\left[\underline{Y},\overline{Y}\right]\right)=\max\left\{\left|\underline{X}-\underline{Y}\right|,\left|\overline{X}-\overline{Y}\right|\right\}.$$

It is known that $(\mathbb{R}_{\mathcal{I}},d)$ is a complete metric space [1].

Now, we give the definitions of basic interval arithmetic operations for the intervals X and Y as follows [30]:

$$\begin{aligned}
X+Y &= \left[\underline{X}+\underline{Y},\overline{X}+\overline{Y}\right],\\
X-Y &= \left[\underline{X}-\overline{Y},\overline{X}-\underline{Y}\right],\\
X.Y &= \left[\min S,\max S\right] \text{ where } S=\left\{\underline{X}\ \underline{Y},\underline{X}\ \overline{Y},\overline{X}\underline{Y},\overline{X}\ \overline{Y}\right\},\\
X/Y &= \left[\min T,\max T\right] \text{ where } T=\left\{\underline{X}/\underline{Y},\underline{X}/\overline{Y},\overline{X}/\underline{Y},\overline{X}/\overline{Y}\right\} \text{ and } 0\notin Y.
\end{aligned}$$

Scalar multiplication of the interval X is defined by

$$\lambda X=\lambda\left[\underline{X},\overline{X}\right]=\begin{cases}\left[\lambda\underline{X},\lambda\overline{X}\right], & \lambda>0,\\ \{0\}, & \lambda=0,\\ \left[\lambda\overline{X},\lambda\underline{X}\right], & \lambda<0,\end{cases}$$

where $\lambda\in\mathbb{R}$. For $\lambda=-1$, the interval

$$-X:=(-1)X=[-\overline{X},-\underline{X}]$$

gives the opposite of the interval X.

The subtraction is given by

$$X-Y=X+(-Y)=[\underline{X}-\overline{Y},\overline{X}-\underline{Y}].$$

In general, $-X$ is not additive inverse for X, i.e., $X - X \neq 0$. Equality is provided when $\underline{X} = \overline{X}$.

The definitions of operations lead to a number of algebraic properties which allows $\mathbb{R}_{\mathcal{I}}$ to be a quasilinear space [24]. They can be listed as follows [23]-[25], [29]:

(1) (Associativity of addition) $(X+Y)+Z = X+(Y+Z)$ for all $X,Y,Z \in \mathbb{R}_{\mathcal{I}}$,
(2) (Additive element) $X+0 = 0+X = X$ for all $X \in \mathbb{R}_{\mathcal{I}}$,
(3) (Commutativity of addition) $X+Y = Y+X$ for all $X,Y \in \mathbb{R}_{\mathcal{I}}$,
(4) (Cancellation law) $X+Z = Y+Z \Longrightarrow X = Y$ for all $X,Y,Z \in \mathbb{R}_{\mathcal{I}}$,
(5) (Associativity of multiplication) $(X.Y).Z = X.(Y.Z)$ for all $X,Y,Z \in \mathbb{R}_{\mathcal{I}}$,
(6) (Commutativity of multiplication) $X.Y = Y.X$ for all $X,Y \in \mathbb{R}_{\mathcal{I}}$,
(7) (Unit element) $X.1 = 1.X = X$ for all $X \in \mathbb{R}_{\mathcal{I}}$,
(8) (Associate law) $\lambda(\mu X) = (\lambda\mu)\, X$ for all $X \in \mathbb{R}_{\mathcal{I}}$ and all $\lambda, \mu \in \mathbb{R}$,
(9) (First distributive law) $\lambda(X+Y) = \lambda X + \lambda Y$ for all $X,Y \in \mathbb{R}_{\mathcal{I}}$ and all $\lambda \in \mathbb{R}$,
(10) (Second distributive law) $(\lambda+\mu)X = \lambda X + \mu X$ for all $X \in \mathbb{R}_{\mathcal{I}}$ and all $\lambda, \mu \in \mathbb{R}$ with $\lambda\mu \geq 0$.

