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FRACTIONAL SOBOLEV EXTENSION AND IMBEDDING

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ABSTRACT. Let Ω be a domain of $\mathbb{R}^n$ with $n \geq 2$ and denote by $W^{s,p}(\Omega)$ the fractional Sobolev space for $s \in (0, 1)$ and $p \in (0, \infty)$. We prove that the following are equivalent:

(i) there exists a constant $C_1 > 0$ such that for all $x \in \Omega$ and $r \in (0, 1]$,
$$|B(x, r) \cap \Omega| \geq C_1 r^n;$$
(ii) Ω is a $W^{s,p}$-extension domain for all $s \in (0, 1)$ and all $p \in (0, \infty)$;
(iii) Ω is a $W^{s,p}$-extension domain for some $s \in (0, 1)$ and some $p \in (0, \infty)$;
(iv) Ω is a $W^{s,p}$-imbedding domain for all $s \in (0, 1)$ and all $p \in (0, \infty)$;
(v) Ω is a $W^{s,p}$-imbedding domain for some $s \in (0, 1)$ and some $p \in (0, \infty)$.

1. INTRODUCTION

Let $n \geq 2$ and Ω be a domain (namely, connected open subset) of $\mathbb{R}^n$. For $s \in (0, 1)$ and $p \in (0, \infty)$, define the *fractional Sobolev space on the domain* Ω as

$$W^{s,p}(\Omega) \equiv \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{n/p+s}} \in L^p(\Omega \times \Omega) \right\} \tag{1.1}$$

with the norm

$$\|u\|_{W^{s,p}(\Omega)} \equiv \left(\int_\Omega |u(x)|^p \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p}, \tag{1.2}$$

which is also called a *Aronszajn, Gagliardo* or *Slobodeckij space* in the literature after the names of those who introduced them almost simultaneously; see [1, 6, 26]. The fractional Sobolev spaces are special cases of Besov and Triebel-Lizorkin spaces; for a comprehensive treatment and their applications in different subjects see [16, 17, 21, 22, 27–30] and the references therein.

Due to the applications, it attracts a lot of attention to extend fractional Sobolev (and also Besov and Triebel-Lizorkin) functions on a domain to the entire $\mathbb{R}^n$ continuously; see [4, 16, 17, 23–25, 27, 29, 30]. We say that $\Omega \subset \mathbb{R}^n$ is a $W^{s,p}$-*extension domain* if every function $u \in W^{s,p}(\Omega)$ can be extended to a function $\widetilde{u} \in W^{s,p}(\mathbb{R}^n)$ continuously, that is, $\widetilde{u}(x) = u(x)$ for all $x \in \Omega$, and there exists a constant $C = C(n, p, s, \Omega)$ such that $\|\widetilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{s,p}(\Omega)}$. Jonsson and Wallin [17] (and also Shvartsman [24]) essentially proved that a regular domain must be a $W^{s,p}$-extension domain for all $0 < s < 1$ and all $p \geq 1$. Recall that Ω is called a *regular domain* (also called a *plump domain*) if it satisfies the *measure*

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density condition: there exists a constant $C_1 > 0$ such that for all $x \in \Omega$ and all $r \in (0, 1]$,

$$|B(x, r) \cap \Omega| \geq C_1 r^n. \tag{1.3}$$

However, an arbitrary domain is not necessarily a $W^{s,p}$-extension domain. For example, $\Omega \equiv \{x = (x_1, x_2) \in \mathbb{R}^2 : \ 0 < x_1 < 1, \ 0 < x_2 < x_1^2\}$ is not a $W^{s,p}$-extension domain for all $s \in (0, 1)$ and all $p \in (2/s, \infty)$, as we show by hand in Remark 1.5 below (without using Theorems 1.1 and 1.2). Such a fact was well understood earlier by mathematicians.

The main aim of this paper is to characterize the $W^{s,p}$-extension domains for all $s \in (0, 1)$ and all $p \in (0, \infty)$ as below, and hence, give an answer to a question by Nezza, Palatucci and Valdinoci (see [21, Section 5]).

Theorem 1.1. *Let $n \geq 2$ and Ω be a domain of $\mathbb{R}^n$. Then the following are equivalent:*

(i) Ω is a regular domain;

(ii) Ω is a $W^{s,p}$-extension domain for all $s \in (0, 1)$ and all $p \in (0, \infty)$;

(iii) Ω is a $W^{s,p}$-extension domain for some $s \in (0, 1)$ and some $p \in (0, \infty)$.

