# Fractional Quantum Hall Effect and Nonabelian Statistics 

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#### Abstract

It is argued that fractional quantum Hall effect wavefunctions can be interpreted as conformal blocks of two-dimensional conformal field theory. Fractional statistics can be extended to nonabelian statistics and examples can be constructed from conformal field theory. The Pfaffian state is related to the 2D Ising model and possesses fractionally charged excitations which are predicted to obey nonabelian statistics.


## § 1. Introduction

_-Particle statistics and conformal field theory -_
This paper is a brief overview of some aspects of the relationship between the theories of the fractional quantum Hall effect (FQHE) ${ }^{1)}$ and two-dimensional conformal fields (CFT), ${ }^{2}$ ) which has been explored in more detail in Ref. 3). The present section describes general aspects of particle statistics, especially nonabelian statistics; the second section uses the Laughlin wavefunctions as an example of the relation with CFT; the third section presents a case of nonabelian statistics in a FQHE system (the Pfaffian state).

The notion of particle statistics in quantum mechanics usually refers to the action of the permutation group $S_{n}$ on the wavefunction for a collection of $n$ identical, indistinguishable particles: the wavefunction is taken to transform as a definite representation of this group, which interchanges particles. The usual examples are Bose statistics and Fermi statistics, which are the trivial and alternating representations, respectively. A more modern approach prefers to exchange particles along some definite paths in space, and allows for the possibility that the wavefunction be not well-defined when particles coincide, so that intersecting exchange paths must be omitted. Topology then enters the subject, and while it turns out for spatial dimension $d>2$ that the group of exchanges still reduces to $S_{n}$, for $d=2$ the group of topologically distinct exchanges is Artin's braid group $\mathscr{B}_{n}{ }^{4}$ ) The infinite group $\mathscr{B}_{n}$ can be generated by a set of elementary exchanges $B_{i}, i=1, \cdots n-1$ which simply exchange particles $i, i+1$ along a path not enclosing any other particles. The most familiar representation of $\mathscr{B}_{n}$ is fractional statistics, where each $B_{i}$ acts on the wavefunction as $e^{i \theta}$ ( $\theta$ real) and so is a one-dimensional, abelian representation; this includes Bose and Fermi statistics as special cases.

We will use the term "nonabelian statistics" when particle wavefunctions transform as a nonabelian representation of the permutation or (especially) of the braid
group, and we have introduced the term "nonabelions" for such particles. ${ }^{3)}$ In this case, not all the representatives of the $B_{i}$ commute, and must be matrices acting on vector wavefunctions. Thus, even when the positions and quantum numbers of the particles have been specified, the wavefunction is not unique but is a member of a vector space (a subspace of the Hilbert space). The vector space may be finite dimensional but its dimension will grow with the number of particles. Physical observables are invariant under exchanges, so cannot distinguish which state in this space the system is in, just as the phase of the wavefunction cannot be directly measured in the usual (one dimensional) case. In the latter case, differences or changes of phase, and interference effects, can, however, be observed, and analogously, nonabelian statistics can manifest itself in physical effects.

For the permutation group, nonabelian statistics is an old idea known as "parastatistics". It appears, however, that parastatistics cannot be realized in a local theory in $d>2$ in a nontrivial way (the literature containing this result is reviewed in Ref. 5)). The issue is whether the global degeneracy and nonlocal effects of exchanges are compatible with the natural physical idea of locality of interactions, as realized for example in quantum field theory governed by a local Lagrangian density.

An analogous analysis for the braid group in $d=2$ has been carried out in Ref. 5). It involves not only the matrices $B_{i}$ but also the notion of "fusion rules". A theory will usually contain several distinct types of particles, each with its own $B$ matrices, and taking a particle of one type around one of another type will produce a matrix effect like that of $B^{2}$. Fusion of two or more particles into a composite produces either annihilation of the particles or else a particle of some new type with its own statistics properties. The matrix elements for these processes, denoted $F$, describe the fusion rules and will have to satisfy consistency relations involving $B$ 's. For example, taking a third particle around a pair before or after they fuse should give the same result, since the third particle is far from the fusing pair and so cannot distinguish the close pair from their composite (locality). Frohlich et al. ${ }^{5}$ ) conclude that nonabelian statistics can be acceptable in two dimensions. Indeed, local actions are known that produce nonabelian properties: they are Chern-Simons terms for nonabelian gauge fields, and the particle worldlines are represented as Wilson lines. ${ }^{6}$ ) Note that as in the abelian case, the statistics effects can be transferred between the wavefunctions and the Hamiltonian by a singular gauge transformation; the general discussion above was for the gauge choice where statistics is exhibited in the wavefunction, i.e. no Chern-Simons-type vector potentials in the Hamiltonian.