Besides these properties, the distributive law is not always valid for intervals. For example, $X = [1,2], Y = [2,3]$, $Z = [-2,-1]$

$$X.(Y+Z) = [0,4]$$

whereas

$$X.Y + X.Z = [-2,5].$$

But, this law holds in certain cases. If $YZ > 0$, then

$$X.(Y+Z) = X.Y + X.Z.$$

What's more, one of the set properties is the inclusion "$\subseteq$" that is given by

$$X \subseteq Y \Longleftrightarrow \underline{Y} \leq \underline{X} \text{ and } \overline{X} \leq \overline{Y}.$$

Also, one has the following property which is called an inclusion isotonic of interval operations ([30, p. 34]).

Let $\odot$ be the addition, multiplication, subtraction, or division. If X, Y, Z, and T are intervals such that

$$X \subseteq Y \quad \text{and } Z \subseteq T,$$

then the following relation is valid

$$X \odot Z \subseteq Y \odot T.$$

The following remark is about that scalar multiplication preserve the inclusion.

Remark 2.1. Let $X, Y \in \mathbb{R}_I$ and $\lambda \in \mathbb{R}$. If $X \subseteq Y$, then $\lambda X \subseteq \lambda Y$.

Proof. It is clear from the above property with $Z = T = [\lambda, \lambda]$. □

2.1. **Integral of interval-valued functions.** In this section, the notion of integral is mentioned for interval-valued functions. Before the definition of integral, the necessary concepts will be given as follows.

A function F is said to be an interval-valued function of t on $[a,b]$ if it assigns a nonempty interval to each $t \in [a,b]$

$$F(t) = \left[\underline{F}(t), \overline{F}(t)\right],$$

where $\underline{F}$ and $\overline{F}$ are real-valued functions.

A partition of $[a,b]$ is any finite ordered subset P having the form

$$P : a = t_0 < t_1 < ... < t_n = b.$$

The mesh of a partition P is defined by

$$mesh(P) = \max\{t_i - t_{i-1} : i = 1, 2, ..., n\}.$$

We denote by $\mathcal{P}([a,b])$ the set of all partitions of $[a,b]$. Let $\mathcal{P}(\delta,[a,b])$ be the set of all $P \in \mathcal{P}([a,b])$ such that $mesh(P) < \delta$. Choose an arbitrary point ξ_i in interval $[t_{i-1}, t_i]$, $i = 1, 2, ..., n$ and we define the sum

$$S(F, P, \delta) = \sum_{i=1}^{n} F(\xi_i)\left[t_i - t_{i-1}\right],$$

where $F : [a,b] \to \mathbb{R}_{\mathcal{I}}$. We call $S(F,P,\delta)$ a Riemann sum of F corresponding to $P \in \mathcal{P}(\delta,[a,b])$.

Definition 2.2 ([11], [38], [39]). A function $F : [a,b] \to \mathbb{R}_{\mathcal{I}}$ is called interval Riemann integrable (IR-integrable) on $[a,b]$ if there exists $A \in \mathbb{R}_I$ such that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d\left(S(F,P,\delta), A\right) < \varepsilon$$

for every Riemann sum S of F corresponding to each $P \in \mathcal{P}(\delta,[a,b])$ and independent of choice of $\xi_i \in [t_{i-1}, t_i]$ for $1 \le i \le n$. In this case, A is called the IR-integral of F on $[a,b]$ and is denoted by

$$A = (IR)\int_a^b F(t)dt.$$

The collection of all functions that are IR-integrable on $[a,b]$ will be denoted by $\mathcal{IR}_{([a,b])}$.

The following theorem gives a relation between IR-integrable and Riemann integrable (R-integrable) ([30, p. 131]).

Theorem 2.3. *Let $F : [a,b] \to \mathbb{R}_{\mathcal{I}}$ be an interval-valued function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$. $F \in \mathcal{IR}_{([a,b])}$ if and only if $\underline{F}(t)$, $\overline{F}(t) \in \mathcal{R}_{([a,b])}$ and*

$$(IR)\int_a^b F(t)dt = \left[(R)\int_a^b \underline{F}(t)dt, (R)\int_a^b \overline{F}(t)dt\right],$$

where $\mathcal{R}_{([a,b])}$ denotes the R-integrable function.

It is easily seen that if $F(t) \subseteq G(t)$ for all $t \in [a,b]$, then $(IR)\int_a^b F(t)dt \subseteq (IR)\int_a^b G(t)dt$.

