Filomat 30:8 (2016), 2177–2189
DOI 10.2298/FIL1608177B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Fractional Optimal Control Problem for Differential System with Control Constraints

G. M. Bahaa[*a]

[a]*Department of Mathematics, Faculty of Science, Taibah University, Al-Madinah Al-Munawwarah, Saudi-Arabia*

Abstract. In this paper, the fractional optimal control problem for differential system is considered. The fractional time derivative is considered in Riemann-Liouville sense. Constraints on controls are imposed. Necessary and sufficient optimality conditions for the fractional Dirichlet and Neumann problems with the quadratic performance functional are derived. Some examples are analyzed in details.

Introduction

Fractional calculus deals with the generalization of differentiation and integration of non-integer orders. In recent years, it has played a significant role in physics, chemistry, biology, electronics, and control theory. Extensive treatment and various applications of the fractional calculus are discussed in (Agrwal 2002; 2004; 2007). It has been demonstrated that Fractional Order Differential Equations (FODEs) model dynamic systems and processes more accurately than integer order differential equations do, and fractional controllers perform better than integer order controllers (see, for example, (Jarad et. al. 2010; 2012) and (Mophou 2011a; 2011b) and the papers and references therein).

Various minimization problems associated with integer order optimal control of second order distributed parameter systems were studied for example in (Lions, 1971). Also they define on spaces of function with an infinite number of variables are initiated and proved in (Bahaa, 2003; 2005; 2008), (Bahaa & Tharwat 2011; 2012b), (Kotarski & EL-Saify & Bahaa 2002b) and (Bahaa and El-Shatery 2013).

In the area of calculus of variations and optimal control of fractional differential equations little has been done compared to a differential equations with integer time derivatives. In (Agrwal 2002; 2004; 2007), Agrawal presented a general formulation and solution scheme for the fractional optimal control problems involving first and second order operators. The formulation was obtained by means of the fractional variation principle and the lagrange multiplier technique. In (Mophou 2011a; 2011b), Mophou used the classical control theory to a fractional diffusion equation involving second order operator (Lapalce operator) in a bounded domain with and without a state constraints.

In this paper, we extend the previous results. We consider here a different type of equations, namely, fractional partial differential equations involving second order operators. The existence and uniqueness of solutions for such equations were proved. Fractional optimal control is characterized by the adjoint problem. By using this characterization, particular properties of fractional optimal control are proved.

2010 *Mathematics Subject Classification*. 46C05, 49J20, 93C20

Keywords. Fractional optimal control problems; Fractional differential systems; Dirichlet and Neumann conditions; Existence and uniqueness of solutions; Riemann-Liouville sense; Caputo derivative.

Received: 24 May 2014; Accepted: 09 August 2014

Communicated by Predrag Stanimirović

Email addresses: Bahaa_gmhotmail.com (G. M. Bahaa*), Permanent address: Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt (G. M. Bahaa*)

This paper is organized as follows. In section 1, we introduce some definitions and preliminary results. In section 2, we formulate the fractional Dirichlet problem. In section 3, we show that our fractional optimal control problem holds and gives the optimality system for the optimal control. In section 4, we formulate the fractional Neumann problem. In section 5, the minimization problem is formulated and we state some illustrated examples. In section 6, we discuss the controllability of the fractional Dirichlet problem.

1. Some Basic Definitions

The object of this section is to give the definition of some fractional integrals and fractional derivatives of function in the Riemann-Liouville sense.

Let $n \in N^*$ and Ω be a bounded open subset of R^n with a smooth boundary Γ of class C^2. For a time $T > 0$, we set $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$.

Definition 1.1. *(see [Agrawal, 2002; 2004, Mophou 2011a; 2011b and Ahmad, Ntouyas, 2013]). Let $f : R^+ \to R$ be a continuous function on R^+ and $\beta > 0$. Then the expression*

$$I_+^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)ds,\ t > 0,$$

is called the Riemann-Liouville integral of order β.

Definition 1.2. *(see [Agrawal, 2002; 2004, Mophou 2011a; 2011b and Ahmad, Ntouyas, 2013]). Let $f : R^+ \to R$. The Riemann-Liouville fractional derivative of order β of f is defined by:*

$$D_+^\beta f(t) = \frac{1}{\Gamma\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\beta-1} f(s)ds,\ t > 0,$$

where $\beta \in (n-1, n), n \in N$.

Definition 1.3. *(see [Agrawal, 2002; 2004 and Mophou 2011a; 2011b]). Let $f : R^+ \to R$. The left Caputo fractional derivative of order β of f is defined by*

$$D_0^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} f^{(n)}(s)ds, \quad t > 0,$$

where $\beta \in (n-1, n), n \in N$.

The Caputo fractional derivative is a sort of regularization in the time origin for the Riemann-Liouville fractional derivative.

Lemma 1.1. *(see [Agrawal, 2002; 2004 and Mophou 2011a; 2011b]). Let $T > 0, u \in C^m([0, T]), p \in (m-1, m), m \in N$ and $v \in C^1([0, T])$. Then for $t \in [0, T], 0 < \beta \le 1$ the following properties hold*

$$D_+^p v(t) = \frac{d}{dt} I_+^{1-p} v(t), \quad m = 1,$$
$$D_+^p I_+^p v(t) = v(t);$$

$$I_+^p D_0^p u(t) = u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^{(k)}(0);$$

$$\lim_{t\to 0^+} D_0^p u(t) = \lim_{t\to 0^+} I_+^p u(t) = 0.$$

From now on we set

$$D^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_t^T (s-t)^{-\beta} f'(s)ds.$$

Remark 1.2. *(see [Agrawal, 2002; 2004 and Mophou 2011a; 2011b]).* $-D^{\beta} f(t)$ *is the so-called right fractional Caputo derivative. It represents the future state of* $f(t)$*. For more details on the derivative we refer to [Agrawal, 2002; 2004 and Mophou 2011a; 2011b]. Note also that when* $T = +\infty, D^{\beta} f(t)$ *is the Weyl fractional integral of order* β *of* f'.

