1482. Fractional order generalized thermoelasticity with variable thermal conductivity

Hamdy M. Youssef[1], Ibrahim A. Abbas[2]
[1]Mechanics Department, Faculty of Engineering, Umm Al-Qura University,
P.O. 5555, Makkah, Saudi Arabia
[1]Mathematics Department, Faculty of Education, Alexandria University, Alexandria, Egypt
[2]Mathematics Department, Faculty of Science and Arts – Khulais,
King Abdulaziz University, Jeddah, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Sohag University, Sohag, Egypt
[2]Corresponding author
E-mail: [1]*youssefanne2005@gmail.com*, [2]*ibrabbas7@yahoo.com*

Abstract. In this work, the consideration of variable thermal conductivity as a linear function of temperature has been taken into account in the context of fractional order generalized thermoelasticity (Youssef's model). The governing equations have been derived and used to solve the one-dimensional problems of an elastic half-space. The solution has been induced in the Laplace transform domain and applying for thermal shock half-space on the bounding plane when it is rigid. The numerical inversion of the Laplace transform has been calculated numerically by using Tzou method and the results have been represented in figures with some comparisons to stand on the effect of the fractional order parameter and the variability of the thermal conductivity on all the studied fields.

Keywords: fractional order, generalized thermoelasticity, variable thermal conductivity.

1. Introduction

Biot introduced the first coupled theory of thermoelasticity in which the equations of heat conduction and elasticity became coupled and agree with physical behavior of the elastic materials, where any change of the temperature gives a certain amount of strain in an elastic body and vice versa [1]. The equations of this theory consist of the equation of motion, which is a hyperbolic partial differential equation, and the heat equation, which is parabolic and implies that if an elastic medium is extending to infinity subjected to a thermal or mechanical disturbance, the effect will vanish instantaneously at infinity, which contradicts physical experiments. Hence, a new equation of energy with hyperbolic type is needed.

Lord and Shulman constructed the theory of generalized thermoelasticity with one relaxation time for the particular case of an isotropic body [2]. Dhaliwal and Sherief extended this theory to include the anisotropic case [3]. According to this theory, a modified law of heat conduction including both the heat flux and its time rate of change, which is called Cattaneo's heat conduction, replaces the conventional Fourier's law. Now, the heat equation is hyperbolic and hence it introduces finite speeds of propagation for mechanical and thermal waves.

The physical property of a solid body related to application of heat energy is defined as a thermal property. Thermal properties explain the response of a material to the application of heat and one of the most important thermal properties is the thermal conductivity. Thermal conductivity K is ability of a material to transport heat energy through it from high temperature region to low temperature region and it is a microstructure sensitive property its value range for metals 20-400, for ceramics 2-50 and for polymers order of 0.3. Heat is transported in two ways – electronic contribution, vibrational (phonon) contribution. In metals, electronic contribution is very high. Thus metals have higher thermal conductivities. Thermal conductivity of iron initially decreases then increases slightly; thermal conductivity decreases with increase in temperature for aluminium; while it increases for platinum and gold [4]. Godfrey has reported decreases of up to 45 % in the thermal conductivity of various samples of silicon nitride between (1 and 400.0 C).

The question arises: what are the effects of these variations on the stress and displacement distributions in metal components [5]?

Youssef used the state-space approach to solve a problem of generalized thermoelasticity for an infinite material with a spherical cavity and variable thermal conductivity subjected to ramp-type heating [6].

Recently, the fractional calculus has been used to modify many existing models of physical processes. Abel is the first one who gives an application of fractional derivatives by applied fractional calculus in the solution of an integral equation. The connection between fractional derivatives and the theory of linear viscoelasticity has been found by Caputo [7]. The theory of fractional calculus introduced mathematical modeling to simulate the polymers materials successfully.

A new formula of heat conduction has been considered in the context of the fractional integral operator definition by Youssef [8]. This new consideration generated the fractional order generalized thermoelasticity which was cited by Youssef who approved the uniqueness of its solutions [9].

