Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 17, NUMBER 2 (2014) (Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

FRACTIONAL RELAXATION WITH TIME-VARYING COEFFICIENT

Roberto Garra [1], Andrea Giusti [2], Francesco Mainardi [3], Gianni Pagnini [4]

Abstract

From the point of view of the general theory of the hyper-Bessel operators, we consider a particular operator that is suitable to generalize the standard process of relaxation by taking into account both memory effects of power law type and time variability of the characteristic coefficient. According to our analysis, the solutions are still expressed in terms of functions of the Mittag-Leffler type as in case of fractional relaxation with constant coefficient but exhibit a further stretching in the time argument due to the presence of Erdélyi-Kober fractional integrals in our operator. We present solutions, both singular and regular in the time origin, that are locally integrable and completely monotone functions in order to be consistent with the physical phenomena described by non-negative relaxation spectral distributions.

MSC 2010: Primary 26A33; Secondary 33E12, 34A08, 76A10
Key Words and Phrases: fractional derivatives, fractional relaxation, Mittag–Leffler functions, fractional power of operators, hyper-Bessel differential operators

1. Introduction

The applications of fractional calculus in the mathematical theory of the relaxation processes have a long history and have gained great interest in different fields of the applied science. These models, usually applied for

pp. 424–439 , DOI: 10.2478/s13540-014-0178-0 Springer

anomalous (that is non-exponential) relaxation in linear viscoelastic and dielectric media, consider memory effects by replacing in the governing evolution equation the ordinary derivatives with fractional derivatives but keeping constant the coefficients, see e.g. [3, 16] and references therein. In this paper we discuss some mathematical results about fractional relaxation models with power law time-varying coefficients. We apply the McBride-Lamb theory of the fractional powers of Bessel-type operators. By doing so, we can give an explicit representation of the fractional order operator in terms of Erdélyi-Kober and Hadamard integrals. For this we refer the reader to the books [11, 17] and references recalled in the following.

In order to be consistent with most relaxation processes described by non-negative relaxation spectral distributions, we concentrate our attention to the cases for which the solutions are locally integrable and completely monotone functions. Our solutions turn out to be both singular and regular in the time origin. The regularization is carried out with a Caputo-like counterpart of our operators. The solutions are still represented via functions of the Mittag-Leffler type with one or two parameters as in the standard fractional relaxation with constant coefficient but with a further time stretching due to the Erdélyi-Kober integrals involved in our operator.

The plan of the paper is the following. In Section **2** we first provide the definition of the time operator $(t^\theta d/dt)^\alpha$ that for $0 < \alpha \le 1$ and $\theta \in \mathbb{R}$ is assumed by us to generalize the usual fractional derivative (both in the Riemann-Liouville and Caputo sense) in the more common fractional relaxation equation with constant coefficient often investigated in the literature, see for example the survey by Gorenflo and Mainardi [8]. Our definition is justified in the framework of the theory of fractional powers of hyper-Bessel operators recalled in Appendix (Section 5) along with the relevant references, for reader's convenience. Indeed our main purpose is to find the solutions of the relaxation-type equation obtained by our operator that turns out to be related to the Erdélyi-Kober and Hadamard integrals. Because of the effect of these integrals on the power laws we find the power series representations of the solutions (both singular and regular in the time origin) that we easily recognize as functions of the Mittag-Leffler type.

In Section 3 we point out the relevance of the completely monotonicity to ensure that solutions will be suitable to represent physical relaxation processes with a non-negative spectral distribution. In view of this requirement we devote our attention to investigate the zones in the parameter plane $\{\alpha, \theta\}$ where the solutions are both locally integrable and completely monotone. We then exhibit some illustrative plots of the solutions versus time in a few case-studies in order to remark the role of the parameter θ

with respect to the case of fractional relaxation with constant coefficient ($\theta = 0$).

Section 4 is reserved to the concluding remarks and directions to future work.

2. Fractional operators and differential equations of relaxation type with time-varying coefficient

In this section we consider fractional differential equation with time-varying coefficient of the form

$$\left(t^\theta \frac{d}{dt}\right)^\alpha u(t) = -\lambda u(t), \quad 0 < \alpha \leq 1, \ \ \theta \in \mathbb{R}, \ t \geq 0. \tag{2.1}$$

By means of the theory of fractional powers of the general hyper-Bessel operators (see Appendix), we have an explicit representation of the operator $\left(t^\theta \frac{d}{dt}\right)^\alpha$ in terms of the Erdélyi–Kober integrals, that is

$$\left(t^\theta \frac{d}{dt}\right)^\alpha f(t) = \begin{cases} (1-\theta)^\alpha t^{-(1-\theta)\alpha} I_{1-\theta}^{0,-\alpha} f(t), & \text{if } \theta < 1, \\ (\theta-1)^\alpha I_{1-\theta}^{-1,-\alpha} t^{(1-\theta)\alpha} f(t), & \text{if } \theta > 1. \end{cases} \tag{2.2}$$

The case $\theta = 1$ will be considered separately.

The first result that we are going to consider is the following.

THEOREM **2.1**. *The function*

$$u(t) = t^{(\alpha-1)(1-\theta)} E_{\alpha,\alpha}\left(-\frac{\lambda t^{\alpha(1-\theta)}}{(1-\theta)^\alpha}\right) \tag{2.3}$$

solves the fractional relaxation equation (2.1).

