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Fractional powers of dissipative operators

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Introduction

The object of the present paper is to investigate the properties of the fractional powers A^{α} of linear operators A in a Hilbert space $\mathfrak{H}$, when $-A$ is closed and *maximal dissipative* in the sense of Phillips [**15, 16**]. $-A$ is said to be dissipative if $\mathrm{Re}(Au, u)\geqq 0$ for every $u\in\mathfrak{D}[A]$, and $-A$ is maximal dissipative if it has no proper dissipative extension. It is known (see [**15**]) that a closed, maximal dissipative operator is densely defined, that $-A$ is closed and maximal dissipative if and only if $-A^{*}$ is, and also if and only if $-A$ is the infinitesimal generator of a *contraction semi-group* $\{\exp(-tA)\}_{0<t<\infty}$, that is, $\|\exp(-tA)\|\leqq 1$.

Following a suggestion due to Friedrichs [**4**], we shall say that A is *accretive* if $-A$ is dissipative. In what follows we shall be concerned with accretive rather than with dissipative operators.

The fractional powers A^{α} can be defined in a natural way, at least for $0\leqq\alpha\leqq 1$, if A is closed and maximal accretive, and A^{α} are again closed and maximal accretive. Such fractional powers have been defined for a more general class of linear operators in Banach spaces by several authors (see, among others, Balakrishnan [**1, 2**], Glushko and Krein [**5**], Kato [**9**], Krasnosel'skii and Pustylnik [**12**], Krasnosel'skii and Sobolevskii [**13**], Sobolevskii [**17**], Solomiak [**18**], Yosida [**19**]).

One of the important results to be proved in the present paper is that, if A is closed and maximal accretive, A^{α} and $A^{*\alpha}$ are *comparable* for $0\leqq\alpha<1/2$; by this we mean that A^{α} and $A^{*\alpha}$ have the same domain $\mathfrak{D}_{\alpha}$ and that the ratios $\|A^{*\alpha}u\|/\|A^{\alpha}u\|$ for $u\in\mathfrak{D}_{\alpha}$ are bounded from above and from below by positive constants. Another result is that A^{α} and $A^{*\alpha}$ have an *acute angle* for $0\leqq\alpha<1/2$; by this is meant that $\mathrm{Re}(A^{\alpha}u, A^{*\alpha}u)/\|A^{\alpha}u\|\,\|A^{*\alpha}u\|$ is bounded from below by a positive constant (see Sobolevskii [**17**]). These results are remarkable in view of the fact that nothing is assumed for the relationship between the domains of A and A^{*} themselves or for the angle between A and A^{*}.

It follows from these results that $H_{\alpha}=(A^{\alpha}+A^{*\alpha})/2$ is nonnegative self-adjoint and that it is comparable, and has acute angle, with both A^{α} and

$A^{*\alpha}$. H_α may be regarded as the *real part* of A^α (and also of $A^{*\alpha}$). This suggests the possibility that A^α and $A^{*\alpha}$ be also comparable with H^α, where H is the "real part" of A. But the real part of A cannot be defined without further restrictions on A; in any case the simple definition $H=(A+A^*)/2$ fails because the intersection of $\mathfrak{D}[A]$ and $\mathfrak{D}[A^*]$ need not be wide enough.

A reasonable definition of the real part of A is furnished by the theory of *sesquilinear forms* in $\mathfrak{H}$, not for all maximal accretive A but for an important subclass of such operators. A sesquilinear form $\phi[u,v]$ is a complex-valued function defined for all u, v of a linear subset $\mathfrak{D}[\phi]$ of $\mathfrak{H}$, the *domain* of ϕ, such that $\phi[u,v]$ is linear in u and semilinear (conjugate-linear) in v. It is known (and will be proved below for completeness) that, under certain general conditions, including the one that $\mathrm{Re}\,\phi[u,u]\geqq 0$, there exists a closed, maximal accretive operator A such that $\mathfrak{D}[A]\subset\mathfrak{D}[\phi]$ and $(Au,v)=\phi[u,v]$ for $u\in\mathfrak{D}[A]$ and $v\in\mathfrak{D}[\phi]$. An accretive operator A associated in this way with a sesquilinear form will be called *regularly accretive*. If we now set $f[u,u]=\mathrm{Re}\,\phi[u,u]$, f can be extended to a symmetric (Hermitian) sesquilinear form (polar form) $f[u,v]$ with $\mathfrak{D}[\phi]=\mathfrak{D}[f]$, and the operator H associated with f in the above sense is nonnegative selfadjoint. This H is by definition the real part of A. Thus the real part of A is defined whenever A is regularly accretive.

We can now prove that, with H thus defined, H^α is comparable with any one of A^α, $A^{*\alpha}$ and H_α for $0\leqq\alpha<1/2$.

Our further study is concerned with the change of A^α when A is subjected to a small change (perturbation theory). To this end, it is convenient to assume that A is regularly accretive and express the change of A in terms of the change of the associated sesquilinear form ϕ. Under certain conditions, including that $\mathfrak{D}[\phi]$ is unchanged, it can be shown that $\mathfrak{D}[A^\alpha]$ is unchanged and that the change of $A^\alpha u$ is small if $0\leqq\alpha<1/2$ and if the change of $\phi[u]$ is small for all $u\in\mathfrak{D}[\phi]$. An important special case is that in which $\phi=\phi(t)$ depends on a complex parameter t *analytically*; in this case it follows that $A(t)^\alpha$ has a constant domain and $A(t)^\alpha u$ depends on t analytically.

Most of the results stated above are subject to the restriction $0\leqq\alpha<1/2$. There are examples showing that their extension to the case $\alpha>1/2$ is in general impossible, but it is not known whether or not $\alpha=1/2$ can be included. We have some partial results for this case (see § 5), but there remain many unsettled questions.

It appears that these results have their own interest, but their investigation by the present author has been motivated by the study of the *abstract evolution equation*

$$du/dt=-A(t)u+f(t)\,, \qquad 0\leqq t\leqq T\,, \tag{E}$$

in which the unknown $u=u(t)$ and the inhomogeneous term $f(t)$ are elements of $\mathfrak{H}$ and the coefficient $A(t)$ is a regularly accretive operator, all depending on t. It can be shown that (E) has a unique strict solution for an arbitrary initial value $u(0)$, if $A(t)^{\alpha}$ has a constant domain for some $\alpha=1/m$, $m=1,2,3,\cdots$, and if $A(t)$ and $f(t)$ satisfy certain smoothness conditions. The results on the perturbation of the fractional powers of regularly accretive operators described above now show that these assumptions are satisfied for $m=3,4,\cdots$, if the sesquilinear form $\phi(t)$ associated with $A(t)$ has constant domain and changes smoothly with t in a certain sense. In particular it can be shown that $u(t)$ is analytic if $\phi(t)$ and $f(t)$ are analytic. In this way the results of the present paper have an important application to the theory of the evolution equation (E).

The general theory of the evolution equation (E), which is developed in a more general setting in the case of a Banach space, as well as the application of the results of the present paper to (E), will be dealt with in a separate paper of the author [**11**]. The present paper contains the material necessary for this application, but it contains other results too and can be read independently of this application.

1. Fractional powers of dissipative (accretive) operators.

Let $\mathfrak{H}$ be a Hilbert space. In this section we consider closed, maximal accretive operators in $\mathfrak{H}$ and their fractional powers. The definition and elementary properties of dissipative and accretive operators are stated in Introduction, and those are all that we need in the following. The definition and fundamental properties of the fractional powers of linear operators are summarized in Appendix at the end of the present paper, together with some lemmas needed in the text.

Let A be closed and maximal accretive. Then A is of type $(\pi/2,1)$ (see Appendix), so that the fractional power A^{α} is defind and of type $(\pi\alpha/2,1)$ for $0\leqq\alpha\leqq 1$. In particular A^{α} is also closed and maximal accretive and, moreover, $-A^{\alpha}$ is the infinitesimal generator of an *analytic* semi-group $\{\exp(-tA^{\alpha})\}$ for $0\leqq\alpha<1$. Since A^* is also closed and maximal accretive, $A^{*\alpha}$ are defined and enjoy similar properties.

Our first problem concerns the relationship between A^{α} and $A^{*\alpha}$. We introduce the operators

$$H_{\alpha}=\frac{1}{2}(A^{\alpha}+A^{*\alpha}), \qquad K_{\alpha}=\frac{1}{2i}(A^{\alpha}-A^{*\alpha}). \tag{1.1}$$

H_{α} and K_{α} have domain $\mathfrak{D}[A^{\alpha}]\cap\mathfrak{D}[A^{*\alpha}]$, and at first it is not clear whether this domain is larger than $\{0\}$. But it will be seen that it is large enough if

$0 \leqq \alpha < 1/2$. In fact we shall prove

THEOREM 1.1. *Let A be closed and maximal accretive. For any α such that $0 \leqq \alpha < 1/2$, we have $\mathfrak{D}[A^\alpha] = \mathfrak{D}[A^{*\alpha}] = \mathfrak{D}[H_\alpha] = \mathfrak{D}[K_\alpha] \equiv \mathfrak{D}_\alpha$. H_α is nonnegative selfadjoint and K_α is symmetric. For any $u \in \mathfrak{D}_\alpha$ we have*

$$\|K_\alpha u\| \leqq \tan \frac{\pi\alpha}{2} \|H_\alpha u\|, \tag{1.2}$$

$$\left(1 - \tan \frac{\pi\alpha}{2}\right) \|H_\alpha u\| \leqq \|A^\alpha u\| \leqq \left(1 + \tan \frac{\pi\alpha}{2}\right) \|H_\alpha u\| \tag{1.3}$$

$$\|A^{*\alpha} u\| \leqq \tan \frac{\pi(1+2\alpha)}{4} \|A^\alpha u\|, \tag{1.4}$$

$$\mathrm{Re}\,(A^\alpha u, A^{*\alpha} u) \geqq \cos \pi\alpha \|A^\alpha u\| \|A^{*\alpha} u\|, \tag{1.5}$$

$$\mathrm{Re}\,(A^\alpha u, H_\alpha u) \geqq \frac{(\cos \pi\alpha)^{1/2}}{\cos \frac{\pi\alpha}{2}} \|A^\alpha u\| \|H_\alpha u\|, \tag{1.6}$$

and similar inequalities with A^α and $A^{*\alpha}$ exchanged.

REMARK. (1.2) to (1.4) imply that A^α, $A^{*\alpha}$ and H_α are mutually "comparable." (1.5) and (1.6) imply that any two of these three operators form an "acute angle." As is seen from the example given below, A^α and $A^{*\alpha}$ need not be comparable for $\alpha > 1/2$.

PROOF. The proof will be given in several steps.

I. First we make the additional assumption that A is bounded and $\mathrm{Re}\,(Au, u) \geqq \delta(u, u)$ with a constant $\delta > 0$, so that A^{-1} is also bounded. Then A^α is defined for all *complex* numbers α by means of, for example, the Dunford integral

$$A^\alpha = \frac{1}{2\pi i} \int_C \lambda^\alpha (\lambda - A)^{-1} d\lambda, \tag{1.7}$$

where the integration contour C is a simple closed curve enclosing the spectrum of A but excluding the negative real axis and the origin; such a curve can be found since $\lambda = 0$ belongs to the resolvent set of A. It is easy to see that for $0 < \alpha < 1$ (1.7) coincides with the A^α defined before (see Appendix). It follows from (1.7) that A^α is an *entire function* of α. Similarly, $A^{*\alpha}$ is an entire function of α, so that the same is true with H_α and K_α if these are defined by (1.1) for all complex numbers α.

Now we have

$$\|H_\alpha u\|^2 - \|K_\alpha u\|^2 = \mathrm{Re}\,(A^\alpha u, A^{*\alpha} u) = \mathrm{Re}\,(A^{\alpha+\bar{\alpha}} u, u), \tag{1.8}$$

since $(A^{*\alpha})^* = A^{\bar{\alpha}}$ as is seen from (1.7) by choosing C symmetric with respect to the real axis. It follows from (1.8) that

$$\|K_\alpha u\| \leqq \|H_\alpha u\| \quad \text{for} \quad -1/2 \leqq \mathrm{Re}\,\alpha \leqq 1/2; \tag{1.9}$$

this is obvious for $0 \leqq \operatorname{Re} \alpha \leqq 1/2$ since A^{β} is accretive for $0 \leqq \beta \leqq 1$ as noted above, while for $-1/2 \leqq \operatorname{Re} \alpha \leqq 0$ it suffices to note that A^{-1} is accretive with A by

$$\operatorname{Re}(A^{-1}u, u) = \operatorname{Re}(A^{-1}u, AA^{-1}u) \geqq \delta \| A^{-1}u \|^2 \geqq \delta \| A \|^{-2} \| u \|^2 \geqq 0 . \tag{1.10}$$

(1.8) implies also that, writing $\xi = \operatorname{Re} \alpha$,

$$\| H_\alpha u \|^2 \geqq \operatorname{Re}(A^{2\xi}u, u) \geqq \delta^{2\xi} \| u \|^2 \quad \text{for} \quad 0 \leqq \xi \leqq 1/2 \tag{1.11}$$

in virtue of Lemma A6 of Appendix. Noting (1.10), we have similarly

$$\| H_\alpha u \|^2 \geqq \operatorname{Re}(A^{2\xi}u, u) \geqq (\delta \| A \|^{-2})^{2|\xi|} \| u \|^2 , \ -1/2 \leqq \xi \leqq 0 . \tag{1.12}$$

These inequalities show that H_α has bounded inverse H_α^{-1} for $|\operatorname{Re} \alpha| \leqq 1/2$. H_α^{-1} has domain $\mathfrak{H}$ for real α, for H_α is then selfadjoint. Since H_α is continuous in α in norm, it follows that H_α^{-1} has domain $\mathfrak{H}$ for all α with $|\operatorname{Re} \alpha| \leqq 1/2$. Thus H_α^{-1} is also holomorphic for $|\operatorname{Re} \alpha| \leqq 1/2$.

