Fractional Klein–Gordon equations and related stochastic processes

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Outline

Part Zero: Prologue

Klein–Gordon equation

The classical Klein–Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) u(x,t) = -\frac{m^2 c^4}{\hbar^2} u(x,t), \tag{1}$$

emerges from the quantum relativistic energy equation

$$E^2 = p^2 c^2 + m^2 c^4, \tag{2}$$

inserting the quantum mechanical operators for energy and momentum, i.e. $E = i\hbar\frac{\partial}{\partial t}$ and $p = -i\hbar\frac{\partial}{\partial x}$, where c is the light velocity and $\hbar$ the Planck constant.

Fractional Klein–Gordon equation

We consider the fractional Klein–Gordon equations of the form

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right)^{\alpha} u(x,t) = -\lambda^2 u(x,t), \qquad \alpha \in (0,1], \tag{3}$$

and search a solution of the type $f(\sqrt{c^2 t^2 - x^2})$.
By means of the transformation

$$w = \sqrt{c^2 t^2 - x^2},$$

equation (3) takes the form

$$\left(\frac{d^2}{dw^2} + \frac{1}{w}\frac{d}{dw}\right)^{\alpha} u(w) = \frac{\lambda^2}{c^{2\alpha}} u(w), \tag{4}$$

Main objectives:

- Analytical treatment of the fractional Klein–Gordon equation by means of the McBride–Lamb theory of fractional powers of hyper-Bessel-type operators.
- In view of the relation between the fundamental solution of the Klein–Gordon equation and the probability law of the telegraph process, we try to generalize the telegraph process.
- (Further work about random flights governed by fractional Klein–Gordon-type equations)

Part One: Fractional powers of hyper-Bessel operators

Fractional power of hyper-Bessel operators

What is the operator $\left(\frac{d^2}{dw^2} + \frac{1}{w}\frac{d}{dw}\right)^\alpha$ appearing in (4)?

In a series of works, Adam McBride (1975, 1979, 1982), studied fractional powers of the generalized hyper-Bessel-type operator

$$L = x^{a_1} D x^{a_2} \dots x^{a_n} D x^{a_{n+1}}, \tag{5}$$

where n is an integer number, $a_1, \dots, a_{n+1}$ are complex numbers and $D = d/dx$.

Main results of the McBride theory

Lemma 1: The operator L in (5) can be written as

$$Lf = m^n x^{a-n} \prod_{k=1}^{n} x^{m-mb_k} D_m x^{mb_k} f, \qquad (6)$$

where

$$D_m := \frac{d}{dx^m} = m^{-1} x^{1-m} \frac{d}{dx}.$$

The constants appearing in (6) are defined as

$$a = \sum_{k=1}^{n+1} a_k, \qquad m = |a-n|, \qquad b_k = \frac{1}{m}\left(\sum_{i=k+1}^{n+1} a_i + k - n\right).$$

Example: The operator

$$L = \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} = \frac{1}{x^2}\left(x\frac{d}{dx}x\frac{d}{dx}\right),$$

is a special case of (5) with $a_1 = -1$, $a_2 = 1$, $a_3 = 0$, $n = m = 2$, $a = 0$, $b_1 = b_2 = 0$. By Lemma 1, we have that

$$L = \frac{4}{x^2}\prod_{k=1}^{2} x^{2-2b_k} D_2 x^{2b_k} = \frac{4}{x^2}\left(x^2 D_2\right)\left(x^2 D_2\right)$$
$$= \frac{4}{x^2}\left(\frac{x}{2}\frac{d}{dx}\right)\left(\frac{x}{2}\frac{d}{dx}\right) = \frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2}.$$

Hereafter we assume that the operator L defined in (5) acts on the functional space

$$F_{p,\mu} = \{f \colon x^{-\mu} f(x) \in F_p\}, \tag{7}$$

where

$$F_p = \{f \in C^\infty \colon x^k d^k f/dx^k \in L^p, k = 0, 1, \dots\}, \tag{8}$$

for $1 \le p < \infty$ and for any complex number μ.