As a partial example of nonabelian statistics, we present the following system. In the example, we may create particles $\sigma$, but only in even numbers. The $\sigma$ particles have no internal quantum numbers. For two particles, the vector space for fixed positions $z_{1}, z_{2}$ is one dimensional, and the wavefunction is $\left(z_{1}-z_{2}\right)^{-1 / 8}$, so one might imagine that we have abelian fractional statistics $\theta / \pi=-1 / 8$. However, for four particles, the vector space is two-dimensional. If we place particles at 0,1 and $\infty$ (which in this example can always be done through a global conformal transformation), the wavefunctions are functions only of the remaining coordinate $z$, and (choosing a basis) are

$$
\psi_{ \pm}(z)=(z(1-z))^{-1 / 8} \sqrt{1 \pm \sqrt{1-z}}
$$

In the $z$-plane, there are branch points at $z=0,1$. As $z$ is analytically continued around $z=1$, we see that $\psi_{ \pm}$are interchanged. This operation corresponds to two exchanges of the particle at $z$ with that at 1 , i.e. to $B^{2}$. Other double exchanges involving $z$ are diagonal in this basis. To exhibit $B$ itself we would have to perform a conformal transformation to obtain $z \mapsto 1,1 \mapsto z$. Thus we have a single multibranched function whose two sheets are linearly independent functions of $z$, and so the $\sigma$ particles behave as nonabelions. For general number $2 n$ of $\sigma$ particles the wavefunctions form a $2^{n-1}$-dimensional space.

The example arises from a particular conformal field theory (CFT), the twodimensional Ising model at its critical point. We next describe some general features of CFT. ${ }^{7)}$ In a $1+1$-dimensional system with short range interactions (or a local lagrangian) at a critical point, one has not only scale invariance but also conformal invariance, an infinite-parameter group under which correlation functions transform covariantly. Exactly at the critical point (so that corrections due to slowly decaying irrelevant or marginal operators are omitted) the correlation functions of a collection of fields $\left\{\phi_{i_{r}}\left(\boldsymbol{x}_{r}\right)\right\}$ can be split in the form (taking the "diagonal" case for simplicity):

$$
\left\langle\prod_{r=1}^{n} \phi_{i r}\left(\boldsymbol{x}_{r}\right)\right\rangle=\sum_{p}\left|\mathscr{F}_{p ; i_{1} \cdots i_{n}}\left(z_{1}, \cdots z_{n}\right)\right|^{2}
$$

where $z_{r}=x_{r}+i y_{r}$. The conformal block functions $\mathscr{F}_{p}$ are multibranched functions, analytic in their arguments, except possibly when two $z$ 's coincide. The variable $p$ labelling different functions runs over a finite set in a rational CFT. As the $z$ 's are varied so as to exchange some $\phi_{i}$ 's, the functions $\mathscr{F}_{p}$ are analytically continued to different sheets, but can be expressed as $z$ independent linear combinations of the original functions through some braiding matrices $B$ as we saw above. In this way, the correlation function can be single valued.

Another operation that can be performed on the correlation functions is the operator product expansion. As the arguments $z_{1}, z_{2}$ of two fields approach one another, the operators merge into a linear combination of single operators:

$$
\phi_{i}(z) \phi_{j}(w) \sim \sum_{k} C_{i j}^{k}(z-w) \phi_{k}(w)
$$

as $z \rightarrow w$, where $\phi_{k}$ is some new field of type $k$ and $C_{i j}^{k}(z-w)$ is a singular coefficient function. This operation can be used to define some new matrices $F$, the fusion matrix that describes which fields $k$ appear in the product of $i$ and $j$. In Ref. 8) the consistency conditions that must be satisfied by $B, F$ were analyzed; these same conditions emerged later in the work of Frohlich et al. Thus CFT provides many examples of nonabelian statistics, when conformal blocks are interpreted as wavefunctions. In the following we show that this idea applies directly to FQHE states.