Thus (1.9) is equivalent with

$$\| K_\alpha H_\alpha^{-1} \| \leqq 1 , \qquad |\operatorname{Re} \alpha| \leqq 1/2 . \tag{1.13}$$

Consider now the operator-valued function $T(\alpha) = K_\alpha H_\alpha^{-1} / \tan \frac{\pi\alpha}{2}$. $T(\alpha)$ is holomorphic in α in the strip $|\operatorname{Re} \alpha| \leqq 1/2$, for K_α has a zero at $\alpha = 0$. Since $|\tan \frac{\pi\alpha}{2}| = 1$ on the boundary of this strip (the points with $\operatorname{Im} \alpha = \pm\infty$ being included in this boundary), it follows from (1.13) that $\| T(\alpha) \|$ is bounded by 1 on the boundary. According to the maximum principle, it follows that $\| T(\alpha) \|$ is bounded by 1 in the whole strip. Restricting α to the real interval $0 \leqq \alpha \leqq 1/2$, this proves (1.2) under the stated additional assumptions.

(1.3) follows from (1.2) by noting that $A^\alpha = H_\alpha + iK_\alpha$ and (1.4) follows from (1.3) and similar inequalities with A replaced by A^*. To prove (1.5), we substitute (1.1) into (1.2), obtaining $\| A^\alpha u - A^{*\alpha} u \| \leqq \tan \frac{\pi\alpha}{2} \| A^\alpha u + A^{*\alpha} u \|$, which gives $2 \operatorname{Re}(A^\alpha u, A^{*\alpha} u) \geqq \cos \pi\alpha [\| A^\alpha u \|^2 + \| A^{*\alpha} u \|^2] \geqq 2 \cos \pi\alpha \| A^\alpha u \| \| A^{*\alpha} u \|$. Similarly, the substitution of $iK_\alpha = A^\alpha - H_\alpha$ into (1.2) gives $2 \operatorname{Re}(A^\alpha u, H_\alpha u) \geqq \| A^\alpha u \|^2 + \left(1 - \tan^2 \frac{\pi\alpha}{2}\right) \| H_\alpha u \|^2 \geqq 2\left(1 - \tan^2 \frac{\pi\alpha}{2}\right)^{1/2} \| A^\alpha u \| \| H_\alpha u \|$. This proves (1.6).

II. Next we consider unbounded A but still assume that a bounded inverse A^{-1} exists. Set

$$J_n = (1 + n^{-1}A)^{-1} , \qquad A_n = AJ_n = n(1 - J_n) , \qquad n = 1, 2, 3, \cdots . \tag{1.14}$$

J_n are bounded with $\| J_n \| \leqq 1$ since A is of type $(\pi/2, 1)$ (see Appendix). Hence A_n are also bounded, and it follows from $(A_n u, u) = (AJ_n u, (1 + n^{-1}A) J_n u) = (AJ_n u, J_n u) + n^{-1} \| A_n u \|^2$ that A_n are accretive and $\| A_n \| \leqq n$. Furthermore, we have $A_n^{-1} = A^{-1} + n^{-1}$ so that A_n^{-1} are uniformly bounded. Thus the inequalities (1.2) to (1.6) are satisfied if A is replaced by A_n and H_α, K_α respectively by $H_{n\alpha}$, $K_{n\alpha}$ constructed from A_n by (1.1). We shall now show that the same inequalities are obtained for A by taking the limit $n\to\infty$, with the necessary domain characterization.

To this end we first note that

$$A_n^\alpha = A^\alpha J_n^\alpha \supset J_n^\alpha A^\alpha, \qquad 0\leqq\alpha\leqq 1. \tag{1.15}$$

Here $J_n^\alpha=(1+n^{-1}A)^{-\alpha}$ exists since $1+n^{-1}A$ is maximal accretive with A. (1.15) follows from the identity $J_n^\alpha=(A^{-1}A_n)^\alpha=A^{-\alpha}A_n^\alpha=A_n^\alpha A^{-\alpha}$, which is in turn a simple consequence of the "operational calculus" mentioned in Appendix (note that A^{-1} and $A^{-\alpha}$ are bounded).

In letting $n\to\infty$, we note that

$$\|J_n^\alpha\|\leqq 1, \qquad \text{strong}\lim_{n\to\infty} J_n^\alpha=1, \qquad 0\leqq\alpha\leqq 1. \tag{1.16}$$

The first inequality of (1.16) follows from Lemma A1 (Appendix), while the second relation can be proved easily by making use of the formula (A1) (set $\lambda=0$, replace A by $1+n^{-1}A$ and let $n\to\infty$, noting the principle of dominated convergence).

Suppose now that $u\in\mathfrak{D}[A^\alpha]$. Then $A_n^\alpha u=J_n^\alpha A^\alpha u$ by (1.15), and so $\|A_n^\alpha u\|\leqq\|A^\alpha u\|$, $A_n^\alpha u\to A^\alpha u$, $n\to\infty$, by (1.16). But we have, for $0\leqq\alpha\leqq 1/2$,

$$\|A_n^{*\alpha}u\|\leqq c'_\alpha\|A_n^\alpha u\|\leqq c'_\alpha\|A^\alpha u\|, \qquad c_\alpha'=\tan\frac{\pi(1+2\alpha)}{4}, \tag{1.17}$$

for (1.4) has been proved for A_n. (1.17) shows that the sequence $\|A_n^{*\alpha}u\|$ is bounded. Furthermore, we have for any $v\in\mathfrak{D}[A^\alpha]$

$$(A_n^{*\alpha}u, v)=(u, A_n^\alpha v)\to(u, A^\alpha v), \qquad n\to\infty. \tag{1.18}$$

(Note that $A_n^{*\alpha}=A_n^{\alpha *}$, see (A9) of Appendix.) Since $\mathfrak{D}[A^\alpha]$ is dense in $\mathfrak{H}$, we see that $A_n^{*\alpha}u$ is weakly convergent. Let w be its weak limit. Then (1.18) gives $(w,v)=(u,A^\alpha v)$. Since this is true for every $v\in\mathfrak{D}[A^\alpha]$, we conclude that $u\in\mathfrak{D}[A^{\alpha *}]=\mathfrak{D}[A^{*\alpha}]$. Then the same argument as given above shows that $A_n^{*\alpha}u\to A^{*\alpha}u$ even strongly. In view of the symmetric relationship between A and A^*, we have thus proved that $\mathfrak{D}[A^{*\alpha}]=\mathfrak{D}[A^\alpha]\equiv\mathfrak{D}_\alpha$ and that $A_n^\alpha u\to A^\alpha u$, $A_n^{*\alpha}u\to A^{*\alpha}u$ for $u\in\mathfrak{D}_\alpha$.

Now the operators H_α and K_α defined by (1.1) have likewise the domain $\mathfrak{D}_\alpha$ and are symmetric, and we have $H_{n\alpha}u\to H_\alpha u$, $K_{n\alpha}u\to K_\alpha u$ for $u\in\mathfrak{D}_\alpha$. The inequalities (1.2) to (1.6) are thus established as the limit of the corresponding inequalities already proved for A_n.

III. Let us now remove the assumption that A has a bounded inverse. For any $\varepsilon>0$, $A+\varepsilon$ is closed and maximal accretive with A and $(A+\varepsilon)^{-1}$ is bounded. Thus we have $\mathfrak{D}[(A+\varepsilon)^\alpha]=\mathfrak{D}[(A^*+\varepsilon)^\alpha]$ for $0\leqq\alpha<1/2$ and the inequalities (1.2) to (1.6) hold for A replaced by $A+\varepsilon$. But we know (see Lemma A2 of Appendix) that $\mathfrak{D}[(A+\varepsilon)^{\alpha}]=\mathfrak{D}[A^{\alpha}]$ and $(A+\varepsilon)^{\alpha}u\rightarrow A^{\alpha}u$, $\varepsilon\rightarrow 0$, for $u\in\mathfrak{D}[A^{\alpha}]$ and similarly for A^*. Hence these inequalities are extended to A by taking the limit $\varepsilon\rightarrow 0$.

IV. It remains to show that H_{α} is selfadjoint (that H_{α} is nonnegative is obvious since A^{α} is accretive). Since H_{α} is symmetric and nonnegative, it suffices to show that $1+H_{\alpha}$ has range $\mathfrak{H}$. Since $\|K_{\alpha}u\|\leqq c_{\alpha}\|H_{\alpha}u\|\leqq c_{\alpha}\|(1+H_{\alpha})u\|$ with $c_{\alpha}=\tan\frac{\pi\alpha}{2}<1$, there is a bounded operator B_{α} such that $K_{\alpha}=B_{\alpha}(1+H_{\alpha})$ and $\|B_{\alpha}\|\leqq c_{\alpha}$. Then $1+A^{\alpha}=1+H_{\alpha}+iK_{\alpha}=(1+iB_{\alpha})(1+H_{\alpha})$. But $1+A^{\alpha}$ has range $\mathfrak{H}$ since A^{α} is closed and maximal accretive. Since $1+iB_{\alpha}$ is an automorphism of $\mathfrak{H}$ in virtue of $\|B_{\alpha}\|<1$, it follows that $1+H_{\alpha}$ has range $\mathfrak{H}$. This completes the proof of Theorem 1.1.

EXAMPLE. Let $\mathfrak{H}=L^2(0,\infty)$ and let A be the differential operator $A=d/dx$ with the boundary condition $u(0)=0$. As is well known, A is maximal accretive (if the differentiation d/dx is interpreted appropriately). The adjoint of A is $A^*=-d/dx$ with no boundary condition. Thus $\mathfrak{D}[A]$ is a proper subset of $\mathfrak{D}[A^*]$. Nevertheless, Theorem 1.1 shows that A^{α} and $A^{*\alpha}$ have the same domain for $0\leqq\alpha<1/2$. It is not easy to verify this directly, for A^{α} and $A^{*\alpha}$ are rather complicated operators.

We shall now show that $\mathfrak{D}[A]\neq\mathfrak{D}[A^*]$ for $1/2<\alpha\leqq 1$. As is well known, $(\lambda+A)^{-1}$ and $(\lambda+A^*)^{-1}$ are integral operators given by

$$(\lambda+A)^{-1}u(x)=\int_0^x e^{-\lambda(x-y)}u(y)dy\,,$$

$$(\lambda+A^*)^{-1}u(x)=\int_x^{\infty} e^{-\lambda(y-x)}u(y)dy\,.$$

Application of (A1) of Appendix (set $\lambda=0$, replace A by $A+\lambda$) gives

$$(\lambda+A)^{-\alpha}u(x)=\Gamma(\alpha)^{-1}\int_0^x e^{-\lambda(x-y)}(x-y)^{\alpha-1}u(y)dy\,,$$

$$(\lambda+A^*)^{-\alpha}u(x)=\Gamma(\alpha)^{-1}\int_x^{\infty} e^{-\lambda(y-x)}(y-x)^{\alpha-1}u(y)dy\,,$$

where $\lambda>0$ and $0<\alpha<1$. If $1/2<\alpha<1$, it follows that

$$|(\lambda+A)^{-\alpha}u(x)|^2\leqq\Gamma(\alpha)^{-2}\Big[\int_0^x e^{-2\lambda(x-y)}(x-y)^{2\alpha-2}dy\Big]\cdot$$

$$\cdot\Big[\int_0^x|u(y)|^2dy\Big]\leqq\text{const. }x^{2\alpha-1}\,.$$

Therefore, for any $u\in L^2(0,\infty)$, $(\lambda+A)^{-\alpha}u(x)$ is locally bounded and tends to zero for $x\rightarrow 0$. Since $\mathfrak{D}[A^{\alpha}]=\mathfrak{D}[(A+\lambda)^{\alpha}]=\mathfrak{R}[(A+\lambda)^{-\alpha}]$ by Lemma A2 of Appendix, any $v\in\mathfrak{D}[A^{\alpha}]$ must satisfy the boundary condition $\lim v(x)=0$. On the other hand, $\mathfrak{D}[A^{*\alpha}] \supset \mathfrak{D}[A^*]$ and a function $w \in \mathfrak{D}[A^*]$ need not satisfy such a boundary condition. Hence $\mathfrak{D}[A^{*\alpha}] \neq \mathfrak{D}[A^\alpha]$. But it is not known to the author whether $A^{1/2}$ and $A^{*1/2}$ are comparable.

THEOREM 1.2. *Let A be bounded with domain $\mathfrak{H}$ and accretive. Then*

$$\| H_1^\alpha u \| \leqq \| H_\alpha u \| , \qquad 0 \leqq \alpha \leqq 1/2 , \tag{1.19}$$

$$\| H_1^\alpha u \| \leqq \left(1 - \tan \frac{\pi\alpha}{2}\right)^{-1} \| A^\alpha u \| , \qquad 0 \leqq \alpha \leqq 1/2 . \tag{1.20}$$

REMARK. If A is bounded, H_1 defined by (1.1) with $\alpha = 1$ is also bounded and symmetric, hence selfadjoint. Since $H_1 \geqq 0$, H_1^α can be constructed. This is in general not true if A is unbounded.

