Research Article

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Fractional Hermite-Hadamard-type inequalities for interval-valued co-ordinated convex functions

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Abstract: In this work, we introduce the notions about the Riemann-Liouville fractional integrals for interval-valued functions on co-ordinates. We also establish Hermite-Hadamard and some related inequalities for co-ordinated convex interval-valued functions by applying the newly defined fractional integrals. The results of the present paper are the extension of several previously published results.

Keywords: fractional integrals, Hermite-Hadamard inequality, interval-valued functions

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1 Introduction

The Hermite-Hadamard inequality discovered by Hermite and Hadamard (see, e.g., [1], [2, p. 137]) is one of the most well-established inequalities in the theory of convex functions having a geometrical interpretation and many applications. The Hermite-Hadamard inequality states that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it can be easily done by using Jensen's inequality. Hermite-Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied (see, for example, [3–26] and references therein).

On the other hand, interval analysis is a particular case of set-valued analysis, which is the study of sets in the spirit of mathematical analysis and general topology. It was introduced as an attempt to handle the

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interval uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena. An old example of interval enclosure is Archimede's method which is related to computing of the circumference of a circle. In 1966, the first book related to interval analysis was given by Moore, who is known as the first user of intervals in computational mathematics [27]. After his book, several scientists started to investigate the theory and applications of interval arithmetic. Nowadays, because of its applications, interval analysis is a useful tool in various areas which are interested intensely in uncertain data. You can see applications in computer graphics, experimental and computational physics, error analysis, robotics, and many others.

In addition, several important inequalities (Hermite-Hadamard, Ostrowski, etc.) have been studied for interval-valued functions in recent years. In [28,29], Chalco-Cano et al. obtained Ostrowski-type inequalities for interval-valued functions by using Hukuhara derivative for interval-valued functions. However, inequalities were studied for more general set-valued maps. In [30–32], the authors proved different variants of the Hermite-Hadamard inequalities for interval-valued functions.

The purpose of this paper is to complete the Riemann-Liouville integrals for interval-valued functions and to obtain Hermite-Hadamard inequalities via these integrals. Furthermore, Hermite-Hadamard-type inequalities are given using these integrals.

2 Preliminaries

In this section, we recall some basic definitions, results, notions, and properties, which are used throughout the paper. We denote $\mathbb{R}_I^+$ the family of all positive intervals of $\mathbb{R}$. The Hausdorff distance between $[\underline{X}, \overline{X}]$ and $[\underline{Y}, \overline{Y}]$ is defined as

$$d([\underline{X}, \overline{X}], [\underline{Y}, \overline{Y}]) = \max\{|\underline{X} - \underline{Y}|, |\overline{X} - \overline{Y}|\}.$$

The $(\mathbb{R}_I, d)$ is a complete metric space. For more details and basic notations on interval-valued functions, see [33,34].

In [27], Moore introduced the Riemann integral for interval-valued functions. The set of all Riemann integrable interval-valued functions and real-valued functions on $[a, b]$ are denoted by $\mathcal{IR}_{([a,b])}$ and $\mathcal{R}_{([a,b])}$, respectively. The following theorem gives a relation between (IR)-integrable and Riemann integrable (R-integrable) (see [33, p. 131]):

Theorem 1. *Let $F : [a, b] \to \mathbb{R}_I$ be an interval-valued function such that $F(t) = [\underline{F}(t), \overline{F}(t)]$. $F \in \mathcal{IR}_{([a,b])}$ if and only if $\underline{F}(t), \overline{F}(t) \in \mathcal{R}_{([a,b])}$, and*

$$(IR)\int_a^b F(t)\mathrm{d}t = \left[(R)\int_a^b \underline{F}(t)\mathrm{d}t, (R)\int_a^b \overline{F}(t)\mathrm{d}t\right].$$

In [34,35], Zhao et al. introduced a class of convex interval-valued functions as follows:

Definition 1. Let $h : J \subset \mathbb{R} \to \mathbb{R}$ be a positive function. We say that $F : I \subset \mathbb{R} \to \mathbb{R}_I^+$ is a h-convex interval-valued function, if for all $x, y \in I$ and $t \in (0, 1)$, we have

$$h(t)F(x) + h(1-t)F(y) \subseteq F(tx + (1-t)y). \tag{2}$$

With $SX(h, [a, b], \mathbb{R}_I^+)$ we will show the set of all h-convex interval-valued functions.

The usual notion of convex interval-valued function corresponds to relation (2) with $h(t) = t$, see [30]. Furthermore, if we take $h(t) = t^s$ in (2), then Definition 1 gives the convex interval-valued function defined by Breckner, see [36].

In [34], Zhao et al. obtained the following Hermite-Hadamard inequality for h-convex interval-valued functions:

Theorem 2. [34] *Let $F:[a,b]\to\mathbb{R}_I^+$ be an interval-valued function such that $F(t)=[\underline{F}(t),\overline{F}(t)]$ and $F\in\mathcal{IR}_{([a,b])}$, $h:[0,1]\to\mathbb{R}$ be a non-negative function and $h\left(\frac{1}{2}\right)\neq 0$. If $F\in SX(h,[a,b],\mathbb{R}_I^+)$, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}F\left(\frac{a+b}{2}\right)\supseteq\frac{1}{b-a}(IR)\int_a^b F(x)\mathrm{d}x\supseteq[F(a)+F(b)]\int_0^1 h(t)\mathrm{d}t. \tag{3}$$

Remark 1.

(i) If $h(t)=t$, then (3) reduces to the following result:

$$F\left(\frac{a+b}{2}\right)\supseteq\frac{1}{b-a}(IR)\int_a^b F(x)\mathrm{d}x\supseteq\frac{F(a)+F(b)}{2}, \tag{4}$$

which is obtained by [30].

(ii) If $h(t)=t^s$, then (3) reduces to the following result:

$$2^{s-1}F\left(\frac{a+b}{2}\right)\supseteq\frac{1}{b-a}(IR)\int_a^b F(x)\mathrm{d}x\supseteq\frac{F(a)+F(b)}{s+1},$$

which is obtained by [37].

In [38], Budak et al. gave the fractional version of Hermite-Hadamard inequalities for convex interval-valued functions as follows:

Theorem 3. *If $F:[a,b]\to\mathbb{R}_I^+$ is a convex interval-valued function such that $F(t)=[\underline{F}(t),\overline{F}(t)]$ and $\alpha>0$, then we have*

$$F\left(\frac{a+b}{2}\right)\supseteq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a+}^{\alpha}F(b)+J_{b-}^{\alpha}F(a)]\supseteq\frac{F(a)+F(b)}{2}. \tag{5}$$

Theorem 4. *If $F,G:[a,b]\to\mathbb{R}_I^+$ are two convex interval-valued functions such that $F(t)=[\underline{F}(t),\overline{F}(t)]$ and $G(t)=[\underline{G}(t),\overline{G}(t)]$, then for $\alpha>0$ we have*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a+}^{\alpha}F(b)G(b)+J_{b-}^{\alpha}F(a)G(a)]\supseteq\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)M(a,b)+\frac{\alpha}{(\alpha+1)(\alpha+2)}N(a,b) \tag{6}$$

and

$$\begin{aligned}2F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right)\supseteq&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a+}^{\alpha}F(b)G(b)+J_{b-}^{\alpha}F(a)G(a)]\\&+\frac{\alpha}{(\alpha+1)(\alpha+2)}M(a,b)+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)N(a,b),\end{aligned} \tag{7}$$

where

$$M(a,b)=F(a)G(a)+F(b)G(b)\quad\text{and}\quad N(a,b)=F(a)G(b)+F(b)G(a).$$

3 Fractional integral of interval-valued functions

In this section, we introduce the notions of fractional double integral for interval-valued functions and recall some basic definitions of interval-valued integrals. We also give the definition of interval-valued convex functions on co-ordinates. In the sequel of the paper, $\Delta = [a, b] \times [c, d]$.