## § 2. Example <br> - Laughlin states -

In this section we show that both the fractional statistics of quasiparticles in the

Laughlin states ${ }^{9}$ and the actual wavefunctions are related to CFT conformal blocks. The construction given here is from Ref. 3), but parts of it have been obtained independently by others.

In 1+1-dimensional Euclidean spacetime, define a free scalar field by its correlator,

$$
\left\langle\varphi(z) \varphi\left(z^{\prime}\right)\right\rangle=-\log \left(z-z^{\prime}\right),
$$

where the $\log$ is complex, so the field creates right-moving excitations only. All correlators can be obtained from this one using Wick's theorem. Now consider the function

$$
\left\langle\prod_{i=1}^{N} e^{i \sqrt{q} \varphi\left(z_{i}\right)} e^{-i \sqrt{q} \int d^{2} z^{\prime} \bar{\rho} \varphi\left(z^{\prime}\right)}\right\rangle
$$

where $\bar{\rho}=1 / 2 \pi q, q$ is an integer, and the integral is taken over a disk of area $2 \pi q N$ centered at the origin. We expand and contract using $(2 \cdot 1)$. Each exponential is assumed normal ordered, i.e., $\varphi$ 's from the same exponential are not to be contracted together. The result is ${ }^{10)}$

$$
\prod_{i<j}\left(z_{i}-z_{j}\right)^{q} \exp \left(-\frac{1}{2 \pi} \sum_{i} \int d^{2} z^{\prime} \log \left(z_{i}-z^{\prime}\right)\right)
$$

In the last factor, the real part of the $\log$ gives $\exp \left(-1 / 4 \sum_{i}\left|z_{i}\right|^{2}\right)$ for $z_{i}$ inside the disk. Apart from the remaining phase, we now recognize the function as Laughlin's wavefunction for particles in the lowest Landau level at filling factor $\nu=1 / q$. Since the vertex operators $\exp (i \sqrt{q} \varphi(z))$ are associated with Coulomb charges, it is clear that we have a holomorphic version of Laughlin's 2D plasma picture, ${ }^{9)}$ which gives the wavefunction and not just its modulus squared!

The imaginary part of the $\log$ in $(2 \cdot 3)$ winds by $2 \pi d^{2} z^{\prime}$ as each $z_{i}$ goes round each point $z^{\prime}$ in the integration region, so is highly singular. ${ }^{11)}$ It contributes a pure phase to the function, which can be gauged away by an equally singular gauge transformation. This phase describes a uniform magnetic field, which we identify as the physical magnetic field in the FQHE problem. A nice way to see this is to consider not the phase itself but the change as one particle describes a closed loop C. Clearly the exponent changes by

$$
-\frac{1}{2 \pi} \int d^{2} z^{\prime} 2 \pi i
$$

where the integral is over the area enclosed by $C$, so the phase change is $1 / 2 \pi$ times the area of the loop, which is just the Berry phase for adiabatic transport of a particle of charge 1 in a field of strength $1 / 2 \pi$. In all the following equations, the singular phase will be implicitly gauged away.

In a similar way, one can obtain quasihole wavefunctions as (for two quasiholes):

$$
\begin{align*}
& \left\langle e^{(i \sqrt{q}) \varphi(z)} e^{(i / \sqrt{q}) \varphi(w)} \prod_{i=1}^{N} e^{i \sqrt{q} \varphi\left(z_{i}\right)} e^{-i \sqrt{q} \int d^{2} z^{\prime} \bar{\rho} \varphi\left(z^{\prime}\right)}\right\rangle \\
& \quad=(z-w)^{1 / q} \prod_{k}\left(z-z_{k}\right)\left(w-z_{k}\right) \prod_{i<j}\left(z_{i}-z_{j}\right)^{q} \exp \left(-\frac{1}{4} \sum_{i}\left|z_{i}\right|^{2}-\frac{1}{4}|z|^{2}-\frac{1}{4}|w|^{2}\right) .
\end{align*}
$$

The first factor gives explicitly the fractional statistics, $\theta / \pi=1 / q$. Apparently, the constructions from CFT correlators give results in the gauge where all Berry phases -background magnetic field as well as fractional statistics - appear in the wavefunction, not as a vector potential (connection) in the Hamiltonian. This will be important to us in the next section when we move on to nonabelian statistics (adiabatic transport).