Lemma 1.3. *(Green's formula) (see [Agrawal, 2002; 2004, Mophou 2011a; 2011b], Pavlovic, 2009). Let* $0 < \beta \leq 1$. *Then for any* $\phi \in C^{\infty}(\overline{Q})$ *we have*

$$\int_0^T \int_{\Omega} (D_+^{\beta} y(x,t) + \mathcal{A} y(x,t))\phi(x,t) dxdt = \int_{\Omega} \phi(x,T) I_+^{1-\beta} y(x,T) dx - \int_{\Omega} \phi(x,0) I_+^{1-\beta} y(x,0^+) dx$$
$$+ \int_0^T \int_{\partial\Omega} y \frac{\partial \phi}{\partial \nu_{\mathcal{A}}} d\Gamma dt - \int_0^T \int_{\partial\Omega} \frac{\partial y}{\partial \nu_{\mathcal{A}}} \phi d\Gamma dt + \int_0^T \int_{\Omega} y(x,t)(-D^{\beta}\phi(x,t) + \mathcal{A}^* \phi(x,t)) dxdt.$$

where $\mathcal{A}$ *is a given operator which is defined by (2.4) below and*

$$\frac{\partial y}{\partial \nu_{\mathcal{A}}} = \sum_{i,j=1}^{n} a_{ij} \frac{\partial y}{\partial x_j} cos(n, x_j) \quad on\, \Gamma,$$

$cos(n, x_j)$ *is the i-th direction cosine of n,n being the normal at* Γ *exterior to* Ω.

We also introduce the space

$$\mathcal{W}(0,T) := \{y : y \in L^2(0,T; H_0^1(\Omega)), D_+^{\beta} y(t) \in L^2(0,T; H_0^{-1}(\Omega))\}$$

in which a solution of a differential systems is contained. The spaces considered in this paper are assumed to be real.

2. Fractional Dirichlet Problem for Differential System

Let us consider the fractional partial differential equations:

$$D_+^{\beta} y(t) + \mathcal{A} y(t) = f(t), \quad t \in [0,T], \tag{2.1}$$

$$I_+^{1-\beta} y(0^+) = y_0, \quad x \in \Omega, \tag{2.2}$$

$$y(x,t) = 0, \quad x \in \Gamma, t \in (0,T), \tag{2.3}$$

where $0 < \beta < 1$, $y_0 \in H^2(\Omega) \bigcap H_0^1(\Omega)$, the function f belongs to $L^2(Q)$. The fractional integral $I_+^{1-\beta}$ and the derivative D_+^{β} are understand here in the Riemann-Liouville sense, Ω has the same properties as in section 1 and $I_+^{1-\beta} y(0^+) = \lim_{t \to 0^+} I_+^{1-\beta} y(t)$. The operator $\mathcal{A}$ in the state equation (2.1) is a second order operator given by

$$\mathcal{A} y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x) \frac{\partial y}{\partial x_j}\Big) + a_0(x) y, \tag{2.4}$$

where $a_{ij}, i, j = 1, 2, ..., n,$ be given function on Ω with the properties

$$a_0(x), a_{ij}(x) \in L^{\infty}(\Omega) \quad \text{(with real values)},$$

$$a_0(x) \geq \alpha > 0, \quad \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \alpha(\xi_1^2 + ... + \xi_n^2), \quad \forall \xi \in R^n,$$

almost everywhere on Ω. The operator $\mathcal{A} \in \mathcal{L}\Big(H_0^1(\Omega), H_0^{-1}(\Omega)\Big)$.

For this operator we define the bilinear form as follows:

Definition 2.1. *For each $t \in]0, T[$, we define a family of bilinear forms $\pi(t; y, \phi)$ on $H_0^1(\Omega)$ by:*

$$\pi(t; y, \phi) = (\mathcal{A}y, \phi)_{L^2(\Omega)}, \quad y,\ \phi \in H_0^1(\Omega), \tag{2.5}$$

where $\mathcal{A}$ maps $H_0^1(\Omega)$ onto $H_0^{-1}(\Omega)$ and takes the form (2.4). Then

$$\begin{aligned}\pi(t; y, \phi) &= \Big(\mathcal{A}y, \phi\Big)_{L^2(\Omega)}\\ &= \Big(-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial y}{\partial x_j}\Big)+a_0(x)y, \phi(x)\Big)_{L^2(\Omega)}\\ &= \int_\Omega\sum_{i,j=1}^{n} a_{ij}\frac{\partial}{\partial x_i}y(x)\frac{\partial}{\partial x_j}\phi(x)dx + \int_\Omega a_0(x)y(x)\phi(x)dx.\end{aligned}$$

Lemma 2.1. *The bilinear form $\pi(t; y, \phi)$ is coercive on $H_0^1(\Omega)$ that is*

$$\pi(t; y, y) \geq \lambda\, \|y\|^2_{H_0^1(\Omega)}, \quad \lambda > 0. \tag{2.6}$$