In [39], Lupulescu defined the following interval-valued left-sided Riemann-Liouville fractional integral.

Definition 2. Let $F : [a, b] \to \mathbb{R}_{\mathcal{I}}$ be an interval-valued function such that $F(t) = [\underline{F}(t), \overline{F}(t)]$ and let $\alpha > 0$. The interval-valued left-sided Riemann-Liouville fractional integral of a function f is defined by

$$\mathcal{J}^{\alpha}_{a+}F(x) = \frac{1}{\Gamma(\alpha)}(IR)\int_a^x (x-s)^{\alpha-1}F(t)\mathrm{d}t, \quad x > a,$$

where Γ is the Euler Gamma function.

Based on the definition of Lupulescu, Budak et al. in [38] gave the definition of interval-valued right-sided Riemann-Liouville fractional integral of the function by

$$\mathcal{J}^{\alpha}_{b-}F(x) = \frac{1}{\Gamma(\alpha)}(IR)\int_x^b (s-x)^{\alpha-1}F(t)\mathrm{d}t, \quad x < b,$$

where Γ is the Euler Gamma function.

Theorem 5. *If $F : [a, b] \to \mathbb{R}_{\mathcal{I}}$ is an interval-valued function such that $F(t) = [\underline{F}(t), \overline{F}(t)]$, then we have*

$$\mathcal{J}^{\alpha}_{a+}\overline{F}(x) = [I^{\alpha}_{a+}\underline{F}(x), I^{\alpha}_{a+}\overline{F}(x)]$$

and

$$\mathcal{J}^{\alpha}_{b-}f(x) = [I^{\alpha}_{b-}\underline{F}(x), I^{\alpha}_{b-}\overline{F}(x)].$$

Now we recall the concept of interval-valued double integral given by Zhao et al. in [40]:

Theorem 6. [40] *Let $F : \Delta = [a, b] \times [c, d] \to \mathbb{R}_{\mathcal{I}}$. Then F is called ID-integrable on Δ with ID-integral $U = (ID)\iint_{\Delta} F(t, s)\mathrm{d}A$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$d(S(F, P, \delta, \Delta)) < \varepsilon$$

for any $P \in \mathcal{P}(\delta, \Delta)$. The collection of all ID-integrable functions on Δ will be denoted by $\mathcal{ID}_{(\Delta)}$. For more details about the notations used here, one can read [40].

Theorem 7. [40] *Let $\Delta = [a, b] \times [c, d]$. If $F : \Delta \to \mathbb{R}_{\mathcal{I}}$ is ID-integrable on Δ, then we have*

$$(ID)\iint_{\Delta} F(s, t)\mathrm{d}A = (IR)\int_a^b (IR)\int_c^d F(s, t)\mathrm{d}s\mathrm{d}t.$$

Example 1. Let $F : \Delta = [0, 1] \times [1, 2] \to \mathbb{R}^{+}_{\mathcal{I}}$ be defined by

$$F(s, t) = [st, s + t],$$

then $F(s, t)$ is integrable on Δ and $(ID)\iint_{\Delta} F(s, t)\mathrm{d}A = \left[\frac{3}{4}, 2\right]$.

By applying the concepts of Lupulescu [39] and Zhao et al. [40] about interval-valued integrals, we can define interval-valued Riemann-Liouville double fractional integral of the function $F(x, y)$ by

Definition 3. Let $F \in L_1([a, b] \times [c, d])$. The Riemann-Liouville integrals $\mathcal{J}^{\alpha,\beta}_{a+,c+}, \mathcal{J}^{\alpha,\beta}_{a+,d-}, +\mathcal{J}^{\alpha,\beta}_{b-,c+}$, and $\mathcal{J}^{\alpha,\beta}_{b-,d-}$ of order $\alpha, \beta > 0$ with $a, c \geq 0$ are defined by

$$\mathcal{J}^{\alpha,\beta}_{a+,c+}F(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_a^x\int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1}F(t,s)\mathrm{d}s\mathrm{d}t, \quad x > a,\ y > c,$$
$$\mathcal{J}^{\alpha,\beta}_{a+,d-}F(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_a^x\int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1}F(t,s)\mathrm{d}s\mathrm{d}t, \quad x > a,\ y > d,$$
$$\mathcal{J}^{\alpha,\beta}_{b-,c+}F(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_x^b\int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1}F(t,s)\mathrm{d}s\mathrm{d}t, \quad x < b,\ y > c,$$
$$\mathcal{J}^{\alpha,\beta}_{b-,d-}F(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_x^b\int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1}F(t,s)\mathrm{d}s\mathrm{d}t, \quad x < b,\ y < d,$$

respectively.

Definition 4. [41] A function $F : \Delta \to \mathbb{R}^+_{\mathcal{I}}$ is said to be interval-valued co-ordinated convex function, if the following inequality holds

$$F(tx + (1-t)y, su + (1-s)w) \supseteq tsF(x,u) + t(1-s)F(x,w) + s(1-t)F(y,u) + (1-s)(1-t)F(y,w),$$

for all $(x, y), (u, w) \in \Delta$ and $s, t \in [0, 1]$.

Lemma 1. [41] *A function $F : \Delta \to \mathbb{R}^+_{\mathcal{I}}$ is interval-valued convex on co-ordinates if and only if there exist two functions $F_x : [c, d] \to \mathbb{R}^+_{\mathcal{I}}$, $F_x(w) = F(x, w)$ and $F_y : [a, b] \to \mathbb{R}^+_{\mathcal{I}}$, $F_y(u) = F(y, u)$ are interval-valued convex.*

It is easy to prove that an interval-valued convex function is interval-valued co-ordinated convex but the converse may not be true. For this we can see the following example.

Example 2. An interval-valued function $f : [0, 1]^2 \to \mathbb{R}^+_{\mathcal{I}}$ defined as $F(x, y) = [xy, (6 - e^x)(6 - e^y)]$ is interval-valued convex on co-ordinates but is not interval-valued convex on $[0, 1]^2$.

Proposition 1. [41] *If $F, G : \Delta \to \mathbb{R}^+_{\mathcal{I}}$ are two interval-valued co-ordinated convex functions on Δ and $\alpha \geq 0$, then $F + G$ and αF are interval-valued co-ordinated convex functions.*

Proposition 2. [41] *If $F, G : \Delta \to \mathbb{R}^+_{\mathcal{I}}$ are two interval-valued co-ordinated convex functions on Δ, then (FG) is interval-valued co-ordinated convex function on Δ.*

4 Main results

In this section, we establish Hermite-Hadamard integral inequalities for interval-valued co-ordinated convex functions by applying interval-valued double fractional integral. We also present inequalities of Hermite-Hadamard-type for the product of interval-valued co-ordinated convex functions.