We now return to the $2+1$-dimensional world of electrons in two dimensions in a strong magnetic field and describe briefly the order parameter picture of the FQHE, following Ref. 12) (see also Refs. 13) and 14)). In words, a composite of 1 electron and $q$ quasiholes (or "vortices", or "flux quanta") at filling factor $\nu=1 / q$, (i) is a boson; (ii) sees no net effective magnetic field. When both (i) and (ii) are true, the composite can "Bose condense", i.e. have long-range order. For the Laughlin states, (i) and (ii) follow from the calculations in Ref. 15).

More formally, let $\phi^{\dagger}$ create an electron in the lowest Landau level (LLL), and let

$$
U(z)=\prod_{i=1}^{N}\left(z-z_{i}\right)
$$

be Laughlin's quasihole operator (acting in the LLL). If $\left|0_{L} ; N\right\rangle$ is the normalized Laughlin state for $N$ particles, then one can show ${ }^{12)}$

$$
\lim _{\left|z-z^{\prime}\right| \rightarrow \infty} \lim _{N \rightarrow \infty}\left\langle 0_{L} ; N\right| U^{\dagger}(z)^{q} \psi(z) \psi^{\dagger}\left(z^{\prime}\right) U\left(z^{\prime}\right)^{q}\left|0_{L} ; N\right\rangle e^{-1 / 4|z|^{2}-1 / 4\left|z^{\prime}\right|^{2}}=\bar{\rho}
$$

Given our choice of order parameter operator, the Laughlin state is the state with the most order. In fact, we have ${ }^{12)}$

$$
\left|0_{L} ; N\right\rangle=\left(\int d^{2} z \psi^{\dagger}(z) U(z)^{q} e^{-1 /\left.4|z|\right|^{2}}\right)^{N}|0\rangle
$$

where $|0\rangle$ is the vacuum (no electrons). This says that the Laughlin state is precisely a Bose condensation of the composite bosons.

The order parameter in a general state obeys ${ }^{12)}$ a system of Landau-Ginzburg-Chern-Simons equations, ${ }^{16)}$ which also involve internal vector and scalar potentials. Similar equations are found in another, more field theoretic approach ${ }^{14,17)}$ which has been reviewed recently by Zhang. ${ }^{18)}$ The ideas have been extended ${ }^{17,19), 20)}$ to the hierarchy scheme ${ }^{21) \sim 24)}$ and its generalizations, ${ }^{19)}$ which give states with abelian statistics for all filling factors $p / q<1$. These also have interpretations as CFT correlators. ${ }^{3), 19)}$

## § 3. Pfaffian state and nonabelions

Consider the wavefunction ${ }^{3)}$

$$
\Psi_{\mathrm{Pf}}\left(z_{1}, \cdots, z_{N}\right)=\operatorname{Pfaff}\left(\frac{1}{z_{i}-z_{j}}\right)_{i<j}\left(z_{i}-z_{j}\right)^{q} e^{-1 / 4 \Sigma\left|z_{i}\right|^{2}}
$$

The Pfaffian is defined by

$$
\operatorname{Pfaff} M_{i j}=\frac{1}{2^{L / 2}(L / 2)!} \sum_{\sigma \in S_{L}} \operatorname{sgn} \sigma \prod_{k=1}^{L / 2} M_{\sigma(2 k-1), \sigma(2 k)}
$$

for an $L \times L$ antisymmetric matrix whose elements are $M_{i j}$, or as the square root of the determinant of $M$; $S_{n}$ is the permutation group on $n$ objects. $\Psi_{\mathrm{Pf}}$ can be regarded as a wavefunction for spinless or spin polarized electrons in the LLL if $q>0$ is even, since then it is antisymmetric; the filling factor is $\nu=1 / q$. Note that $\operatorname{Pfaff}\left(\left(z_{i}-z_{j}\right)^{-r}\right)$, $r$ odd, $<q$ would also give a valid wavefunction.