Lemma 2: Let r be a positive integer, $a < n$, $f \in F_{p,\mu}$ and

$b_k \in A_{p,\mu,m} := \{\eta \in \mathbb{C} \colon \Re(m\eta + \mu) + m \neq 1/p - ml,\ l = 0, 1, 2, \ldots\}$,

Then

$$L^r f = m^{nr} x^{-mr} \prod_{k=1}^{n} I_m^{b_k, -r} f, \tag{9}$$

where, for $\alpha > 0$ and $\Re(m\eta + \mu) + m > 1/p$

$$I_m^{\eta,\alpha} f = \frac{x^{-m\eta - m\alpha}}{\Gamma(\alpha)} \int_0^x (x^m - u^m)^{\alpha - 1} u^{m\eta} f(u)\, d(u^m), \tag{10}$$

and for $\alpha \leq 0$

$$I_m^{\eta,\alpha} f = (\eta + \alpha + 1) I_m^{\eta,\alpha+1} f + \frac{1}{m} I_m^{\eta,\alpha+1} \left(x \frac{d}{dx} f \right). \tag{11}$$

The fractional integrals $I_m^{\eta,\alpha}$ are Erdélyi–Kober-type operators.

Fractional extension

Definition 1: Let $m = n - a > 0$, η any complex number, $b_k \in A_{p,\mu,m}$, for $k = 1, \ldots, n$. Then, for any $f(x) \in F_{p,\mu}$

$$L^{\eta} f = m^{n\eta} x^{-m\eta} \prod_{k=1}^{n} I_m^{b_k, -\eta} f, \tag{12}$$

In order to understand the key-role played by the operator D_m, we remark that the following equality holds

$$(D_m)^{\eta} f = \frac{m}{\Gamma(n-\eta)} (D_m)^n \int_0^x (x^m - u^m)^{n-\eta-1} u^{m-1} f(u)\, du. \tag{13}$$

Then it is possible to prove Lemma 2, considering the relation between negative powers of D_m and Erdélyi–Kober integrals.

Part two: Applications to the fractional Klein–Gordon equation

The fractional Klein–Gordon equation of the form

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)^{\alpha} u(x,t) = -\lambda^2 u(x,t), \qquad \alpha \in (0,1], \tag{14}$$

is reduced to

$$\left(\frac{d^2}{dw^2} + \frac{1}{w}\frac{d}{dw}\right)^{\alpha} u(w) = \frac{\lambda^2}{c^{2\alpha}} u(w), \tag{15}$$

by means of the transformation

$$w = \sqrt{c^2t^2 - x^2}.$$

Theorem 1: A solution of (14) can be written as

$$u_\alpha(x,t) = (\sqrt{c^2t^2 - x^2})^{2\alpha-2} \sum_{k=0}^{\infty} \left(\frac{\lambda}{2^\alpha c^\alpha}\right)^{2k} (-1)^k \frac{(\sqrt{c^2t^2 - x^2})^{2\alpha k}}{[\Gamma(\alpha k + \alpha)]^2}, \tag{16}$$

and for $\alpha = 1$, it reduces to

$$u_1(x,t) = J_0\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right), \qquad |x| < ct.$$

Part three: Telegraph process

The classical symmetric telegraph process is defined as

$$\mathcal{T}(t) = V(0) \int_0^t (-1)^{\mathcal{N}(s)} ds, \qquad t \geq 0, \tag{17}$$

where $V(0)$ is a two-valued random variable ($\pm c$) independent of the Poisson process $\mathcal{N}(t)$, $t \geq 0$. The telegraph process is a *finite-velocity random motion* where changes of direction are governed by $\mathcal{N}(t)$.
The absolutely continuous component of the distribution of the telegraph process is given by (e.g. De Gregorio et al. (2005))

$$\begin{aligned} & P\{\mathcal{T}(t) \in dx\}/dx \\ & = \frac{e^{-\lambda t}}{2c} \left[\lambda I_0 \left(\frac{\lambda}{c} \sqrt{c^2 t^2 - x^2} \right) + \frac{\partial}{\partial t} I_0 \left(\frac{\lambda}{c} \sqrt{c^2 t^2 - x^2} \right) \right], \end{aligned} \tag{18}$$

with $|x| < ct$ and the singular component is

$$P\{\mathcal{T}(t) = \pm ct\} = \frac{e^{-\lambda t}}{2}.$$

The absolutely continuous component of the distribution of the telegraph process is the solution to the Cauchy problem

$$\begin{cases} \frac{\partial^2 p}{\partial t^2} + 2\lambda \frac{\partial p}{\partial t} = c^2 \frac{\partial^2 p}{\partial x^2}, \\ p(x,0) = \delta(x), \\ \frac{\partial p}{\partial t}(x,t)\Big|_{t=0} = 0. \end{cases} \tag{19}$$

By means of the transformation $p(x,t) = e^{-\lambda t} u(x,t)$, equation (19) is converted into the Klein–Gordon-type equation

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) u(x,t) = \lambda^2 u(x,t). \tag{20}$$

To give a fractional generalization of the telegraph process we consider the relation between a solution of the fractional Klein–Gordon equation and the distribution of a more general process.