Proof. It is well known that the ellipticity of $\mathcal{A}$ is sufficient for the coerciveness of $\pi(t; y, \phi)$ on $H_0^1(\Omega)$. Since

$$\pi(t; y, \phi) = \int_\Omega\sum_{i,j=1}^{n} a_{ij}\frac{\partial}{\partial x_i}y(x)\frac{\partial}{\partial x_j}\phi(x)dx + \int_\Omega a_0(x)y(x)\phi(x)dx,$$

then we get

$$\begin{aligned}\pi(t; y, y) &= \int_\Omega\sum_{i,j=1}^{n} a_{ij}\frac{\partial}{\partial x_i}y(x)\frac{\partial}{\partial x_j}y(x)dx + \int_\Omega a_0(x)y(x)y(x)dx\\ &= \sum_{i,j=1}^{n} a_{ij}\|\frac{\partial}{\partial x_i}y(x)\|^2_{L^2(\Omega)} + \|y(x)\|^2_{L^2(\Omega)}\\ &\geq \lambda\|y\|^2_{H_0^1(\Omega)},\ \lambda > 0.\end{aligned}$$

□

Also we assume that $\forall y, \phi \in H_0^1(\Omega)$ the function $t \to \pi(t; y, \phi)$ is continuously differentiable in $]0, T[$ and the bilinear form $\pi(t; y, \phi)$ is symmetric,

$$\pi(t; y, \phi) = \pi(t; \phi, y) \qquad \forall y, \phi \in H_0^1(\Omega). \tag{2.7}$$

The equations (2.1) - (2.3) constitute a fractional Dirichlet problem. First by using the Lax-Milgram lemma, we prove sufficient conditions for the existence of a unique solution of the mixed initial-boundary value problem (2.1) - (2.3).

Lemma 2.2. *(see [Agrawal, 2002; 2004 and Mophou 2011a; 2011b]) (Fractional Green's formula). Let y be the solution of system (2.1)-(2.3). Then for any $\phi \in C^\infty(\overline{Q})$ such that $\phi(x, T) = 0$ in Ω and $\phi = 0$ on Σ, we have*

$$\int_0^T\int_\Omega (D_+^\beta y(x,t) + \mathcal{A}y(x,t))\phi(x,t)dxdt = -\int_\Omega \phi(x,0)I_+^{1-\beta}y(x,0^+)dx$$
$$+\int_0^T\int_{\partial\Omega} y\frac{\partial\phi}{\partial\nu}d\Gamma dt - \int_0^T\int_{\partial\Omega}\frac{\partial y}{\partial\nu}\phi d\Gamma dt + \int_0^T\int_\Omega y(x,t)(-D^\beta\phi(x,t) + \mathcal{A}^*\phi(x,t))dxdt.$$

Lemma 2.3. *If (2.6) and (2.7) hold, then the problem (2.1)-(2.3) admits a unique solution $y \in \mathcal{W}(0, T)$.*

Proof. See [Lions, 1971]. From the coerciveness condition (2.6) and using the Lax-Milgram lemma, there exists a unique element $y(t) \in H_0^1(\Omega)$ such that

$$(D_+^\beta y(t), \phi)_{L^2(Q)} + \pi(t; y, \phi) = L(\phi) \qquad \text{for all} \quad \phi \in H_0^1(\Omega), \tag{2.8}$$

which equivalent to there exists a unique solution $y(t) \in H_0^1(\Omega)$ for

$$(D_+^\beta y(t), \phi)_{L^2(Q)} + (\mathcal{A}y(t), \phi)_{L^2(Q)} = L(\phi) \qquad \text{for all} \quad \phi \in H_0^1(\Omega),$$

i.e. for

$$\left(D_+^\beta y(t) + \mathcal{A}y(t), \phi(x)\right)_{L^2(Q)} = L(\phi),$$

which can be written as

$$\int_Q (D_+^\beta y(t) + \mathcal{A}y(t))\phi(x)dxdt = L(\phi) \qquad \text{for all} \quad \phi \in H_0^1(\Omega). \tag{2.9}$$

This know as the variational fractional Dirichlet problem, where $L(\phi)$ is a continuous linear form on $H_0^1(\Omega)$ and takes the form

$$L(\phi) = \int_Q f\phi\, dxdt + \int_\Omega y_0\phi(x,0)dx, \quad f \in L^2(Q), y_0 \in L^2(\Omega). \tag{2.10}$$

Then equation (2.9) is equivalent to

$$\int_Q (D_+^\beta y(t) + \mathcal{A}y(t))\phi(x)dxdt = \int_Q f\phi\, dxdt + \int_\Omega y_0\phi(x,0)dx \qquad \text{for all} \quad \phi \in H_0^1(\Omega),$$

that is, the PDE

$$D_+^\beta y(t) + \mathcal{A}y(t) = f, \tag{2.11}$$

"tested" against $\phi(x)$.