The idea behind the construction of this wavefunction was the following. From the calculations in Ref. 15), it follows that, in an incompressible fluid state of electrons at filling factor $1 / q$, the composite $\psi^{\dagger}(z) U(z)^{q}$ is a neutral boson if $q$ is odd (as mentioned in § 2) and is a neutral fermion if $q$ is even. Hence, for $q$ odd the operator can Bose condense, which gives the Laughlin states. For $q$ even, on the other hand, it cannot condense singly, but pairs of fermions can condense, as in the BCS theory of superconductivity. For the spinless or spin-polarized case, the pairing function must be of odd parity to satisfy Fermi statistics. Thus a possible function is

$$
\left(\int d^{2} z d^{2} w \frac{1}{z-w} \phi^{\dagger}(z) U^{q}(z) \phi^{\dagger}(w) U^{q}(w) e^{-1 / 4\left(|z|^{2}+|w|^{2}\right\rangle}\right)^{N / 2}|0\rangle
$$

Writing out this function in coordinate representation yields (3•1). The Pfaffian is precisely the sum of products of fermion pairs, antisymmetrized over all distinct ways of pairing. More generally, the pairing function $(z-w)^{-1}$ can be replaced by $(z-w)^{-r}$ with $r$ odd, provided $r<q$, as noted above. This construction was also inspired by noticing that the Haldane-Rezayi (HR) spin-singlet state for electrons at $\nu=1 / q, q$ even, ${ }^{25)}$ can be written in the form involving $\operatorname{det}\left(z_{i}{ }^{\uparrow}-z_{j} \downarrow\right)^{-2}$. The real-space form of the spin singlet BCS wavefunction for spin-(1/2) fermions is just such a determinant of an even parity pairing function, as is well known (it is perhaps less well known that the analogous result for spinless or spin-polarized fermions is a Pfaffian). Thus the HR state is a condensate of pairs of spin-(1/2) composite neutral fermions, $\psi_{\sigma}{ }^{\dagger}(z) U(z)^{q}$ (Ref. 3))!

The pairing picture suggests two kinds of possible excitations. One is the analogue of the BCS quasiparticle, obtained by adding composite fermions, or by "breaking pairs". Thus a state with such a neutral fermion excitation localized at $z$ is obtained by acting on the ground state with $\psi^{\dagger}(z) U(z)^{q}$. (In the HR state, one likewise obtains spin-(1/2) fermions.) These excitations have no analogue in the Laughlin states since there the bosons are already condensed; the only excitations, other than the collective density mode, ${ }^{26}$ are Laughlin's quasiparticles, which are fractionally charged. In the order parameter picture, these are vortices, i.e., the order parameter winds in phase by a multiple of $2 \pi$ around each quasiparticle, which thus resemble flux quanta in a superconductor, the flux quantum being $\Phi_{0}=h c / e$ since the order parameter carries charge 1 from the single electron that it contains. The
flux $\Phi$ determines the charge $e^{*}$ through the quantized Hall relation, $e^{*}=\nu\left(\Phi / \Phi_{0}\right)$. Similarly, in the paired states, one expects to find excitations corresponding to flux quantized in multiples of $(1 / 2) \Phi_{0}$, since the order parameter contains two electrons, and these will have charge in multiples of $(2 q)^{-1}$.

These ideas motivated the following wavefunction ${ }^{3}$ for a pair of quasiholes in the Pfaffian state:

$$
\begin{align*}
\Psi_{\text {Pfaff }+ \text { qholes }}\left(z_{1}, \cdots, z_{N} ; v_{1}, v_{2}\right)= & \left\{\sum_{\sigma \in S_{N}} \frac{\operatorname{sgn} \sigma \prod_{k=1}^{N / 2}\left[\left(z_{\sigma(2 k-1)}-v_{1}\right)\left(z_{\sigma(2 k)}-v_{2}\right)+\left(v_{1} \leftrightarrow v_{2}\right)\right]}{\left(v_{1}-v_{2}\right)^{1 / 8-1 / 4 q}\left(z_{\sigma(1)}-z_{\sigma(2)}\right) \cdots\left(z_{\sigma(N-1)}-z_{\sigma(N)}\right)}\right\} \\
& \times \prod_{i<j}\left(z_{i}-z_{j}\right)^{q} e^{-1 / 4 \Sigma_{i}\left|z_{i}\right|^{2}-1 / 4\left|v_{1}\right|^{2}-1 / 4\left|v_{2}\right|^{2}}
\end{align*}
$$