P r o o f. By using the fact that (see Lemma **5.3** in Appendix)

$$I_m^{\eta,\alpha} t^\beta = \frac{\Gamma\left(\eta + \frac{\beta}{m} + 1\right)}{\Gamma\left(\alpha + \eta + 1 + \frac{\beta}{m}\right)} t^\beta, \tag{2.4}$$

we have that

$$\left(t^\theta \frac{d}{dt}\right)^\alpha t^\beta = (1-\theta)^\alpha t^{-(1-\theta)\alpha} I_{1-\theta}^{0,-\alpha} t^\beta = (1-\theta)^\alpha \frac{\Gamma(\frac{\beta}{1-\theta}+1)}{\Gamma(\frac{\beta}{1-\theta}+1-\alpha)} t^{\beta-(1-\theta)\alpha}. \tag{2.5}$$

Then, we have

$$\left(t^\theta \frac{d}{dt}\right)^\alpha u(t) = (1-\theta)^\alpha \sum_{k=0}^\infty \frac{(-\lambda)^k}{(1-\theta)^{\alpha k}} \frac{t^{(1-\theta)k\alpha-(1-\theta)}}{\Gamma(k\alpha)} \tag{2.6}$$

$$= -\lambda \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{(1-\theta)^{\alpha k}} \frac{t^{(1-\theta)k\alpha+(\alpha-1)(1-\theta)}}{\Gamma(k\alpha+\alpha)} = -\lambda u(t),$$

as claimed. □

In analogy with the classical theory of fractional calculus operators, we can also define a regularized Caputo-like counterpart of the operator (2.2), that is given by

$$^C\left(t^\theta \frac{d}{dt}\right)^\alpha f(t) = \left(t^\theta \frac{d}{dt}\right)^\alpha f(t) - \frac{f(0^+)}{(1-\theta)^{-\alpha}} \frac{t^{-(1-\theta)\alpha}}{\Gamma(1-\alpha)}, \tag{2.7}$$

where $f(0) = f(t=0)$ is the initial condition.

Caputo-type generalized fractional derivatives of rather general form, including also fractional powers of hyper-Bessel-type operators, have been recently introduced by Kiryakova and Luchko [13], see also the so-called Gelfond-Leontiev generalized differentiation with respect to the Mittag-Leffler function, [11, Ch.2].

We are now able to prove the following

THEOREM **2.2**. *The function*

$$u(t) = E_{\alpha,1}\left(-\frac{\lambda t^{\alpha(1-\theta)}}{(1-\theta)^\alpha}\right) \tag{2.8}$$

solves the fractional Cauchy problem

$$\begin{cases} ^C\left(t^\theta \frac{d}{dt}\right)^\alpha u(t) = -\lambda u(t), \quad \alpha \in (0,1), \quad \theta < 1, \ t \geq 0, \\ u(0) = 1. \end{cases} \tag{2.9}$$

P r o o f. First of all, we recall that, in this case,

$$^C\left(t^\theta \frac{d}{dt}\right)^\alpha u(t) = \left(t^\theta \frac{d}{dt}\right)^\alpha u(t) - (1-\theta)^\alpha \frac{t^{-(1-\theta)\alpha}}{\Gamma(-\alpha+1)}, \tag{2.10}$$

so that the fractional equation becomes

$$\left(t^\theta \frac{d}{dt}\right)^\alpha u(t) = -\lambda u(t) + (1-\theta)^\alpha \frac{t^{-(1-\theta)\alpha}}{\Gamma(-\alpha+1)}. \tag{2.11}$$

By using (2.4), we have that

$$\left(t^\theta \frac{d}{dt}\right)^\alpha t^\beta = (1-\theta)^\alpha t^{-(1-\theta)\alpha} I_{1-\theta}^{0,-\alpha} t^\beta = (1-\theta)^\alpha \frac{\Gamma(\frac{\beta}{1-\theta}+1)}{\Gamma(\frac{\beta}{1-\theta}+1-\alpha)} t^{\beta-(1-\theta)\alpha}. \tag{2.12}$$

Then, we have that

$$\left(t^{\theta}\frac{d}{dt}\right)^{\alpha} E_{\alpha,1}\left(-\frac{\lambda t^{\alpha(1-\theta)}}{(1-\theta)^{\alpha}}\right) = (1-\theta)^{\alpha}\sum_{k=0}^{\infty}\frac{(-\lambda)^{k}}{(1-\theta)^{\alpha k}}\frac{t^{(1-\theta)k\alpha-(1-\theta)\alpha}}{\Gamma(k\alpha+1-\alpha)}$$

$$= -\lambda\sum_{k=-1}^{\infty}\frac{(-\lambda)^{k}}{(1-\theta)^{\alpha k}}\frac{t^{(1-\theta)k\alpha}}{\Gamma(k\alpha+1)} = -\lambda f(t) + \frac{1}{(1-\theta)^{-\alpha}}\frac{t^{-(1-\theta)\alpha}}{\Gamma(-\alpha+1)},$$

as claimed. □

An analogous result for the case $\theta > 1$ can be achieved with the same reasoning of Theorem 2.2. However this case has not a physical interest in this framework.

