Fractional Partial Random Differential Equations with State-Dependent Delay

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Abstract. In the present paper we provide some existence results for the Darboux problem of partial fractional random differential equations with state-dependent delay by applying the measure of noncompactness and a random fixed point theorem with stochastic domain.

1 Introduction

Fractional calculus is generalization of ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus, starting from some speculations of G.W. Leibniz (1967) and L. Euler (1730) and since then, it has continued to be developed up to nowadays. Integral equations are one of the most useful mathematical tools in both pure and applied analysis. This is particulary true of problems in mechanical vibrations and the related fields of engineering and mathematical physics. we can find numerous applications of differential and integral equation of fractional order in finance, hydrology, biophysics, thermodynamics,

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control theory, statistical mechanics, astrophysics, cosmology and bioengineering ([14, 26, 38, 39, 42]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [7, 8], Baleanu *et al.* [14], Kilbas *et al.* [29], the papers of Abbas *et al.* [1–6, 9, 10], Ahmad and Nieto [11], Czlapinski [21, 22], Kilbas and Marzan [28], Stanek [41], Vityuk and Golushkov [44], and the references therein.

The theory of functional differential equations has emerged as an important branch of nonlinear analysis. Differential delay equations, or functional differential equations, have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay; see for instance the books by Hale and Verduyn Lunel [25], Kolmanovskii and Myshkis [30], Smith [40], and Wu [46], and the references therein. However, complicated situations in which the delay depends on the unknown functions have been considered in recent years. These equations are frequently called equations with state-dependent delay, see, for instance [15–17, 20, 23, 35, 36, 45].

The nature of a dynamic system in engineering or natural sciences depends on the accuracy of the information we have concerning the parameters that describe that system. If the knowledge about a dynamic system is precise then a deterministic dynamical system arises. Unfortunately in most cases the available data for the description and evaluation of parameters of a dynamic system are inaccurate, imprecise or confusing. In other words, evaluation of parameters of a dynamical system is not without uncertainties. When our knowledge about the parameters of a dynamic system are of statistical nature, that is, the information is probabilistic, the common approach in mathematical modeling of such systems is the use of random differential equations or stochastic differential equations. Random differential equations, as natural extensions of deterministic ones, arise in many applications and have been investigated by many mathematicians. We refer the reader to the monographs [18, 31, 43]. The initial value problems for fractional differential with random parameters have been studied by Lupulescu and Ntouyas [33]. The basic tool in the study of the problems for random fractional differential equations is to treat it as a fractional differential equation in some appropriate Banach space. In [34], the authors proved the existence results for a random fractional equation under a Carathéodory condition.

In this paper, we discuss the existence of random solutions for the following fractional partial random differential equations with state-dependent

delay

$$(^cD_0^r u)(x, y, w) = f(x, y, u_{(\rho_1(x,y,u_{(x,y)}),w),\rho_2(x,y,u_{(x,y)}),w))}, w),$$

$$\text{if } J := [0, a] \times [0, b], w \in \Omega, \tag{1.1}$$

$$u(x, y, w) = \phi(x, y, w), \text{ if } (x, y) \in \tilde{J} := [-\alpha, a] \times [-\beta, b] \backslash (0, a] \times (0, b], w \in \Omega, \tag{1.2}$$

$$u(x, 0, w) = \varphi(x, w), \ x \in [0, a], \ u(0, y, w) = \psi(y, w), \ y \in [0, b], w \in \Omega, \tag{1.3}$$

where $\alpha, \beta, a, b > 0$, $^cD_0^r$ is the standard Caputo's fractional derivative of order $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, $(\Omega, \mathcal{A})$ is a measurable space, $f : J \times C \times \Omega \to E$, $\rho_1, \rho_2 : J \times C \times \Omega \to E$ are given functions, $\phi : \tilde{J} \times \Omega \to E$ is a given continuous function, $\varphi : [0, a] \times \Omega \to E, \psi : [0, b] \times \Omega \to E$ are given absolutely continuous functions with $\varphi(x, w) = \phi(x, 0, w)$, $\psi(y, w) = \phi(0, y, w)$ for each $x \in [0, a]$, $y \in [0, b]$, $w \in \Omega$, $(E, \| \cdot \|_E)$ a separable Banach space, and $C := C([-\alpha, 0] \times [-\beta, 0], E)$ is the space of continuous functions on $[-\alpha, 0] \times [-\beta, 0]$.