Extension properties play important roles in applications; in particular, they can be used to establish some imbedding properties. A domain $\Omega \in \mathbb{R}^n$ is said to be a *$W^{s,p}$-imbedding domain* if the following holds:

(a) when $sp < n$, there exists a constant $C > 0$ such that for all $u \in W^{s,p}(\Omega)$, we have $u \in L^{np/(n-sp)}(\Omega)$ and $\|u\|_{L^{np/(n-sp)}(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)}$;

(b) when $sp = n$, there exist constants C_3, $C > 0$ such that for all $u \in W^{s,p}(\Omega)$ and all balls B,

$$\inf_{c\in\mathbb{R}} \int_{B\cap\Omega} \exp\left(C_3 \frac{|u(x) - c|}{\|u\|_{W^{s,n/s}(\Omega)}}\right)^{n/(n-s)} dx \leq C|B|; \tag{1.4}$$

(c) when $sp > n$, there exists a constant $C > 0$ such that for all $u \in W^{s,p}(\Omega)$ and every pair of $x, y \in \Omega$, we have $|u(x) - u(y)| \leq C\|u\|_{W^{s,p}(\Omega)}|x - y|^{s-n/p}$.

We also have the following results.

Theorem 1.2. *Let $n \geq 2$ and Ω be a domain of $\mathbb{R}^n$. Then the following are equivalent:*

(i) Ω is a regular domain;

(iv) Ω is a $W^{s,p}$-imbedding domain for all $s \in (0, 1)$ and all $p \in (0, \infty)$;

(v) Ω is a $W^{s,p}$-imbedding domain for some $s \in (0, 1)$ and some $p \in (0, \infty)$.

The proofs of Theorems 1.1 and 1.2 will be given in Section 2. We borrow some ideas from [13, 14, 18, 24, 32].

Case (i)⇒(ii). If $p \in [1, \infty)$, the proof has already been given by [16, 24] via constructing an extension operator with the mean value $u_X = \frac{1}{|X|}\int_X u(z)\,dz$. Recall that the procedure to construct an extension operator was essentially known after [18]. If $p \in (0, 1)$, the mean value u_X makes no sense since a function $u \in W^{s,p}(\Omega)$ may fail to be local integrable. So the extension operator in [14, 24] with the mean value does not work here. To overcome the possible non-integrability, we improve the extension operator in [14, 24] by replacing the mean vale u_X with the median value $m_u(X)$ defined in (2.2) below; this is the main novelty of this paper. The point is that the median value $m_u(X)$ is well defined for arbitrary measurable functions and enjoys the nice properties (2.5) and (2.6); see Lemma 2.2. The extension operator with the median value works for all $p \in (0, \infty)$ as shown in Section 2.

Case (ii)⇒(iv) or (iii)⇒(v). When $n = sp$, we need Lemma 2.4 below; when $sp < n$ and $p \in (0, 1)$, we need to use property (2.6) of the median value; the other cases are well known (see for example [21, Theorems 6.7 and 8.2]).

Case (v)⇒(i). We first control the $W^{s,p}(\Omega)$-norms of test functions by using the volume of the ball $B(x, r) \cap \Omega$ in Lemma 2.4 below. Then with a suitable slicing of the ball $B(x, r) \cap \Omega$ and iteration, we obtain a lower bound Cr^n for $|B(x, r) \cap \Omega|$ and hence give (1.3). We should point out that the idea to derive the measure density property from the imbedding was originally invented by Hajłasz, Koskela and Tuoninen [13] for Sobolev $W^{1,p}$-extension domains with $p \in [1, \infty)$. Here we adapt their arguments to the setting of the fractional Sobolev $W^{s,p}$-extension.

Finally, we make some remarks. The first remark says that the geometric characterizations of $W^{s,p}$-extension/-imbedding domains have some jumps both when p goes from $p < \infty$ to $p = \infty$ for fixed $s \in (0, 1)$ and when s goes from $s < 1$ to $s = 1$ for fixed $p \in [1, \infty]$. In the second remark, we state some related results. The third remark focuses on a domain which is not a $W^{s,p}$-extension domain for all $s \in (0, 1)$ and all $p > 2/s$.