Youssef solved one dimensional problem in the context of the fractional order generalized thermoelasticity and discussed the effects of the fractional order parameter on all the studied fields in [8] and with Al-Leheabi in [9]. Youssef and Al-Leheabi get the variational principal of the theorem [10]. Kumar et al. discussed the plane deformation due to thermal source in fractional order thermoelastic media [11], while Abo-Dahab and Abbas solved a problem of LS model on thermal shock problem of generalized magneto-thermoelasticity for an infinitely long annular cylinder with variable thermal conductivity [12]. Abbas and Abo-Dahab got the numerical solution of thermal shock problem for generalized magneto-thermoelasticity for an infinitely long annular cylinder with variable thermal conductivity [13].

The counterparts of our problem in the contexts of the thermoelasticity theories have been considered by Sharma and Marin [14] and Marin et al. [15, 16].

In the present work, the solution of thermoelastic medium thermal loaded by thermal shock is developed in the context of fractional order generalized thermoelasticity with variable thermal conductivity. Laplace transform methods are used to determine the temperature, the displacement, the strain and the stress. The effect of the variability of thermal conductivity has been studied together with the fractional order parameter and represented graphically with some comparisons.

2. Basic formulations

In the context of the fractional order generalized thermoelasticity, the displacement and the thermal fields as well as the stress-strain-temperature relations for a linear homogeneous and isotropic medium are in the forms [8, 9].

The equations of motion take the form:

$$\mu\, u_{i,jj} + (\lambda + \mu)\, u_{j,ji} + F_i - \gamma\, \theta_{,i} = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad i,j = 1,2,3, \tag{1}$$

where λ, μ are Lame's constants, u_i is the displacement component, F_i is the body force component, $\gamma = \alpha_T(3\lambda + 2\mu)$ and α_T is the thermal expansion, T is the temperature of the body, T_o is the reference temperature, $\theta = (T - T_o)$ is the increment of the temperature, ρ is the density, the subscript i and j give the order of the component tensor of a function in three dimensions and $u_{i,j} = \partial u_i / \partial x_j$ which means the derivative of the component of the displacement u_i with respect to the x_i-axis.

The heat equations take the form [7]:

$$I^{\alpha-1}(K\,\theta_{,i})_{,i} = \left(1 + \tau_o \frac{\partial}{\partial t}\right)\left(\frac{K}{\kappa}\frac{\partial\theta}{\partial t} + T_o\gamma\frac{\partial u_{j,j}}{\partial t}\right),$$

$$\begin{cases} 0 < \alpha < 1, & \text{for weak conductivity,} \\ \alpha = 1, & \text{for normal conductivity,} \\ 1 < \alpha \le 2, & \text{for strong conductivity,} \end{cases} \tag{2}$$

where τ_o is the relaxation time, K is the thermal conductivity, κ is called the thermal diffusivity, and the fractional integral operator defined as follows [8, 9]:

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)\ d\tau, \quad 0 < \alpha \le 2, \tag{3}$$

where $\Gamma(\alpha)$ is the Gamma function.

The constitutive equations take the form:

$$\sigma_{ij} = \mu(u_{i,j} + u_{j,i}) + (\lambda\,u_{i,i} - \gamma\,\theta)\delta_{ij}, \quad i,j = 1,2,3, \tag{4}$$

where σ_{ij} is the stress tensor and δ_{ij} is the Kronecker delta function.

The strain-displacement equations take the form:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i,j = 1,2,3. \tag{5}$$

For one-dimensional half-space $0 \le x < \infty$ of an elastic medium as in Fig. 1 without any body forces with variable thermal conductivity and displacement components $u_i = (u(x,t),0,0)$, the equations in (1), (2), (4) and (5) take the forms:

$$(\lambda + 2\mu)\frac{\partial^2 u(x,t)}{\partial x^2} - \gamma\frac{\partial\theta(x,t)}{\partial x} = \rho\frac{\partial^2 u(x,t)}{\partial t^2}, \tag{6}$$