If $u \in C([-\alpha, a] \times [-\beta, b], E)$; $a, b, \alpha, \beta > 0$ then for any $(x, y) \in J$ define $u_{(x,y)}$ by

$$u_{(x,y)}(s, t, w) = u(x + s, y + t, w), \text{ for } (s, t) \in [-\alpha, 0] \times [-\beta, 0].$$

Here $u_{(x,y)}(\cdot, \cdot, w)$ represents the history of the state u.

This paper initiates the study of Darboux problem for fractional random partial differential equations with state-dependent delay.

2   Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Denote $L^1(J)$ the space of Bochner-integrable functions $u : J \to E$ with the norm

$$\|u\|_{L^1} = \int_0^a \int_0^b \|u(x, y)\|_E \, dy \, dx.$$

$L^\infty(J)$ the Banach space of measurable functions $u : J \to \mathbb{R}$ which are essentially bounded.
As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into E.

Let β_E be the σ-algebra of Borel subsets of E. A mapping $v : \Omega \to E$ is said to be measurable if for any $B \in \beta_E$, one has

$$v^{-1}(B) = \{w \in \Omega : v(w) \in B\} \subset \mathcal{A}.$$

To define integrals of sample paths of random process, it is necessary to define a jointly measurable map.

Definition 2.1. *A mapping $T : \Omega \times E \to E$ is called jointly measurable if for any $B \in \beta_E$, one has*

$$T^{-1}(B) = \{(w, v) \in \Omega \times E : T(w, v) \in B\} \subset \mathcal{A} \times \beta_E,$$

where $\mathcal{A} \times \beta_E$ is the direct product of the σ-algebras $\mathcal{A}$ and β_E those defined in Ω and E respectively.

Lemma 2.1. *Let $T : \Omega \times E \to E$ be a mapping such that $T(\cdot, v)$ is measurable for all $v \in E$, and $T(w, \cdot)$ is continuous for all $w \in \Omega$. Then the map $(w, v) \mapsto T(w, v)$ is jointly measurable.*

Definition 2.2. *A function $f : J \times E \times \Omega \to E$ is called random Carathéodory if the following conditions are satisfied:*

(i) *The map $(x, y, w) \to f(x, y, u, w)$ is jointly measurable for all $u \in E$, and*

(ii) *The map $u \to f(x, y, u, w)$ is continuous for almost all $(x, y) \in J$ and $w \in \Omega$.*

Let $T : \Omega \times E \to E$ be a mapping. Then T is called a random operator if $T(w, u)$ is measurable in w for all $u \in E$ and it is expressed as $T(w)u = T(w, u)$. In this case we also say that $T(w)$ is a random operator on E. A random operator $T(w)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $T(w, u)$ is continuous (resp. compact, totally bounded and completely continuous) in u for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [27].

Definition 2.3. [24] *Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y and C be a mapping from Ω into $\mathcal{P}(Y)$. A mapping $T : \{(w, y) : w \in \Omega, y \in C(w)\} \to Y$ is called random operator with stochastic domain C if C is measurable (i.e., for all closed $A \subset Y$, $\{w \in \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $y \in Y$, $\{w \in \Omega : y \in C(w), T(w, y) \in D\}$ is*

measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $y : \Omega \to Y$ is called random (stochastic) fixed point of T if for $P-$almost all $w \in \Omega$, $y(w) \in C(w)$ and $T(w)y(w) = y(w)$ and for all open $D \subset Y$, $\{w \in \Omega : y(w) \in D\}$ is measurable.

Let $\mathcal{M}_X$ denote the class of all bounded subsets of a metric space X.

Definition 2.4. *Let X be a complete metric space. A map $\alpha : \mathcal{M}_X \to [0, \infty)$ is called a measure of noncompactness on X if it satisfies the following properties for all $B, B_1, B_2 \in \mathcal{M}_X$.*

(MNC.1) $\alpha(B) = 0$ if and only if B is precompact (Regularity),

(MNC.2) $\alpha(B) = \alpha(\overline{B})$ (Invariance under closure),

(MNC.3) $\alpha(B_1 \cup B_2) = \max\{\alpha(B_1), \alpha(B_2)\}$ (Semi-additivity).