The factor in brackets can be rewritten as

$$
\left(v_{1}-v_{2}\right)^{1 / 4 q-1 / 8} \operatorname{Pfaff}\left(\frac{\left(z_{i}-v_{1}\right)\left(z_{j}-v_{2}\right)+\left(v_{1} \leftrightarrow v_{2}\right)}{z_{i}-z_{j}}\right) .
$$

(The reason for the assumed form of the exponent of ( $v_{1}-v_{2}$ ) will be explained below.) The extra factor in each term of the Pfaffian resembles a pair of Laughlin quasihole operators, except that in each term each factor acts on only one member of each pair of fermions. Therefore, in an average sense, each quasihole is a half flux and carries charge $1 / 2 q$. Note that this construction cannot produce one quasihole. One way to see that this is impossible is that working on a compact geometry like the sphere, the total flux is quantized in units of $\Phi_{0}$. Another feature of $(3 \cdot 5)$ is that as $v_{1} \rightarrow v_{2}$, we recover a Laughlin quasihole of charge $1 / q$.

While our construction of the Pfaffian state was motivated by order parameter considerations, we can also give an interpretation using conformal field theory, which then suggests the full structure of the system of excited quasihole states. Introduce free, massless real (Majorana) fermions $\chi$ in $1+1$ dimensions:

$$
\left\langle\chi(z) \chi\left(z^{\prime}\right)\right\rangle=\frac{1}{z-z^{\prime}} .
$$

Then using also the free scalar field as before, construct

$$
\left\langle\prod_{i=1}^{N} x\left(z_{i}\right) e^{i \sqrt{\varphi} \varphi\left(z_{i}\right)} e^{-i \sqrt{q} \int d^{2} z^{\prime} \bar{\rho} \varphi\left(z^{\prime}\right)}\right\rangle
$$

This reproduces the FQHE wavefunction (3•1). To reproduce the two quasihole function (3.5), we should understand the CFT we are dealing with. Majorana fermions arise naturally in the two-dimensional Ising model at its critical point. The fermions, in conjunction with their leftmoving partners $\bar{\chi}(\bar{z})$, represent the energy density fluctuation $\varepsilon(z, \bar{z})=\bar{\chi}(\bar{z}) \chi(z)$ which hence has dimension $x=1$ and so the correlation length exponent is $\nu=(2-x)^{-1}=1$. The other fields in the Ising model are the spin fields, i.e. the Ising spin itself. These are operators which produce a square root branch point in the fermi field. ${ }^{7)}$ With the latter we can reproduce the two quasihole state:

$$
\Psi_{\mathrm{Pfaff}+\mathrm{qholes}}=\left\langle\sigma\left(v_{1}\right) e^{(i / 2 \sqrt{q}) \varphi\left(v_{1}\right)} \sigma\left(v_{2}\right) e^{(i / 2 \sqrt{q}) \varphi\left(v_{2}\right)} \prod_{i=1}^{N} \chi\left(z_{i}\right) e^{i \sqrt{\varphi} \varphi\left(z_{z}\right)} e^{-i \sqrt{\varphi} \int d^{2} z^{\prime} \bar{\rho} \varphi\left(z^{\prime}\right)}\right\rangle
$$

The square roots produced by the spin fields $\sigma$ are cancelled by those produced by the "half fluxes" $\exp \left((i / 2 \sqrt{q}) \varphi\left(v_{1}\right)\right)$, so the electron wavefunctions are single-valued. The factor $\left(v_{1}-v_{2}\right)^{-1 / 8}$ in $(3 \cdot 5)$ is now explained; it was inserted to make the equality with the correlator hold, and reflects the fact that the conformal weight of the spin field is $1 / 16$ (and hence $\eta=1 / 4$ in the Ising model).