$$I^{\alpha-1}\frac{\partial}{\partial x}\left[K(\theta)\frac{\partial\theta(x,t)}{\partial x}\right] = \left(1 + \tau_o\frac{\partial}{\partial t}\right)\left(\frac{K(\theta)}{\kappa}\frac{\partial\theta(x,t)}{\partial t} + T_o\gamma\frac{\partial e(x,t)}{\partial t}\right), \quad 0 < \alpha \le 2, \tag{7}$$

$$\sigma_{xx}(x,t) = (\lambda + 2\mu)e(x,t) - \gamma\theta(x,t), \tag{8}$$

$$e(x,t) = \frac{\partial u(x,t)}{\partial x}. \tag{9}$$

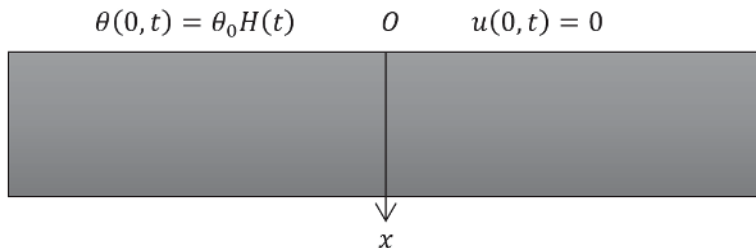$$\theta(0,t) = \theta_0 H(t) \qquad O \qquad u(0,t) = 0$$

Fig. 1. One-dimensional half-space of an elastic medium

We will consider the thermal conductivity has the following linear form of thermo-dynamical temperature as follows [6]:

$$K(\theta) = K_o(1 + K_1\theta), \tag{10}$$

where K_o is constant and gives the thermal conductivity of the material when it does not depend

on θ, and K_1 small constant parameter.

Now, we will use the following mappings [6]:

$$\vartheta(x,t) = \frac{1}{K_o} \int_0^{\theta(x,t)} K(\xi)d\xi.$$ (11)

Differentiating Eq. (6) with respect to x yields:

$$K_o \frac{\partial \vartheta(x,t)}{\partial x} = K(\theta) \frac{\partial \theta(x,t)}{\partial x}.$$ (12)

Differentiating again the above equation with respect to x we obtain:

$$K_o \frac{\partial^2 \vartheta(x,t)}{\partial x^2} = \frac{\partial}{\partial x}\left[K(\theta)\frac{\partial \theta(x,t)}{\partial x}\right].$$ (13)

Differentiating Eq. (11) with respect to the time t, we get:

$$K_o \frac{\partial \vartheta(x,t)}{\partial t} = K(\theta) \frac{\partial \theta(x,t)}{\partial t}.$$ (14)

Substituting from Eqs. (13) and (14) in Eq. (7), we obtain:

$$I^{\alpha-1}\frac{\partial^2 \vartheta(x,t)}{\partial x^2} = \left(\frac{\partial}{\partial t} + \tau_o \frac{\partial^2}{\partial t^2}\right)\left[\frac{1}{\kappa}\vartheta(x,t) + \frac{\gamma T_o}{K_o}\frac{\partial u(x,t)}{\partial x}\right], \quad 0 < \alpha \leq 2.$$ (15)

From Eqs. (10) and (12) we have:

$$\frac{\partial \theta}{\partial x} = \frac{K_o}{K(\theta)}\frac{\partial \vartheta}{\partial x} = \frac{1}{(1 + K_1\theta)}\frac{\partial \vartheta}{\partial x} \approx (1 - K_1\theta)\frac{\partial \vartheta}{\partial x} \approx \frac{\partial \vartheta}{\partial x}.$$ (16)

Thus Eq. (6) will take the form:

$$(\lambda + 2\mu)\frac{\partial^2 u(x,t)}{\partial x^2} - \gamma \frac{\partial \vartheta(x,t)}{\partial x} = \rho \frac{\partial^2 u(x,t)}{\partial t^2}.$$ (17)

From Eqs. (10) and (12), we obtain:

$$\vartheta(x,t) = \theta(x,t) + \frac{K_1}{2}[\theta(x,t)]^2.$$ (18)

Hence, we have:

$$\theta(x,t) = \frac{-1 + \sqrt{1 + 2K_1 \vartheta(x,t)}}{K_1}.$$ (19)