For more details on measure of noncompactness and its properties see [12, 13].

Example 2.1. In every metric space X, the map $\phi : \mathcal{M}_X \to [0, \infty)$ with $\phi(B) = 0$ if B is relatively compact and $\phi(B) = 1$ otherwise is a measure of noncompactness, the so-called discrete measure of noncompactness [[13], Example1, p. 19].

Let $\theta = (0, 0)$, r_1, $r_2 > 0$ and $r = (r_1, r_2)$. For $f \in L^1(J)$, the expression

$$(I_\theta^r f)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} f(s, t) dt ds,$$

is called the left-sided mixed Riemann-Liouville integral of order r, where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi - 1} e^{-t} dt; \ \xi > 0.$$

In particular,

$$(I_\theta^\theta u)(x, y) = u(x, y), \ (I_\theta^\sigma u)(x, y) = \int_0^x \int_0^y u(s, t) dt ds; \text{ for almost all } (x, y) \in J,$$

where $\sigma = (1, 1)$.
For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0, \infty)$, when $u \in L^1(J)$. Note also that when $u \in \mathcal{C}$, then $(I_\theta^r u) \in \mathcal{C}$, moreover

$$(I_\theta^r u)(x, 0) = (I_\theta^r u)(0, y) = 0; \ x \in [0, a], \ y \in [0, b].$$

By $1 - r$ we mean $(1 - r_1, 1 - r_2) \in [0, 1) \times [0, 1)$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$, the mixed second order partial derivative.

Definition 2.5. [44] *Let $r \in (0, 1] \times (0, 1]$ and $u \in L^1(J)$. The Caputo fractional-order derivative of order r of u is defined by the expression*

$$^c D_\theta^r u(x, y) = (I_\theta^{1-r} D_{xy}^2 u)(x, y).$$

The case $\sigma = (1, 1)$ is included and we have

$$(^c D_\theta^\sigma u)(x, y) = (D_{xy}^2 u)(x, y); \text{ for almost all } (x, y) \in J.$$

Lemma 2.2. [19] *If Y is a bounded subset of Banach space X, then for each $\epsilon > 0$, there is a sequence $\{y_k\}_{k=1}^\infty \subset Y$ such that*

$$\alpha(Y) \leq 2\alpha(\{y_k\}_{k=1}^\infty) + \epsilon.$$

Lemma 2.3. [37] *If $\{u_k\}_{k=1}^\infty \subset L^1(J)$ is uniformly integrable, then $\alpha(\{u_k\}_{k=1}^\infty)$ is measurable and for each $(x, y) \in J$,*

$$\alpha\left(\left\{\int_0^x \int_0^y u_k(s, t) dt ds\right\}_{k=1}^\infty\right) \leq 2 \int_0^x \int_0^y \alpha(\{u_k(s, t)\}_{k=1}^\infty) dt ds.$$

Lemma 2.4. [32] *Let F be a closed and convex subset of a real Banach space, let $G : F \to F$ be a continuous operator and $G(F)$ be bounded. If there exists a constant $k \in [0, 1)$ such that for each bounded subset $B \subset F$,*

$$\alpha(G(B)) \leq k\alpha(B),$$

then G has a fixed point in F.

3 Existence Results

We need the following lemma:

Lemma 3.1. [2,7] *Let $h \in L^1(J)$. Then the linear problem*

$$\begin{cases} ^c D_\theta^r u(x, y) = h(x, y); & \text{for a.a. } (x, y) \in J := [0, a] \times [0, b], \\ u(x, 0) = \varphi(x); & x \in [0, a], \\ u(0, y) = \psi(y); & y \in [0, b], \\ \varphi(0) = \psi(0). \end{cases}$$

has a unique solution u given by

$$u(x,y) = \mu(x,y) + I_\theta^r h(x,y); \quad \textit{for a.a. } (x,y) \in J,$$

where

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

Let us assume that the function f is random Carathéodory on $J \times C \times \Omega$. From Lemma 3.1, we have the following lemma.