Let us multiply both sides in (2.11) by ϕ and applying Green's formula (Lemma 2.2), we have

$$\int_Q (D_+^\beta y + \mathcal{A}y)\phi dxdt = \int_Q f\phi dxdt,$$

$$-\int_\Omega \phi(x,0)I_+^{1-\beta}y(x,0^+)dx + \int_0^T\int_{\partial\Omega} y\frac{\partial\phi}{\partial\nu}d\Gamma dt - \int_0^T\int_{\partial\Omega}\frac{\partial y}{\partial\nu}\phi d\Gamma dt$$

$$+\int_0^T\int_\Omega y(x,t)(-D^\beta\phi(x,t) + \mathcal{A}^*\phi(x,t))dxdt = \int_Q f\phi dxdt$$

whence comparing with (2.9), (2.10)

$$\int_\Omega \phi(x,0)I_+^{1-\beta}y(x,0^+)dx - \int_0^T\int_{\partial\Omega} y\frac{\partial\phi}{\partial\nu}d\Gamma dt = \int_\Omega y_0\phi(x,0)dx.$$

From this we deduce (2.2) and (2.3). □

3. Optimization Theorem and the Control Problem

For a control $u \in L^2(Q)$ the state $y(u)$ of the system is given by

$$D_+^\beta y + \mathcal{A}y(u) = u, \quad (x,t) \in Q \tag{3.1}$$

$$y(u)|_\Sigma = 0, \tag{3.2}$$

$$I_+^{1-\beta} y(x,0;u) = y_0(x), \quad x \in \Omega. \tag{3.3}$$

The observation equation is given by

$$z(u) = y(u), \tag{3.4}$$

The cost function $J(v)$ is given by

$$J(v) = \int_Q (y(v) - z_d)^2 dxdt + (Nv, v)_{L^2(Q)}$$

where z_d is a given element in $L^2(\Sigma)$ and $N \in \mathcal{L}(L^2(Q), L^2(Q))$ is hermitian positive definite operator:

$$(Nu, u) \geq c\|u\|^2_{L^2(Q)}, \; c > 0. \tag{3.5}$$

Control Constraints: We define U_{ad} (set of admissible controls) is closed, convex subset of $U = L^2(Q)$.
Control Problem: We want to minimize J over U_{ad} i.e. find u such that

$$J(u) = \inf_{v \in U_{ad}} J(v). \tag{3.6}$$

Under the given considerations we have the following theorem:

Theorem 3.1. *The problem (3.6) admits a unique solution given by (3.1)-(3.3) and*

$$\int_Q (p(u) + Nu)(v - u)\, dxdt \geq 0, \tag{3.7}$$

where $p(u)$ is the adjoint state.

Proof. By similar manner as in ([Mophou, 2011a] Proposition 4.1 and Theorem 4.2) and (Lions, 1971), the control $u \in U_{ad}$ is optimal if and only if

$$J'(u)(v - u) \geq 0 \quad \text{for all } v \in U_{ad}$$

The above condition, when explicitly calculated for this case, gives

$$(y(u) - z_d, y(v) - y(u))_{L^2(Q)} + (Nu, v - u)_{L^2(Q)} \geq 0$$

i.e.

$$\int_Q (y(u) - z_d)(y(v) - y(u))dxdt + (Nu, v - u)_{L^2(Q)} \geq 0. \tag{3.8}$$

For the control $u \in L^2(Q)$ the adjoint state $p(u) \in L^2(Q)$ is defined by

$$-D^\beta p(u) + \mathcal{A}^* p(u) = y(u) - z_d, \quad \text{in } Q, \tag{3.9}$$

$$p(u) = 0, \qquad \text{on } \Sigma, \tag{3.10}$$

$$p(x,T;u)=0, \qquad \text{in } \Omega, \tag{3.11}$$

where $\mathcal{A}^*$ is the adjoint operator for the operator $\mathcal{A}$, which given by

$$\mathcal{A}^*p=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\Big(a_{ij}(x)\frac{\partial p}{\partial x_i}\Big)+a_0(x)p.$$

Now, multiplying the equation (3.9) by $(y(v)-y(u))$ and applying Green's formula, we obtain

$$\begin{aligned}\int_Q (y(u)-z_d)(y(v)-y(u))dxdt &= \int_Q(-D^{\beta}p(u)+\mathcal{A}^*p(u))(y(v)-y(u))\,dxdt\\ &=-\int_\Omega p(x,0)I_+^{1-\beta}(y(v;x,0^+)-y(u;x,0^+))dx\\ &+\int_\Sigma p(u)(\frac{\partial y(v)}{\partial\nu_{\mathcal{A}}}-\frac{\partial y(u)}{\partial\nu_{\mathcal{A}}})\,d\Sigma\\ &-\int_\Sigma\frac{\partial p(u)}{\partial\nu_{\mathcal{A}}}(y(v)-y(u))\,d\Sigma\\ &+\int_Q p(u)(D_+^{\beta}+\mathcal{A})(y(v)-y(u))\,dxdt.\end{aligned}$$

Since from (3.1), (3.2) we have

$$(D_+^{\beta}+\mathcal{A})(y(v)-y(u))=v-u, \quad y(u)|_\Sigma=0, \quad p(u)|_\Sigma=0.$$

Then we obtain

$$\int_Q(y(u)-z_d)(y(v)-y(u))dxdt=\int_Q p(u)(v-u)dxdt,$$

and hence (3.8) is equivalent to

$$\int_Q p(u)(v-u)\,dxdt+(Nu,v-u)_{L^2(Q)}\ge 0$$

i.e.

$$\int_Q(p(u)+Nu)(v-u)\,dxdt\ge 0$$

which completes the proof. □

4. Fractional Neumann Problem for Differential System

From (1.5) we can show that the bilinear form (2.5) is coercive in $H^1(\Omega)$ that is

$$\pi(y,y)\ge c\|y\|^2_{H^1(\Omega)}, \quad c>0 \qquad \text{for all} \quad y\in H^1(\Omega). \tag{4.1}$$

From the above coerciveness condition (4.1) and using the Lax-Milgram lemma we have the following lemma which define the Neumann problem for the operator $\mathcal{A}$ with $\mathcal{A}\in\mathcal{L}(H^1(\Omega),H^{-1}(\Omega))$ and enables us to obtain the state of our control problem.