In [47], Zhao et al. introduced a kind of convex interval-valued function as follows.

Definition 2.4. Let $h : [c,d] \to \mathbb{R}$ be a nonnegative function, $(0,1) \subseteq [c,d]$ and $h \neq 0$. We say that $F : [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}$ is an $h-$convex interval-valued function if for all $x, y \in [a,b]$ and $t \in (0,1)$, we have

$$h(t)F(x) + h(1-t)F(y) \subseteq F(tx + (1-t)y). \tag{2.1}$$

$SX(h,[a,b],\mathbb{R}_{\mathcal{I}}^{+})$ will denote the set of all $h-$convex interval-valued functions.

The usual notion of convex interval-valued function corresponds to relation (2.1) with $h(t) = t$ [31], [41]. Also, if $h(t) = t^s$ in (2.1), then Definition 2.4 gives the other convex interval-valued function defined by Breckner [3].

Otherwise, Zhao et al. obtained the following Hermite-Hadamard inequality for interval-valued functions by using $h-$convex [47].

Theorem 2.5. *Let $F : [a,b] \to \mathbb{R}^+_{\mathcal{I}}$ be an interval-valued function such that $F(t) = [\underline{F}(t), \overline{F}(t)]$ and $F \in \mathcal{IR}_{([a,b])}$, let $h : [0,1] \to \mathbb{R}$ be a nonnegative function, and let $h\left(\frac{1}{2}\right) \neq 0$. If $F \in SX(h, [a,b], \mathbb{R}^+_{\mathcal{I}})$, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int\limits_a^b F(x)dx \supseteq [F(a)+F(b)](IR)\int\limits_0^1 h(t)dt. \tag{2.2}$$

Remark 2.6. (i) If $h(t) = t$, (2.2) reduces the following result:

$$F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int\limits_a^b F(x)dx \supseteq \frac{[F(a)+F(b)]}{2}$$

which is obtained by [41].

(ii) If $h(t) = t^s$, (2.2) reduces the following result:

$$2^{s-1} F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int\limits_a^b F(x)dx \supseteq \frac{1}{s+1}[F(a)+F(b)]$$

which is obtained by [33].

3. Main results

In this section, we give an interval-valued right-sided Riemann–Liouville fractional integral of a function F and then we prove the Hermite-Hadamard inequality for convex interval-valued functions by using interval-valued fractional integrals. Also, Hermite-Hadamard-type inequalities for the product two convex interval-valued functions are given.

First, we recall that the Riemann-Lioville fractional integrals are defined as follows [21].

Definition 3.1. Let $f \in L_1[a,b]$. The Riemann-Liouville fractional integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

In [42] Sarikaya et al. gave the Hermite-Hadamard inequality by using fractional integrals as follows.

Theorem 3.2. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \le \frac{f(a)+f(b)}{2}$$

with $\alpha > 0$.

For more information about Riemann-Liouville integrals, see [19], [21], [27], [40].

By considering the Riemann-Liouville integral for real-valued functions, in [23] Lupulescu defined the following interval-valued left-sided Riemann–Liouville fractional integral.

Definition 3.3. Let $F : [a,b] \to \mathbb{R}_{\mathcal{I}}$ be an interval-valued function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$ and $F \in \mathcal{IR}_{([a,b])}$. The interval-valued left-sided Riemann–Liouville fractional integral of function F is defined by

$$\mathcal{J}_{a+}^{\alpha} F(x) = \frac{1}{\Gamma(\alpha)}(IR)\int_{a}^{x} (x-t)^{\alpha-1} F(t)dt, \quad x > a, \alpha > 0,$$

where Γ is an Euler Gamma function.

Based on the definition of Lupulescu, we define the corresponding interval-valued right-sided Riemann–Liouville fractional integral of function F by

$$\mathcal{J}_{b-}^{\alpha} F(x) = \frac{1}{\Gamma(\alpha)}(IR)\int_{x}^{b} (t-x)^{\alpha-1} F(t)dt, \quad x < b, \alpha > 0,$$

where Γ is an Euler Gamma function.