Theorem 8. *If $F:\Delta\to\mathbb{R}_I^+$ is an interval-valued co-ordinated convex function on Δ such that $F(t)=[\underline{F}(t),\overline{F}(t)]$, then the following inequalities hold:*

$$\begin{aligned}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\supseteq\frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left[\mathcal{J}_{a^+}^{\alpha}F\left(b,\frac{c+d}{2}\right)+\mathcal{J}_{b^-}^{\alpha}F\left(a,\frac{c+d}{2}\right)\right]\\&\quad+\frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\left[\mathcal{J}_{c^+}^{\beta}F\left(\frac{a+b}{2},d\right)+\mathcal{J}_{d^-}^{\beta}F\left(\frac{a+b}{2},c\right)\right]\\&\supseteq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}[\mathcal{J}_{a^+,c^+}^{\alpha,\beta}F(b,d)+\mathcal{J}_{a^+,d^-}^{\alpha,\beta}F(b,c)+\mathcal{J}_{b^-,c^+}^{\alpha,\beta}F(a,d)+\mathcal{J}_{b^-,d^-}^{\alpha,\beta}F(a,c)]\\&\supseteq\frac{\Gamma(\alpha+1)}{8(b-a)^{\alpha}}[\mathcal{J}_{a^+}^{\alpha}F(b,c)+\mathcal{J}_{a^+}^{\alpha}F(b,d)+\mathcal{J}_{b^-}^{\alpha}F(a,c)+\mathcal{J}_{b^-}^{\alpha}F(a,d)]\\&\quad+\frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}[\mathcal{J}_{c^+}^{\beta}F(a,d)+\mathcal{J}_{c^+}^{\beta}F(b,d)+\mathcal{J}_{d^-}^{\beta}F(a,c)+\mathcal{J}_{d^-}^{\beta}F(b,c)]\\&\supseteq\frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}.\end{aligned}\tag{8}$$

Proof. Since F is an interval-valued co-ordinated convex function on Δ, it follows that the mapping, $G_y:[a,b]\to\mathbb{R}_I^+$, $G_y\coloneqq F(x,y)$ is an interval-valued convex on $[a,b]$ for all $x\in[a,b]$. From inequality (5), we have:

$$G_y\left(\frac{a+b}{2}\right)\supseteq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\mathcal{J}_{a^+}^{\alpha}G_y(b)+\mathcal{J}_{b^-}^{\alpha}G_y(a)\right]\supseteq\frac{G_y(a)+G_y(b)}{2}.$$

That can be written as,

$$F\left(\frac{a+b}{2},y\right)\supseteq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[\mathcal{J}_{a^+}^{\alpha}F(b,y)+\mathcal{J}_{b^-}^{\alpha}F(a,y)]\supseteq\frac{F(a,y)+F(b,y)}{2}.$$

That is,

$$F\left(\frac{a+b}{2},y\right)\supseteq\frac{\alpha}{2(b-a)^{\alpha}}\left[(IR)\int_a^b(b-t)^{\alpha-1}F(t,y)\mathrm{d}t+(IR)\int_a^b(t-a)^{\alpha-1}F(t,y)\mathrm{d}t\right]\supseteq\frac{F(a,y)+F(b,y)}{2}.\tag{9}$$

Multiplying the both sides of inequality (28) by $\frac{\beta(d-y)^{\beta-1}}{2(d-c)^{\beta}}$ and integrating the resultant one with respect to y over $[c,d]$, we have

$$\begin{aligned}&\frac{\beta}{2(d-c)^{\beta}}(IR)\int_c^d F\left(\frac{a+b}{2},y\right)(d-y)^{\beta-1}\mathrm{d}y\\&\supseteq(IR)\int_a^b\int_c^d(b-t)^{\alpha-1}(d-y)^{\beta-1}F(t,y)\mathrm{d}y\mathrm{d}t+(IR)\int_a^b\int_c^d(t-a)^{\alpha-1}(d-y)^{\beta-1}F(t,y)\mathrm{d}y\mathrm{d}t\\&\supseteq\frac{1}{2}\left[\frac{\beta}{2(d-c)^{\beta}}(IR)\int_c^d F(a,y)(d-y)^{\beta-1}\mathrm{d}y+\frac{\beta}{2(d-c)^{\beta}}(IR)\int_c^d F(b,y)(d-y)^{\beta-1}\mathrm{d}y\right].\end{aligned}\tag{10}$$

Again, multiplying the both sides of inequality (28) by $\frac{\beta(y-c)^{\beta-1}}{2(d-c)^{\beta}}$ and integrating the resultant one with respect to y over $[c,d]$, we have

$$\begin{aligned}&\frac{\beta}{2(d-c)^{\beta}}(IR)\int_c^d F\left(\frac{a+b}{2},y\right)(y-c)^{\beta-1}\mathrm{d}y\\&\supseteq\frac{\alpha\beta}{4(b-a)^{\alpha}(d-c)^{\beta}}\left[(IR)\int_a^b\int_c^d(b-t)^{\alpha-1}(y-c)^{\beta-1}F(t,y)\mathrm{d}y\mathrm{d}t+(IR)\int_a^b\int_c^d(t-a)^{\alpha-1}(y-c)^{\beta-1}F(t,y)\mathrm{d}y\mathrm{d}t\right]\end{aligned}\tag{11}$$

$$\supseteq \frac{1}{2}\left[\frac{\beta}{2(d-c)^{\beta}}(IR)\int_{c}^{d}F(a,y)(y-c)^{\beta-1}\mathrm{d}y+\frac{\beta}{2(d-c)^{\beta}}(IR)\int_{c}^{d}F(b,y)(y-c)^{\beta-1}\mathrm{d}y\right].$$

Moreover, inequality (10) can be written as

$$\begin{aligned}\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}\mathcal{J}_{c^+}^{\beta}F\left(\frac{a+b}{2},d\right)&\supseteq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}[\mathcal{J}_{a^+,c^+}^{\alpha,\beta}F(b,d)+\mathcal{J}_{b^-,c^+}^{\alpha,\beta}F(a,d)]\\&\supseteq\frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\mathcal{J}_{c^+}^{\beta}F(a,d)+\mathcal{J}_{c^+}^{\beta}F(b,d)\end{aligned}\tag{12}$$

and inequality (11) can be written as

$$\begin{aligned}\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}\mathcal{J}_{d^-}^{\beta}F\left(\frac{a+b}{2},c\right)&\supseteq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}[\mathcal{J}_{a^+,d^-}^{\alpha,\beta}F(b,c)+\mathcal{J}_{b^-,d^-}^{\alpha,\beta}F(a,c)]\\&\supseteq\frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\mathcal{J}_{d^-}^{\beta}F(a,c)+\mathcal{J}_{d^-}^{\beta}F(b,c).\end{aligned}\tag{13}$$

Similarly, $H_x:[c,d]\to\mathbb{R}_{\mathcal{I}}^{+}$, $H_x(y)\coloneqq F(x,y)$ is an interval-valued convex function on $[c,d]$ and $y\in[a,b]$, we get

$$\begin{aligned}\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\mathcal{J}_{a^+}^{\alpha}F\left(b,\frac{c+d}{2}\right)&\supseteq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}[\mathcal{J}_{a^+,c^+}^{\alpha,\beta}F(b,d)+\mathcal{J}_{a^+,d^-}^{\alpha,\beta}F(b,c)]\\&\supseteq\frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\mathcal{J}_{a^+}^{\alpha}F(b,c)+\mathcal{J}_{a^+}^{\alpha}F(b,d)\end{aligned}\tag{14}$$

and

$$\begin{aligned}\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\mathcal{J}_{b^-}^{\alpha}F\left(a,\frac{c+d}{2}\right)&\supseteq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}[\mathcal{J}_{b^-,c^+}^{\alpha,\beta}F(a,d)+\mathcal{J}_{b^-,d^-}^{\alpha,\beta}F(a,c)]\\&\supseteq\frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}[\mathcal{J}_{b^-}^{\alpha}F(a,c)+\mathcal{J}_{b^-}^{\alpha}F(a,d)].\end{aligned}\tag{15}$$

Summing inequalities (12), (13), (14), and (15), we obtain second, third, and fourth inequalities of Theorem 8.