Lemma 3.2. *Let $0 < r_1, r_2 \leq 1, \mu(x,y,w) = \varphi(x,w) + \psi(y,w) - \varphi(0,w)$. A function $u \in \Omega \times C_{(a,b)}$ is a solution of the random problem (1.1)-(1.3) if u satisfies condition (1.2) for $(x,y) \in \tilde{J}, w \in \Omega$ and u is a solution of the equation*

$$u(x,y,w) =$$

$$\mu(x,y,w) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}, w) dt ds$$

for $(x,y) \in J, w \in \Omega$

Set $\mathcal{R} := \mathcal{R}_{(\rho_1^-,\rho_2^-)}$

$$= \{(\rho_1(s,t,u,w), \rho_2(s,t,u,w)) : (s,t) \in J, u_{(s,t)}(\cdot,\cdot,w) \in C, w \in \Omega,$$

$$\rho_i(s,t,u,w) \leq 0; \ i = 1,2\}.$$

We always assume that $\rho_i : J \times C \times \Omega \to E; \ i = 1, 2$ are continuous and the function $(s,t) \longmapsto u_{(s,t)}$ is continuous from $\mathcal{R}$ into C.

The following hypotheses will be used in the sequel.

(H_1) The functions $w \mapsto \varphi(x,0,w)$ *and* $w \mapsto \psi(0,y,w)$ are measurable and bounded for a.e. $x \in [0,a]$ and $y \in [0,b]$ respectively,

(H_2) The function Φ is measurable for $(x,y) \in \tilde{J}$

(H_3) The function f is random Carathéodory on $J \times C \times \Omega$,

(H_4) There exist functions $p_1, p_2 : J \times \Omega \to [0,\infty)$ with $p_i(\cdot,w) \in L^\infty(J,[0,\infty)); \ i = 1,2$ such that for each $w \in \Omega$,

$$\|f(x,y,u,w)\|_E \leq p_1(x,y,w) + p_2(x,y,w)\|u\|_C,$$

for all $u \in C$ and a.e. $(x,y) \in J,$

(H_5) For any bounded $B \subset E$,

$$\alpha(f(x, y, B, w)) \leq p_2(x, y, w)\alpha(B), \ for \ a.e. \ (x, y) \in J,$$

(H_6) There exists a random function $R : \Omega \to (0, \infty)$ such that

$$\mu^*(w) + \frac{(p_1^*(w) + p_2^*(w)R(w))a^{r_1}b^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} \leq R(w),$$

where

$$\mu^*(w) = \sup_{(x,y)\in J} \|\mu(x, y, w)\|_E, \ p_i^*(w) = \sup ess_{(x,y)\in J}p_i(x, y, w); \ i = 1, 2,$$

Theorem 3.3. *Assume that hypotheses* $(H_1) - (H_6)$ *hold. If*

$$\ell := \frac{4p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} < 1,$$

then the problem (1.1)-(1.3) has a random solution defined on $[-\alpha, a] \times [-\beta, b]$.

Proof. Define the operator $N : \Omega \times C_{(a,b)} \to C_{(a,b)}$ by

$$(N(w)u)(x, y) = \begin{cases} \phi(x, y, w), & (x, y) \in \tilde{J}, w \in \Omega \\ \mu(x, y, w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1-1}(y - t)^{r_2-1} \\ \times f(s, t, u_{(\rho_1(s,t,u_{(s,t)},w),\rho_2(s,t,u_{(s,t)},w)),w})dtds, \\ & (x, y) \in J, w \in \Omega. \end{cases}$$

$$(3.1)$$

Since the functions φ, ψ and f are absolutely continuous, the function μ and the indefinite integral are absolutely continuous for all $w \in \Omega$ and almost all $(x, y) \in J$. Again, as the map μ is continuous for all $w \in \Omega$ and the indefinite integral is continuous on J, then $N(w)$ defines a mapping $N : \Omega \times C_{(a,b)} \to C_{(a,b)}$. Hence u is a solution for the problem (1.1)-(1.3) if and only if $u = (N(w))u$. We shall show that the operator N satisfies all conditions of Lemma 2.4. The proof will be given in several steps.