Lemma 4.1. *If (4.1) is satisfied then there exists a unique element $y\in H^1(\Omega)$ satisfying Neumann problem*

$$D_+^{\beta}y+\mathcal{A}y=f \qquad \text{in } Q, \tag{4.2}$$

$$\frac{\partial y}{\partial\nu_{\mathcal{A}}}=h \qquad \text{on } \Sigma, \tag{4.3}$$

$$I_+^{1-\beta}y(0^+)=y_0(x),\ x\in\Omega, \tag{4.4}$$

Proof. From the coerciveness condition (4.1) and using the Lax-Milgram lemma, there exists a unique element $y \in H^1(\Omega)$ such that

$$\int_Q y(-D^\beta \psi + \mathcal{A}^* \psi) dxdt = M(\psi) \quad \text{for all} \quad \psi \in H^1(\Omega). \tag{4.5}$$

This know as the fractional Neumann problem, where $M(\psi)$ is a continuous linear form on $H^1(\Omega)$ and takes the form

$$M(\psi) = \int_Q f\psi \, dxdt + \int_\Omega y_0 \psi(x,0) dx - \int_\Sigma h \frac{\partial \psi}{\partial \nu_{\mathcal{A}^*}} d\Sigma, \tag{4.6}$$

$$f \in L^2(Q), y_0 \in L^2(\Omega), h \in H^1(\Sigma).$$

The equation (4.5) is equivalent to

$$D_+^\beta y + \mathcal{A} y = f \qquad \text{on } Q. \tag{4.7}$$

Let us multiply both sides in (4.7) by ψ and applying Green's formula, we have

$$\int_Q (D_+^\beta y + \mathcal{A} y) \psi dxdt = \int_Q f \psi dxdt$$

$$- \int_\Omega \psi(x,0) I_+^{1-\beta} y(x,0^+) dx + \int_0^T \int_{\partial\Omega} y \frac{\partial \psi}{\partial \nu} d\Gamma dt - \int_0^T \int_{\partial\Omega} \frac{\partial y}{\partial \nu} \psi d\Gamma dt$$

$$+ \int_0^T \int_\Omega y(x,t)(-D^\beta \psi(x,t) + \mathcal{A}^* \psi(x,t)) dxdt = \int_Q f \psi dxdt$$

whence comparing with (4.5), (4.6)

$$\int_\Omega \psi(x,0) I_+^{1-\beta} y(x,0^+) dx + \int_0^T \int_{\partial\Omega} \psi \frac{\partial y}{\partial \nu} d\Gamma dt = \int_\Omega y_0 \psi(x,0) dx + \int_0^T \int_{\partial\Omega} h \psi d\Gamma dt.$$

From this we deduce (4.3) and (4.4). □

5. Minimization Theorem and Boundary Control Problem

We consider the space $U = L^2(\Sigma)$ (the space of controls), for every control $u \in U$, the state of the system $y(u) \in H^1(\Omega)$ is given by the solution of

$$D_+^\beta y(u) + \mathcal{A} y(u) = f \quad \text{in } Q, \tag{5.1}$$

$$\frac{\partial y(u)}{\partial \nu_{\mathcal{A}}} = u \quad \text{on } \Sigma, \tag{5.2}$$

$$I_+^{1-\beta} y(x,0;u) = y_0(x), \ x \in \Omega. \tag{5.3}$$

For the observation, we consider the following two cases:

(i)

$$z(u) = y(u) \tag{5.4}$$

(ii) observation of final state

$$z(u) = y(x,T;u) \tag{5.5}$$

Case (i)

The cost function is given by

$$J(v) = \int_Q (y(v) - z_d)^2\, dxdt + (Nv, v)_{L^2(\Sigma)}, \quad z_d \in L^2(Q), \tag{5.6}$$

where $N \in \mathcal{L}(L^2(\Sigma), L^2(\Sigma)), N$ is hermitian positive definite

$$(Nu, u)_{L^2(\Sigma)} \geq c\|u\|^2_{L^2(\Sigma)}, c > 0. \tag{5.7}$$

Control Constraints: We define U_{ad}(set of admissible controls) is closed, convex subset of $U = L^2(\Sigma)$. **Control Problem:** We wish to find

$$\inf_{v \in U_{ad}} J(v). \tag{5.8}$$

Under the given considerations we have the following theorem.

Theorem 5.1. *Assume that (5.7) holds and the cost function being given by (5.6). The optimal control u is characterized by (5.1), (5.2), and (5.3) together with*

$$-D^\beta p(u) + \mathcal{A}^* p(u) = y(u) - z_d \quad in\ Q, \tag{5.9}$$

$$\frac{\partial p(u)}{\partial \nu_{\mathcal{A}^*}} = 0 \qquad on\ \Sigma, \tag{5.10}$$

$$p(x, T; u) = 0, \quad x \in \Omega, \tag{5.11}$$

and the optimality condition is

$$\int_\Sigma (p(u) + Nu)(v - u) d\Sigma \geq 0 \quad \forall\, v \in U_{ad} \tag{5.12}$$

where $p(u)$ is the adjoint state.