Corollary 1. *If $F : [a,b] \to \mathbb{R}_{\mathcal{I}}$ is an interval-valued function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$ with $\underline{F}(t)$, $\overline{F}(t) \in \mathcal{R}_{([a,b])}$, then we have*

$$\mathcal{J}_{a+}^{\alpha} F(x) = \left[J_{a+}^{\alpha} \underline{F}(x), J_{a+}^{\alpha} \overline{F}(x)\right]$$

and

$$\mathcal{J}_{b-}^{\alpha} F(x) = \left[J_{b-}^{\alpha} \underline{F}(x), J_{b-}^{\alpha} \overline{F}(x)\right].$$

Proof. It is obvious from Theorem 2.3. □

Now, we give the Hermite-Hadamard inequality for the convex interval-valued function:

Theorem 3.4. *If $F : [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}$ is a convex interval-valued function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$ and $\alpha > 0$, then we have*

$$F\left(\frac{a+b}{2}\right) \supseteq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\mathcal{J}_{a+}^{\alpha} F(b) + \mathcal{J}_{b-}^{\alpha} F(a)\right] \supseteq \frac{F(a)+F(b)}{2}. \tag{3.1}$$

Proof. Since F is a convex interval-valued function, we have

$$F\left(\frac{x+y}{2}\right) \supseteq \frac{F(x)+F(y)}{2} \tag{3.2}$$

for all $x, y \in [a,b]$. Taking $x = ta + (1-t)b$ and $y = (1-t)a + tb$, $t \in [0,1]$ in (3.2), we have

$$F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{2}\left[F(ta+(1-t)b) + F((1-t)a+tb)\right]. \tag{3.3}$$

Multiplying (3.3) by $t^{\alpha-1}$, $\alpha > 0$, we have

$$t^{\alpha-1}F\left(\frac{a+b}{2}\right) \supseteq \frac{t^{\alpha-1}}{2}\left[F(ta+(1-t)b) + F((1-t)a+tb)\right]. \tag{3.4}$$

Integrating (3.4) on $[0,1]$, we obtain

$$\begin{aligned} &(IR)\int_0^1 t^{\alpha-1}F\left(\frac{a+b}{2}\right)dt \\ &\supseteq \frac{1}{2}\left[(IR)\int_0^1 t^{\alpha-1}F(ta+(1-t)b)dt + (IR)\int_0^1 t^{\alpha-1}F((1-t)a+tb)dt\right]. \end{aligned} \tag{3.5}$$

In the relation (3.5), by using Theorem 2.3, we get

$$\begin{aligned} &(IR)\int_0^1 t^{\alpha-1}F\left(\frac{a+b}{2}\right)dt \\ =\ & \left[(R)\int_0^1 t^{\alpha-1}\underline{F}\left(\frac{a+b}{2}\right)dt, (R)\int_0^1 t^{\alpha-1}\overline{F}\left(\frac{a+b}{2}\right)dt\right] \\ =\ & \left[\underline{F}\left(\frac{a+b}{2}\right)(R)\int_0^1 t^{\alpha-1}dt,\ \overline{F}\left(\frac{a+b}{2}\right)(R)\int_0^1 t^{\alpha-1}dt\right] \\ =\ & \left[\frac{1}{\alpha}\underline{F}\left(\frac{a+b}{2}\right), \frac{1}{\alpha}\overline{F}\left(\frac{a+b}{2}\right)\right] \\ =\ & \frac{1}{\alpha}F\left(\frac{a+b}{2}\right). \end{aligned} \tag{3.6}$$

Moreover, we get

$$
\begin{aligned}
(3.7)\qquad & (IR)\int_0^1 t^{\alpha-1}F(ta+(1-t)b)dt \\
= & \left[(R)\int_0^1 t^{\alpha-1}\underline{F}(ta+(1-t)b)dt,(R)\int_0^1 t^{\alpha-1}\overline{F}(ta+(1-t)b)dt\right] \\
= & \left[\frac{1}{(b-a)^{\alpha}}(R)\int_a^b (b-x)^{\alpha-1}\,\underline{F}(x)dx,\frac{1}{(b-a)^{\alpha}}(R)\int_0^1 (b-x)^{\alpha-1}\,\overline{F}(x)dx\right] \\
= & \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[I_{a+}^{\alpha}\underline{F}(b),I_{a+}^{\alpha}\overline{F}(b)\right] \\
= & \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\mathcal{J}_{a+}^{\alpha}F(b)
\end{aligned}
$$

and similarly

$$(IR)\int_0^1 t^{\alpha-1}F((1-t)a+tb)dt=\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\mathcal{J}_{b-}^{\alpha}F(a). \tag{3.8}$$