PROOF. Again we first assume that $\mathrm{Re}(Au, u) \geqq \delta(u, u)$ for some $\delta > 0$, so that A has a bounded inverse A^{-1}. Then we have (1.11). On the other hand H_1^α is an entire function of α and

$$\| H_1^\alpha u \| = \| H_1^\xi u \| , \tag{1.21}$$

since H_1 is selfadjoint and $H_1 \geqq \delta > 0$. It follows from (1.11) and (1.21) that $\| H_1^\alpha u \| / \| H_\alpha u \| \leqq \delta^{-\xi} \| H_1^\xi u \| / \| u \| \leqq \delta^{-\xi} \| H_1 \|^\xi$. Hence $\| H_1^\alpha H_\alpha^{-1} \| \leqq \delta^{-\xi} \| H_1 \|^\xi$, and the holomorphic function $H_1^\alpha H_\alpha^{-1}$ is bounded in the strip $0 \leqq \mathrm{Re}\, \alpha \leqq 1/2$. But we have a sharper bound to this function on the boundary of this strip, namely

$$\| H_1^\alpha H_\alpha^{-1} \| \leqq 1 \qquad \text{for} \quad \xi = 0 \quad \text{and} \quad \xi = 1/2 . \tag{1.22}$$

In fact, for $\xi = 0$ we have from (1.11) $\| H_\alpha u \|^2 \geqq \| u \|^2 = \| H_1^\alpha u \|^2$ and for $\xi = 1/2$ similarly $\| H_\alpha u \|^2 \geqq \mathrm{Re}\,(Au, u) = (H_1 u, u) = \| H_1^{1/2} u \|^2 = \| H_1^\alpha u \|^2$. These are equivalent with (1.22).

According to a theorem of Phragmén-Lindelöf type, it follows that $\| H_1^\alpha H_\alpha^{-1} \| \leqq 1$ in the whole strip $0 \leqq \mathrm{Re}\, \alpha \leqq 1/2$. This proves (1.19), and (1.20) follows by using (1.3). The additional assumption that $\mathrm{Re}\,(Au, u) \geqq \delta(u, u)$ can be removed by considering $A + \varepsilon$ in place of A and later going to the limit $\varepsilon \to 0$, just as in the proof of Theorem 1.1.

2. Sesquilinear forms and regularly accretive operators.

To proceed further with the study of accretive operators, we introduce the notion of *regularly accretive* operators. This is most conveniently defined in terms of *sesquilinear forms* in $\mathfrak{H}$. In the present section we present a general theory of sesquilinear forms and the associated operators. Some of the results given below are known, but since they are scattered in literature, we find it convenient to give here a unified treatment of them. A detailed theory of *symmetric* sesquilinear forms is given by Friedrichs [**3**] and the author [8]. The following exposition of non-symmetric forms are based on the results of these treatises.

A complex-valued function $\phi[u,v]$ defined for u, v belonging to a linear subset $\mathfrak{D}$ of $\mathfrak{H}$ is called a sesquilinear form if it is linear in u and semilinear in v. $\mathfrak{D}$ is called the *domain* of ϕ and denoted by $\mathfrak{D}[\phi]$. For brevity we write $\phi[u]=\phi[u,u]$. If ϕ is a sesquilinear form, $\phi^*[u,v]=\overline{\phi[v,u]}$ defines another sesquilinear form ϕ^* with $\mathfrak{D}[\phi^*]=\mathfrak{D}[\phi]$. ϕ^* is called the *adjoint form* of ϕ. If $\phi=\phi^*$, ϕ is said to be (Hermitian) *symmetric*.

If ϕ is a symmetric form, $\phi[u]$ is real-valued; if $\phi[u]\geqq 0$ for all $u\in\mathfrak{D}[\phi]$, ϕ is *nonnegative*. For a nonnegative symmetric form ϕ, we have the *Schwarz inequality* $|\phi[u,v]|^2\leqq\phi[u]\phi[v]$. If ϕ is symmetric and bounded in the sense that $|\phi[u]|\leqq M\|u\|^2$ for all $u\in\mathfrak{D}[\phi]$, then we have also $|\phi[u,v]|\leqq M\|u\|\,\|v\|$.

Any sesquilinear form ϕ can be written

$$\phi=f+ig,\qquad f=\frac{1}{2}(\phi+\phi^*),\qquad g=\frac{1}{2i}(\phi-\phi^*),$$

where f and g are symmetric. We have $\mathrm{Re}\,\phi[u]=f[u]$ and $\mathrm{Im}\,\phi[u]=g[u]$. Therefore f and g may be called the *real* and the *imaginary parts* of ϕ, and denoted by $\mathrm{Re}\,\phi$ and $\mathrm{Im}\,\phi$, respectively, although $f[u,v]$ and $g[u,v]$ are not real-valued.

Let us now state without proof some results on symmetric forms (cf. [3] and [8] cited above). A nonnegative, symmetric form f is said to be *closed* if $u_n\in\mathfrak{D}[f]$, $u_n\to u\in\mathfrak{H}$ and $f[u_n-u_m]\to 0$ imply $u\in\mathfrak{D}[f]$ and $f[u_n]\to f[u]$. To each closed, nonnegative symmetric form f with $\mathfrak{D}[f]$ dense in $\mathfrak{H}$, there is associated a unique nonnegative selfadjoint operator H such that $\mathfrak{D}[H]\subset\mathfrak{D}[\phi]$ and $\phi[u,v]=(Hu,v)$ for $u\in\mathfrak{D}[H]$ and $v\in\mathfrak{D}[\phi]$. H is also characterized by the fact that

$$\mathfrak{D}[H^{1/2}]=\mathfrak{D}[\phi]\quad\text{and}\quad\phi[u,v]=(H^{1/2}u,H^{1/2}v)\quad\text{for}\quad u,v\in\mathfrak{D}[\phi]. \tag{2.1}$$

DEFINITION 2.1. *A sesquilinear form ϕ will be said to be regular if* 1) $\mathfrak{D}[\phi]$ *is dense in* $\mathfrak{H}$, 2) $\mathrm{Re}\,\phi$ *is a closed, nonnegative symmetric form and* 3) *there is a* $\beta\geqq 0$ *such that*

$$|\mathrm{Im}\,\phi[u]|\leqq\beta\,\mathrm{Re}\,\phi[u]. \tag{2.2}$$

The smallest number β with the property (2.2) *will be called the index of ϕ.*

Note that (2.2) implies, writing $f=\mathrm{Re}\,\phi$, $g=\mathrm{Im}\,\phi$, that

$$|g[u,v]|\leqq\beta f[u]^{1/2}f[v]^{1/2},\quad|\phi[u,v]|\leqq(1+\beta)f[u]^{1/2}f[v]^{1/2}. \tag{2.3}$$

We now prove a fundamental theorem on regular sesquilinear forms, which generalizes the relationship between a closed nonnegative symmetric form f and the associated selfadjoint operator H stated above.

THEOREM 2.1. *Let ϕ be a regular sesquilinear form. Then there is a unique* *closed, maximal accretive operator* A *with* $\mathfrak{D}[A]\subset\mathfrak{D}[\phi]$ *such that*

$$\phi[u, v]=(Au, v) \quad \textit{for} \quad u\in\mathfrak{D}[A] \quad \textit{and} \quad v\in\mathfrak{D}[\phi]. \tag{2.4}$$

A *will be called the accretive operator associated with the regular sesquilinear form* ϕ. *Similarly,* A^* *is the maximal accretive operator associated with* ϕ^*. *In other words,* $\mathfrak{D}[A^*]\subset\mathfrak{D}[\phi]$ *and* $\phi[u, v]=(u, A^*v)$ *for* $u\in\mathfrak{D}[\phi]$ *and* $v\in\mathfrak{D}[A^*]$.

PROOF. $f=\mathrm{Re}\,\phi$ is nonnegative and closed by hypothesis. As is well known ([**3**, **8**]), it follows that $\mathfrak{D}[\phi]$ becomes a complete Hilbert space, denoted by $\mathfrak{H}_\phi$, which a new inner product and the associated norm are introduced by

$$((u, v))=(u, v)+f[u, v], \quad |||u|||^2=\|u\|^2+f[u]\geqq\|u\|^2. \tag{2.5}$$

Since $|\phi[u, v]|\leqq(1+\beta)|||u|||\,|||v|||$ by (2.3) and (2.5), ϕ is a bounded sesquilinear form on $\mathfrak{H}_\phi$. The same is true with $\phi_0[u, v]=\phi[u, v]+(u, v)$ and we have $\mathrm{Re}\,\phi_0[u]=|||u|||^2$.

Now (w, v) is a bounded semilinear form of $v\in\mathfrak{H}_\phi$ for any fixed $w\in\mathfrak{H}$, for $|(w, v)|\leqq\|w\|\,\|v\|\leqq\|w\|\,|||v|||$. According to a theorem of Lax and Milgram [**9**], there exists a $u\in\mathfrak{H}_\phi$ such that $(w, v)=\phi_0[u, v]$ with $|||u|||\leqq\|w\|$. This defines a bounded linear operator B on $\mathfrak{H}$ to $\mathfrak{H}_\phi$ such that $u=Bw$ and

$$(w, v)=\phi_0[Bw, v]=\phi[Bw, v]+(Bw, v). \tag{2.6}$$

B is invertible; in fact, $Bw=0$ implies $(w, v)=0$ for all $v\in\mathfrak{D}[\phi]$ so that $w=0$ because ϕ is densely defined. Set $A=B^{-1}-1$, B^{-1} and A being considered linear operators in $\mathfrak{H}$. Then $\mathfrak{D}[A]=\mathfrak{D}[B^{-1}]=\mathfrak{R}[B]\in\mathfrak{D}[\phi]$ and (2.6) gives (2.4) by setting $u=Bw$.

Since $\mathrm{Re}\,(Au, u)=\mathrm{Re}\,\phi[u]\geqq 0$, A is accretive. That A is closed and maximal accretive follows from the fact that $A+1=B^{-1}$ has the range $\mathfrak{R}[B^{-1}]=\mathfrak{D}[B]=\mathfrak{H}$; in fact this implies that -1 belongs to the resolvent set of A and, therefore, the whole semi-plane $\mathrm{Re}\,z<0$ must belong to the resolvent set of A, showing that A is maximal accretive.

In particular this implies that $\mathfrak{D}[A]$ is dense in $\mathfrak{H}$. We have actually a stronger result that $\mathfrak{D}[A]$ *is dense in* $\mathfrak{H}_\phi$. In fact, suppose $\mathfrak{D}[A]=\mathfrak{R}[B]$ is not dense in $\mathfrak{H}_\phi$. Then there would exist a $v\neq 0$ of $\mathfrak{H}_\phi$ such that $\phi_0[Bw, v]=0$ for all $w\in\mathfrak{H}$ (the Lax-Milgram theorem) and (2.6) gives $(w, v)=0$, contradicting that $v\neq 0$.

Since ϕ^* is regular with ϕ, we can construct in the same way a closed, maximal accretive operator A' such that $\mathfrak{D}[A']\subset\mathfrak{D}[\phi^*]=\mathfrak{D}[\phi]$ and $\phi^*[v, u]=(A'v, u)$, that is, $\phi[u, v]=(u, A'v)$, for $u\in\mathfrak{D}[\phi]$ and $v\in\mathfrak{D}[A']$. Now let, in particular, $u\in\mathfrak{D}[A]$ and $v\in\mathfrak{D}[A']$. Then we have $(Au, v)=\phi[u, v]=(u, A'v)$. This shows that $A'\subset A^*$. But as A^* is accretive and A' is maximal accretive, we must have $A'=A^*$ and so $A=A'^*$. This proves the last statement of Theorem 2.1 and, at the same time, the uniqueness of the operator A with the properties stated in the theorem. This completes the proof of Theorem 2.1.

DEFINITION 2.2. *An operator A will be said to be regularly accretive if it is associated with a regular sesquilinear form ϕ in the way described in Theorem 2.1. The index of ϕ will be called also the index of A.*

Obviously a regularly accretive operator A must satisfy the inequality

$$|\operatorname{Im}(Au, u)| \leqq \beta \operatorname{Re}(Au, u), \qquad u \in \mathfrak{D}[A], \tag{2.7}$$

for some $\beta \geqq 0$.

THEOREM 2.2. *Let A be regularly accretive with index β. Then the spectrum of A is a subset of the sector S_θ: $|\arg z| \leqq \theta = \arctan \beta < \pi/2$, and the resolvent of A satisfies the inequality*

$$\|(z-A)^{-1}\| \leqq \begin{cases} [|z| \sin(\arg z - \theta)]^{-1}, & \theta < \arg z \leqq \frac{\pi}{2} + \theta, \\ |z|^{-1}, & \arg z > \frac{\pi}{2} + \theta. \end{cases} \tag{2.8}$$

REMARK. This theorem implies that A is of type $(\theta, 1)$. In particular, $-A$ is the infinitesimal generator of an *analytic* semi-group $\{\exp(-tA)\}$, see Appendix.

PROOF. Since A satisfies the inequality (2.7), the *numerical range* of A is a subset of S_θ. Since A is closed and maximal accretive, the spectrum of A is a subset of its numerical range and $\|(z-A)^{-1}\|$ does not exceed the inverse of the distance d of z from the numerical range (note that $|(Au, u)-z| = |((A-z)u, u)| \geqq d$ for $\|u\|=1$ implies $\|(A-z)u\| \geqq d\|u\|$). Hence $\|(z-A)^{-1}\| \leqq d'^{-1}$ where d' is the distance of z from S_θ. This gives (2.8).

It is not known to the author whether (2.7) is sufficient for a closed, maximal accretive operator A to be regularly accretive. The following theorems answer this question only partially.

THEOREM 2.3. *Let A be closed and maximal accretive and let* (2.7) *hold with a $\beta < 1$. Then A is regularly accretive.*

PROOF. Define a sesquilinear from ϕ by $\phi[u, v] = (Au, v)$ with $\mathfrak{D}[\phi] = \mathfrak{D}[A]$. $f = \operatorname{Re}\phi$ is in general not closed, but it has a closed extension. To see this, it suffices to show that $\mathfrak{D}[\phi] \ni u_n \to 0$ and $f[u_n - u_m] \to 0$, $n, m \to \infty$, imply $f[u_n] \to 0$ (see [8]). But $f[u_n - u_m] = f[u_n] + f[u_m] - 2 \operatorname{Re} f[u_n, u_m] = f[u_n] + f[u_m] - \operatorname{Re}(\phi + \phi^*)[u_n, u_m] = f[u_n] + f[u_m] - \operatorname{Re}\phi[u_n, u_m] - \operatorname{Re}\phi[u_m, u_n]$ and $\operatorname{Re}\phi[u_n, u_m] \leqq |\phi[u_n, u_m]| \leqq (1+\beta) f[u_n]^{1/2} f[u_m]^{1/2}$ by (2.3). Hence we have

$$f[u_n - u_m] \geqq f[u_n] + f[u_m] - (1+\beta) f[u_n]^{1/2} f[u_m]^{1/2} - \operatorname{Re}(Au_m, u_n)$$

$$\geqq \Big[1 - \frac{(1+\beta)^2}{4}\Big] f[u_m] - \operatorname{Re}(Au_m, u_n).$$

Let $n \to \infty$ and then $m \to \infty$. Since the left member tends to 0 by hypothesis, the desired result $f[u_m]\to 0$ follows.