Proof. By similar manner as in ([Mophou, 2011a] Proposition 4.1 and Theorem 4.2) and (Lions, 1971), the control $u \in U_{ad}$ is optimal if and only if

$$J'(u)(v - u) \geq 0 \qquad \forall v \in U_{ad} \tag{5.13}$$

that is

$$\Big(y(u) - z_d, y(v) - y(u)\Big)_{L^2(Q)} + (Nu, v - u)_U \geq 0. \tag{5.14}$$

The adjoint state is given by the solution of the adjoint Neumann problem (5.9), (5.10) and (5.11). Now, multiplying the equation in (5.9) by $y(v) - y(u)$ and applying Green's formula, with taking into account the conditions in (5.1), (5.2), we obtain

$$\int_Q (y(u) - z_d)(y(v) - y(u))\, dxdt = \int_Q (-D^\beta p(u) + \mathcal{A}^* p(u))(y(v) - y(u))\, dxdt$$

$$= -\int_\Omega p(x,0) I_+^{1-\beta}(y(v; x, 0^+) - y(u; x, 0^+))dx + \int_\Sigma p(u)(\frac{\partial}{\partial \nu_{\mathcal{A}}} y(v) - \frac{\partial}{\partial \nu_{\mathcal{A}}} y(u))\, d\Sigma - \int_\Sigma \frac{\partial}{\partial \nu_{\mathcal{A}^*}} p(u)(y(v) - y(u))\, d\Sigma$$

$$+ \int_Q p(u)((D_+^\beta + \mathcal{A})(y(v) - y(u))\, dxdt = \int_\Sigma p(u)(v - u)\, d\Sigma. \tag{5.15}$$

Hence we substitute from (5.15) in (5.14), to get

$$\int_\Sigma p(u)(v - u)\, d\Sigma + (Nu, v - u)_{L^2(\Sigma)} \geq 0$$

i.e.

$$\int_\Sigma (p(u) + Nu)(v - u) d\Sigma \geq 0 \quad \forall\, v \in U_{ad}$$

which completes the proof. □

Example 5.1. *In the case of no constraints on the control ($\mathcal{U}_{ad} = \mathcal{U}$). Then (5.12) reduces to*

$$p + Nu = 0 \quad on \;\; \Sigma.$$

The optimal control is obtained by the simultaneous solution of the following system of fractional partial differential equations:

$$D_+^\beta y + \mathcal{A}y = f, \quad -D^\beta p + \mathcal{A}^* p = y - z_d \quad in \;\; Q,$$
$$\frac{\partial y}{\partial \nu_{\mathcal{A}}}|_\Sigma + N^{-1} p|_\Sigma = 0, \quad \frac{\partial p}{\partial \nu_{A^*}} = 0 \quad on \;\; \Sigma,$$
$$I_+^{1-\beta} y(x,0) = y_0(x), \quad p(x,T) = 0 \quad x \in \Omega,$$

further

$$u = -N^{-1}(P|_\Sigma).$$

Example 5.2. *If we take*

$$\mathcal{U}_{ad} = \Big\{ u | u \in L^2(\Sigma), u \geq 0 \quad almost\ everywhere\ on \quad \Sigma \Big\}.$$

The optimal control is obtained by the solution of the fractional problem

$$D_+^\beta y + \mathcal{A}y = f, \quad -D^\beta p + \mathcal{A}^* p = y - z_d \quad in \;\; Q,$$
$$\frac{\partial y}{\partial \nu_{\mathcal{A}}} \geq 0, \quad \frac{\partial p}{\partial \nu_{\mathcal{A}}^*} = 0 \quad on \;\; \Sigma,$$
$$p + N\frac{\partial y}{\partial \nu_{\mathcal{A}}} \geq 0, \quad \frac{\partial y}{\partial \nu_{\mathcal{A}}}[p + N\frac{\partial y}{\partial \nu_{\mathcal{A}}}] = 0 \quad on \;\; \Sigma,$$
$$I_+^{1-\beta} y(x,0) = y_0(x), \quad p(x,T) = 0 \quad x \in \Omega,$$

hence

$$u = \frac{\partial y}{\partial \nu_{\mathcal{A}}}|_\Sigma.$$

Case (ii) *observation of final state*

$$z(u) = y(x,T;u).$$

The cost function is given by

$$J(v) = \int_\Omega (y(x,T;v) - z_d)^2\, dx + (Nv, v)_{L^2(\Sigma)}, \quad z_d \in L^2(\Omega).$$

The adjoint state is defined by

$$-D^\beta p(u) + \mathcal{A}^* p(u) = 0 \quad in \;\; Q,$$
$$\frac{\partial p(u)}{\partial \nu_{\mathcal{A}^*}} = 0 \qquad\qquad on \;\; \Sigma,$$
$$p(x,T;u) = y(x,T;u) - z_d(x), \quad x \in \Omega,$$

and the optimality condition is

$$\int_\Sigma (p + Nu)(v - u) d\Sigma \geq 0 \quad \forall\, v \in U_{ad}, \tag{5.16}$$

where $p(u)$ is the adjoint state.