Substituting the equalities (3.6)-(3.8) in (3.5), then we have

$$\frac{1}{\alpha}F\left(\frac{a+b}{2}\right)\supseteq\frac{\Gamma(\alpha)}{2\,(b-a)^{\alpha}}\left[\mathcal{J}_{a+}^{\alpha}F(b)+\mathcal{J}_{b-}^{\alpha}F(a)\right]. \tag{3.9}$$

Multiplying both sides of (3.9) by α, we obtain the first relation in (3.1).

For the second relation (3.1), by using the convex interval-valued function F, we have

$$F(ta+(1-t)b)\supseteq tF(a)+(1-t)F(b) \tag{3.10}$$

and

$$F((1-t)a+tb)\supseteq(1-t)F(a)+tF(b) \tag{3.11}$$

for $t\in[0,1]$. If we add the relation (3.10) and (3.11), we have

$$F(ta+(1-t)b)+F((1-t)a+tb)\supseteq F(a)+F(b). \tag{3.12}$$

Multiplying both sides of (3.12) by $t^{\alpha-1}$ and integrating on $[0,1]$, we have

$$
\begin{aligned}
&(IR)\int_0^1 t^{\alpha-1}F(ta+(1-t)b)dt+(IR)\int_0^1 t^{\alpha-1}F((1-t)a+tb)dt \\
&\qquad\supseteq(IR)\int_0^1 t^{\alpha-1}\left[F(a)+F(b)\right]dt.
\end{aligned} \tag{3.13}
$$

By Theorem 2.3, we obtain

$$
\begin{aligned}
(3.14)\qquad & (IR)\int_0^1 t^{\alpha-1}\left[F(a)+F(b)\right]dt \\
&= \left[(R)\int_0^1 t^{\alpha-1}\left[\underline{F}(a)+\underline{F}(b)\right]dt,(R)\int_0^1 t^{\alpha-1}\left[\overline{F}(a)+\overline{F}(b)\right]dt\right] \\
&= \left[\left[\underline{F}(a)+\underline{F}(b)\right](R)\int_0^1 t^{\alpha-1}dt,\left[\overline{F}(a)+\overline{F}(b)\right](R)\int_0^1 t^{\alpha-1}dt\right] \\
&= \left[\frac{1}{\alpha}\left[\underline{F}(a)+\underline{F}(b)\right],\frac{1}{\alpha}\left[\overline{F}(a)+\overline{F}(b)\right]\right] \\
&= \frac{1}{\alpha}\left[F(a)+F(b)\right].
\end{aligned}
$$

By substituting the equalities (3.7), (3.8), and (3.14) in (3.13), then we establish

$$
\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[\mathcal{J}_{a+}^{\alpha}F(b)+\mathcal{J}_{b-}^{\alpha}F(a)\right]\supseteq\frac{1}{\alpha}\left[F(a)+F(b)\right]. \tag{3.15}
$$

If we multiply both sides of (3.15) by $\frac{\alpha}{2}$, then we obtain the second relation in (3.1). This completes the proof. $\square$

Theorem 3.5. *If $F,G:[a,b]\to\mathbb{R}_{\mathcal{I}}^{+}$ are two convex interval-valued functions such that $F(t)=\left[\underline{F}(t),\overline{F}(t)\right]$ and $G(t)=\left[\underline{G}(t),\overline{G}(t)\right]$, then for $\alpha>0$ we have*

$$
\begin{aligned}
(3.16)\qquad & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\mathcal{J}_{a+}^{\alpha}F(b)G(b)+\mathcal{J}_{b-}^{\alpha}F(a)G(a)\right] \\
&\supseteq\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)M(a,b)+\frac{\alpha}{(\alpha+1)(\alpha+2)}N(a,b),
\end{aligned}
$$

where

$$M(a,b)=F(a)G(a)+F(b)G(b)$$

and

$$N(a,b)=F(a)G(b)+F(b)G(a).$$

Proof. Since F and G are two convex interval-valued functions for $t\in[0,1]$, we have

$$
F(ta+(1-t)b)\supseteq tF(a)+(1-t)F(b) \tag{3.17}
$$

and

$$
G(ta+(1-t)b)\supseteq tG(a)+(1-t)G(b). \tag{3.18}
$$

Since $F(x),G(x)\in\mathbb{R}_{\mathcal{I}}^{+}$ for all $x\in[a,b]$, from (3.17) and (3.18), we get