Remark **2.1**. We observe that in the limiting case $\alpha = 1$, we have

$$u(t) = \exp\left(\frac{-\lambda t^{1-\theta}}{(1-\theta)}\right), \tag{2.13}$$

that is the analytic solution of the ordinary differential equation

$$t^{\theta}\frac{du}{dt} = -\lambda u, \tag{2.14}$$

as expected.

Remark **2.2**. We observe that in the case $\theta = 0$, from (2.8), we recover the solution of the fractional relaxation equation involving Caputo time-fractional derivatives. Indeed in this case the function (2.8) becomes the eigenfunction of the Caputo derivative.

2.1. The singular case $\theta = 1$

The limiting case $\theta = 1$ brings us to the fractional relaxation equation

$$\left(t\frac{d}{dt}\right)^{\alpha} u(t) = -\lambda u(t), \quad \alpha \in (0,1). \tag{2.15}$$

The explicit form of the operator $\left(t\frac{d}{dt}\right)^{\alpha}$ is well known from the theory of fractional powers of operators. It is the so-called Hadamard fractional derivative of order $\alpha \in (0,1)$, that is (see for example [14])

$$\left(t\frac{d}{dt}\right)^{\alpha} f(t) = \delta\left(\mathcal{J}^{1-\alpha}_{t_0^+} f\right)(t), \quad 0 < \alpha < 1, \quad t_0 \geq 0, \tag{2.16}$$

where $\delta = t\dfrac{d}{dt}$, and

$$\left(\mathcal{J}^{1-\alpha}_{t_0^+} f\right)(t) = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\left(\ln\frac{x}{u}\right)^{-\alpha} f(u)\frac{du}{u}, \quad t_0 \geq 0, \tag{2.17}$$

is the Hadamard fractional integral of order $1-\alpha$. For completeness we recall that the definition of Hadamard fractional derivative for any real $\alpha>0$ is given by

$$\left(t\frac{d}{dt}\right)^{\alpha} f(t)=\delta^{n}\mathcal{J}^{n-\alpha}_{t_0^+}f(t),\quad n-1<\alpha\le n. \tag{2.18}$$

For the aims of this paper, we consider the case $\alpha\in(0,1)$. For the utility of the reader, we recall the following result (see [14], Theorem 8).

THEOREM **2.3**. *A solution of* (2.15), *for* $t_0>0$ *is given by*

$$u(t)=\left[\ln\left(\frac{t}{t_0}\right)\right]^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\ln\frac{t}{t_0}\right)^{\alpha}\right). \tag{2.19}$$

In recent papers, like [9], a regularized Caputo-like Hadamard fractional derivative has been introduced, as follows:

$$^{C}\left(t\frac{d}{dt}\right)^{\alpha} f(t)=\mathcal{J}^{n-\alpha}_{t_0^+}\delta^{n}f(t),\quad \alpha>0,\ n-1<\alpha\le n. \tag{2.20}$$

The relation between (2.20) and (2.18) is given by

$$^{C}\left(t\frac{d}{dt}\right)^{\alpha} f(t)=\left(t\frac{d}{dt}\right)^{\alpha}\left[f(t)-\sum_{k=0}^{n-1}\frac{\delta^{k}y(t_0)}{k!}\left(\ln\frac{t}{t_0}\right)^{k}\right]. \tag{2.21}$$

We now consider the fractional relaxation equation governed by the regularized fractional Hadamard derivative, that is

$$^{C}\left(t\frac{d}{dt}\right)^{\alpha} u(t)=-\lambda u(t),\quad \alpha\in(0,1). \tag{2.22}$$

THEOREM **2.4**. *The solution of* (2.22), *under the initial condition* $u(t_0)=1$ *at initial time* $t_0>0$, *is given by*

$$u(t)=E_{\alpha,1}\left(-\lambda\left(\ln\frac{t}{t_0}\right)^{\alpha}\right). \tag{2.23}$$

P r o o f. Recalling that (see equation (38) in [9])

$$^{C}\left(t\frac{d}{dt}\right)^{\alpha}\left(\ln\frac{t}{t_0}\right)^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}\left(\ln\frac{t}{t_0}\right)^{\beta-\alpha}, \tag{2.24}$$

we have

$$^{C}\left(t\frac{d}{dt}\right)^{\alpha}E_{\alpha,1}\left(-\lambda\left(\ln\frac{t}{t_0}\right)^{\alpha}\right)=\sum_{k=1}^{\infty}\frac{(-\lambda)^{k}(\ln t)^{\alpha k-\alpha}}{\Gamma(\alpha k+1-\alpha)}$$

$$= -\lambda \sum_{k'=0}^{\infty} \frac{(-\lambda)^{k'} (\ln t)^{\alpha k'}}{\Gamma(\alpha k' + 1)} = -\lambda E_{\alpha,1}\left(-\lambda \left(\ln \frac{t}{t_0}\right)^{\alpha}\right).$$

□

3. Completely monotone solutions and numerical results

Let us recall that a real non-negative function $f(t)$ defined for $t \in \mathbb{R}^+$ is said to be completely monotone (CM) if it possesses derivatives $f^{(n)}(t)$ for all $n = 0, 1, 2, 3, ..$ that are alternating in sign, namely

$$(-1)^n f^{(n)}(t) \geq 0, \quad t > 0\,. \tag{3.1}$$

The limit $f^{(n)}(0^+) = \lim_{t \to 0^+} f^{(n)}(t)$ finite or infinite exists. For the existence of the Laplace transform of $f(t)$ (in the classical sense) we require that the function be locally integrable in $\mathbb{R}^+$.