Now, if we consider that $D^{\alpha-1}$ is the fractional derivative as an inverse operator of the integral operator $I^{\alpha-1}$, then we can write Eq. (12) in the form:

$$\frac{\partial^2 \vartheta(x,t)}{\partial x^2} = (D^\alpha + \tau_o \, D^{\alpha+1})\left[\frac{1}{\kappa}\vartheta(x,t) + \frac{\gamma T_o}{K_o}\frac{\partial u(x,t)}{\partial x}\right], \quad 0 < \alpha \leq 2,$$ (20)

where $D^\alpha = \partial^\alpha/\partial t^\alpha$.

For simplicity we use the following non-dimensional variables:

$$x = \sqrt{\left(\frac{\lambda + 2\mu}{\rho}\right)} \frac{x'}{\kappa}, \quad u = \sqrt{\left(\frac{\lambda + 2\mu}{\rho}\right)} \frac{u'}{\kappa}, \quad t = \left(\frac{\lambda + 2\mu}{\rho}\right) \frac{t'}{\kappa},$$

$$\beta = \sqrt{\left(\frac{\lambda + 2\mu}{\mu}\right)}, \quad \tau_o = \left(\frac{\lambda + 2\mu}{\rho}\right) \frac{\tau'_o}{\kappa}, \quad \sigma = \frac{\sigma'}{\mu}, \quad \vartheta = \frac{\vartheta'}{T_o}.$$

Now, we have the following system of coupled equations:

$$\frac{\partial^2 u(x,t)}{\partial x^2} - a \frac{\partial \vartheta(x,t)}{\partial x} = \frac{\partial^2 u(x,t)}{\partial t^2}, \tag{21}$$

$$\frac{\partial^2 \vartheta(x,t)}{\partial x^2} = \left[\frac{\partial^\alpha}{\partial t^\alpha} + \tau_o \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right] \left[\vartheta(x,t) + g \frac{\partial u(x,t)}{\partial x}\right], \quad 0 < \alpha \le 2, \tag{22}$$

where $a = \gamma T_o/\mu \beta^2$ and $g = \gamma \kappa/K_o$.

To solve the above system we have to determine a certain boundary conditions, so we will consider the bounding plane of the half-space ($x = 0$) is thermally shocked and has a rigid foundation, so we have:

$$\theta(0,t) = \theta_0 H(t),$$

where $H(t)$ is the Heaviside unite step function which gives after using Eq. (18) that:

$$\vartheta(0,t) = \theta_0 H(t) + \frac{K_1}{2} [\theta_0 H(t)]^2, \tag{23}$$

$$u(0,t) = 0. \tag{24}$$

3. Governing equations in the Laplace transform domain

Applying the Laplace transforms for Eqs. (21)-(24) define by the formula:

$$\bar{f}(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}dt.$$

Hence, we obtain the following system of differential equations:

$$\frac{d^2\bar{u}(x,s)}{dx^2} - a \frac{d\bar{\vartheta}(x,s)}{dx} = s^2\bar{u}(x,s), \tag{25}$$

$$\frac{d^2\bar{\vartheta}(x,s)}{dx^2} = s^\alpha[1 + s\tau_o] \left[\bar{\vartheta}(x,s) + g \frac{d\bar{u}(x,s)}{dx}\right], \quad 0 < \alpha \le 2, \tag{26}$$

$$\bar{\vartheta}(0,s) = \frac{\theta_0}{s} + \frac{K_1}{2} \frac{\theta_0^2}{s}, \tag{27}$$

$$\bar{u}(0,s) = 0. \tag{28}$$

Eqs. (25) and (26) can be written in a vector-matrix differential equation as follows [20]:

$$\frac{d\vec{V}}{dx} = A\vec{V}, \tag{29}$$

where:

$$\vec{V} = \left[\bar{u} \quad \bar{\vartheta} \quad \frac{d\bar{u}}{dx} \quad \frac{d\bar{\vartheta}}{dx}\right]^T, \tag{30}$$

where:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_{31} & 0 & 0 & c_{34} \\ 0 & c_{42} & c_{43} & 0 \end{bmatrix}, \tag{31}$$

$$c_{31} = s^2, \quad c_{34} = a, \quad c_{42} = s^\alpha(1 + s\tau_o), \quad c_{43} = gs^\alpha(1 + s\tau_o). \tag{32}$$