Let f' be the closure of f (see [8]). In view of (2.3), ϕ can be extended to a form ϕ' with $\mathfrak{D}[\phi']=\mathfrak{D}[f']$ such that $f'=\mathrm{Re}\,\phi'$, the inequality (2.3) being conserved in this extension. ϕ' is obviously regular, and we have $(Au,v)=\phi[u,v]=\phi'[u,v]$ for $u,v\in\mathfrak{D}[A]\subset\mathfrak{D}[\phi']$, and this relation is extended to all $v\in\mathfrak{D}[\phi']$ by continuity. This shows that A is the regularly accretive operator associated with the regular sesquilinear form ϕ' (note the uniqueness of A in Theorem 2.1).

THEOREM 2.4. *Let A be closed and maximal accretive. If $0\leqq\gamma<1$, A^γ is regularly accretive with index $\leqq\tan\frac{\pi\gamma}{2}$.*

PROOF. Let $\alpha=\gamma/2$. If $0\leqq\gamma<1$, we have $0\leqq\alpha<1/2$ so that $\mathfrak{D}[A^\alpha]=\mathfrak{D}[A^{*\alpha}]\equiv\mathfrak{D}_\alpha$ by Theorem 1.1. We now define a sesquilinear form ϕ_α by

$$\phi_\alpha[u,v]=(A^\alpha u, A^{*\alpha}v)\,,\qquad \mathfrak{D}[\phi_\alpha]=\mathfrak{D}_\alpha\,. \tag{2.9}$$

On introducing the operators H_α and K_α by (1.1), this becomes

$$\begin{aligned}&\phi_\alpha=f_\alpha+ig_\alpha\,,\\ &f_\alpha[u,v]=(H_\alpha u, H_\alpha v)-(K_\alpha u, K_\alpha v)\,,\\ &g_\alpha[u,v]=(H_\alpha u, K_\alpha v)+(K_\alpha u, H_\alpha v)\,.\end{aligned} \tag{2.10}$$

The inequality (1.2) now gives $f_\alpha[u]=\|H_\alpha u\|^2-\|K_\alpha u\|^2\leqq\|H_\alpha u\|^2$ and $f_\alpha[u]\geqq(1-c_\alpha^2)\|H_\alpha u\|^2\geqq 0$ with $c_\alpha=\tan\frac{\pi\alpha}{2}$. From this it is easy to show that f_α is nonnegative and closed (note that H_α is selfadjoint). Also $|g_\alpha[u]|\leqq 2\|H_\alpha u\|\cdot\|K_\alpha u\|\leqq 2c_\alpha\|H_\alpha u\|^2\leqq\beta_\alpha f_\alpha[u]$, where $\beta_\alpha=2c_\alpha(1-c_\alpha^2)^{-1}=\tan\pi\alpha=\tan\frac{\pi\gamma}{2}$.

Thus ϕ_α is regular. Let the associated accretive operator be denoted by A_α. We shall show that $A_\alpha=A^{2\alpha}=A^\gamma$, establishing thereby that A^γ is indeed regularly accretive.

Set $A_\varepsilon=A+\varepsilon$, $\varepsilon>0$. Then A_ε^{-1} is bounded and it is easily seen that $A_\varepsilon^{-2\alpha}=(A_\varepsilon^{-\alpha})^2$ (operational calculus). This implies that $A_\varepsilon^{2\alpha}=(A_\varepsilon^\alpha)^2$ and so $(A_\varepsilon^{2\alpha}u,v)=(A_\varepsilon^\alpha u, A^{*}{}_\varepsilon^\alpha v)$ for $u\in\mathfrak{D}[A_\varepsilon^{2\alpha}]\subset\mathfrak{D}[A_\varepsilon^\alpha]$ and $v\in\mathfrak{D}[A_\varepsilon^\alpha]$. But since $\mathfrak{D}[A_\varepsilon^\alpha]=\mathfrak{D}[A^\alpha]$ and $A_\varepsilon^\alpha u\to A^\alpha u$ for $\varepsilon\to 0$ by Lemma A2 (Appendix), we obtain $(A^{2\alpha}u,v)=(A^\alpha u, A^{*\alpha}v)$ for $u\in\mathfrak{D}[A^{2\alpha}]\subset\mathfrak{D}[A^\alpha]\ni v$. On the other hand we have $(A^\alpha u, A^{*\alpha}v)=(u, A_\alpha^{*}v)$ for $u\in\mathfrak{D}[A^\alpha]$ and $v\in\mathfrak{D}[A_\alpha^{*}]\subset\mathfrak{D}[A^\alpha]$ by the definition of A_α. Hence $(A^{2\alpha}u,v)=(u,A_\alpha^{*}v)$ for $u\in\mathfrak{D}[A^{2\alpha}]$ and $v\in\mathfrak{D}[A_\alpha^{*}]$, so that $A^{2\alpha}\subset A_\alpha$. But we have the equality here, for A_α is accretive and $A^{2\alpha}$ is maximal accretive.

3. Fractional powers of regularly accretive operators.

If A is regularly accretive, stronger results can be obtained for the fractional powers A^{α} than given by Theorem 1.1 and 1.2. By definition A is associated with a regular sesquilinear form $\phi=f+ig$ according to Theorem 2.1. Since the symmetric form f is nonnegative and closed, there is associated with f a nonnegative selfadjoint operator H such that (2.1) is true. H will be called the *real part* of A. H is also the real part of A^*, as is seen from Theorem 2.1. H should be distinguished from the H_1 given by (1.1). If A is bounded we have $H=\frac{1}{2}(A+A^*)=H_1$, but in general H_1 is not selfadjoint even if A is regularly accretive.

We shall now show that H^{α} is comparable with A^{α}, $A^{*\alpha}$ and H_{α} for $0\leqq \alpha<1/2$. More precisely, we have

THEOREM 3.1. *Let A be regularly accretive with index β and let H be the real part of A. For each α with $0\leqq\alpha<1/2$, we have $\mathfrak{D}[H^{\alpha}]=\mathfrak{D}[A^{\alpha}]=\mathfrak{D}[A^{*\alpha}]=\mathfrak{D}[H_{\alpha}]=\mathfrak{D}_{\alpha}$ and, besides the inequalities* (1.2) *to* (1.6),

$$(3.1)\qquad \left(1-\tan\frac{\pi\alpha}{2}\right)\|H^{\alpha}u\|\leqq\|A^{\alpha}u\|\leqq\left[1+\left(\frac{\alpha}{\pi}\tan\pi\alpha\right)^{1/2}(\beta+\beta^2)\right]\|H^{\alpha}u\|$$

and similar inequalities with A replaced by A^*.

PROOF. I. First we deduce an identity connecting A and H. Let A be associated with the regular sesquilinear form $\phi=f+ig$ as in Theorem 2.1; then H is associated with f by (2.1). We have by (2.1) and (2.3)

$$(3.2)\qquad |g[u,v]|\leqq\beta\|H^{1/2}u\|\,\|H^{1/2}v\|,\qquad u,v\in\mathfrak{D}[H^{1/2}]=\mathfrak{D}[\phi].$$

Thus $g[u,v]$ is determined by $H^{1/2}u$ and $H^{1/2}v$ and, therefore, may be regarded as a bounded symmetric form of the latter. Hence there is a bounded operator B (not necessarily unique) with domain $\mathfrak{H}$ such that

$$(3.3)\qquad g[u,v]=(BH^{1/2}u,H^{1/2}v),\quad B^*=B,\quad \|B\|\leqq\beta.$$

From (2.1) and (3.3) we have

$$(3.4)\qquad \phi[u,v]=(f+ig)[u,v]=((1+iB)H^{1/2}u,H^{1/2}v).$$

Let $u\in\mathfrak{D}[A]$. Then $\phi[u,v]=(Au,v)$ for all $v\in\mathfrak{D}[\phi]=\mathfrak{D}[H^{1/2}]$ by (2.4). Comparing this with (3.4) and noting the selfadjointness of $H^{1/2}$, we see that $H^{1/2}(1+iB)H^{1/2}u$ exists and is equal to Au. This shows that $A\subset H^{1/2}(1+iB)H^{1/2}$. But it is obvious that the right member is accretive. In view of the fact that A is maximal accretive, we must have an equality in place of the inclusion, that is,

$$(3.5)\qquad A=H^{1/2}(1+iB)H^{1/2}.$$

II. We shall now show that

(3.6) $$A+\lambda=(H+\lambda)^{1/2}(1+iK_\lambda)(H+\lambda)^{1/2}, \qquad \lambda>0,$$
where

(3.7) $$K_\lambda=\left(\frac{H}{H+\lambda}\right)^{1/2}B\left(\frac{H}{H+\lambda}\right)^{1/2}, \qquad K_\lambda^*=K_\lambda,$$

$$\|K_\lambda\|\leqq\|B\|\leqq\beta.$$

To this end, let $u\in\mathfrak{D}[A]\subset\mathfrak{D}[H^{1/2}]$. Using (3.7) and (3.5), we have

$$\begin{aligned}(1+iK_\lambda)(H+\lambda)^{1/2}u&=(H+\lambda)^{1/2}u+i\left(\frac{H}{H+\lambda}\right)^{1/2}BH^{1/2}u\\&=(H+\lambda)^{1/2}u-H(H+\lambda)^{-1/2}u+\left(\frac{H}{H+\lambda}\right)^{1/2}(1+iB)H^{1/2}u\\&=(H+\lambda)^{-1/2}(\lambda+A)u.\end{aligned}$$

Hence $(H+\lambda)^{1/2}(1+iK_\lambda)(H+\lambda)^{1/2}u$ exists and is equal to $(A+\lambda)u$. An argument similar to that used above then gives (3.6).

Taking the inverse of (3.6) and noting that $(1+iK)^{-1}=1-iK-K(1+iK)^{-1}K$, we have (for brevity we write K in place of K_λ)

(3.8) $$\begin{aligned}(A+\lambda)^{-1}&=(H+\lambda)^{-1}-i\frac{H^{1/2}}{H+\lambda}B\frac{H^{1/2}}{H+\lambda}\\&\quad-\frac{H^{1/2}}{H+\lambda}B\left(\frac{H}{H+\lambda}\right)^{1/2}(1+iK)^{-1}\left(\frac{H}{H+\lambda}\right)^{1/2}B\frac{H^{1/2}}{H+\lambda}\\&\equiv(H+\lambda)^{-1}+\frac{H^{1/2}}{H+\lambda}C_\lambda\frac{H^{1/2}}{H+\lambda}\end{aligned}$$

with $\|C_\lambda\|\leqq\beta+\beta^2$. Note that $K^*=K$ implies $\|(1+iK)^{-1}\|\leqq1$, and we have $\|B\|\leqq\beta$, $\|H^{1/2}(H+\lambda)^{-1/2}\|\leqq1$. It follows by Lemma A7 (Appendix) that for $u\in\mathfrak{D}[H^\alpha]$

(3.9) $$\lim_{R\to\infty}\int_0^R[(A+\lambda)^{-1}u-(H+\lambda)^{-1}u]\lambda^\alpha d\lambda=w$$

exists and

$$\|w\|\leqq(\beta+\beta^2)\left(\frac{2\pi\alpha}{\sin 2\pi\alpha}\right)^{1/2}\|H^\alpha u\|.$$

But we have

(3.10) $$\lim_{R\to\infty}\int_0^R[\lambda^{-1}-(H+\lambda)^{-1}]u\lambda^\alpha d\lambda=\frac{\pi}{\sin\pi\alpha}H^\alpha u,$$

as is easily verified by using the spectral representation of H. Hence

(3.11) $$\begin{aligned}v&=\lim_{R\to\infty}\int_0^R[\lambda^{-1}-(A+\lambda)^{-1}]u\lambda^\alpha d\lambda\\&=\lim_{R\to\infty}\int_0^R A(A+\lambda)^{-1}u\lambda^{\alpha-1}d\lambda\end{aligned}$$

exists and

$$\left\| v - \frac{\pi}{\sin \pi\alpha} H^\alpha u \right\| \leqq (\beta+\beta^2)\left(\frac{2\pi\alpha}{\sin 2\pi\alpha}\right)^{1/2} \| H^\alpha u \| . \tag{3.12}$$

An application of Lemma A5 (Appendix) shows, then, that $u \in \mathfrak{D}[A^\alpha]$ and $v = \frac{\pi}{\sin \pi\alpha} A^\alpha u$. Thus $\mathfrak{D}[H^\alpha] \subset \mathfrak{D}[A^\alpha]$ and (3.12) implies that

$$\| A^\alpha u - H^\alpha u \| \leqq \left(\frac{\alpha}{\pi} \tan \pi\alpha\right)^{1/2} (\beta+\beta^2) \| H^\alpha u \| , \qquad u \in \mathfrak{D}[H^\alpha] . \tag{3.13}$$

III. Next we prove the converse inclusion $\mathfrak{D}[A^\alpha] \subset \mathfrak{D}[H^\alpha]$. To this end, we introduce an approximating sequence $\{A_n\}$ of A by setting

$$A_n = H_n^{1/2}(1+iB)H_n^{1/2} , \quad H_n = H(1+n^{-1}H)^{-1} , \qquad n = 1, 2, 3, \cdots . \tag{3.14}$$