From (5), we have

$$F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\supseteq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\mathcal{J}_{a^+}^{\alpha}F\left(b,\frac{c+d}{2}\right)+\mathcal{J}_{b^-}^{\alpha}F\left(a,\frac{c+d}{2}\right)\right]\tag{16}$$

$$F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\supseteq\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}\left[\mathcal{J}_{c^+}^{\beta}F\left(\frac{a+b}{2},d\right)+\mathcal{J}_{d^-}^{\beta}F\left(\frac{a+b}{2},c\right)\right].\tag{17}$$

By adding (16) and (17) and using Theorem 9, we have the first inequality in (8),

$$\begin{aligned}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\supseteq&\frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left[\mathcal{J}_{a^+}^{\alpha}F\left(b,\frac{c+d}{2}\right)+\mathcal{J}_{b^-}^{\alpha}F\left(a,\frac{c+d}{2}\right)\right]\\&+\frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\left[\mathcal{J}_{c^+}^{\beta}F\left(\frac{a+b}{2},d\right)+\mathcal{J}_{d^-}^{\beta}F\left(\frac{a+b}{2},c\right)\right].\end{aligned}$$

At the end, again from (3) and Theorem 9, we have

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\mathcal{J}_{a^+}^{\alpha}F(b,c)+\mathcal{J}_{b^-}^{\alpha}F(a,c)\supseteq\frac{F(a,c)+F(b,c)}{2}\right],\tag{18}$$

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\mathcal{J}_{a^+}^{\alpha}F(b,d)+\mathcal{J}_{b^-}^{\alpha}F(a,d)\supseteq\frac{F(a,d)+F(b,d)}{2}\right],\tag{19}$$

$$\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}\mathcal{J}_{c^+}^{\beta}F(a,d)+\mathcal{J}_{d^-}^{\beta}F(a,c)\supseteq\frac{F(a,c)+F(a,d)}{2},\tag{20}$$

$$\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}\mathcal{J}_{c^+}^{\beta}F(b,d)+\mathcal{J}_{d^-}^{\beta}F(b,c)\supseteq\frac{F(b,c)+F(b,d)}{2}.\tag{21}$$

If we add (18), (19), (20), and (21) and then multiplying by $\frac{1}{4}$, we have the last inequality of Theorem 8.

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{8(b-a)^{\alpha}}[\mathcal{J}_{a^+}^{\alpha}F(b,c)+\mathcal{J}_{a^+}^{\alpha}F(b,d)+\mathcal{J}_{b^-}^{\alpha}F(a,c)+\mathcal{J}_{b^-}^{\alpha}F(a,d)]\\&+\frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}[\mathcal{J}_{c^+}^{\beta}F(a,d)+\mathcal{J}_{c^+}^{\beta}F(b,d)+\mathcal{J}_{d^-}^{\beta}F(a,c)+\mathcal{J}_{d^-}^{\beta}F(b,c)]\\&\supseteq\frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4},\end{aligned}$$

and the proof is completed. □

Remark 2. If we choose $\alpha=\beta=1$ in Theorem 8, then we have

$$\begin{aligned}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\supseteq\frac{1}{2}\left[\frac{1}{b-a}(IR)\int_a^b F\left(x,\frac{c+d}{2}\right)\mathrm{d}x+\frac{1}{d-c}(IR)\int_c^d F\left(\frac{a+b}{2},y\right)\mathrm{d}y\right]\\&\supseteq\frac{1}{(b-a)(d-c)}(IR)\int_a^b\int_c^d F(x,y)\mathrm{d}y\mathrm{d}x\\&\supseteq\frac{1}{4}\left[\frac{1}{b-a}(IR)\int_a^b[F(x,c)+F(x,d)]\mathrm{d}x+\frac{1}{d-c}(IR)\int_c^d[F(a,y)+F(b,y)]\mathrm{d}y\right]\\&\supseteq\frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4},\end{aligned}$$

which is proved by Zhao et al. in [41].

Remark 3. If $\underline{F}(t)=\overline{F}(t)$ in Theorem 8, then Theorem 8 reduces to the result of Sarikaya [42, Theorem 4].

Theorem 9. *If $F,G:\Delta\to\mathbb{R}_{\mathcal{I}}^{+}$ are two interval-valued co-ordinated convex functions on Δ such that $F(t)=[\underline{F}(t),\overline{F}(t)]$ and $G(t)=[\underline{G}(t),\overline{G}(t)]$, then the following inequality holds:*

$$\begin{aligned}&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\\&\times[\mathcal{J}_{a+,c+}^{\alpha,\beta}F(b,d)G(b,d)+\mathcal{J}_{a+,d-}^{\alpha,\beta}F(b,c)G(b,c)+\mathcal{J}_{b-,c+}^{\alpha,\beta}F(a,d)G(a,d)+\mathcal{J}_{b-,d-}^{\alpha,\beta}F(a,c)G(a,c)]\\&\supseteq\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)K(a,b,c,d)\\&\quad+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\frac{\alpha}{(\alpha+1)(\alpha+2)}L(a,b,c,d)\\&\quad+\frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)M(a,b,c,d)\\&\quad+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\alpha}{(\alpha+1)(\alpha+2)}N(a,b,c,d),\end{aligned}\tag{22}$$

where

$$K(a,b,c,d)=F(a,c)G(a,c)+F(b,c)G(b,c)+F(a,d)G(a,d)+F(b,d)G(b,d),$$

$$L(a, b, c, d) = F(a, c)G(b, c) + F(b, c)G(a, c) + F(a, d)G(b, d) + F(b, d)G(a, d),$$
$$M(a, b, c, d) = F(a, c)G(a, d) + F(b, c)G(b, d) + F(a, d)G(a, c) + F(b, d)G(b, c),$$

and

$$N(a, b, c, d) = F(a, c)G(b, d) + F(b, c)G(a, d) + F(a, d)G(b, c) + F(b, d)G(a, c).$$

Proof. Since F and G are interval-valued co-ordinated convex functions on Δ, if we define the mappings $F_x : [c, d] \to \mathbb{R}_I^+$, $F_x(y) = F(x, y)$, and $G_x : [c, d] \to \mathbb{R}_I^+$, $G_x(y) = G(x, y)$, then $F_x(y)$ and $G_x(y)$ are convex functions on $[c, d]$ for all $x \in [a, b]$. If we apply inequality (6) for the convex functions $F_x(y)$ and $G_x(y)$, then we have

$$\frac{\Gamma(\beta+1)}{2(d-c)^\beta}[J_{c+}^\beta F_x(d)G_x(d) + J_{d-}^\beta F_x(c)G_x(c)] \supseteq \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right)[F_x(c)G_x(c) + F_x(d)G_x(d)] + \frac{\beta}{(\beta+1)(\beta+2)}[F_x(c)G_x(d) + F_x(d)G_x(c)]. \tag{23}$$