4. Solution of the vector-matrix differential equation

Let us now proceed to solve Eq. (29) by the eigenvalue approach proposed by [17]. The characteristic equation of the matrix A takes the form:

$$\lambda^4 - (c_{31} + c_{42} + c_{34}c_{43})\lambda^2 + c_{31}c_{42} = 0. \tag{33}$$

The roots of the characteristic Eq. (15) which are also the eigenvalues of matrix A are of the form $\lambda = \pm\lambda_1$, $\lambda = \pm\lambda_2$.

The eigenvector $\vec{X} = [x_1, x_2, x_3, x_4]^T$, corresponding to eigenvalue λ can be calculated as:

$$x_1 = \lambda c_{34}, \quad x_2 = \lambda^2 - c_{31}, \quad x_3 = \lambda x_1, \quad x_4 = \lambda x_2. \tag{34}$$

From Eqs. (33), we can easily calculate the eigenvector $\vec{X}_j$, corresponding to eigenvalue λ_j, $j = 1, 2, 3, 4$.

For further reference, we shall use the following notations:

$$\vec{X}_1 = [\vec{X}]_{\lambda=-\lambda_1}, \quad \vec{X}_2 = [\vec{X}]_{\lambda=-\lambda_2}, \quad \vec{X}_3 = [\vec{X}]_{\lambda=\lambda_1}, \quad \vec{X}_4 = [\vec{X}]_{\lambda=\lambda_2}. \tag{35}$$

The solution of Eq. (26) can be written from as follows:

$$\vec{V} = \sum_{j=1}^{2} B_j \vec{X}_j e^{\lambda_i x} = B_1 \vec{X}_1 e^{-\lambda_1 x} + B_2 \vec{X}_2 e^{-\lambda_2 x}, \tag{36}$$

where the terms containing exponentials of growing nature in the space variable x have been discarded due to the regularity condition of the solution at infinity, B_1 and B_2 are constants to be determined from the boundary condition of the problem. Thus, the field variables can be written for x and s as:

$$\bar{u}(x, s) = \sum_{j=1}^{2} B_j x_3^j e^{-\lambda_j x}, \tag{37}$$

$$\bar{\vartheta}(x, s) = \sum_{j=1}^{2} B_j x_4^j e^{-\lambda_j x}, \tag{38}$$

$$\bar{\theta}(x,s) = \frac{1}{K_1}\left(-1 + \sqrt{1 + 2K_1 \sum_{j=1}^{2} B_j x_4^j e^{-\lambda_j x}}\right), \tag{39}$$

$$\bar{\sigma}(x,s) = \sum_{j=1}^{2} (\lambda_j x_3^j \beta^2 - b x_4^j) B_j e^{-\lambda_j x}. \tag{40}$$

To complete the solution we have to know the constants B_1 and B_2, by using the boundary conditions Eqs. (27) and (28) we obtain:

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{\theta_0}{s}\left(1 + \frac{\theta_0 K_1}{2}\right) \end{pmatrix}, \tag{41}$$

where $R_{11} = x_3^1, R_{12} = x_3^2, R_{21} = x_4^1, R_{22} = x_4^2$.

5. Numerical inversion of the Laplace transform

In order to determine the conductive and thermal temperature, displacement and stress distributions in the time domain, the Riemann-sum approximation method is used to obtain the numerical results. In this method, any function in Laplace domain can be inverted to the time domain as:

$$f(t) = \frac{e^{\kappa t}}{t}\left[\frac{1}{2}\bar{f}(\varepsilon) + \mathrm{Re}\sum_{n=1}^{N}(-1)^n \bar{f}\left(\varepsilon + \frac{in\pi}{t}\right)\right], \tag{42}$$

where Re is the real part and i is imaginary number unit. For faster convergence, numerous numerical experiments have shown that the value of ε satisfies the relation $\varepsilon t \approx 4.7$ [18].