Part Four: Fractional telegraph process

Strategy:

1. Find a solution of the fractional Klein–Gordon equation that generalizes the fundamental solution of the classical Klein–Gordon equation.
2. Extract from this solution the conditional distribution of the generalized process and the probability distribution that governs the number of changes of directions.
3. Find the absolutely continuous and sigular components of the distribution of the fractional telegraph process.

Step 1

Lemma 3: The function

$$F(x,t) = \frac{1}{2c}\frac{\partial}{\partial t}\sum_{k=1}^{\infty}\left(\frac{\lambda}{2^{\alpha}c^{\alpha}}\right)^{2k}\frac{(c^2t^2-x^2)^{\alpha k}}{[\Gamma(\alpha k+1)]^2} \tag{21}$$

solves the fractional Klein–Gordon-type equation

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)^{\alpha} u(x,t) = \lambda^2 u(x,t), \qquad \alpha \in (0,1]. \tag{22}$$

Step 2 The solution (21) can be written as

$$F(x,t)\,dx = E_{\alpha,1}(\lambda t^{\alpha})\sum_{k=1}^{\infty} P\{\mathcal{T}^{\alpha}(t) \in dx | \mathcal{N}^{\alpha}(t) = 2k\} P\{\mathcal{N}^{\alpha}(t) = 2k\},$$

where

$$P\{\mathcal{T}^{\alpha}(t) \in dx | \mathcal{N}^{\alpha}(t) = 2k\} = dx\frac{\left(c^2t^2-x^2\right)^{\alpha k-1}}{(2ct)^{2k\alpha-1}}\frac{\Gamma(2\alpha k)}{[\Gamma(\alpha k)]^2},$$

where $k \geq 1$, $|x| < ct$, and $P\{\mathcal{N}^{\alpha}(t) = 2k\}$ gives the probability of an even number of changes, according to the fractional Poisson process $\mathcal{N}^{\alpha}(t)$.

The fractional Poisson process (introduced by Beghin, Orsingher, 2009), $\mathcal{N}^\alpha(t)$, $t \geq 0$, has the following one-dimensional distribution

$$P\{\mathcal{N}^\alpha(t) = k\} = \frac{1}{E_{\alpha,1}(\lambda t^\alpha)} \frac{(\lambda t^\alpha)^k}{\Gamma(\alpha k + 1)}, \qquad \alpha \in (0, 1],\ k \geq 0.$$

It is called *fractional*, because its probability generating function G_α satisfies the following fractional equation

$$\frac{{}^C\partial^\alpha}{\partial u^\alpha} G_\alpha(u, t) = \lambda t^\alpha G_\alpha(u^\alpha, t^\alpha),$$

where ${}^C\partial^\alpha / \partial u^\alpha$ is the so-called Caputo fractional derivative.

The conditional densities can be found as the laws of the r.v.'s

$$\mathcal{T}^{\alpha}(t) = ct\left[T^{\alpha}_{(n^+)} - (1 - T^{\alpha}_{(n^+)})\right], \tag{23}$$

where $T^{\alpha}_{(n^+)}$ possesses probability density given by

$$f_{T^{\alpha}_{(n^+)}}(w) = \frac{\Gamma(n\alpha)}{\Gamma(n^+\alpha)\Gamma((n-n^+)\alpha)} w^{n^+\alpha - 1}(1-w)^{(n-n^+)\alpha - 1},$$
$$0 < w < 1.$$

The r.v. defined in (23) can be regarded as a rightward displacement of random length of $ct\, T^{\alpha}_{(n^+)}$ and a leftward displacement for the remaining interval of time.
In an analogue way, we can find the conditional distributions when the fractional Poisson process $\mathcal{N}^{\alpha}(t)$ takes an odd number of events.

Theorem 2: The fractional telegraph-type process $\mathcal{T}^\alpha(t)$, $t \geq 0$, has the following probability law

$$p^\alpha(x,t) = \frac{1}{E_{\alpha,1}(\lambda t^\alpha)} \Bigg[ct \sum_{k=1}^{\infty} \left(\frac{\lambda}{2^\alpha c^\alpha} \right)^{2k} \frac{(c^2t^2 - x^2)^{\alpha k - 1}}{\Gamma(\alpha k)\Gamma(\alpha k + 1)} + \\ + \sum_{k=0}^{\infty} \left(\frac{\lambda}{2^\alpha c^\alpha} \right)^{2k+1} \frac{(c^2t^2 - x^2)^{\alpha k + \frac{\alpha - 1}{2}}}{[\Gamma(\alpha k + \frac{1+\alpha}{2})]^2} \Bigg] \\ + \frac{1}{2E_{\alpha,1}(\lambda t^\alpha)} [\delta(x + ct) + \delta(x - ct)], \qquad \alpha \in (0,1].$$

What is the governing equation of the probability law of the fractional telegraph process?