That is,

$$\begin{aligned}&\frac{\beta}{2(d-c)^\beta}\left[(IR)\int_c^d (d-y)^{\beta-1}F(x,y)G(x,y)\mathrm{d}y + (IR)\int_c^d (y-a)^{\beta-1}F(x,y)G(x,y)\mathrm{d}y\right]\\ &\supseteq \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right)[F(x,c)G(x,c) + F(x,d)G(x,d)]\\ &\quad + \frac{\beta}{(\beta+1)(\beta+2)}[F(x,c)G(x,d) + F(x,d)G(x,c)].\end{aligned} \tag{24}$$

Multiplying inequality (24) by $\frac{\alpha}{2(b-a)^\alpha}(b-x)^{\alpha-1}$ and integrating the resultant one with respect to x over $[a, b]$, we obtain

$$\begin{aligned}&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)\beta}[\mathcal{J}_{a+,c+}^{\alpha,\beta}F(b,d)G(b,d) + \mathcal{J}_{a+,d-}^{\alpha,\beta}F(b,c)G(b,c)]\\ &\supseteq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right)[\mathcal{J}_{a+}^\alpha F(b,c)G(b,c) + \mathcal{J}_{a+}^\alpha F(b,d)G(b,d)]\\ &\quad + \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\frac{\beta}{(\beta+1)(\beta+2)}[\mathcal{J}_{a+}^\alpha F(b,c)G(b,d) + \mathcal{J}_{a+}^\alpha F(b,d)G(b,c)].\end{aligned} \tag{25}$$

Similarly, multiplying inequality (24) by $\frac{\alpha}{2(b-a)^\alpha}(x-a)^{\alpha-1}$ and integrating the resultant one with respect to x on $[a, b]$, we have

$$\begin{aligned}&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)\beta}[\mathcal{J}_{b-,c+}^{\alpha,\beta}F(a,d)G(a,d) + \mathcal{J}_{b-,d-}^{\alpha,\beta}F(a,c)G(a,c)]\\ &\supseteq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right)[\mathcal{J}_{b-}^\alpha F(a,c)G(a,c) + \mathcal{J}_{b-}^\alpha F(a,d)G(a,d)]\\ &\quad + \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\frac{\beta}{(\beta+1)(\beta+2)}[\mathcal{J}_{b-}^\alpha F(a,c)G(a,d) + \mathcal{J}_{b-}^\alpha F(a,d)g(a,c)].\end{aligned} \tag{26}$$

From (25) and (26), we get

$$\begin{aligned}&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2(b-a)^\alpha(d-c)\beta}\\ &\times [\mathcal{J}_{a+,c+}^{\alpha,\beta}f(b,d)g(b,d) + \mathcal{J}_{a+,d-}^{\alpha,\beta}F(b,c)G(b,c) + \mathcal{J}_{b-,c+}^{\alpha,\beta}F(a,d)G(a,d) + \mathcal{J}_{b-,d-}^{\alpha,\beta}F(a,c)G(a,c)]\end{aligned} \tag{27}$$

$$\supseteq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)[\mathcal{J}^{\alpha}_{a+}F(b,c)G(b,c)+\mathcal{J}^{\alpha}_{b-}F(a,c)G(a,c)]$$
$$+\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)[\mathcal{J}^{\alpha}_{a+}F(b,d)G(b,d)+\mathcal{J}^{\alpha}_{b-}F(a,d)G(a,d)]$$
$$+\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\frac{\beta}{(\beta+1)(\beta+2)}[\mathcal{J}^{\alpha}_{a+}F(b,c)G(b,d)+\mathcal{J}^{\alpha}_{b-}F(a,c)G(a,d)]$$
$$+\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\frac{\beta}{(\beta+1)(\beta+2)}[\mathcal{J}^{\alpha}_{a+}F(b,d)G(b,c)+\mathcal{J}^{\alpha}_{b-}F(a,d)G(a,c)].$$

For each term of the right hand side of (27), by inequality (6), we have

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a+}F(b,c)G(b,c)+J^{\alpha}_{b-}F(a,c)G(a,c)]\\&\supseteq\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)[F(a,c)G(a,c)+F(b,c)G(b,c)]+\frac{\alpha}{(\alpha+1)(\alpha+2)}[F(a,c)G(b,c)+F(b,c)G(a,c)],\end{aligned}\tag{28}$$

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a+}F(b,d)G(b,d)+J^{\alpha}_{b-}F(a,d)G(a,d)]\\&\supseteq\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)[F(a,d)G(a,d)+F(b,d)G(b,d)]+\frac{\alpha}{(\alpha+1)(\alpha+2)}[F(a,d)G(b,d)+F(b,d)G(a,d)],\end{aligned}\tag{29}$$

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a+}F(b,c)G(b,d)+J^{\alpha}_{b-}F(a,c)G(a,d)]\\&\supseteq\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)[F(a,c)G(a,d)+F(b,c)G(b,d)]+\frac{\alpha}{(\alpha+1)(\alpha+2)}[F(a,c)G(b,d)+F(b,c)G(a,d)],\end{aligned}\tag{30}$$

and

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[\mathcal{J}^{\alpha}_{a+}F(b,d)G(b,c)+\mathcal{J}^{\alpha}_{b-}F(a,d)G(a,c)]\\&\supseteq\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)[F(a,d)G(a,c)+F(b,d)G(b,c)]+\frac{\alpha}{(\alpha+1)(\alpha+2)}[F(a,d)G(b,c)+F(b,d)G(a,c)].\end{aligned}\tag{31}$$

If we substitute (28)–(31) in (27), we obtain the desired result (22). □

Remark 4. If we choose $\alpha=\beta=1$ in Theorem 9, then we have

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\mathrm{d}s\mathrm{d}t\supseteq\frac{1}{9}K(a,b,c,d)+\frac{1}{18}[L(a,b,c,d)+M(a,b,c,d)]+\frac{1}{36}N(a,b,c,d),$$

which is proved by Zhao et al. in [41].

Corollary 1. *If we choose* $G(x,y)=[1,1]$ *for all* $(x,y)\in\Delta$ *in* Theorem 9, *then we have the following inequality*:

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}[\mathcal{J}^{\alpha,\beta}_{a+,c+}F(b,d)+\mathcal{J}^{\alpha,\beta}_{a+,d-}F(b,c)+\mathcal{J}^{\alpha,\beta}_{b-,c+}F(a,d)+\mathcal{J}^{\alpha,\beta}_{b-,d-}F(a,c)]$$
$$\supseteq\frac{1}{4}[F(a,c)+F(b,c)+F(a,d)+F(b,d)].$$

Remark 5. If $\underline{F}(t)=\overline{F}(t)$ in Theorem 9, then Theorem 9 reduces to the result of Budak and Sarikaya [43, Theorem 2.1].