The generalization to $2 n$ of our "half flux" quasiholes now seems self-evident: we should insert $2 n$ of the combinations $\sigma \exp ((i / 2 \sqrt{q}) \varphi)$ into the correlator. However, it is known ${ }^{7}$ that the Ising model part of such an expression is ambiguous; for fixed positions $v_{1}, \cdots v_{2 n}$ of the spin fields there are many different possible correlators (conformal blocks), forming a vector space of dimension $2^{n-1}$ (over the complex numbers) independent of how many fermions $\chi$ are present. To resolve the ambiguity of notation, the spin fields should be replaced by "chiral vertex operators". ${ }^{8)}$ Furthermore, the monodromy of these functions is nonabelian, i.e., analytically continuing $v_{i}$ around $v_{j}$ produces a linear combination of the original branches of the functions, the coefficients being some braiding matrices that do not commute. At present we do not know the explicit functions for $2 n$ spin fields and $N$ fermions in general, except in the case $N=0$. The case $N=0$ may seem strange from the point of view of electron wavefunctions, but it does at least provide an explicit realization of nonabelian statistics, which is just that given in § 1, where the wavefunctions for the " $\sigma$ particles" are just $N=0$ Ising conformal blocks. In fact, the braiding properties of the spin fields are independent of $N$. Note that, when discussing the $N$ fermion functions as electron wavefunctions, we must show that they are $2^{n-1}$ linearly independent functions of the electron coordinates but this is easily done using the operator product expansion of the spin fields as the co-ordinates $v_{1}, \cdots, v_{2 n}$ approach each other in pairs $v_{1} \rightarrow v_{2}, v_{3} \rightarrow v_{4}$, etc., ${ }^{27)}$ even without full knowledge of these functions. The same operator product expansion also shows that as any $v_{i}$ approaches any $v_{j}$, the leading term is equivalent to a Laughlin quasihole of charge $1 / q$, which is clearly a desirable property. This completes our construction, though it remains to check that adiabatic transport of our quasiholes does give the same nonabelian statistics as the monodromy of the conformal blocks. We are confident that this is true because of the existence of $2^{n-1}$ functions and because analogous properties hold in the abelian examples of § 2 .

Before closing, some comments on recent related work. Wen ${ }^{28)}$ has argued that excitations of wavefunctions arising in one of Jain's later constructions ${ }^{24)}$ possess nonabelion excitations, using the point of view of Ref. 3) of wavefunctions as conformal blocks. These examples involve $S U(N)$ symmetry. Greiter, Wen and Wilczek ${ }^{29}$ have studied numerically the vicinity of $\nu=1 / 2$ for certain Hamiltonians in spherical geometry, and find evidence for an incompressible state with the excitations (neutral fermions and charge $1 / 4$ quasiholes) of the Pfaffian state. However, they claim that the quasiholes obey abelian $\theta / \pi=1 / 8$ statistics. Their arguments are based on an "adiabatic heuristic" principle that connects the FQHE state to a paired state of
fermions in zero magnetic field given by the Pfaffian alone. While this picture is extremely similar to the order parameter or condensation picture put forward in previous papers ${ }^{12), 13), 14), 3)}$ and herein, we do not agree with the conclusion that the statistics is abelian. The full structure of the Landau-Ginzburg-Chern-Simons theory of the state must account for the fermion excitations, and not only the paired order parameter. Their conclusion is what one would obtain by neglecting the Pfaffian completely when calculating the statistics of the quasiholes by adiabatic transport $\bar{a}$ $l a$ Ref. 15). In fact, while a direct search for nonabelian statistics must use wavefunctions for 4 quasiholes (which are not given by Greiter et al.), there is already a difference in the apparent abelian statistics for two quasiholes, as shown by our Eq. $(3 \cdot 5)$, which implies $\theta / \pi=1 / 8-1 / 8=0$; this effect would be due to the Pfaffian. These authors also point out that the Pfaffian state is the unique highest density zero energy eigenstate of a certain three-body Hamiltonian.

Clearly, once the possibility of nonabelions is accepted, many questions remain. How would we confirm their statistics in an experimental setting? Is a gas of nonabelions a superfluid, as for anyons? ${ }^{30}$ )

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