For the Bernstein theorem that states a necessary and sufficient condition for the CM, the function $f(t)$ can be expressed as a real Laplace transform of non-negative (generalized) function, namely

$$f(t) = \int_0^{\infty} \mathrm{e}^{-rt}\, K(r)\, \mathrm{d}r\,,\ K(r) \geq 0\,,\ t \geq 0\,. \tag{3.2}$$

For more details see e.g. the survey by Miller & Samko [21]. In physical applications the function $K(r)$ is usually referred to as the *spectral distribution*, in that it is related to the fact that the process governed by the function $f(t)$ with $t \geq 0$ can be expressed in terms of a continuous distribution of elementary (exponential) relaxation processes with frequencies r on the whole range $(0, \infty)$. In the case of the pure exponential $f(t) = \exp(-\lambda t)$ with a given relaxation frequency $\lambda > 0$, we have $K(r; \lambda) = \delta(r - \lambda)$.

Since $\widetilde{f}(s)$ turns to be the iterated Laplace transform of $K(r)$ we recognize that $\widetilde{f}(s)$ is the Stieltjes transform of $K(r)$ and therefore, the spectral distribution can be determined as the inverse Stieltjes transform of $\widetilde{f}(s)$ via the Titchmarsh inversion formula, see e.g. [25, 26],

$$\widetilde{f}(s) = \int_0^{\infty} \frac{K(r)}{s + r}\, \mathrm{d}r\,, \quad K(r) = \mp \pi\, \mathrm{Im}\left[\left.\widetilde{f}(s)\right|_{s = re^{\pm i\pi}}\right]. \tag{3.3}$$

Another relevant class of functions related to complete monotonicity are the Bernstein functions. Let us recall that a real non-negative function $g(t)$ defined for $t \in \mathbb{R}^+$ is said to be Bernstein, if it possesses derivatives $g^{(n)}(t)$ for all $n = 0, 1, 2, 3, ..$ and its first derivative is CM. So, characteristic examples of CM and Bernstein functions are respectively $f(t) = \exp(-t)$ and $g(t) = 1 - \exp(-t)$. Other simple examples are provided by $\phi(t) = t^{\gamma}$ that is CM if $\gamma < 0$, but locally integrable if $-1 < \gamma < 0$ and Bernstein if $0 < \gamma \leq 1$.

We recall two fundamental properties useful in the following:

- Let $f(t)$ be CM and let $g(t)$ be Bernstein, then $f[g(t)]$ is a CM function.
- The product of two CM functions is CM.

Our solutions in Eqs. (2.3) and (2.8) are functions of Mittag-Leffler type. We recall that for the Mittag-Leffler functions in one and two parameters the conditions to be CM on the negative real axis were derived by Pollard [22] in 1948, for $0 < \alpha \leq 1$, and by Schneider [24] in 1996, for $0 < \alpha \leq 1$ and $\beta \geq \alpha$. See also Miller and Samko [20, 21] for further details. As a consequence, it is straightforward to derive the conditions of locally integrability and complete monotonicity for our solutions in (2.3) and (2.8) taking into account the exponents of the powers of t in the argument of the Mittag-Leffler functions and in the pre-factor. Then we have:

THEOREM **3.1**. *The solution in* (2.8) *is locally integrable and CM if*

$$0 < \alpha < 1, \quad \theta < 1, \quad 0 < \alpha(1-\theta) < 1,$$

and the solution in (2.3) *is locally integrable and CM if we add to the previous conditions*

$$-1 < (\alpha - 1)(1-\theta) < 0.$$

We recall that functions t^γ with $\gamma \leq -1$ ($t \in \mathbb{R}^+$) are non-local integrable; they are referred to as *pseudo-functions* in the treatise by Doetsch [7] on Laplace transforms and consequently treated as distributions.

Hereafter we exhibit in the plane $\{\theta, \alpha\}$ the zones of CM of both locally integrable solutions.

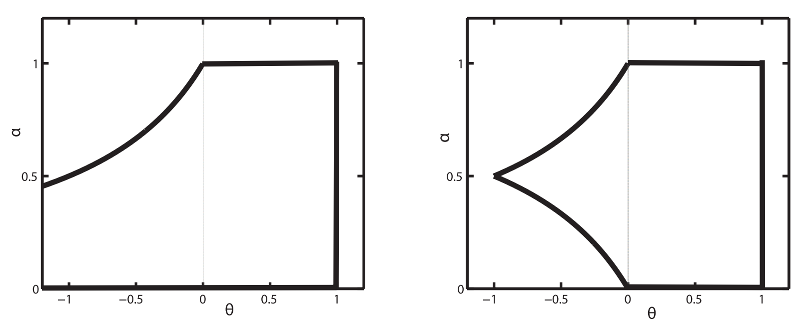

Fig. 1: The zones of CM of the locally integrable solutions (2.8) (left) and (2.3) (right) in the plane $\{\alpha, \theta\}$ limited by thick lines. Whereas the left zone continues indefinitely for $-\infty < \theta < 1$, the right zone is limited to $-1 < \theta < 1$.