H_n are bounded with $\| H_n \| \leqq n$. Hence A_n are also bounded with $\| A_n \| \leqq n(1+\beta)$. Furthermore, A_n are uniformly regularly accretive since

$$| \operatorname{Im} (A_n u, u) | = | (BH_n^{1/2}u, H_n^{1/2}u) | \leqq \beta \| H_n^{1/2} u \|^2 = \beta \operatorname{Re} (A_n u, u) .$$

Since $A_n + A_n^* = 2H_n$, it follows from (1.20) that

$$\| H_n^\alpha u \| \leqq c''_\alpha \| A_n^\alpha u \| , \quad c''_\alpha = \left(1 - \tan \frac{\pi\alpha}{2}\right)^{-1} , \qquad 0 \leqq \alpha < 1/2 . \tag{3.15}$$

For the moment let us assume that H has positive lower bound, so that H^{-1} is bounded. Then $H_n^{-1} = H^{-1} + n^{-1}$ and $A_n^{-1} = H_n^{-1/2}(1+iB)^{-1}H_n^{-1/2}$ are also bounded, and (3.15) implies that $\| H_n^\alpha A_n^{-\alpha} \| \leqq c''_\alpha$. We shall show that the bounded sequence $\{H_n^\alpha A_n^{-\alpha}\}$ is weakly convergent. For this it suffices to show that, for any $u \in \mathfrak{H}$ and $v \in \mathfrak{D}[H^\alpha]$, $(H_n^\alpha A_n^{-\alpha} u, v) = (A_n^{-\alpha} u, H_n^\alpha v) \to (A^{-\alpha} u, H^\alpha v)$. But this follows from $H_n^{-1} = H_n^{-1} + n^{-1} \to H^{-1}$. In fact, this implies $H_n^{-1/2} \to H^{-1/2}$ and so $A_n^{-1} \to A^{-1}$ in norm, whence follows that $A_n^{-\alpha} \to A^{-\alpha}$, while $H_n^\alpha v \to H^\alpha v$ can be proved by noting that $(1+n^{-1}H)^{-\alpha} \to 1$.

Let R be the weak limit of $\{H_n^\alpha A_n^{-\alpha}\}$. It follows from above that $\| R \| \leqq c''_\alpha$ and $(Ru, v) = (A^{-\alpha} u, H^\alpha v)$ for every $v \in \mathfrak{D}[H^\alpha]$. Hence $H^\alpha A^{-\alpha} u$ exists and is equal to Ru for every $u \in \mathfrak{H}$. This implies that $\mathfrak{D}[A^\alpha] \subset \mathfrak{D}[H^\alpha]$ and that $H^\alpha w = RA^\alpha w$, $\| H^\alpha w \| \leqq c''_\alpha \| A^\alpha w \|$ for $w \in \mathfrak{D}[A^\alpha]$. Combined with (3.13), this proves Theorem 3.1 under the additional assumption that H has positive lower bound.

IV. The general case can be dealt with, as we have frequently done, by applying the foregoing results to $A+\varepsilon$ and then letting $\varepsilon \to 0$. There is no difficulty if we note Lemma A2 of Appendix.

4. Perturbation theory.

In this section we consider the change of a regularly accretive operator A and its fractional powers A^α, when the sesquilinear form ϕ that defines A is subjected to a small change. In particular we shall prove the analytic dependence of A^{α} on a parameter t when ϕ depends on t analytically.

Let $\phi=f+ig$ be a regular sesquilinear form with index β, and let $\phi'=f'+ig'$ be another sesquilinear form such that

$$\mathfrak{D}[\phi']=\mathfrak{D}[\phi]=\mathfrak{D} \quad \text{and} \quad |\phi'[u]-\phi[u]|\leqq bf[u] \quad \text{for} \quad u\in\mathfrak{D}, \tag{4.1}$$

where b is a constant. This implies that $|f'[u]-f[u]|\leqq bf[u]$ or

$$(1-b)f[u]\leqq f'[u]\leqq(1+b)f[u]. \tag{4.2}$$

If $b<1$, it follows from (4.2) that the symmetric form f' is nonnegative and closed with f (see [8]). (4.1) implies also that $|g'[u]-g[u]|\leqq bf[u]$, hence $|g'[u]|\leqq|g[u]|+bf[u]\leqq(\beta+b)f[u]\leqq(\beta+b)(1-b)^{-1}f'[u]$. This shows that ϕ' is also regular with index not exceeding $(\beta+b)(1-b)^{-1}$.

It follows also from (4.1) that

$$|(\phi'-\phi)[u,v]|\leqq 2b\,f[u]^{1/2}f[v]^{1/2}=2b\,\|H^{1/2}u\|\,\|H^{1/2}v\|. \tag{4.3}$$

Just as in (3.3), (4.3) implies the existence of a bounded linear operator T such that

$$(\phi'-\phi)[u,v]=(TH^{1/2}u,H^{1/2}v), \qquad u,v\in\mathfrak{D},\ \|T\|\leqq 2b. \tag{4.4}$$

From (3.4) and (4.4) we have

$$\phi'[u,v]=((1+iB+T)H^{1/2}u,H^{1/2}v), \tag{4.5}$$

which gives, as in (3.5) and (3.6),

$$A'=H^{1/2}(1+iB+T)H^{1/2}, \tag{4.6}$$

$$A'+\lambda=(H+\lambda)^{1/2}(1+iK_{\lambda}+X_{\lambda})(H+\lambda)^{1/2}, \tag{4.7}$$

where K_{λ} is given as before by (3.7) and

$$X_{\lambda}=\left(\frac{H}{H+\lambda}\right)^{1/2}T\left(\frac{H}{H+\lambda}\right)^{1/2}, \qquad \|X\|\leqq\|T\|\leqq 2b. \tag{4.8}$$

We now take the inverse of (4.7). Here we substitute the expansion (we write $K_{\lambda}=K$, $X_{\lambda}=X$ for simplicity)

$$(1+iK+X)^{-1}=(1+iK)^{-1}+\sum_{p=1}^{\infty}(-1)^{p}(1+iK)^{-1}[X(1+iK)^{-1}]^{p}. \tag{4.9}$$

Since $\|(1+iK)^{-1}\|\leqq 1$ by $K^{*}=K$, the series (4.9) is absolutely convergent for $\|X\|\leqq 1$, which is the case if $2b<1$. Since A' coincides with A for $X=0$, we thus obtain

$$(A'+\lambda)^{-1}-(A+\lambda)^{-1}=\sum_{p=1}^{\infty}(H+\lambda)^{-1/2}(1+iK)^{-1}X^{(p)}(1+iK)^{-1}(H+\lambda)^{-1/2}, \tag{4.10}$$

where

$$X^{(p)}=(-1)^{p}[X(1+iK)^{-1}]^{p-1}X=\left(\frac{H}{H+\lambda}\right)^{1/2}Y^{(p)}\left(\frac{H}{H+\lambda}\right)^{1/2} \tag{4.11}$$

with $Y^{(1)}=-T$ and

(4.12) $$Y^{(p)}=(-1)^p T\left(\frac{H}{H+\lambda}\right)^{1/2}(1+iK)^{-1}[X(1+iK)^{-1}]^{p-2}\left(\frac{H}{H+\lambda}\right)^{1/2}T, \qquad p\geqq 2.$$

Let us examine the p-th term on the right of (4.10). Here $(1+iK)^{-1}$ can be written in the two different forms

$$(1+iK)^{-1}=1-iK(1+iK)^{-1}=1-\left(\frac{H}{H+\lambda}\right)^{1/2}iB\left(\frac{H}{H+\lambda}\right)^{1/2}(1+iK)^{-1},$$

$$(1+iK)^{-1}=1-(1+iK)^{-1}iK=1-(1+iK)^{-1}\left(\frac{H}{H+\lambda}\right)^{1/2}iB\left(\frac{H}{H+\lambda}\right)^{1/2},$$

in virtue of (3.7). Therefore, noting the form (4.11) of $X^{(p)}$, (4.10) can be written in the form

(4.13) $$(A'+\lambda)^{-1}-(A+\lambda)^{-1}=\sum_{p=1}^{\infty}\frac{H^{1/2}}{H+\lambda}Z^{(p)}\frac{H^{1/2}}{H+\lambda}$$

with

(4.14) $$\begin{aligned}Z^{(p)}= Y^{(p)}&-iB\left(\frac{H}{H+\lambda}\right)^{1/2}(1+iK)^{-1}\left(\frac{H}{H+\lambda}\right)^{1/2}Y^{(p)}\\ &-Y^{(p)}\left(\frac{H}{H+\lambda}\right)^{1/2}(1+iK)^{-1}\left(\frac{H}{H+\lambda}\right)^{1/2}iB\\ &+iB\left(\frac{H}{H+\lambda}\right)^{1/2}(1+iK)^{-1}\left(\frac{H}{H+\lambda}\right)^{1/2}Y^{(p)}\left(\frac{H}{H+\lambda}\right)^{1/2}\\ &\cdot(1+iK)^{-1}\left(\frac{H}{H+\lambda}\right)^{1/2}iB.\end{aligned}$$

Although $Z^{(p)}$ depends on λ, it can be estimated uniformly in λ, using $\|(1+iK)^{-1}\|\leqq 1$, $\left\|\left(\frac{H}{H+\lambda}\right)^{1/2}\right\|\leqq 1$, $\|B\|\leqq\beta$ and $\|X\|\leqq\|T\|\leqq 2b$, as

(4.15) $$\|Z^{(p)}\|\leqq(1+2\|B\|+\|B\|^2)\|Y^{(p)}\|\leqq(1+\beta)^2(2b)^p, \qquad p=1,2,\cdots.$$

Hence

(4.16) $$\left\|\sum_{p=1}^{\infty}Z^{(p)}\right\|\leqq\sum_{p=1}^{\infty}\|Z^{(p)}\|\leqq\frac{(1+\beta)^2 2b}{1-2b}.$$

It follows from (4.13), (4.16) and Lemma A7 (Appendix) that

(4.17) $$\left\|\lim_{R\to\infty}\int_0^R[(A'+\lambda)^{-1}u-(A+\lambda)^{-1}u]\lambda^{\alpha}d\lambda\right\| \leqq\frac{(1+\beta)^2 2b}{1-2b}\left(\frac{2\pi\alpha}{\sin 2\pi\alpha}\right)^{1/2}\|H^{\alpha}u\|,$$

where we assume that $u\in\mathfrak{D}[H^{\alpha}]$ with $0\leqq\alpha<1/2$. If we further restrict u

to $\mathfrak{D}[A]\subset\mathfrak{D}[A^{\alpha}]=\mathfrak{D}[H^{\alpha}]$ (see Theorem 3.1), we have $\int_0^{\infty}[\lambda^{-1}-(A+\lambda)^{-1}]u\lambda^{\alpha}d\lambda=\frac{\pi}{\sin\pi\alpha}A^{\alpha}u$ by Lemma A4. Therefore, an argument similar to the one applied to deduce (3.13) leads to the result that $u\in\mathfrak{D}[A'^{\alpha}]$ and

$$\|A'^{\alpha}u-A^{\alpha}u\|\leqq\frac{(1+\beta)^2 2b}{1-2b}\Big(\frac{\alpha}{\pi}\tan\pi\alpha\Big)^{1/2}\|H^{\alpha}u\|. \tag{4.18}$$

In virtue of (3.1), this gives

$$\|A'^{\alpha}u-A^{\alpha}u\|\leqq\frac{(1+\beta)^2 2b}{1-2b}\Big(1-\tan\frac{\pi\alpha}{2}\Big)^{-1}\Big(\frac{\alpha}{\pi}\tan\pi\alpha\Big)^{1/2}\|A^{\alpha}u\|. \tag{4.19}$$

So far this was proved only for $u\in\mathfrak{D}[A]$. Since, however, $\mathfrak{D}[A]$ is a core of A^{α} (see Lemma A3 of Appendix), (4.19) is extended to all $u\in\mathfrak{D}[A^{\alpha}]$, the inclusion $\mathfrak{D}[A^{\alpha}]\subset\mathfrak{D}[A'^{\alpha}]$ being implied.

Actually we have $\mathfrak{D}[A^{\alpha}]=\mathfrak{D}[A'^{\alpha}]$. To see this, we regard A' as the unperturbed operator and A the perturbed one. Then we have $|\phi[u]-\phi'[u]|\leqq bf[u]\leqq b(1-b)^{-1}f'[u]$ by (4.2). If $2b(1-b)^{-1}<1$, the above result is applicable with A and A' exchanged, so that $\mathfrak{D}[A'^{\alpha}]=\mathfrak{D}[A^{\alpha}]$. If $2b(1-b)\geqq 1$, we consider a family of forms $\phi(t)$, $0\leqq t\leqq 1$, defined by $\phi(t)=\phi+t(\phi'-\phi)$. Then $f(t)=\mathrm{Re}\,\phi(t)$ are mutually comparable by $(1-tb)f[u]\leqq f(t)[u]\leqq(1+tb)f[u]$, and we have $|\phi(t')[u]-\phi(t)[u]|\leqq|t'-t|\,b\,f[u]\leqq|t'-t|\,b(1-b)^{-1}f(t)[u]$. Thus we see that $\mathfrak{D}[A(t')^{\alpha}]=\mathfrak{D}[A(t)^{\alpha}]$ for $|t'-t|<(1-b)/2b$. Hence $\mathfrak{D}[A(t)^{\alpha}]$ must be constant for $0\leqq t\leqq 1$, in particular $\mathfrak{D}[A^{\alpha}]=\mathfrak{D}[A'^{\alpha}]$.