Step 1: $N(w)$ is a random operator with stochastic domain on $C_{(a,b)}$. Since $f(x, y, u, w)$ is random Carathéodory, the map $w \to f(x, y, u, w)$ is measurable in view of Definition 2.1. Similarly, the function $(s, t) \mapsto (x - s)^{r_1-1}(y - t)^{r_2-1}f(s, t, u_{(s,t)}, w)$ is measure as the product of a continuous and a measurable function. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$w \mapsto \mu(x, y, w)+$$

$$\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))},w) dt ds,$$

is measurable. As a result, N is a random operator on $\Omega \times C_{(a,b)}$ into $C_{(a,b)}$.

Let $W : \Omega \to \mathcal{P}(C_{(a,b)})$ be defined by

$$W(w) = \{u \in C_{(a,b)} : \|u\|_\infty \le R(w)\},$$

with $R(\cdot)$ is chosen appropriately. From instance, we assume that

$$R(w) \ge \frac{\mu^* + p_1^*(w)\frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}{1 - p_2^*(w)\frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}.$$

Clearly, $W(w)$ is bounded, closed, convex and solid for all $w \in \Omega$. Then W is measurable by Lemma 17 of [24]. Let $w \in \Omega$ be fixed, then from (H_4), for any $u \in W(w)$, we get

$$\|(N(w)u)(x,y)\|_E$$
$$\le \|\mu(x,y,w)\|_E +$$
$$\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \|f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))},w)\|_E dt ds$$
$$\le \|\mu(x,y,w)\|_E + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} p_1(s,t,w) dt ds$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} p_2(s,t,w) \|u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\|_E dt ds$$
$$\le \mu^*(w) + \frac{p_1^*(w)}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dt ds$$
$$+ \frac{p_2^*(w)R(w)}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dt ds$$
$$\le \mu^*(w) + \frac{(p_1^*(w) + p_2^*(w)R(w))a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}$$
$$\le R(w).$$

Therefore, N is a random operator with stochastic domain W and $N(w) : W(w) \to N(w)$. Furthermore, $N(w)$ maps bounded sets into bounded sets in $C_{(a,b)}$.

Step 2: $N(w)$ is continuous.
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C_{(a,b)}$. Then, for each $(x,y) \in J$ and $w \in \Omega$, we have

$$\|(N(w)u_n)(x,y) - (N(w)u)(x,y)\|_E \le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}$$
$$\times \|f(s,t,u_{n(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))},w) - f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))},w)\|_E dt ds.$$

Using the Lebesgue dominated convergence theorem, we get

$$\|N(w)u_n - N(w)u\|_\infty \to 0 \text{ as } n \to \infty.$$

As a consequence of Steps 1 and 2, we can conclude that $N(w) : W(w) \to N(w)$ is a continuous random operator with stochastic domain W, and $N(w)(W(w))$ is bounded.

Step 3: For each bounded subset B of $W(w)$ we have

$$\alpha(N(w)B) \leq \ell\alpha(B).$$

Let $w \in \Omega$ be fixed. From Lemmas 2.2 and 2.3, for any $B \subset W$ and any $\epsilon > 0$, there exists a sequence $\{u_n\}_{n=0}^\infty \subset B$, such that for all $(x, y) \in J$, we have

$$\alpha((N(w)B)(x,y))$$
$$= \alpha\left(\left\{\mu(x,y) + \right.\right.$$
$$\left.\left. \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))},w)dtds; \; u \in B\right\}\right)$$
$$\leq 2\alpha\left(\left\{\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u_{n(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))},w)dtds\right\}_{n=1}^\infty\right) + \epsilon$$
$$\leq 4\int_0^x \int_0^y \alpha\left(\left\{\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u_{n(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))},w)\right\}_{n=1}^\infty\right) dtds + \epsilon$$
$$\leq 4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \alpha\left(\{f(s,t,u_{n(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))},w)\}_{n=1}^\infty\right) dtds + \epsilon$$
$$\leq 4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)\alpha\left(\{u_{n(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}\}_{n=1}^\infty\right) dtds + \epsilon$$
$$\leq \left(4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dsdt\right) \alpha\left(\{u_n\}_{n=1}^\infty\right) + \epsilon$$
$$\leq \left(4\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s,t,w)dtds\right) \alpha(B) + \epsilon$$
$$\leq \frac{4p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\alpha(B) + \epsilon$$
$$= \ell\alpha(B) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, then

$$\alpha(N(B)) \leq \ell\alpha(B).$$

It follows from Lemma 2.4 that for each $w \in \Omega$, N has at least one fixed point in W. Since $\bigcap_{w \in \Omega} intW(w) \neq \emptyset$, and a measurable selector of $intW$ exists,