Theorem 10. *If $F, G : \Delta \to \mathbb{R}_I^+$ are two interval-valued co-ordinated convex functions on Δ such that $F(t) = [\underline{F}(t), \overline{F}(t)]$ and $G(t) = [\underline{G}(t), \overline{G}(t)]$, then the following Hermite-Hadamard-type inequality holds*

$$\begin{aligned}
&4F\left(\frac{a+b}{2}, \frac{c+d}{2}\right)G\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\\
&\supseteq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\\
&\quad\times [\mathcal{J}^{\alpha,\beta}_{a+,c+}F(b,d)G(b,d) + \mathcal{J}^{\alpha,\beta}_{a+,d-}F(b,c)G(b,c) + \mathcal{J}^{\alpha,\beta}_{b-,c+}F(a,d)G(a,d) + \mathcal{J}^{\alpha,\beta}_{b-,d-}F(a,c)G(a,c)]\\
&\quad+ \left[\frac{\alpha}{2(\alpha+1)(\alpha+2)} + \frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\right]K(a,b,c,d)\\
&\quad+ \left[\frac{1}{2}\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right) + \frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]L(a,b,c,d)\\
&\quad+ \left[\frac{1}{2}\left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right) + \frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]M(a,b,c,d)\\
&\quad+ \left[\frac{1}{4} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]N(a,b,c,d),
\end{aligned} \tag{32}$$

where $K(a,b,c,d)$, $L(a,b,c,d)$, $M(a,b,c,d)$, and $N(a,b,c,d)$ are defined as in Theorem 9.

Proof. Since F and G are interval-valued co-ordinated convex functions on Δ, by inequality (7), we have

$$\begin{aligned}
&2F\left(\frac{a+b}{2}, \frac{c+d}{2}\right)G\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\\
&\supseteq \frac{\alpha}{2(b-a)^\alpha}\left[(IR)\int_a^b (b-x)^{\alpha-1}F\left(x, \frac{c+d}{2}\right)G\left(x, \frac{c+d}{2}\right)\mathrm{d}x + (IR)\int_a^b (x-a)^{\alpha-1}F\left(x, \frac{c+d}{2}\right)G\left(x, \frac{c+d}{2}\right)\mathrm{d}x\right]\\
&\quad+ \frac{\alpha}{(\alpha+1)(\alpha+2)}\left[F\left(a, \frac{c+d}{2}\right)G\left(a, \frac{c+d}{2}\right) + F\left(b, \frac{c+d}{2}\right)G\left(b, \frac{c+d}{2}\right)\right]\\
&\quad+ \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left[F\left(a, \frac{c+d}{2}\right)G\left(b, \frac{c+d}{2}\right) + F\left(b, \frac{c+d}{2}\right)G\left(a, \frac{c+d}{2}\right)\right]
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
&2F\left(\frac{a+b}{2}, \frac{c+d}{2}\right)G\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\\
&\supseteq \frac{\beta}{2(d-c)^\beta}\left[(IR)\int_c^d (d-y)^{\beta-1}F\left(\frac{a+b}{2}, y\right)G\left(\frac{a+b}{2}, y\right)\mathrm{d}y + (IR)\int_c^d (y-c)^{\beta-1}F\left(\frac{a+b}{2}, y\right)G\left(\frac{a+b}{2}, y\right)\mathrm{d}y\right]\\
&\quad+ \frac{\beta}{(\beta+1)(\beta+2)}\left[F\left(\frac{a+b}{2}, c\right)G\left(\frac{a+b}{2}, c\right) + F\left(\frac{a+b}{2}, d\right)G\left(\frac{a+b}{2}, d\right)\right]\\
&\quad+ \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right)\left[F\left(\frac{a+b}{2}, c\right)G\left(\frac{a+b}{2}, d\right) + F\left(\frac{a+b}{2}, d\right)G\left(\frac{a+b}{2}, c\right)\right].
\end{aligned} \tag{34}$$

From (33) and (34), we get

$$\begin{aligned}
&8F\left(\frac{a+b}{2}, \frac{c+d}{2}\right)G\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\\
&\supseteq \frac{\alpha}{2(b-a)^\alpha}\left[\int_a^b (b-x)^{\alpha-1}2F\left(x, \frac{c+d}{2}\right)G\left(x, \frac{c+d}{2}\right)\mathrm{d}x + (IR)\int_a^b (x-a)^{\alpha-1}2F\left(x, \frac{c+d}{2}\right)G\left(x, \frac{c+d}{2}\right)\mathrm{d}x\right]
\end{aligned} \tag{35}$$

$$\times \frac{\beta}{2(d-c)^{\beta}}\left[(IR)\int_{c}^{d}(d-y)^{\beta-1}2F\left(\frac{a+b}{2},y\right)G\left(\frac{a+b}{2},y\right)dy+\int_{c}^{d}(y-c)^{\beta-1}2F\left(\frac{a+b}{2},y\right)G\left(\frac{a+b}{2},y\right)dy\right]$$
$$+\frac{\alpha}{(\alpha+1)(\alpha+2)}\left[2F\left(a,\frac{c+d}{2}\right)G\left(a,\frac{c+d}{2}\right)+2F\left(b,\frac{c+d}{2}\right)G\left(b,\frac{c+d}{2}\right)\right]$$
$$+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left[2F\left(a,\frac{c+d}{2}\right)G\left(b,\frac{c+d}{2}\right)+2F\left(b,\frac{c+d}{2}\right)G\left(a,\frac{c+d}{2}\right)\right]$$
$$+\frac{\beta}{(\beta+1)(\beta+2)}\left[2F\left(\frac{a+b}{2},c\right)G\left(\frac{a+b}{2},c\right)+2F\left(\frac{a+b}{2},d\right)G\left(\frac{a+b}{2},d\right)\right]$$
$$+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left[2F\left(\frac{a+b}{2},c\right)G\left(\frac{a+b}{2},d\right)+2F\left(\frac{a+b}{2},d\right)G\left(\frac{a+b}{2},c\right)\right].$$

Since the mappings $F_x : [c, d] \to \mathbb{R}_{\mathcal{I}}^{+}$, $F_x(y) = F(x, y)$ and $G_x : [c, d] \to \mathbb{R}_{\mathcal{I}}^{+}$, $G_x(y) = G(x, y)$ are convex interval-valued, by applying inequality (7), we have

$$\begin{aligned}2F\left(a,\frac{c+d}{2}\right)G\left(a,\frac{c+d}{2}\right)\supseteq{}&\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}[\mathcal{J}_{c+}^{\beta}F(a,d)G(a,d)+\mathcal{J}_{d-}^{\beta}F(a,c)G(a,c)]\\&+\frac{\beta}{(\beta+1)(\beta+2)}[F(a,c)G(a,c)+F(a,d)G(a,d)]\\&+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)[F(a,c)G(a,d)+F(a,d)G(a,c)],\end{aligned}\tag{36}$$

$$\begin{aligned}2F\left(b,\frac{c+d}{2}\right)G\left(b,\frac{c+d}{2}\right)\supseteq{}&\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}[\mathcal{J}_{c+}^{\beta}F(b,d)G(b,d)+\mathcal{J}_{d-}^{\beta}F(b,c)G(b,c)]\\&+\frac{\beta}{(\beta+1)(\beta+2)}[F(b,c)G(b,c)+F(b,d)G(b,d)]\\&+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)[F(b,c)G(b,d)+F(b,d)G(b,c)],\end{aligned}\tag{37}$$

$$\begin{aligned}2F\left(a,\frac{c+d}{2}\right)G\left(b,\frac{c+d}{2}\right)\supseteq{}&\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}[\mathcal{J}_{c+}^{\beta}F(a,d)G(b,d)+\mathcal{J}_{d-}^{\beta}F(a,c)G(b,c)]\\&+\frac{\beta}{(\beta+1)(\beta+2)}[F(a,c)G(b,c)+F(a,d)G(b,d)]\\&+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)[F(a,c)G(b,d)+F(a,d)G(b,c)],\end{aligned}\tag{38}$$

and

$$\begin{aligned}2F\left(b,\frac{c+d}{2}\right)G\left(a,\frac{c+d}{2}\right)\supseteq{}&\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}[\mathcal{J}_{c+}^{\beta}F(b,d)G(a,d)+\mathcal{J}_{d-}^{\beta}F(b,c)G(a,c)]\\&+\frac{\beta}{(\beta+1)(\beta+2)}[F(b,c)G(a,c)+F(b,d)G(a,d)]\\&+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)[F(b,c)G(a,d)+F(b,d)G(a,c)].\end{aligned}\tag{39}$$