Then, for illustration we show some plots of the solutions in (2.8) for fixed $\alpha = 0.25, 0.50, 0.75, 1$ with a few values of θ included in the zone of CM in order to remark the role of the parameter θ with respect to the case of fractional relaxation ($\theta = 0$) with constant coefficient, investigated in [8].

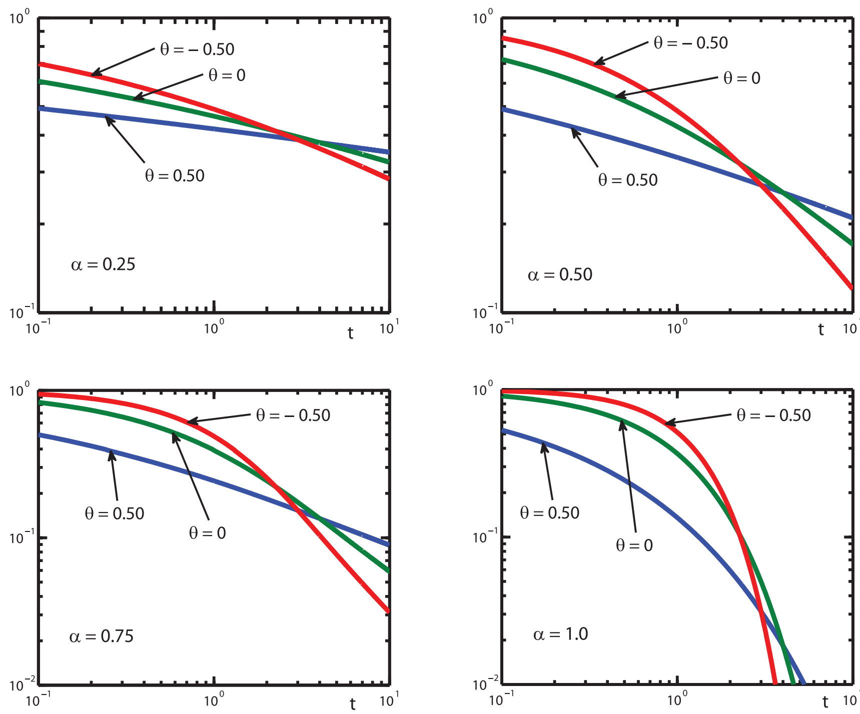

Fig. 2: Log-log plots of the locally integrable CM solutions, regular at $t = 0$ provided by (2.8), versus time for selected values of $\{\alpha, \theta\}$. We note a power law decay for $\alpha < 1$ and an exponential decay for $\alpha = 1$.

4. Conclusions and further work

Based on the general theory of fractional powers of hyper-Bessel operators, we have introduced and discussed a particular integro-differential operator characterized by two parameters $\{\alpha, \theta\}$ that is suitable to generalize the process of standard relaxation by taking into account both memory effects and time variability of the characteristic coefficient. To be consistent with most relaxation processes, we have required the local integrability and complete monotonicity of the solutions that turn out to be functions of the Mittag-Leffler type involving the two characteristic parameters.

Further work can be done still in relation to our model equation (2.1). For example, the solutions of (2.1) for $1 < \alpha \leq 2$ can be studied in the framework of models of fractional oscillations with a time varying coefficient.

Note that the results presented in Theorems **2.1**, **2.2** can be considered as an interesting realization in explicit form of the more abstract results from Kiryakova et al. [1], [12] for differential equations involving fractional multi-order analogues of the hyper-Bessel operators L (see Appendix), presented in terms of the so-called multi-index Mittag-Leffler functions.

Namely, one could consider noteworthy cases of equations, generalizing (2.1) and having the form

$$\mathfrak{L}u(t) := t^{a_1} D^{\alpha_1}\, t^{a_2} D^{\alpha_2} \ldots t^{a_n} D^{\alpha_n}\, t^{a_{n+1}}\, u(t) = -\lambda u(t), \qquad D = d/dt, \quad (4.1)$$

with the so-called *fractional hyper-Bessel operators* $\mathfrak{L}$, introduced in [11, Ch.5]. In Kiryakova et al. [12], [1], etc. it is proved that the *multi-index Mittag-Leffler functions solve the differential equation* (4.1) of fractional multi-order $(\alpha_1, ..., \alpha_n)$, thus replacing the role of the Delerue hyper-Bessel functions in the case of hyper-Bessel differential equations $L\,u(t) = -\lambda u(t)$ of order $n = (1, ..., 1)$, with the operators (5.1) considered below. Much work has to be done to show the relevance of these exact results in the numerous applications of fractional operators in mathematical physics.

Finally, a probabilistic analysis of the obtained results can be of interest in relation to generalized fractional Poisson processes and Mittag-Leffler waiting time distributions, based the recent results by Beghin [2].

5. Appendix: On the theory of fractional powers of hyper-Bessel operators

In this section we recall some useful results from the theory of fractional powers of the so-called hyper-Bessel operators.