Theorem 3: The function

$$f(x,t) = E_{\alpha,1}(\lambda t^{\alpha}) P\{\mathcal{T}^{\alpha}(t) \in dx\}, \qquad x \in (-ct, +ct),$$

where $P\{\mathcal{T}^{\alpha}(t) \in dx\}$ represents the absolutely continuous component of the distribution of the fractional telegraph process $\mathcal{T}^{\alpha}(t)$, $t \geq 0$, is a solution to the non-homogeneous fractional Klein–Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right)^{\alpha} u_{\alpha}(x,t) = \lambda^2 u_{\alpha}(x,t) + \lambda 2^{\alpha} c^{\alpha} \frac{(\sqrt{c^2 t^2 - x^2})^{-\alpha-1}}{[\Gamma(\frac{1-\alpha}{2})]^2}.$$

Remark: In the case $\alpha = 1$, we recover in Theorem 2 the distribution of the classical telegraph process. Moreover, by Theorem 3, for $\alpha = 1$, we have that

$$f(x,t) = E_{1,1}(\lambda t) P\{\mathcal{T}(t) \in dx\} = e^{\lambda t} P\{\mathcal{T}(t) \in dx\}$$

solves the classical Klein–Gordon-type equation

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) u_\alpha(x,t) = \lambda^2 u_\alpha(x,t),$$

as expected.

References

L. Beghin, E. Orsingher. Fractional Poisson processes and related planar random motions, *Electronic Journal of Probability*, 14(61):1790–1826, (2009)

A. De Gregorio, E. Orsingher, L. Sakhno. Motions with finite velocity analyzed with order statistics and differential equations, *Theory of Probability and Mathematical Statistics*, 71:63–79, (2005)

R. Garra, E. Orsingher, F. Polito, Fractional Klein-Gordon equations and related stochastic processes, *preprint*, **arXiv:1308.2139**, (2013)

A.C. McBride. A theory of fractional integration for generalized functions, *SIAM Journal on Mathematical Analysis*, 6(3):583–599, (1975)

A.C. McBride. *Fractional calculus and integral transforms of generalised functions*, Pitman, London, (1979)

A.C. McBride. Fractional Powers of a Class of Ordinary Differential Operators, *Proceedings of the London Mathematical Society*, 3(45):519–546, (1982)

Proof of Theorem 1:
The Bessel operator

$$L_B = \frac{d^2}{dw^2} + \frac{1}{w}\frac{d}{dw}$$

appearing in (15) is a special case of L, when $n = 2$, $a_1 = -1$, $a_2 = 1$, $a_3 = 0$. By Definition 1 and Lemma 2 we have that $m = 2$, $b_1 = b_2 = 0$ and thus

$$(L_B)^\alpha f(w) = 4^\alpha w^{-2\alpha} I_2^{0,-\alpha} I_2^{0,-\alpha} f(w). \tag{24}$$

By simple calculations we have that

$$\begin{aligned}(L_B)^\alpha w^\beta &= 4^\alpha w^{-2\alpha} I_2^{0,-\alpha} I_2^{0,-\alpha} w^\beta \\ &= 4^\alpha \left[\frac{\Gamma\left(\frac{\beta}{2}+1\right)}{\Gamma\left(1-\alpha+\frac{\beta}{2}\right)}\right]^2 w^{\beta-2\alpha}.\end{aligned} \tag{25}$$

Let us write the function (16) in the new variable w, i.e.

$$u_\alpha(w) = w^{2\alpha-2} \sum_{k=0}^{\infty} \left(\frac{\lambda}{2^\alpha c^\alpha} \right)^{2k} (-1)^k w^{2\alpha k} \frac{1}{[\Gamma(\alpha k + \alpha)]^2}, \quad (26)$$

By applying now the operator $(L_B)^\alpha$ to the function (26) we have that (being $\beta = 2\alpha k + 2\alpha - 2$)

$$(L_B)^\alpha \left(w^{2\alpha-2} \sum_{k=0}^{\infty} (-1)^k \left(\frac{\lambda}{2^\alpha c^\alpha} w^\alpha \right)^{2k} \frac{1}{[\Gamma(\alpha k + \alpha)]^2} \right) \quad (27)$$

$$= -\frac{\lambda^2}{c^{2\alpha}} u_\alpha(w),$$

and going back to the variables (x, t), we obtain the claimed result.