Similarly, since the mappings $F_y : [a, b] \to \mathbb{R}_{\mathcal{I}}^{+}$, $F_y(x) = F(x, y)$, and $G_y : [a, b] \to \mathbb{R}_{\mathcal{I}}^{+}$, $G_y(x) = G(x, y)$ are convex interval-valued, by applying inequality (7), we have

$$\begin{aligned}2F\left(\frac{a+b}{2},c\right)G\left(\frac{a+b}{2},c\right)\supseteq{}&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[\mathcal{J}_{a+}^{\alpha}F(b,c)G(b,c)+\mathcal{J}_{b-}^{\alpha}F(a,c)G(a,c)]\\&+\frac{\alpha}{(\alpha+1)(\alpha+2)}[F(a,c)G(a,c)+F(b,c)G(b,c)]\\&+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)[F(a,c)G(b,c)+F(b,c)G(a,c)],\end{aligned}\tag{40}$$

$$2F\left(\frac{a+b}{2}, d\right)G\left(\frac{a+b}{2}, d\right) \supseteq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[\mathcal{J}^{\alpha}_{a+}F(b,d)G(b,d) + \mathcal{J}^{\alpha}_{b-}F(a,d)G(a,d)] + \frac{\alpha}{(\alpha+1)(\alpha+2)}[F(a,d)G(a,d) + F(b,d)G(b,d)] + \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)[F(a,d)G(b,d) + F(b,d)G(a,d)], \tag{41}$$

$$2F\left(\frac{a+b}{2}, c\right)G\left(\frac{a+b}{2}, d\right) \supseteq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[\mathcal{J}^{\alpha}_{a+}F(b,c)G(b,d) + \mathcal{J}^{\alpha}_{b-}F(a,c)G(a,d)] + \frac{\alpha}{(\alpha+1)(\alpha+2)}[F(a,c)G(a,d) + F(b,c)G(b,d)] + \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)[F(a,c)G(b,d) + F(b,c)G(a,d)] \tag{42}$$

and

$$2F\left(\frac{a+b}{2}, d\right)G\left(\frac{a+b}{2}, c\right) \supseteq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[\mathcal{J}^{\alpha}_{a+}F(b,d)G(b,c) + \mathcal{J}^{\alpha}_{b-}F(a,d)G(a,c)] + \frac{\alpha}{(\alpha+1)(\alpha+2)}[F(a,d)G(a,c) + F(b,d)G(b,c)] + \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)[F(a,d)G(b,c) + F(b,d)G(a,c)]. \tag{43}$$

On the other hand, by applying inequality (7), we get

$$\begin{aligned}
&\frac{\alpha}{2(b-a)^{\alpha}}(IR)\int_a^b (b-x)^{\alpha-1}2F\left(x, \frac{c+d}{2}\right)G\left(x, \frac{c+d}{2}\right)\mathrm{d}x \\
&\supseteq \frac{\alpha\beta}{4(b-a)^{\alpha}(d-c)^{\beta}}\left[(IR)\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}F(x,y)G(x,y)\mathrm{d}y\mathrm{d}x \right.\\
&\quad \left. + (IR)\int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1}F(x,y)G(x,y)\mathrm{d}y\mathrm{d}x\right] \\
&\quad + \frac{\beta}{(\beta+1)(\beta+2)}\frac{\alpha}{2(b-a)^{\alpha}}(IR)\int_a^b (b-x)^{\alpha-1}[F(x,c)G(x,c) + F(x,d)G(x,d)]\mathrm{d}x \\
&\quad + \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right)\frac{\alpha}{2(b-a)^{\alpha}}(IR)\int_a^b (b-x)^{\alpha-1}[F(x,c)G(x,d) + F(x,d)G(x,c)]\mathrm{d}x \\
&= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}[\mathcal{J}^{\alpha,\beta}_{a+,c+}F(b,d)G(b,d) + \mathcal{J}^{\alpha,\beta}_{a+,d-}F(b,c)G(b,c)] \\
&\quad + \frac{\beta}{(\beta+1)(\beta+2)}\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[\mathcal{J}^{\alpha}_{a+}F(b,c)G(b,c) + \mathcal{J}^{\alpha}_{a+}F(b,d)G(b,d)] \\
&\quad + \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right)\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[\mathcal{J}^{\alpha}_{a+}F(b,c)G(b,d) + \mathcal{J}^{\alpha}_{a+}F(b,d)G(b,c)].
\end{aligned} \tag{44}$$

Similarly, we also have

$$\begin{aligned}
&\frac{\alpha}{2(b-a)^{\alpha}}(IR)\int_a^b (x-a)^{\alpha-1}2F\left(x, \frac{c+d}{2}\right)G\left(x, \frac{c+d}{2}\right)\mathrm{d}x \\
&\supseteq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}[\mathcal{J}^{\alpha,\beta}_{b-,c+}F(a,d)G(a,d) + \mathcal{J}^{\alpha,\beta}_{b-,d-}F(a,c)G(a,c)]
\end{aligned} \tag{45}$$

$$+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[\mathcal{J}^{\alpha}_{b-}F(a,c)G(a,c)+\mathcal{J}^{\alpha}_{b-}F(a,d)G(a,d)]$$
$$+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[\mathcal{J}^{\alpha}_{b-}F(a,c)G(a,d)+\mathcal{J}^{\alpha}_{b-}F(a,d)G(a,c)],$$

$$\begin{aligned}&\frac{\beta}{2(d-c)^{\beta}}(IR)\int_{c}^{d}(d-y)^{\beta-1}2F\left(\frac{a+b}{2},y\right)G\left(\frac{a+b}{2},y\right)\mathrm{d}y\\&\supseteq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}[\mathcal{J}^{\alpha,\beta}_{a+,c+}F(b,d)G(b,d)+\mathcal{J}^{\alpha,\beta}_{b-,c+}F(a,d)G(a,d)]\\&+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}[\mathcal{J}^{\beta}_{c+}F(a,d)G(a,d)+\mathcal{J}^{\beta}_{c+}F(b,d)G(b,d)]\\&+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}[\mathcal{J}^{\beta}_{c+}F(a,d)G(b,d)+\mathcal{J}^{\beta}_{c+}F(b,d)G(a,d)],\end{aligned}\tag{46}$$

and

$$\begin{aligned}&\frac{\beta}{2(d-c)^{\beta}}\int_{c}^{d}(y-c)^{\beta-1}2F\left(\frac{a+b}{2},y\right)G\left(\frac{a+b}{2},y\right)\mathrm{d}y\\&\supseteq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}[\mathcal{J}^{\alpha,\beta}_{a+,d-}F(b,c)G(b,c)+\mathcal{J}^{\alpha,\beta}_{b-,d-}F(a,c)G(a,c)]\\&+\frac{\alpha}{(\alpha+1)(\alpha+2)}\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}[\mathcal{J}^{\beta}_{d-}F(a,c)G(a,c)+\mathcal{J}^{\beta}_{d-}F(b,c)G(b,c)]\\&+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}[\mathcal{J}^{\beta}_{d-}F(a,c)G(b,c)+\mathcal{J}^{\beta}_{d-}F(b,c)G(a,c)].\end{aligned}\tag{47}$$