The linear singular differential operators with variable coefficients of arbitrary integer order $n = 1, 2, 3, ...$ of the form

$$L = t^{a_1} D t^{a_2} D \ldots t^{a_n} D t^{a_{n+1}}, \quad 0 < t < \infty, \qquad (5.1)$$

where $a_1, \ldots, a_{n+1}$ are real numbers and $D = d/dt$, have been introduced and considered by Dimovski [4], Kiryakova [10], McBride [19], and called later by the name hyper-Bessel operators in the works by Kiryakova as [11]. The reason is because the hyper-Bessel functions of Delerue (variants of the generalized hypergeometric function ${}_0F_{n-1}$, reducing to the Bessel functions ${}_0F_1$ for $n = 2$) are the solutions (forming f.s.s. around $t = 0$) of the equations $Lu(t) = -\lambda u(t)$.

The operator L generalizes the 2nd order Bessel differential operator, the n-th order hyper-Bessel operator of Ditkin and Prudnikov [6]

$$L_{B_n} = t^{-n} \underbrace{t\frac{d}{dt} t\frac{d}{dt} \ldots t\frac{d}{dt}}_{n \text{ times}},$$

as well as many other differential operators considered by different authors (see e.g. Refs. in [11], [19]) and appearing very often in differential equations of mathematical physics, specially in the theory of potential, viscoelasticity, etc.

McBride [19], on whose results we base in this paper, considers the operator L defined in (5.1) to act on the functional space

$$F_{p,\mu} = \{f \colon t^{-\mu} f(t) \in F_p\}, \tag{5.2}$$

where

$$F_p = \{f \in C^{\infty} \colon t^k d^k f/dt^k \in L^p, k = 0, 1, \dots\}, \tag{5.3}$$

for $1 \leq p < \infty$ and for any complex number μ, for details see [18, 17]. Dimovski and Kiryakova, and other authors made their studies on L in weighted spaces of continuous or Lebesgue integrable functions. Especially, the fractional powers L^{α} can be defined and represented by means of convolutions ([4]) or in terms of the Meijer G-function ([19], [10], [11]) for functions of the form $f(t) = t^p \widetilde{f}(t)$ with $p > \max\limits_{1\leq k\leq n} [(a-n)(b_k+1)]$, $\widetilde{f} \in C[0, \infty)$.

The following lemma gives an alternative representation of the operator L, in the denotations of McBride.

Lemma **5.1**. *The operator L in (5.1) can be written as*

$$Lf = m^n t^{a-n} \prod_{k=1}^{n} t^{m-mb_k} D_m t^{mb_k} f, \tag{5.4}$$

where

$$D_m := \frac{d}{dt^m} = m^{-1} t^{1-m} \frac{d}{dt}.$$

The constants appearing in (5.4) are defined as

$$a = \sum_{k=1}^{n+1} a_k, \quad m = |a-n|, \quad b_k = \frac{1}{m}\left(\sum_{i=k+1}^{n+1} a_i + k - n\right), \quad k = 1, \dots, n.$$

For the proof, see Lemma 3.1, page 525 of [19].

In the analysis of the integer power (as well as of the fractional power) of the operator L, a key role is played by the differentiation D_m with respect to t^m, appearing in (5.4).

Lemma **5.2**. *Let r be a positive integer, $f \in F_{p,\mu}$ and*

$$b_k \in A_{p,\mu,m} := \{\eta \in \mathbb{C} \colon \Re(m\eta+\mu)+m \neq 1/p - ml,\ l = 0, 1, 2, \dots\},\ k = 1, \dots, n.$$

Then

$$L^r f = \begin{cases} m^{nr} t^{-mr} \prod_{k=1}^{n} I_m^{b_k,-r} f, & \text{if } a-n<0, \\ \\ m^{nr} \prod_{k=1}^{n} I_m^{b_k-1,-r} t^{mr} f, & \text{if } a-n>0, \end{cases} \tag{5.5}$$

where for $\alpha > 0$ *and* $\Re(m\eta+\mu)+m > 1/p$, *we mean the Erdélyi-Kober fractional integral*

$$I_m^{\eta,\alpha} f = \frac{t^{-m\eta-m\alpha}}{\Gamma(\alpha)} \int_0^t (t^m-u^m)^{\alpha-1} u^{m\eta} f(u)\, d(u^m), \tag{5.6}$$

and for $\alpha \leq 0$, *the interpretation is via the integro-differential operator*

$$I_m^{\eta,\alpha} f = (\eta+\alpha+1) I_m^{\eta,\alpha+1} f + \frac{1}{m} I_m^{\eta,\alpha+1}\left(t\frac{d}{dt}f\right). \tag{5.7}$$

For the proof of the lemma consult, for example [19], page 525.

Note that the expression for $\alpha \leq 0$ in (5.7) is called in the works of Kiryakova (as [11]) and Luchko et al. (as [27]) as Erdélyi-Kober fractional derivative denoted by $D_m^{\eta,\alpha}$, namely, we mean $I_m^{\eta,-\alpha} := D_m^{\eta+\alpha,-\alpha}$. In the same terms, Kiryakova writes the operators (5.4), (5.5) as the products

$$Lf = m^n t^{-m} \prod_{k=1}^{n} [D_m^{b_k,1}] f, \quad L^r f = m^{nr} t^{-mr} \prod_{k=1}^{n} [D_m^{b_k,r}] f,$$

see details in [11, Ch.1, Ch.3, Ch.5].