Lemma 2.4 implies that N has a stochastic fixed point, i.e., the problem (1.1)-(1.3) has at least one random solution on $C_{(a,b)}$. □

4 An Example

Let $E = \mathbb{R}$, $\Omega = (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Consider the following partial functional random differential equation of the form

$$(^cD_0^r u)(x, y) = \frac{|u(x - \sigma_1(u(x, y, w)), y - \sigma_2(u(x, y, w)), w)| + 2}{e^{x+y+4}(1 + w^2 + |u(x - \sigma_1(u(x, y, w)), y - \sigma_2(u(x, y, w)), w)|)}$$

$$\text{if } (x, y) \in [0, 1] \times [0, 1], \tag{4.1}$$

$$u(x, y, w) = x \sin w + y^2 \cos w, \ (x, y) \in [-1, 1] \times [-2, 1] \backslash (0, 1] \times (0, 1], \ w \in \Omega, \tag{4.2}$$

$$u(x, 0, w) = x \sin w; \ x \in [0, 1], \ u(0, y, w) = y^2 \cos w; \ y \in [0, 1], \ w \in \Omega, \tag{4.3}$$

where $\sigma_1 \in C(\mathbb{R}, [0, 1])$, $\sigma_2 \in C(\mathbb{R}, [0, 2])$. Set

$$\rho_1(x, y, \varphi, w) = x - \sigma_1(\varphi(0, 0, w)),$$

$$\rho_2(x, y, \varphi, w) = y - \sigma_2(\varphi(0, 0, w)),$$

where $(x, y) \in J, \varphi(\cdot, \cdot, w) \in C([-1, 0] \times [-2, 0], \mathbb{R}), w \in \Omega$,

$$f(x, y, u, w) = \frac{|u| + 2}{(e^{x+y+4})(1 + w^2 + |u|)}, \ (x, y) \in [0, 1] \times [0, 1],$$

$$u \in C([-1, 0] \times [-2, 0], \mathbb{R}).$$

The functions

$$w \mapsto \varphi(x, 0, w) = x \sin w \ , \ w \mapsto \psi(0, y, w) = y^2 \cos w,$$

$$w \mapsto \Phi(x, y, w) = x \sin w + y^2 \cos w$$

are measurable and bounded with

$$|\varphi(x, 0, w)| \leq 1, \ |\psi(0, y, w)| \leq 1,$$

hence, condition (H_1) is satisfied.

Clearly, the map $(x, y, w) \mapsto f(x, y, u, w)$ is jointly continuous for all $u \in \mathbb{R}$ and hence jointly measurable for all $u \in \mathbb{R}$. Also the map $u \mapsto f(x, y, u, w)$ is

continuous for all $(x, y) \in J$ and $w \in \Omega$. So the function f is Carathéodory on $[0, 1] \times [0, 1] \times \mathbb{R} \times \Omega$.

For each $\varphi \in C([-1, 0] \times [-2, 0], \mathbb{R})$, $(x, y) \in [0, 1] \times [0, 1]$ and $w \in \Omega$, we have

$$|f(x, y, u, w)| \leq 1 + \frac{1}{e^4}|u|.$$

Hence condition (H_4) is satisfied with

$$p_1(x, y, w) = p_1^* = 1, \quad p_2(x, y, w) = p_2^* = \frac{1}{e^4}.$$

Also, the conditions (H_5) is satisfied.

We shall show that condition $\ell < 1$ holds with $a = b = 1$. Indeed, for each $(r_1, r_2) \in (0, 1] \times (0, 1]$ we have

$$
\begin{aligned}
\ell &= \frac{4p_2^* a^{r_1} b^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} \\
&= \frac{4}{e^4 \Gamma(1 + r_1)\Gamma(1 + r_2)} \\
&< 1.
\end{aligned}
$$

Consequently, Theorem 3.3 implies that the problem $(4.1) - (4.3)$ has a random solution defined on $[-1, 1] \times [-2, 1]$.

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