(3.19)

$$
\begin{aligned}
F(ta+(1-t)b)G(ta+(1-t)b) \quad\supseteq\quad & t^{2}F(a)G(a)+(1-t)^{2}F(b)G(b) \\
& +t(1-t)\left[F(a)G(b)+F(b)G(a)\right].
\end{aligned}
$$

Similarly, as F and G are convex interval-valued functions, we have

$$F\left((1-t)\,a+tb\right)G\left((1-t)\,a+tb\right) \supseteq (1-t)^2\,F\left(a\right)G(a)+t^2F\left(b\right)G(b) +t\left(1-t\right)\left[F(a)G(b)+F(b)G(a)\right]. \tag{3.20}$$

By adding (3.19) and (3.20), we obtain

$$\begin{aligned} & F\left(ta+(1-t)\,b\right)G\left(ta+(1-t)\,b\right)+F\left((1-t)\,a+tb\right)G\left((1-t)\,a+tb\right) \\ \supseteq\; & \left[t^2+(1-t)^2\right]\left[F\left(a\right)G(a)+F\left(b\right)G(b)\right]+2t\left(1-t\right)\left[F(a)G(b)+F(b)G(a)\right] \\ =\; & \left[2t^2-2t+1\right]M(a,b)+2t\left(1-t\right)N(a,b). \end{aligned} \tag{3.21}$$

Multiplying both sides of (3.21) by $t^{\alpha-1}$ and integrating on $[0,1]$, we have

$$\begin{aligned} & (IR)\int_0^1 t^{\alpha-1}F(ta+(1-t)b)G\left(ta+(1-t)\,b\right)dt \\ & +(IR)\int_0^1 t^{\alpha-1}F((1-t)a+tb)G\left((1-t)\,a+tb\right)dt \\ \supseteq\; & (IR)\int_0^1\left[2t^{\alpha+1}-2t^{\alpha}+t^{\alpha-1}\right]M(a,b)dt+(IR)\int_0^1 2t^{\alpha}\left(1-t\right)N(a,b)dt. \end{aligned} \tag{3.22}$$

By Theorem 2.3, we obtain

$$(IR)\int_0^1 t^{\alpha-1}F(ta+(1-t)b)G\left(ta+(1-t)\,b\right)dt=\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\mathcal{J}_{a+}^{\alpha}F(b)G(b) \tag{3.23}$$

and

$$(IR)\int_0^1 t^{\alpha-1}F((1-t)a+tb)G\left((1-t)\,a+tb\right)dt=\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\mathcal{J}_{b-}^{\alpha}F(a)G(a). \tag{3.24}$$

Similarly, by using Theorem 2.3, one can show that

$$(IR)\int_0^1\left[2t^{\alpha+1}-2t^{\alpha}+t^{\alpha-1}\right]M(a,b)dt=\frac{2}{\alpha}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)\left(\alpha+2\right)}\right)M(a,b) \tag{3.25}$$

and

$$(IR)\int_0^1 2t^{\alpha}\left(1-t\right)N(a,b)dt=\frac{2}{(\alpha+1)\left(\alpha+2\right)}N(a,b). \tag{3.26}$$

By substituting the equalities (3.23)-(3.26) in (3.22), then we establish

$$\begin{aligned}(3.27)\qquad &\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[\mathcal{J}_{a+}^{\alpha}F(b)G(b)+\mathcal{J}_{b-}^{\alpha}F(a)G(a)\right]\\ &\supseteq \frac{2}{\alpha}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)M(a,b)+\frac{2}{(\alpha+1)(\alpha+2)}N(a,b).\end{aligned}$$