6. Numerical results and discussion

The copper material was chosen for purposes of numerical evaluations and the constants of the problem were taken as following [6]:

$\lambda_o = 7.76 \times 10^{10}(kg)(m)^{-1}(s)^{-2}, \quad \mu_o = 3.86 \times 10^{10}(kg)(m)^{-1}(s)^{-2},$
$T_0 = 293(K), \quad K_o = 3.68 \times 10^2(kg)(m)(K)^{-1}(s)^{-3}, \quad \kappa = 0.9748 \times 10^{-4}(m)^2(s)^{-2},$
$\theta_o = 1, \quad g = 1.6, \quad a = 0.0105, \quad \beta^2 = 3.99, \quad \tau_o = 0.02.$

The computations were carried out for $t = 0.2$ and the temperature, the stress, the displacement and the strain distributions are represented graphically for wide range of x ($0 \leq x \leq 3$) and for different values of the fractional order parameter α ($\alpha = 0.5, 1.0, 1.5$) when $K_1 = 0.0$ which gives the normal case of thermal conductivity and when $K_1 = -0.5$ which gives variable thermal conductivity. In Figs. 2 and 3, we found that, the parameter α and K_1 have significant effects on the temperature distributions. Fig. 2 shows that, the temperature values starting form 1.0 as the boundary condition been set, and then the waves decrease till vanish at large value of x and when the value of the parameter α increases, the lack of temperature faster when values for distance less than. In Figs. 4 and 5, we found that, the parameter α and K_1 have significant effects on the stress distributions. When the value of the parameter α increases the lack of stress faster when values for distance less than. The absolute value of the maximum points of the stress increase when the value of the parameter α increases, while it decreases when the value of the parameter K_1 increases.
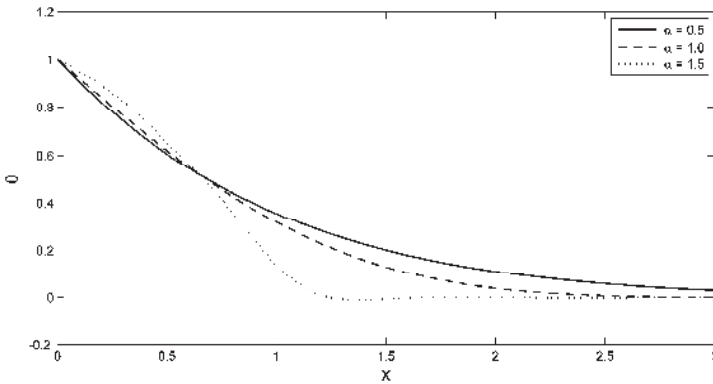
Fig. 2. The temperature distribution with different value of fractional order parameter when $K_1 = 0.0$
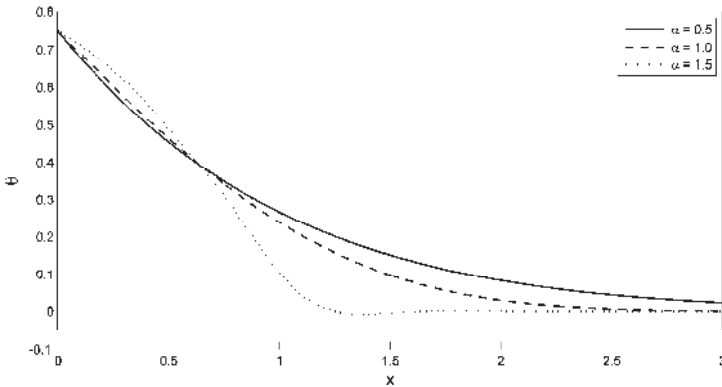
Fig. 3. The temperature distribution with different value of fractional order parameter when $K_1 = -0.5$
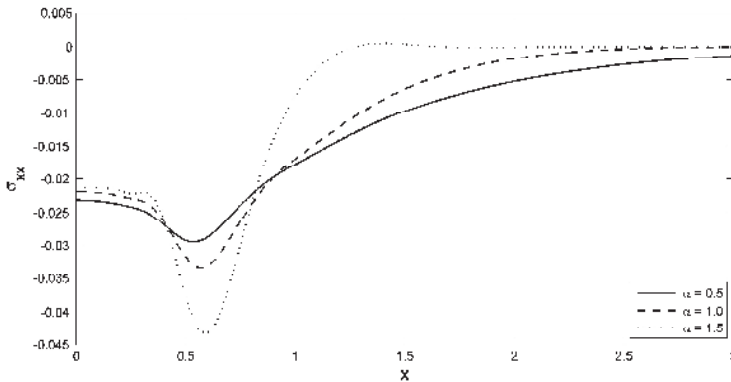
Fig. 4. The stress distribution with different value of fractional order parameter when $K_1 = 0.0$