Thus we have proved the following

THEOREM 4.1. *Let $\phi=f+ig$ be a regular sesquilinear form with index β. Let $\phi'=f'+ig'$ be another sesquilinear form such that* (4.1) *holds with $0\leqq b<1$. Then ϕ' is also regular. Let A and A' be the regularly accretive operators associated with ϕ and ϕ', respectively. Then we have $\mathfrak{D}[A^{\alpha}]=\mathfrak{D}[A'^{\alpha}]$ for $0\leqq\alpha<1/2$. Furthermore, there is a constant M_{α}, depending only on α, such that*

$$\|A'^{\alpha}u-A^{\alpha}u\|\leqq(1+\beta)^2M_{\alpha}\frac{2b}{1-2b}\|A^{\alpha}u\|,\qquad u\in\mathfrak{D}[A^{\alpha}], \tag{4.20}$$

provided $b<1/2$. If in particular A has a bounded inverse, the operator $A'^{\alpha}A^{-\alpha}$ is bounded with domain $\mathfrak{H}$, with

$$\|A'^{\alpha}A^{-\alpha}-1\|\leqq(1+\beta)^2M_{\alpha}\frac{2b}{1-2b}\quad \textit{for}\quad b<1/2. \tag{4.21}$$

REMARK. The fact that $\mathfrak{D}[A^{\alpha}]=\mathfrak{D}[A'^{\alpha}]$ can be proved more simply if one uses the Heinz inequality. Since f' and f have the same domain $\mathfrak{D}$, the associated selfadjoint operators H' and H have the property that $\mathfrak{D}[H'^{1/2}]=\mathfrak{D}[H^{1/2}]$ by (2.1). Then it follows from the Heinz inequality (see [**6**, **7**]) that $\mathfrak{D}[H'^{\alpha}]=\mathfrak{D}[H^{\alpha}]$ for $0\leqq\alpha\leqq 1/2$. The desired result then follows immediately from Theorem 3.1. This proof does not assume that $\phi'-\phi$ is small; it suffices to assume that ϕ and ϕ' are both regular and have the same domain.

Suppose now that a family of regular sesquilinear forms $\phi(t)$ is given, with the associated regularly accretive operators $A(t)$, where t is a real or complex parameter. Then Theorem 4.1 shows that any continuity property of $\phi(t)$ is inherited by $A(t)^\alpha$. We have, for example,

THEOREM 4.2. *Let $\phi(t)=f(t)+ig(t)$ be Hölder continuous in the sense that $\mathfrak{D}[\phi(t)]=\mathfrak{D}$ is independent of t and*

$$|\phi(t)[u]-\phi(s)[u]| \leqq M|t-s|^\varepsilon f(s)[u], \qquad u\in\mathfrak{D}, \tag{4.22}$$

for any s, t in the parameter domain. Then $A(t)^\alpha$ has constant domain $\mathfrak{D}_\alpha$ for $0\leqq\alpha<1/2$ and is Hölder continuous in t in the sense that

$$\|A(t)^\alpha u-A(s)^\alpha u\| \leqq M'_\alpha |t-s|^\varepsilon \|A(s)^\alpha u\|, \qquad u\in\mathfrak{D}_\alpha, \tag{4.23}$$

at least for sufficiently small $|t-s|$. If in particular $A(t)$ has bounded inverse for each t, $A(t)^\alpha A(s)^{-\alpha}$ is bounded with

$$\|A(t)^\alpha A(s)^{-\alpha}-1\| \leqq M'_\alpha |t-s|^\varepsilon. \tag{4.24}$$

The most interesting case for application is that in which $\phi(t)$ is analytic in t. We have

THEOREM 4.3. *Let $\phi(t)$ be holomorphic in a domain Δ of the complex t-plane, in the sense that $\mathfrak{D}[\phi(t)]=\mathfrak{D}$ is independent of t and $\phi(t)[u]$ is holomorphic for $t\in\Delta$ for each $u\in\mathfrak{D}$. Furthermore, let $f(t)=\mathrm{Re}\,\phi(t)$ be strictly positive for each t in the sense that $f(t)[u]\geqq\delta(t)\|u\|^2$ with $\delta(t)>0$. Then $A(t)^\alpha$ is holomorphic for $0\leqq\alpha<1/2$ in the sense that $\mathfrak{D}[A(t)^\alpha]=\mathfrak{D}_\alpha$ is independent of t and $A(t)^\alpha u$ is holomorphic for $t\in\Delta$ for each $u\in\mathfrak{D}_\alpha$. $A(t)^\alpha A(s)^{-\alpha}$ and $A(t)^{-\alpha}$ are bounded operator functions holomorphic for $t\in\Delta$ for each fixed s.*

PROOF. Without loss of generality we may assume that $0\in\Delta$. $\mathfrak{D}$ becomes a complete Hilbert space $\mathfrak{H}_0$ by the introduction of the new inner product $((u,v))=f_0[u,v]$ and the corresponding norm $|||u|||\geqq\delta_0^{1/2}\|u\|$, where $f_0=f(0)$ and $\delta_0=\delta(0)$. If $\phi(t)$ is considered a form on $\mathfrak{H}_0$, it is a bounded form depending holomorphically on t, so that it can be expanded into a power series of t near $t=0$:

$$\phi(t)=\sum_{n=0}^{\infty} t^n\phi_n, \tag{4.25}$$

where each ϕ_n is a bounded form on $\mathfrak{H}_0$ with bound majorized by, say, kc^{n-1} with some constants k and c (a consequence of the principle of uniform boundedness). This implies, as in (4.4),

$$\phi_n[u,v]=(T_nH_0^{1/2}u, H_0^{1/2}v), \qquad \|T_n\|\leqq kc^{n-1}, \tag{4.26}$$

where H_0 is the selfadjoint operator associated with f_0 by (2.1). Thus

$$(\phi(t)-\phi(0))[u,v]=(T(t)H_0^{1/2}u, H_0^{1/2}v) \tag{4.27}$$

with

(4.28) $$T(t)=\sum_{n=1}^{\infty}t^n T_n\,.$$

Now substitute $\phi(0)$, $\phi(t)$, $T(t)$, H_0 for ϕ, ϕ', T, H of (4.4), respectively. An examination of the formulas (4.4) to (4.16) then shows that the operator $Z^{(p)}$ becomes a power series with the majorizing series $(1+\beta_0)^2\left(\frac{kt}{1-ct}\right)^p$, where β_0 is the index of $\phi(0)$. Thus we see from (4.18) that $A(t)^\alpha u-A(0)^\alpha u$ has the majorizing series $(1+\beta_0)^2\frac{kt}{1-(k+c)t}\left(\frac{\alpha}{\pi}\tan\pi\alpha\right)^{1/2}\|H_0{}^\alpha u\|$. This result is immediately extended to every $u\in\mathfrak{D}_\alpha$ as before. Thus $A(t)^\alpha u$ is holomorphic near $t=0$. Since $t=0$ is not a distinguished point of Δ, $A(t)^\alpha u$ is holomorphic for $t\in\Delta$. Since $A(t)^{-1}$ are bounded by $\|A(t)^{-1}\|\leqq\delta(t)^{-1}$, $A(t)^{-\alpha}$ are also bounded and the last assertion of the theorem is a simple consequence of the preceding result.

5. The case $\alpha=1/2$.

Most of the theorems obtained above for the fractional powers A^α are concerned with the case $0\leqq\alpha<1/2$. In view of the example given after Theorem 1.1, it is in general impossible to extend them to $\alpha>1/2$, but the question naturally arises whether these results are valid for $\alpha=1/2$. Unfortunately, the questions are open in most of these theorems. But we have at least partial answers to them.

THEOREM 5.1. *Let A be closed and maximal accretive. Then $\mathfrak{D}_{1/2}=\mathfrak{D}[A^{1/2}]\cap\mathfrak{D}[A^{*1/2}]$ is a core of both $A^{1/2}$ and $A^{*1/2}$. $H_{1/2}$ given by* (1.1) *with $\alpha=1/2$ is selfadjoint.*

PROOF. I. Set

(5.1) $$B_n=A^{1/2}(1+n^{-1}A^{1/2})^{-1}\,,\qquad n=1,2,3,\cdots.$$

Since $A^{1/2}$ is closed and maximal accretive, B_n are defined on $\mathfrak{H}$ and bounded with $\|B_n\|\leqq n$ (cf. (1.14)). Hence we may write

(5.2) $$B_n=P_n+iQ_n\,,\quad P_n^*=P_n\,,\quad Q_n^*=Q_n\,.$$

That $A^{1/2}$ is accretive implies the same for B_n (cf. (1.14)) so that $P_n\geqq0$. Actually we have, more strongly,

(5.3) $$0\leqq P_1\leqq P_2\leqq P_3\leqq\cdots.$$

To see this we note that, for $m\leqq n$,

(5.4) $$B_n-B_m=(m^{-1}-n^{-1})A(1+n^{-1}A^{1/2})^{-1}(1+m^{-1}A^{1/2})^{-1}\,,$$

$$((B_n-B_m)u,u)=(m^{-1}-n^{-1})(Av,(1+m^{-1}A^{1/2})(1+n^{-1}A^{1/2})v)$$

$$= (m^{-1}-n^{-1})[(Av, v)+(m^{-1}+n^{-1})(Av, A^{1/2}v) + m^{-1}n^{-1}\|Av\|^2],$$

with $v=(1+n^{-1}A^{1/2})^{-1}(1+m^{-1}A^{1/2})^{-1}u$. But $\text{Re}(Av, v)\geqq 0$ and $\text{Re}(Av, A^{1/2}v)\geqq 0$ since A and $A^{1/2}$ are accretive. Hence $((P_n-P_m)u, u)=\text{Re}((B_n-B_m)u, u)\geqq 0$.

II. As is well known (see, for example, Kato [7, 8]), (5.3) implies

(5.5) $$\lambda^{-1}\geqq(\lambda+P_1)^{-1}\geqq(\lambda+P_2)^{-1}\geqq\cdots, \qquad \geqq 0,$$

and, therefore, there is a bounded selfadjoint operator R, $0\leqq R\leqq\lambda^{-1}$, depending on λ, such that

(5.6) $$\text{strong}\lim_{n\to\infty}(\lambda+P_n)^{-1}=R.$$

We shall show that R is invertible. To this end we note that

$$(\lambda+B_n)^{-1}=(\lambda+P_n)^{-1/2}(1+iQ_n')^{-1}(\lambda+P_n)^{-1/2}$$

$$\text{with}\quad Q_n'=(\lambda+P_n)^{-1/2}Q_n(\lambda+P_n)^{-1/2},$$

$$\|(\lambda+B_n)^{-1}u\|\leqq\lambda^{-1/2}\|(\lambda+P_n)^{-1/2}u\|$$

$$(\text{since}\quad \|(1+iQ_n')^{-1}\|\leqq 1 \qquad \text{by}\quad Q_n'^*=Q_n')$$

and therefore

(5.7) $$\limsup_{n\to\infty}\|(\lambda+B_n)^{-1}u\|\leqq\lambda^{-1/2}\|R^{1/2}u\|;$$

note that (5.6) implies $\text{strong}\lim_{n\to\infty}(\lambda+P_n)^{-1/2}=R^{1/2}$. On the other hand, (5.1) gives $B_nv\to A^{1/2}v$ if $v\in\mathfrak{D}[A^{1/2}]$. Hence

(5.8) $$(\lambda+B_n)^{-1}u-(\lambda+A^{1/2})^{-1}u=(\lambda+B_n)^{-1}(A^{1/2}-B_n)(\lambda+A^{1/2})^{-1}u\to 0$$

for every $u\in\mathfrak{H}$ (note that $\|(\lambda+B_n)^{-1}\|\leqq\lambda^{-1}$).

It follows from (5.7) and (5.8) that $\|(\lambda+A^{1/2})^{-1}u\|\leqq\lambda^{-1/2}\|R^{1/2}u\|$. Thus $Ru=0$ implies $R^{1/2}u=0$ and so $(\lambda+A^{1/2})^{-1}u=0$, $u=0$. This shows that R is invertible. Set $P=R^{-1}-\lambda$ so that $R=(\lambda+P)^{-1}$. P is selfadjoint with $P\geqq 0$, since $0\leqq R\leqq\lambda^{-1}$. (That P is independent of λ will be seen later. For the moment λ is fixed.) We have thus proved that

(5.9) $$(\lambda+P_n)^{-1}\geqq(\lambda+P)^{-1},\quad (\lambda+P_n)^{-1}\to(\lambda+P)^{-1}\quad \text{strongly}.$$

III. We next prove that

(5.10) $$\|(\lambda+P_n)^{1/2}(\lambda+P)^{-1/2}\|\leqq 1,$$

(5.11) $$\text{strong}\lim_{n\to\infty}(\lambda+P_n)^{1/2}(\lambda+P)^{-1/2}=1.$$

To this end, set $X_n=(\lambda+P_n)^{1/2}(\lambda+P)^{-1/2}$. (5.9) shows that $\|(\lambda+P)^{-1/2}u\|\leqq\|(\lambda+P_n)^{-1/2}u\|$ or $\|X_n^*\|\leqq 1$. Hence $\|X_n\|\leqq 1$, which proves (5.10). Furthermore, we have $\|(X_n^*-1)(\lambda+P)^{-1/2}u\|=\|X_n^*[(\lambda+P)^{-1/2}u-(\lambda+P_n)^{-1/2}u]\|\to 0$ (see the remark after (5.7)). Since the range of $(\lambda+P)^{-1/2}$ is dense and $\|X_n^*\|\leqq 1$, it follows that $X_n^*\to 1$ strongly. This implies that $X_n\to 1$ weakly and, in view of $\|X_n\| \leqq 1$, even strongly. This proves (5.11).