Further, it is possible to give a definition and representation of the fractional powers $L^\alpha, \alpha > 0$ of the operator L. By means of operational calculus the operators L^α have been introduced by Dimovski [4], Dimovski and Kiryakova [5] and further used by Kiryakova as a base of her generalized fractional calculus [11].

However, in this paper we use the denotations and the results as exposed in the McBride-Lamb theory of the fractional powers of operators, [19, p.527] and [15].

Definition **5.1**. Let α be any real number, $b_k \in A_{p,\mu,m}$, for $k = 1,\ldots,n$. Then, for any $f(t) \in F_{p,\mu}$

$$L^\alpha f = \begin{cases} m^{n\alpha} t^{-m\alpha} \prod_{k=1}^{n} I_m^{b_k,-\alpha} f, & \text{if } a-n<0, \\ \\ m^{n\alpha} \prod_{k=1}^{n} I_m^{b_k-1,-\alpha} t^{m\alpha} f, & \text{if } a-n>0. \end{cases} \tag{5.8}$$

In this paper we will consider however only $\alpha \in \mathbb{R}^+$.

The relation between the two lemmas above emerges directly from the analysis of the mathematical connection between the power of the operator D_m and the generalized fractional integrals $I_m^{\eta,\alpha}$, as we are going to discuss. In order to understand this relationship we introduce the following operator [18]

$$I_m^\alpha f = \frac{m}{\Gamma(\alpha)} \int_0^t (t^m - u^m)^{\alpha-1} u^{m-1} f(u)\, du, \qquad \alpha > 0, \tag{5.9}$$

which is connected to the Erdélyi-Kober fractional integral (5.6) by means of the simple relation

$$I_m^\alpha f = t^{m\alpha} I_m^{0,\alpha} f,$$

valid for all $\alpha \in \mathbb{R}$.

It is quite simple to prove that

$$\begin{aligned} I_m^\alpha f &= (D_m) I_m^{\alpha+1} f \\ &= \frac{m}{\Gamma(\alpha+1)} D_m \int_0^t (t^m - u^m)^\alpha u^{m-1} f(u)\, du = \underbrace{D_m \dots D_m}_{r \text{ times}} I_m^{\alpha+r} f. \end{aligned}$$

If $\alpha = -r$, we have that

$$I_m^{-r} f = \underbrace{D_m \dots D_m}_{r \text{ times}} I_m^0 f = (D_m)^r f.$$

For a real number α, the same relationship is extended in the form

$$I_m^{-\alpha} f = (D_m)^\alpha f. \tag{5.10}$$

Since the semigroup property holds for the Erdélyi–Kober operator (5.9), we have that

$$\begin{aligned} (D_m)^\alpha f &= (D_m)^n (D_m)^{\alpha-n} f = I_m^{-n} I_m^{n-\alpha} f \\ &= \frac{m}{\Gamma(n-\alpha)} (D_m)^n \int_0^t (t^m - u^m)^{n-\alpha-1} u^{m-1} f(u)\, du. \end{aligned} \tag{5.11}$$

Finally we observe that, for $m = 1$ we recover the definition of Riemann–Liouville fractional derivative, [23].

A well-known result from fractional calculus, see e.g. [23] that is used in this paper, is given by the following lemma.

Lemma **5.3**. *Let be* $\eta + \frac{\beta}{m} + 1 > 0$, $m \in \mathbb{N}$, *we have that*

$$I_m^{\eta,\alpha}\, t^\beta = \frac{\Gamma\left(\eta + \frac{\beta}{m} + 1\right)}{\Gamma\left(\alpha + \eta + 1 + \frac{\beta}{m}\right)}\, t^\beta. \tag{5.12}$$

Finally we observe that the operator appearing in (2.1) is a special case of (5.8) with $n = 1$, $a_1 = \theta$, $a_2 = 0$, $b_1 = 0$, $a = \theta$ and $m = |1 - \theta|$. Then by using Definition **5.1** we have (2.2).

Acknowledgements

The authors are grateful to Prof. Virginia Kiryakova for the useful comments and relevant references.

References

[1] I. Ali, V. Kiryakova, S. Kalla, Solutions of fractional multi-order integral and differential equations using a Poisson-type transform. *J. Math. Anal. and Appl.* **269**, No 1 (2002), 172–199; DOI:10.1016/S0022-247X(02)00012-4.

[2] L. Beghin, Fractional relaxation equations and Brownian crossing probabilities of a random boundary. *Advances in Applied Probability* **44** (2012), 479–505.

[3] E. Capelas de Oliveira, F. Mainardi, J. Vaz Jr., Models based on Mittag-Leffler functions for anomalous relaxation in dielectrics. *J. Eur. Phys.- Special Topics* **193**, No 1 (2011), 161–171. Revised version in E-print `arXiv:1106.1761`.

[4] I. Dimovski, On an operational calculus for a differential operator. *Compt. Rendues de l'Acad. Bulg. des Sci.* **21**, No 6 (1966), 513–516.