Example 5.3. *In the case of no constraints on the control ($\mathcal{U}_{ad} = \mathcal{U}$). Then (5.16) reduces to*

$$p + Nu = 0 \quad on \ \Sigma.$$

The optimal control is obtained by the simultaneous solution of the following system of fractional partial differential equations:

$$D_+^\beta y + \mathcal{A}y = f, \quad -D^\beta p + \mathcal{A}^* p = 0 \ \ in \ \ Q,$$
$$\frac{\partial y}{\partial \nu_{\mathcal{A}}}|_\Sigma + N^{-1} p|_\Sigma = 0, \quad \frac{\partial p}{\partial \nu^*_{\mathcal{A}}} = 0 \quad on \ \ \Sigma,$$
$$I_+^{1-\beta} y(x, 0) = y_0(x), \quad p(x, T) = y(x, T; u) - z_d(x), \quad x \in \Omega,$$

further

$$u = -N^{-1}(P|_\Sigma).$$

Example 5.4. *If we take*

$$\mathcal{U}_{ad} = \Big\{ u | u \in L^2(\Sigma), u \geq 0 \quad almost\ everywhere\ on \quad \Sigma \Big\}.$$

Then (5.16) is equivalent to

$$u \geq 0, \quad p(u) + Nu \geq 0, \quad u(p(u) + Nu) = 0 \quad on\ \Sigma.$$

6. Controllability

This section is devoted to study the controllability of the fractional differential system (3.1),(3.2), and (3.3). We begin by the following definition.

Definition 6.1 (Lions, 1971). *The system whose state is defined by (3.1),(3.2), and (3.3) is said to be controllable if as u is varied without any constraints, the observation $Cy(u)$ generates a dense (affine) subspace of the space of observations.*

Let us consider the the case of section 3. Hence for a control $u \in L^2(Q)$ the state of the system $y(u)$ is given by

$$D_+^\beta y(u) + \mathcal{A}y(u) = u, \quad (x, t) \in Q \tag{6.1}$$

$$y(u)|_\Sigma = 0, \tag{6.2}$$

$$I_+^{1-\beta} y(x, 0; u) = y_0(x), \quad x \in \Omega. \tag{6.3}$$

The observation $y(y)$ is in $L^2(Q)$ and given by

$$z(u) = y(u). \tag{6.4}$$

As u ranges over $L^2(Q)$, $y(u)$ generates a dense (affine) subspace of $L^2(Q)$; hence the system is controllable.

To see this, let us first remark that by translation we may always reduce the problem to the case where $y_0(x) = 0$.

Let $\psi \in L^2(Q)$ be the orthogonal to the subspace generated by $y(u)$;

$$\int_Q y(u)\psi dxdt = 0 \quad \forall u. \tag{6.5}$$

We consider ξ as the solution of

$$-D^\beta \xi + \mathcal{A}^* \xi = \psi, \quad (x, t) \in Q \tag{6.6}$$

$$\xi|_{\Sigma} = 0, \tag{6.7}$$

$$\xi(x,T) = 0, \quad x \in \Omega. \tag{6.8}$$

Then

$$\begin{aligned}\int_Q \psi y(u)dxdt &= \int_Q (-D^{\beta}\xi + \mathcal{A}^*\xi)y(u)dxdt \\ &= -\int_\Omega \xi(x,0)I_+^{1-\beta}y(u;x,0^+)dx + \int_\Sigma \xi\frac{\partial y(u)}{\partial\nu_{\mathcal{A}}}d\Sigma \\ &- \int_\Sigma \frac{\partial\xi}{\partial\nu_{\mathcal{A}}}y(u)\,d\Sigma + \int_Q \xi(D_+^{\beta} + \mathcal{A})y(u)\,dxdt \\ &= \int_Q \xi udxdt = 0 \quad \forall u;\end{aligned}$$

hence $\xi = 0$ and hence $\psi = 0$.

Remark 6.1. *We can also study by a similar manner the controllability of the system whose state is given by (5.1), (5.2), and (5.3).*

Remark 6.2. *If we take $\beta = 1$ in the previews sections we obtain the classical results in the optimal control with integer derivatives.*

Conclusions

An analytical scheme for fractional optimal control of differential systems is considered. The fractional derivatives was defined in the Riemann-Liouville sense. The analytical results were given in terms of Euler-Lagrange equations for the fractional optimal control problems. The formulation presented and the resulting equations are very similar to those for classical optimal control problems. The optimization problem presented in this paper constitutes a generalization of the optimal control problem of parabolic systems with Dirichlet and Neumann boundary conditions considered in (Lions, 1971) to fractional optimal control problem for second order systems. Also the main result of the paper contains necessary and sufficient conditions of optimality for second order systems that give characterization of optimal control (Theorems 3.1 and 5.1).

Also it is evident that by modifying:

- the boundary conditions, (Dirichlet, Neumann, mixed, etc.),
- the nature of the control (distributed, boundary, etc.),
- the nature of the observation (distributed, boundary, etc.),
- the initial differential system,
- the number of variables (finite number of variables, infinite number of variables systems, etc.),
- the type of equation (elliptic, parabolic, hyperbolic, etc.),
- the order of equation (second order, Schrödinger, infinite order, etc.),
- the type of control (optimal control problem, time-optimal control problem, etc.),

an many of variations on the above problem are possible to study with the help of (Lions, 1971) and Dubovitskii-Milyutin formalisms (Bahaa, 2003; 2005; 2007; 2008; 2012a,b,; 2013), (Bahaa and Tharwat, 2011; 2012a,b), (Bahaa and El-Shatery 2013), (Bahaa and El-Gohany 2013a, 2013b). Those problems need further investigations and form tasks for future research. These ideas mentioned above will be developed in forthcoming papers.