IV. B_n^2 is accretive, for $B_n^2 = A(1+n^{-1}A^{1/2})^{-2}$ and $\mathrm{Re}(B_n^2 u, u) \geqq 0$ follows as in (5.4). Since $B_n^2 = (P_n+iQ_n)^2 = P_n^2 - Q_n^2 + i(P_nQ_n + Q_nP_n)$, we have $P_n^2 \geqq Q_n^2$, in other words,

$$\|Q_n u\| \leqq \|P_n u\| . \tag{5.12}$$

Hence

$$\lambda + B_n = \lambda + P_n + iQ_n = (1+S_n)(\lambda + P_n), \quad \text{with} \quad S_n = iQ_n(\lambda+P_n)^{-1}, \quad \|S_n\| \leqq 1 . \tag{5.13}$$

$\|S_n\| \leqq 1$ follows from (5.12): $\|S_n u\| = \|Q_n(\lambda+P_n)^{-1}u\| \leqq \|P_n(\lambda+P_n)^{-1}u\| \leqq \|u\|$. Therefore, we can choose a subsequence $\{n'\}$ of $1, 2, 3, \cdots$ such that weak $\lim S_n' = S$ exists. Writing (5.13) in the form $(\lambda+P_n)^{-1} = (\lambda+B_n)^{-1}(1+S_n)$ and taking the adjoint, we have $(\lambda+P_n)^{-1} = (1+S_n^*)(\lambda+B_n^*)^{-1}$. We now let $n \to \infty$ along the subsequence $\{n'\}$. Noting (5.9) and that $(\lambda+B_n^*)^{-1} \to (\lambda+A^{*1/2})^{-1}$ strongly for the same reason as in (5.8), we obtain $(\lambda+P)^{-1} = (1+S^*)(\lambda+A^{*1/2})^{-1}$. Reverting to the adjoint, this gives $(\lambda+P)^{-1} = (\lambda+A^{1/2})^{-1}(1+S)$. This implies

$$(\lambda+A^{1/2})(\lambda+P)^{-1} = 1+S, \quad \mathfrak{D}[A^{1/2}] \supset \mathfrak{D}[P] . \tag{5.14}$$

Incidentally, this shows that the weak limit S of $\{S_n'\}$ is independent of the choice of the subsequence $\{n'\}$, so that the original sequence $\{S_n\}$ is itself weakly convergent to S.

Let $u \in \mathfrak{D}[P] \subset \mathfrak{D}[A^{1/2}]$. Then we have

$$\begin{aligned} \mathrm{Re}((\lambda+A^{1/2})u, u) &= \lim \mathrm{Re}((\lambda+B_n)u, u) = \lim ((\lambda+P_n)u, u) \\ &= \lim \|(\lambda+P_n)^{1/2}u\|^2 = \lim \|X_n(\lambda+P)^{1/2}u\|^2 \\ &= \|(\lambda+P)^{1/2}u\|^2 = ((\lambda+P)u, u) \end{aligned}$$

where X_n is as above. This implies that $\mathrm{Re}(A^{1/2}u, u) = (Pu, u)$. Since the selfadjoint operator P is determined by the values of (Pu, u) for all $u \in \mathfrak{D}[P]$, it follows that P is independent of λ used in its definition.

It follows also from (5.14) that $(\lambda+A^{1/2})u = (1+S)(\lambda+P)u$ for $u \in \mathfrak{D}[P]$. Hence $A^{1/2}u = Pu + iQu$, where we set $iQ = S(\lambda+P)$ with $\mathfrak{D}[Q] = \mathfrak{D}[P]$. Q is also independent of λ and the above result $\mathrm{Re}(A^{1/2}u, u) = (Pu, u)$ shows that Q is symmetric. Since $\|S\| \leqq 1$ by (5.13), we have $\|Qu\| \leqq \|(\lambda+P)u\|$ and, letting $\lambda \to 0$, we have $\|Qu\| \leqq \|Pu\|$. Thus we have

$$A^{1/2} \supset P + iQ, \quad \|Qu\| \leqq \|Pu\| . \tag{5.15}$$

V. Applying the same arguments as above to $B_n^* = P - iQ_n$, we arrive at the result $A^{*1/2} \supset P - iQ$. Combined with (5.15), this gives $A^{1/2} + A^{*1/2} \supset 2P$. Here we must have equality instead of inclusion, for the left member is symmetric and the right member selfadjoint. Hence

$$P = \frac{1}{2}(A^{1/2} + A^{*1/2}) = H_{1/2}, \quad \mathfrak{D}[P] = \mathfrak{D}[A^{1/2}] \cap \mathfrak{D}[A^{*1/2}] = \mathfrak{D}_{1/2}, \tag{5.16}$$

see (1.1).

Finally we show that $\mathfrak{D}[P]$ is a core of $A^{1/2}$. This is equivalent to that $(\lambda+A^{1/2})\mathfrak{D}[P]$ is dense in $\mathfrak{H}$ for $\lambda>0$. Since $\mathfrak{D}[P]$ is the range of $(\lambda+P)^{-1}$, it suffices by (5.14) to show that $1+S$ has a dense range, or, that $1+S^*$ has nullity zero. Since $\|S\|\leqq 1$, this is the case if and only if $1+S$ has nullity zero. But this is obvious from (5.14). With this result, the proof of Theorem 5.1 is complete.

REMARK 1. We do not know whether or not $\mathfrak{D}[A^{1/2}]=\mathfrak{D}[A^{*1/2}]$ in Theorem 5.1. This is perhaps not true in general. But the question is open even when A is regularly accretive. In this case it appears reasonable to suppose that both $\mathfrak{D}[A^{1/2}]$ and $\mathfrak{D}[A^{*1/2}]$ coincide with $\mathfrak{D}[H^{1/2}]=\mathfrak{D}[\phi]$, where H is the real part of A and ϕ is the regular sesquilinear form which defines A according to Theorem 2.1. But all that we know are $\mathfrak{D}[\phi]\supset\mathfrak{D}[A]\subset\mathfrak{D}[A^{1/2}]\supset\mathfrak{D}[P]$ and a similar chain of inclusions with A replaced by A^*.

REMARK 2. If $A=H$ is selfadjoint in Theorem 5.1, the question raised above is answered in the affirmative, for we have $\mathfrak{D}[\phi]=\mathfrak{D}[H^{1/2}]$, see (2.1). The question is still open, however, whether or not Theorems 4.2 and 4.3 are true with $\alpha=1/2$ when $A(t)$ are selfadjoint for real t, although it is true that $\mathfrak{D}[A(t)^{1/2}]$ is independent of t as long as t is real. Thus it must be stated that our knowledge is quite unsatisfactory regarding the case $\alpha=1/2$.

Under these circumstances, it would be of some interest to consider the problem in a *finite-dimensional* Hilbert space. Then there is no problem concerning the domains of $A^{1/2}$ and $A^{*1/2}$, but all the same we can ask whether or not they are comparable. The answer turns out to be in the affirmative, but the constants in the estimates apparently depend on the dimensionality of the space. We have namely

THEOREM 5.2. *Let A be an accretive operator in a finite-dimensional Hilbert space $\mathfrak{H}$*, dim $\mathfrak{H}=m<\infty$. *Then we have*

$$\|A^{*1/2}u\|\leqq 2^m\|A^{1/2}u\|, \qquad u\in\mathfrak{H}, \tag{5.17}$$

and a similar inequality with A and A^* exchanged.

REMARK. It is not known whether the factor 2^m can be replaced by a constant independent of m. If this were possible, $A^{1/2}$ and $A^{*1/2}$ would be comparable in the general infinite-dimensional case. Otherwise, it would follow (by considering an appropriate direct sum of operators) that $A^{1/2}$ and $A^{*1/2}$ are not necessarily comparable in the general case.

PROOF. Again we may assume that A^{-1} exists; the general case can be dealt with by considering $A+\varepsilon$ and going to the limit $\varepsilon\to 0$. Then $H_{1/2}{}^{-1}$ exists by the proof of Theorem 1.1 and $A^{1/2}=H_{1/2}+iK_{1/2}=(1+iB)H_{1/2}$ with $B=K_{1/2}H_{1/2}{}^{-1}$, where $\|B\|\leqq 1$ by (1.13). Now all the eigenvalues of B are real, for B is the product of a symmetric operator $K_{1/2}$ with a positive-definite symmetric operator $H_{1/2}$. Hence $|\det(1+iB)|\geqq 1$ and (see Lemma 1 of Appendix of Kato [**10**])

(5.18) $$\|(1+iB)^{-1}\|\leqq\|1+iB\|^{m-1}/|\det(1+iB)|\leqq 2^{m-1}.$$

Since $A^{*1/2}=(1-iB)H_{1/2}$ as above, we have $A^{*1/2}A^{-1/2}=(1-iB)(1+iB)^{-1}$ and $\|A^{*1/2}A^{-1/2}\|\leqq\|1-iB\|\,\|(1+iB)^{-1}\|\leqq 2^{m}$, which is equivalent to (5.17).

Appendix

We collect here several lemmas to be used in the text, mostly concerned with the fractional powers A^{α} of a linear operator A in a *Banach space* $\mathfrak{X}$. Such a fractional power was defined, for example, in [**9**].

For convenience we begin with a brief summary of the results of [**9**]. A linear operator A in $\mathfrak{X}$ is said to be of type (ω, M) if i) A is densely defined and closed, ii) the resolvent set of $-A$ contains the sector $|\arg\lambda|<\pi-\omega$ of the complex plane and $\lambda(\lambda+A)^{-1}$ is uniformly bounded in each smaller sector $|\arg\lambda|<\pi-\omega-\varepsilon$, $\varepsilon>0$, with $\|\lambda(\lambda+A)^{-1}\|\leqq M$ for $\lambda>0$.

If $\mathfrak{X}$ is a Hilbert space, A is of type $(\pi/2, 1)$ if and only if A is closed and maximal accretive.

If A is of type (ω, M), the fractional power A^{α}, $0<\alpha<1$, can be defined indirectly by

(A1) $$(\lambda+A^{\alpha})^{-1}=\frac{\sin\pi\alpha}{\pi}\int_0^{\infty}\frac{\mu^{\alpha}}{\lambda^2+2\lambda\mu^{\alpha}\cos\pi\alpha+\mu^{2\alpha}}(\mu+A)^{-1}d\mu,$$

which is valid at least for $\lambda>0$. A^{α} is an operator of type $(\alpha\omega, M)$. If $\alpha\omega<\pi/2$, $-A^{\alpha}$ is the infinitesimal generator of an *analytic* semi-group, that is, the semi-group $T_{t,\alpha}=\exp(-tA^{\alpha})$ has an analytic continuation to the sector $|\arg t|<\frac{\pi}{2}-\alpha\omega$, $T_{t,\alpha}$ and $tdT_{t,\alpha}/dt$ being uniformly bounded for $|\arg t|<\frac{\pi}{2}-\alpha\omega-\varepsilon$, $\varepsilon>0$.

If we make the additional assumption that A^{-1} is bounded, (A1) is equivalent to the Dunford integral

(A2) $$(\lambda+A^{\alpha})^{-1}=-\frac{1}{2\pi i}\int_C(\lambda+z^{\alpha})^{-1}(z-A)^{-1}dz,$$

where the integration path C runs in the resolvent set of A from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$, $\omega<\theta\leqq\pi$, avoiding the negative real axis and 0. In this case (A1) and (A2) are valid also for $\lambda=0$, and various formulas of "operational calculus" can be deduced by the manipulation of Dunford integrals.

We now prove several lemmas that are not stated in [**9**]. *In what follows it is always assumed that A is of type $\left(\frac{\pi}{2}, M\right)$ and that* $0<\alpha<1$. We note that $A+\varepsilon$ is also of type $\left(\frac{\pi}{2}, M\right)$ if $\varepsilon>0$; this is a direct consequence of the assumption.

LEMMA A1. *For any* $\varepsilon>0$ *we have*

(A3) $$\|(1+\varepsilon A)^{-\alpha}\|\leqq M.$$

Note that $1+\varepsilon A$ *is also of type* $\left(\frac{\pi}{2}, M\right)$ *so that* $(1+\varepsilon A)^{\alpha}$ *exists.*

PROOF. Since $(1+\varepsilon A)^{-1}$ is bounded, (A1) is valid for $\lambda=0$ if A is replaced by $1+\varepsilon A$. Since $(\mu+1+\varepsilon A)^{-1}\leqq M(\mu+1)^{-1}$, it follows that

$$\|(1+\varepsilon A)^{-\alpha}\|\leqq\frac{\sin\pi\alpha}{\pi}\int_0^\infty\mu^{-\alpha}M(\mu+1)^{-1}d\mu=M.$$

LEMMA A2. *We have* $\mathfrak{D}[(A+\varepsilon)^{\alpha}]=\mathfrak{D}[A^{\alpha}]$ *and*

(A4) $$\|(A+\varepsilon)^{\alpha}u-A^{\alpha}u\|\leqq c\ \varepsilon^{\alpha}\|u\|,\qquad u\in\mathfrak{D}[A^{\alpha}],$$

where the constant c *depends only on* α *and* M.