If we substitute (36)–(47) in (35), we get

$$\begin{aligned}&8F\left(\frac{a+b}{2},\frac{c+d}{2}\right)G\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\&\supseteq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(2b-a)^{\alpha}(d-c)^{\beta}}[\mathcal{J}^{\alpha,\beta}_{a+,c+}F(b,d)G(b,d)+\mathcal{J}^{\alpha,\beta}_{a+,d-}F(b,c)G(b,c)\\&+\mathcal{J}^{\alpha,\beta}_{b-,c+}F(a,d)G(a,d)+\mathcal{J}^{\alpha,\beta}_{b-,d-}F(a,c)G(a,c)]\\&+\frac{2\alpha}{(\alpha+1)(\alpha+2)}\left\{\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}[\mathcal{J}^{\beta}_{c+}F(a,d)G(a,d)+\mathcal{J}^{\beta}_{c+}F(b,d)G(b,d)]\right.\\&\left.+\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}[\mathcal{J}^{\beta}_{d-}F(a,c)G(a,c)+\mathcal{J}^{\beta}_{d-}F(b,c)G(b,c)]\right\}\\&+2\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left\{\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}[\mathcal{J}^{\beta}_{c+}F(a,d)G(b,d)+\mathcal{J}^{\beta}_{c+}F(b,d)G(a,d)]\right.\\&\left.+\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}[\mathcal{J}^{\beta}_{d-}F(a,c)G(b,c)+\mathcal{J}^{\beta}_{d-}F(b,c)G(a,c)]\right\}\\&+\frac{2\beta}{(\beta+1)(\beta+2)}\left\{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[\mathcal{J}^{\alpha}_{a+}F(b,c)G(b,c)+\mathcal{J}^{\alpha}_{a+}F(b,d)G(b,d)]\right.\\&\left.+\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[\mathcal{J}^{\alpha}_{b-}F(a,c)G(a,c)+\mathcal{J}^{\alpha}_{b-}F(a,d)G(a,d)]\right\}\\&+2\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left\{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[\mathcal{J}^{\alpha}_{a+}F(b,c)G(b,d)+\mathcal{J}^{\alpha}_{a+}F(b,d)G(b,c)]\right.\\&\left.+\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[\mathcal{J}^{\alpha}_{b-}F(a,c)G(a,d)+\mathcal{J}^{\alpha}_{b-}F(a,d)G(a,c)]\right\}\end{aligned}\tag{48}$$

$$+ \frac{2\alpha}{(\alpha+1)(\alpha+2)}\frac{\beta}{(\beta+1)(\beta+2)}K(a,b,c,d) + \frac{2\alpha}{(\alpha+1)(\alpha+2)}\left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right)M(a,b,c,d)$$
$$+ \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\frac{2\beta}{(\beta+1)(\beta+2)}L(a,b,c,d)$$
$$+ 2\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right)N(a,b,c,d).$$

Using inequality (6), we have

$$\begin{aligned}&\frac{\Gamma(\beta+1)}{2(d-c)^\beta}[\mathcal{J}^\beta_{c+}F(a,d)G(a,d) + \mathcal{J}^\beta_{c+}F(b,d)G(b,d)] + \frac{\Gamma(\beta+1)}{2(d-c)^\beta}[\mathcal{J}^\beta_{d-}F(a,c)G(a,c) + \mathcal{J}^\beta_{d-}F(b,c)G(b,c)]\\ &\quad\supseteq \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right)K(a,b,c,d) + \frac{\beta}{(\beta+1)(\beta+2)}M(a,b,c,d),\end{aligned} \tag{49}$$

$$\begin{aligned}&\frac{\Gamma(\beta+1)}{2(d-c)^\beta}[\mathcal{J}^\beta_{c+}F(a,d)G(b,d) + \mathcal{J}^\beta_{c+}F(b,d)G(a,d)] + \frac{\Gamma(\beta+1)}{2(d-c)^\beta}[\mathcal{J}^\beta_{d-}F(a,c)G(b,c) + \mathcal{J}^\beta_{d-}F(b,c)G(a,c)]\\ &\quad\supseteq \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right)L(a,b,c,d) + \frac{\beta}{(\beta+1)(\beta+2)}N(a,b,c,d),\end{aligned} \tag{50}$$

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[\mathcal{J}^\alpha_{a+}F(b,c)G(b,c) + \mathcal{J}^\alpha_{a+}F(b,d)G(b,d)] + \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[\mathcal{J}^\alpha_{b-}F(a,c)G(a,c) + \mathcal{J}^\alpha_{b-}F(a,d)G(a,d)]\\ &\quad\supseteq \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)K(a,b,c,d) + \frac{\alpha}{(\alpha+1)(\alpha+2)}L(a,b,c,d),\end{aligned} \tag{51}$$

and

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[\mathcal{J}^\alpha_{a+}F(b,c)G(b,d) + \mathcal{J}^\alpha_{a+}F(b,d)G(b,c)] + \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[\mathcal{J}^\alpha_{b-}F(a,c)G(a,d) + \mathcal{J}^\alpha_{b-}F(a,d)G(a,c)]\\ &\quad\supseteq \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)M(a,b,c,d) + \frac{\alpha}{(\alpha+1)(\alpha+2)}N(a,b,c,d).\end{aligned} \tag{52}$$

If we substitute (49)–(52) in (48), and divide the resulting inequality by 2, then we obtain the desired result (32). This completes the proof. □

Remark 6. If we choose $\alpha = \beta = 1$ in Theorem 10, then we have

$$\begin{aligned}4F\left(\frac{a+b}{2}, \frac{c+d}{2}\right)G\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \supseteq &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(t,s)G(t,s)\mathrm{d}s\mathrm{d}t + \frac{5}{36}K(a,b,c,d)\\ &+ \frac{7}{36}[L(a,b,c,d) + M(a,b,c,d)] + \frac{2}{9}N(a,b,c,d),\end{aligned}$$

given by Zhao et al. in [41].

Corollary 2. *If we choose* $G(x,y) = [1,1]$ *in* Theorem 10, *then we have the following inequality*:

$$\begin{aligned}4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \supseteq &\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}[\mathcal{J}^{\alpha,\beta}_{a+,c+}F(b,d) + \mathcal{J}^{\alpha,\beta}_{a+,d-}F(b,c) + \mathcal{J}^{\alpha,\beta}_{b-,c+}F(a,d) + \mathcal{J}^{\alpha,\beta}_{b-,d-}F(a,c)]\\ &+ \frac{3}{4}[F(a,c) + F(a,d) + F(b,c) + F(b,d)].\end{aligned}$$

Remark 7. If $\underline{F}(t) = \overline{F}(t)$ in Theorem 10, then Theorem 10 reduces to the result of Budak and Sarikaya [43, Theorem 2.2].

5 Conclusion

In this paper, we presented ideas about interval-valued fractional integrals on co-ordinates. We have used newly described fractional integrals to prove some new Hermite-Hadamard-type inequalities for co-ordinated convex interval-valued functions. It is a fascinating and novel problem that the future researchers may find similar inequalities for various types of convexity in their study.

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