In Fig. 6 and 7, we noticed that, the parameter α and K_1 have significant effects on the displacement distributions. When the value of the parameter α increases the lack of displacement faster when values for distance less than. The value of the maximum points of the displacement increase when the value of the parameter α increases, while it decreases when the value of the parameter K_1 increases. In Figs. 8 and 9, we noticed that, the parameter α and K_1 have significant effects on the strain distributions. When the value of the parameter α increases the lack of strain faster when values for distance less than. The absolute value of the maximum points of the strain increase when the value of the parameter α increases, while it decreases when the value of the parameter K_1 increases.
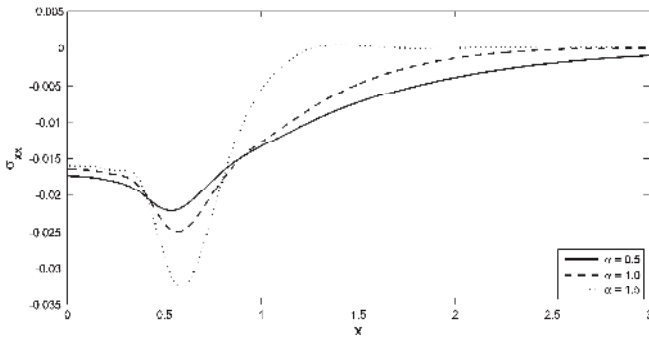
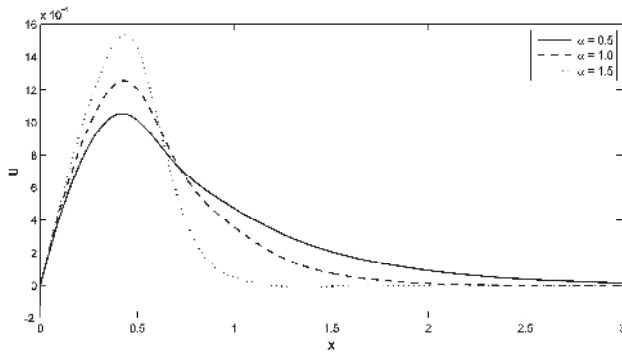**Fig. 5.** The stress distribution with different value of fractional order parameter when $K_1 = -0.5$

Fig. 6. The displacement distribution with different value of fractional order parameter when $K_1 = 0.0$
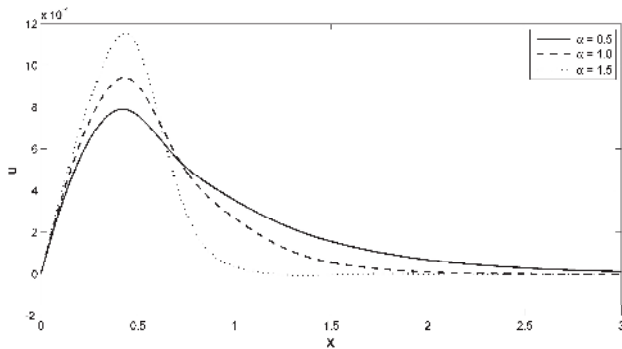