PROOF. For brevity we write $A_\varepsilon=A+\varepsilon$ in this proof. We have $A_\varepsilon^{-\alpha}A_\varepsilon^{-(1-\alpha)}=A_\varepsilon^{-(1-\alpha)}A_\varepsilon^{-\alpha}=A_\varepsilon^{-1}$; this is a result of the "operational calculus" mentioned above (note that A_ε^{-1}, $A_\varepsilon^{-\alpha}$ etc. are bounded for $\varepsilon>0$). Hence follows that $A_\varepsilon^{-(1-\alpha)}A_\varepsilon\subset A_\varepsilon^{\alpha}$. Now (A1) is true for $\lambda=0$ if A is replaced by A_ε. Replacing α by $1-\alpha$ and applying both members of (A1) to $A_\varepsilon u$ where $u\in\mathfrak{D}[A]$, we thus obtain

(A5) $$A_\varepsilon^{\alpha}u=\frac{\sin\pi\alpha}{\pi}\int_0^\infty\mu^{\alpha-1}(\mu+A_\varepsilon)^{-1}A_\varepsilon u d\mu,\quad\varepsilon>0,\qquad u\in\mathfrak{D}[A].$$

Subtracting from (A5) a similar expression with ε replaced by η, $0<\eta<\varepsilon$, we have

$$A_\varepsilon^{\alpha}u-A_\eta^{\alpha}u=\frac{\sin\pi\alpha}{\pi}\Big[\int_0^\delta\mu^{\alpha-1}(\mu+A_\varepsilon)^{-1}A_\varepsilon u\,d\mu-\int_0^\delta\mu^{\alpha-1}(\mu+A_\eta)^{-1}A_\eta u\,d\mu$$
$$+(\varepsilon-\eta)\int_\delta^\infty\mu^{\alpha}(\mu+A_\varepsilon)^{-1}(\mu+A_\eta)^{-1}u\,d\mu\Big].$$

Noting that $\|(\mu+A_\varepsilon)^{-1}A_\varepsilon u\|\leqq(1+M)\|u\|$ and $\|(\mu+A_\varepsilon)^{-1}\|\leqq M\mu^{-1}$, this gives

$$\|A_\varepsilon^{\alpha}u-A_\eta^{\alpha}u\|\leqq\frac{\sin\pi\alpha}{\pi}\Big[2(1+M)\int_0^\delta\mu^{\alpha-1}d\mu+M^2(\varepsilon-\eta)\int_\delta^\infty\mu^{\alpha-2}d\mu\Big]\|u\|$$
$$=\frac{\sin\pi\alpha}{\pi}[2(1+M)\alpha^{-1}\delta^{\alpha}+M^2(\varepsilon-\eta)(1-\alpha)^{-1}\delta^{\alpha-1}]\|u\|.$$

Taking $\delta=(\varepsilon-\eta)M^2/2(1+M)$, we obtain

$$\|A_\varepsilon^{\alpha}u-A_\eta^{\alpha}u\|\leqq c(\varepsilon-\eta)^{\alpha}\|u\|,\ \ 0<\eta<\varepsilon,\ \ u\in\mathfrak{D}[A];$$

$c=2^{1-\alpha}M^{2\alpha}(1+M)^{1-\alpha}\sin\pi\alpha/\pi\alpha(1-\alpha)$ depends only on α and M. This shows that $\lim_{\varepsilon\to 0}A_\varepsilon^{\alpha}u=Bu$ exists for $u\in\mathfrak{D}[A]$, and that

(A6) $$\|A_\varepsilon^{\alpha}u-Bu\|\leqq c\varepsilon^{\alpha}\|u\|,\quad\varepsilon>0,\qquad u\in\mathfrak{D}[A].$$

Now $\mathfrak{D}[A]$ is a *core* of A_ε^{α}; by this we mean that the closure of the restriction of A_ε^{α} to $\mathfrak{D}[A]$ is A_ε^{α} itself. To see this, it suffices to note that, for any $v\in\mathfrak{D}[A_\varepsilon^\alpha]$, the sequence $v_n=(1+n^{-1}A_\varepsilon)^{-1}v\in\mathfrak{D}[A_\varepsilon]=\mathfrak{D}[A]$ has the property that $v_n\to v$ and $A_\varepsilon^\alpha v_n=(1+n^{-1}A_\varepsilon)^{-1}A_\varepsilon^\alpha v\to A_\varepsilon^\alpha v$, $n\to\infty$. Here we used the relation $A_\varepsilon^\alpha(1+n^{-1}A_\varepsilon)^{-1}v=(1+n^{-1}A_\varepsilon)^{-1}A_\varepsilon^\alpha v$. This follows from $(1+n^{-1}A_\varepsilon)^{-1}A_\varepsilon^{-\alpha}w=A_\varepsilon^{-\alpha}(1+n^{-1}A_\varepsilon)^{-1}w$ with $w=A_\varepsilon^\alpha v$, and this is in turn a consequence of the "operational calculus" (note that $A_\varepsilon^{-\alpha}$ is bounded).

The fact that $\mathfrak{D}[A]$ is a core of A_ε^α, combined with (A6), shows that the closure B^{**} of B exists (B was defined only on $\mathfrak{D}[A]$) and has the same domain with A_ε^α. At the same time the inequality (A6) is extended to all $u\in\mathfrak{D}[B^{**}]$, with B replaced by B^{**}. Thus $\mathfrak{D}[A_\varepsilon^\alpha]=\mathfrak{D}[B^{**}]$ is independent of ε and $A_\varepsilon^\alpha u\to B^{**}u$ uniformly for $\|u\|\leq 1$, $u\in\mathfrak{D}[B^{**}]$. From this it follows easily that $(B^{**}+\lambda)^{-1}$ exists as a bounded operator with domain $\mathfrak{H}$ for $\lambda>0$ and that $(A_\varepsilon^\alpha+\lambda)^{-1}\to(B^{**}+\lambda)^{-1}$ for $\varepsilon\to 0$ in the uniform operator topology. Since, however, we know that $(A_\varepsilon^\alpha+\lambda)^{-1}\to(A^\alpha+\lambda)^{-1}$ (see [**9**]), it follows that $B^{**}=A^\alpha$. This completes the proof of Lemma A2.

LEMMA A3. *$\mathfrak{D}[A]$ is a core of A^α.*

PROOF. The proof is contained in the proof of Lemma A2, for $A^\alpha=B^{**}$ is the closure of B and B was the restriction of $B^{**}=A^\alpha$ to $\mathfrak{D}[A]$.

LEMMA A4. *For each $u\in\mathfrak{D}[A]$ we have*

$$A^\alpha u=\frac{\sin\pi\alpha}{\pi}\int_0^\infty\lambda^{\alpha-1}A(\lambda+A)^{-1}u\,d\lambda\,, \tag{A7}$$

the integral being absolutely convergent.

PROOF. Since $\|A(A+\lambda)^{-1}\|\leq 1+M$, the integral is absolutely convergent near $\lambda=0$. If $u\in\mathfrak{D}[A]$, $\|A(A+\lambda)^{-1}u\|=\|(A+\lambda)^{-1}Au\|\leq M\lambda^{-1}\|Au\|$ so that the integral is absolutely convergent also for $\lambda\to\infty$. Furthermore, we know that (A7) is valid if A is replaced by $A_\varepsilon=A+\varepsilon$, see (A5). Thus (A7) follows from (A5) by going to the limit $\varepsilon\to 0$, noting Lemma A2 and that $A_\varepsilon(\lambda+A_\varepsilon)^{-1}u\to A(\lambda+A)^{-1}u$ (dominated convergence).

REMARK. The author does not know whether (A7) is true for all $u\in\mathfrak{D}[A^\alpha]$, though it could be proved that it is true for $u\in\mathfrak{D}[A^\beta]$ with $\alpha<\beta\leq 1$. But the converse is true in a certain sense, as is seen from

LEMMA A5. *Let $\mathfrak{X}$ be reflexive. If $u\in\mathfrak{X}$ is such that*

$$\text{weak}\lim_{R\to\infty}\int_0^R\lambda^{\alpha-1}A(\lambda+A)^{-1}u\,d\lambda=v \tag{A8}$$

exists, then $u\in\mathfrak{D}[A^\alpha]$ and v is equal to $\frac{\pi}{\sin\pi\alpha}A^\alpha u$.

PROOF. If $\mathfrak{X}$ is reflexive, the adjoint A^* of A is an operator (in the adjoint space $\mathfrak{X}^*$) of type $\left(\frac{\pi}{2}, M\right)$, so that $A^{*\alpha}$ is defined and it is obvious from (A1) that $(\lambda+A^{*\alpha})^{-1}=(\lambda+A^\alpha)^{-1*}$ for $\lambda>0$. Hence

$$A^{*\alpha}=A^{\alpha *}\,. \tag{A9}$$

Let $f \in \mathfrak{D}[A^*]$. Then we have by Lemma A4 an expression of $A^{*\alpha}f$ given by (A7) with A and u replaced by A^* and f respectively, so that

$$(u, A^{*\alpha}f) = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1}(u, A^*(\lambda+A^*)^{-1}f)\, d\lambda \tag{A10}$$
$$= \frac{\sin \pi\alpha}{\pi} \lim_{R\to\infty} \int_0^R \lambda^{\alpha-1}(A(\lambda+A)^{-1}u, f)\, d\lambda$$
$$= \frac{\sin \pi\alpha}{\pi} (v, f)\,.$$

Since $\mathfrak{D}[A^*]$ is a core of $A^{*\alpha} = A^{\alpha *}$ by Lemma A3, (A10) shows that $u \in \mathfrak{D}[A^{\alpha **}] = \mathfrak{D}[A^\alpha]$ and that $A^\alpha u = \frac{\sin \pi\alpha}{\pi} v$.

The following lemmas are concerned with operators in a Hilbert space.

Lemma A6. *Let A be a closed, maximal accretive operator in a Hilbert space $\mathfrak{X}$ and let* $\mathrm{Re}\,(Au, u) \geqq \delta(u, u)$ *for some $\delta > 0$ and all $u \in \mathfrak{D}[A]$. Then* $\mathrm{Re}\,(A^\alpha u, u) \geqq \delta^\alpha(u, u)$ *for all $u \in \mathfrak{D}[A^\alpha]$.*

Proof. It suffices to prove the assertion for $u \in \mathfrak{D}[A]$, for $\mathfrak{D}[A]$ is a core of A^α and, therefore, the result can be extended to all $u \in \mathfrak{D}[A^\alpha]$ by continuity. If $u \in \mathfrak{D}[A]$, we have the expression (A7) for $A^\alpha u$. Thus it suffices to show that

$$\mathrm{Re}\,(A(\lambda+A)^{-1}u, u) \geqq \delta(\lambda+\delta)^{-1}(u, u)\,. \tag{A11}$$

Since $A(\lambda+A)^{-1} = 1 - \lambda(\lambda+A)^{-1}$, (A11) follows from $\mathrm{Re}((\lambda+A)^{-1}u, u) \leqq (\lambda+\delta)^{-1}(u, u)$. The last inequality is obvious since we have $\mathrm{Re}\,((\lambda+A)u, u) \geqq (\lambda+\delta)(u, u)$, which implies $\| (\lambda+A)^{-1} \| \leqq (\lambda+\delta)^{-1}$.

Lemma A7. *Let $\mathfrak{X}, \mathfrak{X}'$ be two Hilbert spaces, let H, H' be nonnegative selfadjoint operators acting in $\mathfrak{X}, \mathfrak{X}'$ respectively and let B_λ be a bounded linear operator on $\mathfrak{X}$ to $\mathfrak{X}'$, depending on a parameter $\lambda > 0$ continuously (in the strong sense, say) and uniformly bounded by $\| B_\lambda \| \leqq M$. Then for any $u \in \mathfrak{D}[H^\alpha]$ $(0 \leqq \alpha < 1/2)$,*

$$w' = \text{strong} \lim_{R\to\infty} \int_{1/R}^R \frac{H'^{1/2}}{H'+\lambda} B_\lambda \frac{H^{1/2}}{H+\lambda} u \lambda^\alpha d\lambda \in \mathfrak{X}' \tag{A12}$$

exists and

$$\| w' \| \leqq \left(\frac{2\pi\alpha}{\sin 2\pi\alpha}\right)^{1/2} M \| H^\alpha u \|\,. \tag{A13}$$

Remark. If $\alpha > 0$, $\int_{1/R}^R$ in (A12) may be replaced by $\int_0^R$, for the integral is absolutely convergent at $\lambda \to 0$.

Proof. For any $v' \in \mathfrak{X}'$, we have

$$\left| \int_a^b \left(\frac{H'^{1/2}}{H'+\lambda} B_\lambda \frac{H^{1/2}}{H+\lambda} u, v' \right) \lambda^\alpha d\lambda \right|^2 \tag{A14}$$
$$\leqq M^2 \left(\int_a^b \left\| \frac{H^{1/2}}{H+\lambda} u \right\|^2 \lambda^{2\alpha} d\lambda \right) \left(\int_a^b \left\| \frac{H'^{1/2}}{H'+\lambda} v' \right\|^2 d\lambda \right).$$

But it is easily seen that the last factor of (A14) does not exceed $\int_0^\infty \left\| \frac{H'^{1/2}}{H'+\lambda} v' \right\|^2 d\lambda = \| v' \|^2$. Hence

(A15) $$\left\| \int_a^b \frac{H'^{1/2}}{H'+\lambda} B_\lambda \frac{H^{1/2}}{H+\lambda} u \lambda^\alpha d\lambda \right\|^2 \leqq M^2 \int_a^b \left\| \frac{H^{1/2}}{H+\lambda} u \right\|^2 \lambda^{2\alpha} d\lambda$$
$$= M^2 \int_0^\infty \mu d(E_\mu u, u) \int_a^b \frac{\lambda^{2\alpha}}{(\mu+\lambda)^2} d\lambda$$
$$= M^2 \int_0^\infty \mu^{2\alpha} d(E_\mu u, u) \int_{a/\mu}^{b/\mu} \frac{\lambda^{2\alpha}}{(1+\lambda)^2} d\lambda ,$$

where $H = \int_0^\infty \mu dE_\mu$ is the spectral representation of H. Since $\int_0^\infty \frac{\lambda^{2\alpha}}{(1+\lambda)^2} d\lambda = \frac{2\pi\alpha}{\sin 2\pi\alpha} < \infty$ for $0 \leqq \alpha < 1/2$, it follows that the right member of (A15)$\to 0$ for $b \geqq a \to \infty$ by bounded convergence, provided $u \in \mathfrak{D}[H^\alpha]$ so that $\int_0^\infty \mu^{2\alpha} d(E(\mu)u, u) = \| H^\alpha u \|^2 < \infty$. Similar results hold also for $0 < a \leqq b \to 0$. This proves (A12). On letting $a \to 0$, $b \to \infty$ in (A15), and noting the estimates given above, we obtain (A13).

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