[5] I. Dimovski, V. Kiryakova, Transmutations, convolutions and fractional powers of Bessel-type operators via Meijer's G-function. In: *Proc. Intern. Conf. "Complex Analysis and Applications, Varna' 1983"*, Bulg. Acad. Sci., Sofia, 45–66.

[6] V.A. Ditkin, A.P. Prudnikov, On the theory of operational calculus, generated by the Bessel equation. *Zhournal Vych. Mat. i Mat. Fiziki* **3**, No 2 (1963), 223–238 (In Russian).

[7] G. Doetsch, *Introduction to the Theory and Application of the Laplace Transformation*. Springer Verlag, Berlin (1974).

[8] R. Gorenflo and F. Mainardi, Fractional calculus: integral and differential equations of fractional order. In: A. Carpinteri and F. Mainardi (Editors), *Fractals and Fractional Calculus in Continuum Mechanics*, Springer Verlag, Wien (1997), 223–276. [E-print: `arXiv:0805.3823`].

[9] F. Jarad, R. Abdeljawad and D. Baleanu, Caputo-type modification of the Hadamard fractional derivatives. *Advances in Difference Equations* **2012** (2012), 1–8.

[10] V. Kiryakova, An application of Meijer's G-function to Bessel type operators. In: *Proc. Internat. Conf. "Constr. Function Theory, Varna 1981"*, Bulg. Acad. Sci., Sofia, 457–462.

[11] V. Kiryakova, *Generalized Fractional Calculus and Applications.* Longman – J. Wiley, Harlow – N. York (1994).

[12] V. Kiryakova, Multiple (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus. *J. Comput. Appl. Mathematics* **118** (2000), 241–259; DOI:10.1016/S0377-0427(00)00292-2.

[13] V. Kiryakova, Yu. Luchko, Riemann-Liouville and Caputo type multiple Erdélyi-Kober operators. *Central European J. of Physics* **11**, No 10 (2013), 1314–1336; DOI:10.2478/s11534-013-0217-1.

[14] A.A. Kilbas and A.A. Titioura, Nonlinear differential equations with Marchaud-Hadamard-type fractional derivative in the weighted space of summable functions. *Mathematical Modelling and Analysis* **12**, No 3 (2007), 343–356.

[15] W. Lamb, *Fractional Powers of Operators on Fréchet Spaces with Applications.* Ph.D. Thesis, University of Strathclyde, Glasgow, 1980.

[16] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity.* Imperial College Press – World Sci., London - Singapore (2010).

[17] A.C. McBride, *Fractional Calculus and Integral Transforms of Generalised Functions.* Pitman, London (1979).

[18] A.C. McBride, A theory of fractional integration for generalized functions. *SIAM J. on Mathematical Analysis* **6**, No 3 (1975), 583–599.

[19] A.C. McBride, Fractional powers of a class of ordinary differential operators. *Proc. London Mathematical Society (3)* **45** (1982), 519–546.

[20] K.S. Miller and S.G. Samko, A note on the complete monotonicity of the generalized Mittag-Leffler function. *Real Anal. Exchange* **23** (1997), 753–755.

[21] K.S. Miller and S.G. Samko, Completely monotonic functions. *Integral Transforms and Special Functions* **12** (2001), 389–402.

[22] H. Pollard, The completely monotonic character of the Mittag-Leffler function $E_\alpha(-x)$. *Bull. Amer. Math. Soc.* **54** (1948), 1115–1116.

[23] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives.* Gordon and Breach Science, Yverdon (1993).

[24] W.R. Schneider, Completely monotone generalized Mittag-Leffler functions. *Expositiones Mathematicae* **14** (1996), 3–16.

[25] E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals.* Oxford University Press, Oxford (1937).

[26] D.V. Widder, *The Laplace Transform.* Princeton University Press, Princeton (1946).

[27] S. Yakubovich, Yu. Luchko, *The Hypergeometric Approach to Integral Transforms and Convolutions.* Ser.: Mathematics and Its Applications 287, Kluwer Acad. Publ., Dordrecht-Boston-London (1994).

[1] *Dipartimento di Scienze di Base e Applicate per l'Ingegneria*
"Sapienza" Università di Roma
Via A. Scarpa 14, Roma, ITALY
e-mail: roberto.garra@sbai.uniroma1.it

[2] *Department of Physics, University of Bologna*
Via Irnerio 46, Bologna, ITALY
e-mail: andrea.giusti91@gmail.com

[3] *Department of Physics, University of Bologna, and INFN*
Via Irnerio 46, Bologna, ITALY
e-mail: francesco.mainardi@bo.infn.it *Received: December 18, 2013*

[4] *BCAM - Basque Center for Applied Mathematics*
Alameda de Mazarredo 14
E - 48009 Bilbao, Basque Country - SPAIN
&
Ikerbasque, Basque Foundation for Science
Alameda Urquijo 36-5, Plaza Bizkaia
E - 48011 Bilbao, Basque Country - SPAIN
e-mail: gpagnini@bcamath.org

Please cite to this paper as published in:

Fract. Calc. Appl. Anal., Vol. **17**, No 2 (2014), pp. 424–439;
DOI: 10.2478/s13540-014-0178-0