If we multiply both sides of (3.27) by $\frac{\alpha}{2}$, then we obtain the second relation in (3.16). This completes the proof. □

Theorem 3.6. *If $F,G:[a,b]\to\mathbb{R}_{\mathcal{I}}^{+}$ are two convex interval-valued functions such that $F(t)=\left[\underline{F}(t),\overline{F}(t)\right]$ and $G(t)=\left[\underline{G}(t),\overline{G}(t)\right]$, then for $\alpha>0$ we have*

$$\begin{aligned}(3.28)\qquad &2F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right)\\ \supseteq\ &\frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}\left[\mathcal{J}_{a+}^{\alpha}F(b)G(b)+\mathcal{J}_{b-}^{\alpha}F(a)G(a)\right]\\ &+\frac{\alpha}{(\alpha+1)(\alpha+2)}M(a,b)+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)N(a,b),\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are defined as in Theorem 3.5.

Proof. For $t\in[0,1]$, we can write

$$\frac{a+b}{2}=\frac{(1-t)a+tb}{2}+\frac{ta+(1-t)b}{2}.$$

Since F and G are two convex interval-valued functions, we have

$$\begin{aligned}(3.29)\ \ &F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right)\\ =\ \ &F\left(\frac{(1-t)a+tb}{2}+\frac{ta+(1-t)b}{2}\right)G\left(\frac{(1-t)a+tb}{2}+\frac{ta+(1-t)b}{2}\right)\\ \supseteq\ \ &\frac{1}{4}\left[F((1-t)a+tb)+F(ta+(1-t)b)\right]\left[G((1-t)a+tb)+G(ta+(1-t)b)\right]\\ =\ \ &\frac{1}{4}\left[F((1-t)a+tb)G((1-t)a+tb)+F(ta+(1-t)b)G(ta+(1-t)b)\right]\\ &+\frac{1}{4}\left[F((1-t)a+tb)G(ta+(1-t)b)+F(ta+(1-t)b)G((1-t)a+tb)\right]\\ \supseteq\ \ &\frac{1}{4}\left[F((1-t)a+tb)G((1-t)a+tb)+F(ta+(1-t)b)G(ta+(1-t)b)\right]\\ &+\frac{1}{4}\left[2t(1-t)M(a,b)+\left[(1-t)^{2}+t^{2}\right]N(a,b)\right].\end{aligned}$$

Multiplying by $t^{\alpha-1}$ both sides of the inequality (3.29) and then integrating on $[0,1]$ the result obtained, we get

$$
\begin{aligned}
&(IR)\int_0^1 t^{\alpha-1}F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right)dt \\
\supseteq\ & \frac{1}{4}(IR)\int_0^1 t^{\alpha-1}F((1-t)a+tb)G((1-t)a+tb)dt \\
&+\frac{1}{4}(IR)\int_0^1 t^{\alpha-1}F(ta+(1-t)b)G(ta+(1-t)b)dt \\
&+\frac{1}{2}(IR)\int_0^1 t^{\alpha}(1-t)M(a,b)+\frac{1}{4}(IR)\int_0^1 t^{\alpha-1}\left[(1-t)^2+t^2\right]N(a,b)dt.
\end{aligned}
$$

That is,

$$
\begin{aligned}
&\frac{1}{\alpha}F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right) \\
\supseteq\ & \frac{1}{4}\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\mathcal{J}_{b-}^{\alpha}F(a)G(a)+\frac{1}{4}\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\mathcal{J}_{a+}^{\alpha}F(b)G(b) \\
&+\frac{1}{2(\alpha+1)(\alpha+2)}M(a,b)+\frac{1}{2\alpha}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)N(a,b).
\end{aligned}
\tag{3.30}
$$

If we multiply both sides of the inequality (3.30) by 2α, we obtain the desired result. □

Remark 3.7. Theorems 3.5 and 3.6 generalize Theorems 2.1 and 2.4 in [6], respectively.

Remark 3.8. If we choose $\alpha = 1$ in Theorems 3.5 and 3.6, then we obtain the same results by taking $h_1(t) = h_2(t) = t$ in Theorems 4.5 and 4.6 which are proved by Zhao et al. in [48].

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