Fig. 7. The displacement distribution with different value of fractional order parameter when $K_1 = -0.5$
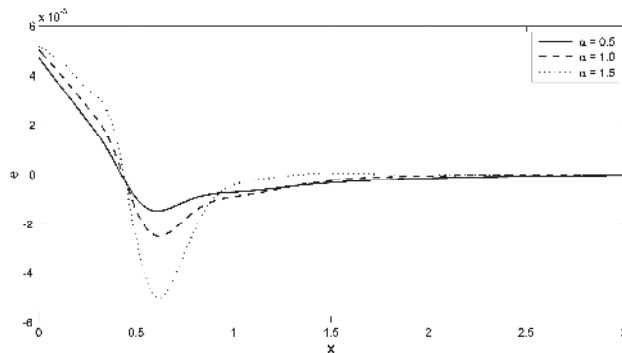

Fig. 8. The strain distribution with different value of fractional order parameter when $K_1 = 0.0$
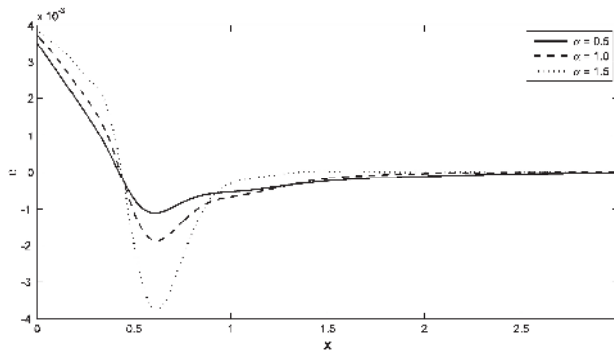
Fig. 9. The strain distribution with different value of fractional order parameter when $K_1 = -0.5$

7. Conclusion

In this work, the solution of thermoelastic medium thermal loaded by thermal shock has been developed in the context of fractional order generalized thermoelasticity with variable thermal conductivity. The effect of the variability of thermal conductivity has been studied together with the fractional order parameter and represented graphically with some comparisons and we obtain that; the fractional order parameter and the variability of the thermal conductivity play a vital role on the speed of the propagation each of the thermal wave and the mechanical wave, increasing the value of the fractional order parameter makes the value of the thermal wave and the mechanical wave vanishes in shorter distance from the bounding plane, changing of the thermal conductivity causes significant changing of the values of the maximum points of the thermal wave and the mechanical wave.

Acknowledgements

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Hamdy M. M. Youssef was born on Abril 22, 1971, Professor of Applied Mathematics, Mechanics Department, Faculty of Engineering and Islamic Architecture, Umm Al-Qura University, KSA. Home Address: Mathematics Department, Faculty of Education, Alexandria University, Egypt. B.Sc. of Mathematics in May 1992, M.Sc. of Applied Mathematics in May 1997 and Ph.D. of Applied Mathematics, Faculty of Science, Alexandria University in March 2003. Editor-in-Chief Journal of Thermoelasticity and consultant of ICRS – Supervisor of Scientific Chairs, UQU. Member of the Egyptian Mathematical Society. Member of the Scientists Society in Egypt. Member of the American Nano Society (116616). Member of the American Mathematical Society (YSHMMA). Member of the European Mechanics Society (EM 120046). Chosen as a candidate for inclusion in 2006-2007 (9th) Edition of Who's who in Science and Engineering. 2000 Outstanding Intellectuals of the 21st Century, 2006, Cambridge University. 21st Century Award for Achievement in Science and Engineering, Cambridge University, 2006. Subject interest: applied mathematics, thermoelasticity, nanomechanics. More than 80 published papers in international journals. Published 15 papers in international conferences.

Ibrahim A. Abbas was born on November 20, 1971 in Sohag, Egypt. In 2004, he received Ph.D. in Mathematics at South Valley University, Egypt. Member of the Egyptian Mathematical Society. He is an associate professor of Applied Mathematics at Sohag University, Egypt and he is currently an associate professor of Applied Mathematics at King Abdulaziz University, Saudi Arabia. He works in the field of theory of thermoelasticity and fluid mechanics by finite element method and eigenvalue approach. He has published over 120 papers in international journals and 8 conference papers. His more detailed CV can be found in "Who's Who in Science and Engineering".