Acknowledgements

The author would like to express his gratitude to the anonymous reviewers for their very valuable remarks and comments.

References

[1] O. P. Agrawal, Formulation of Euler-Lagrange equations for fractional variational problems, Journal of Mathematical Analysis and Applications, 272 (2002), 368–379.

[2] O. P. Agrawal, A general formulation and solution scheme for fractional optimal control problems, Nonlinear Dynamics, 38 (2004), 323–337.

[3] O. P. Agrawal, D. Baleanu, A Hamiltonian formulation and direct numerical scheme for fractional optimal control problems, Journal of Vibration and Control, 13 (9-10) (2007), 1269–1281.

[4] B. Ahmad, S. K. Ntouyas, Existence of solutions for fractional differential inclusions with four-point nonlocal Riemann-Liouville type integral boundary conditions, Filomat, 27 (6) (2013), 1027–1036.

[5] G. M. Bahaa, Quadratic Pareto optimal control of parabolic equation with state-control constraints and an infinite number of variables, IMA J. Math. Control and Inform. 20 (2003), 167–178.

[6] G. M. Bahaa, Time-optimal control problem for parabolic equations with control constraints and infinite number of variables, IMA J. Math. Control and Inform. 22 (2005), 364–375.

[7] G. M. Bahaa, Optimal control for cooperative parabolic systems governed by Schrödinger operator with control constraints, IMA J. Math. Control and Inform. 24 (2007), 1–12.

[8] G. M. Bahaa, Optimal control problems of parabolic equations with an infinite number of variables and with equality constraints, IMA J. Math. Control and Inform. 25 (2008), 37–48.

[9] G. M. Bahaa, Optimality conditions for cooperative parabolic systems governed by Schrödinger operators with control constraints, Asian-European Journal of Mathematics, 1 (2) (2008), 131–146.

[10] G. M. Bahaa, Boundary control problem of infinite order distributed hyperbolic systems involving time lags, Intelligent Control and Automation, 3 (3) (2012), 211–221.

[11] G. M. Bahaa, Optimality conditions for infinite order distributed parabolic systems with multiple time delays given in integral form, Journal of Applied Mathematics, 2012 (2012), 25 pages.

[12] G. M. Bahaa, W. Kotarski, Optimality conditions for $n \times n$ infinite order parabolic coupled systems with control constraints and general performance index, IMA J. Math. Control and Inform. 25 (2008), 49–57.

[13] G. M. Bahaa, M. M. Tharwat, Optimal control problem for infinite variables hyperbolic systems with time lags, Archives of Control Sciences, ACS, 21 (4) (2011), 373-393.

[14] G. M. Bahaa, M. M. Tharwat, Time-optimal control of infinite order parabolic system with time lags given in integral form, Journal of Information and Optimization Sciences, 33 (2&3) (2012), 233–258.

[15] G. M. Bahaa, M. M. Tharwat, Optimal boundary control for infinite variables parabolic systems with time lags given in integral form, Iranian Journal Of Science & Technology, IJST A3 (2012), 277–291.

[16] G. M. Bahaa, Fatemah El-Shatery, Optimal control problems for elliptic systems with control constraints and infinite number of variables, International Journal of Applied Mathematics & Statistics, 34 (4) (2013), 60–72.

[17] G. M. Bahaa, Eman El-Gohany, Optimal control problem for $n \times n$ infinite order parabolic systems, International Journal of Applied Mathematics & Statistics, 36 (6) (2013), 26–41.

[18] G. M. Bahaa, Eman El-Gohany, Optimal control problem for $n \times n$ infinite order elliptic systems, International Journal of Applied Mathematics & Statistics, 39 (9) (2013), 24–34.

[19] D. Baleanu, S. I. Muslih, Lagrangian formulation on classical fields within Riemann-Liouville fractional derivatives, Phys. Scr. 72 (2-3) (2005), 119–121.

[20] D. Baleanu, T. Avkar, Lagrangian with linear velocities within Riemann-Liouville fractional derivatives, Nuovo Cimnto B, 119 (2004), 73–79.

[21] H. A. El-Saify, G. M. Bahaa, Optimal control for $n \times n$ hyperbolic systems involving operators of infinite order, Mathematica Slovaca, 52 (4) (2002), 409–424.

[22] F. Jarad, T. Maraba, D. Baleanu, Fractional variational optimal control problems with delayed arguments, Nonlinear Dyn. 62 (2010), 609–614.

[23] F. Jarad, T. Maraba, D. Baleanu, Higher order fractional variational optimal control problems with delayed arguments, Applied Mathematics and Computation, 218 (2012), 9234–9240.

[24] W. Kotarski, H. A. El-Saify, G. M Bahaa, Optimal control of parabolic equation with an infinite number of variables for non-standard functional and time delay, IMA J. Math. Control and Inform. 19 (4) (2002), 461–476.

[25] J. L. Lions, Optimal control of systems governed by partial differential equations, Springer-Verlag, Band (1971), 170.

[26] G. M. Mophou, Optimal control of fractional diffusion equation, Computers and Mathematics with Applications, 61 (2011), 68–78.

[27] G. M. Mophou, Optimal control of fractional diffusion equation with state constraints. Computers and Mathematics with Applications, 62 (2011), 1413–1426.

[28] M. Pavlović, Green's formula and the Hardy-Stein identities, Filomat, 23 (3) (2009), 135–153.