Remark 1.3. At the endpoint case $s \in (0, 1)$ and $p = \infty$, and case $s = 1$ and $p \in [1, \infty]$, the geometric characterizations of $W^{s,p}$-extension/-imbedding domains are quite different from Theorems 1.1 and 1.2. Precisely,

Case $s \in (0, 1)$ and $p = \infty$. We can define $W^{s,\infty}(\Omega)$ exactly by (1.1) with the norm

$$\|u\|_{W^{s,\infty}(\Omega)} \equiv \|u\|_{L^\infty(\Omega)} + \sup_{x,\, y\in\Omega,\, x\neq y} \frac{|u(x) - u(y)|}{|x - y|^s}.$$

Then every domain is a $W^{s,\infty}$-extension domain. Indeed, notice that $W^{s,\infty}(\Omega)$ is exactly the space of Hölder continuous functions of order s, and can be viewed as the Lipschitz space with respect to the distance $|\cdot - \cdot|^s$. By the McShane extension (see for example [15, Section 2.2]), every function $u \in W^{s,\infty}$ can be extended to a function $\overline{u}$, defined by

$$\overline{u}(x) = \sup_{z\in\Omega}[u(z) + L|x - z|^s]$$

for all $x \in \mathbb{R}^n$, where $L = \sup_{x,\, y\in\Omega,\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^s}$. Set

$$\widetilde{u} = \min\{\|u\|_{L^\infty(\Omega)}, \max\{-\|u\|_{L^\infty(\Omega)}, \overline{u}\}\}.$$

It is easy to check that $\widetilde{u} = u$ on Ω, and, moreover, we obtain $\widetilde{u} \in W^{s,\infty}(\mathbb{R}^n)$ with $\|\widetilde{u}\|_{W^{s,\infty}(\mathbb{R}^n)} \le \|u\|_{W^{s,\infty}(\Omega)}$ as desired. We omit the details.

Case $s = 1$ and $p \in [1, \infty]$. Define $W^{1,p}(\Omega)$ as the classical Sobolev space, that is, $W^{1,p} = \{u \in L^p(\mathbb{R}^n) : \nabla u \in L^p(\Omega)\}$ with the norm $\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$, where ∇u denotes the distributional gradient of u. Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. Then Gehring and Martio [8] proved that Ω is a $W^{1,\infty}$-extension domain if and only if it is quasiconvex. Goldshtein, Latfullin and Vodop'yanov proved that Ω is a $W^{1,2}$-extension domain if and only if it is a uniform domain; see [11, 12, 31] and also [18]. When $p \in (2, \infty)$, Buckley, Koskela and Shvartsman proved that Ω is a $W^{1,p}$-extension/-imbedding domain if and only if it is a weak $(p - 2)/(p - 1)$-cigar domain; see [3, 19, 25]. When $p \in [1, 2)$, Ω

is a $W^{1,p}$-imbedding domain if and only if it is a John domain; see [2]. Higher dimensional analogies were also established therein.

Remark 1.4. The following results are closely relevant to our Theorems 1.1 and 1.2. In [13, 14], Hajłasz, Koskela and Tuominen first proved that the $W^{m,p}$-extension/-imbedding domains satisfy the measure density property (1.3) for all $m \in \mathbb{N}$ and $p \in [1, \infty)$. With the aid of (1.3), they further give a characterization of Sobolev $W^{1,p}$-extension domains for all $p \in [1, \infty)$. Rychkov [23] and Triebel [29] consider the extensions and restrictions of Besov and Triebel-Lizorkin spaces on Lipschitz domains. Moreover, Shvartsman [24] considered the extensions and restrictions of Besov and Triebel-Lizorkin spaces on regular domains. This, as well as the extension of Besov spaces on regular sets established by Jonsson and Wallin [17], also works for $W^{s,p}(\Omega)$ when $s \in (0, 1)$ and $p \in [1, \infty]$.

Remark 1.5. In this remark, we check by hand (without using Theorems 1.1 and 1.2) that the domain $\Omega \equiv \{x = (x_1, x_2) \in \mathbb{R}^2 : \ 0 < x_1 < 1, \ 0 < x_2 < x_1^2\}$ is not a $W^{s,p}$-extension domain for all $s \in (0, 1)$ and all $p > 2/s$. Such a fact was already well understood by mathematicians.

Let $s - 3/p < \alpha < s - 2/p$, and set $u(x) = |x|^\alpha$. Then $u \in W^{s,p}(\Omega)$. Indeed, by $|u(x) - u(y)| \sim |x|^{\alpha-1}|x - y|$, if $|x - y| \le |x|/2$ and $|u(x) - u(y)| \le |x - y|^\alpha$, we write

$$\|u\|^p_{W^{s,p}(\Omega)} \lesssim \int_\Omega |x|^{\alpha p}\, dx + \int_\Omega \int_{B(x,\,|x|/2)} \frac{|x|^{p(\alpha-1)}}{|x-y|^{2-p+sp}}\, dy\, dx + \int_\Omega \int_{\Omega\setminus B(x,\,|x|/2)} \frac{1}{|x-y|^{2+sp-\alpha p}}\, dy\, dx.$$

Observing that when $x \neq 0$, $\int_{B(x,\,|x|/2)} \frac{1}{|x-y|^{2-p+sp}}\, dy \lesssim |x|^{p-sp}$ and

$$\int_{\mathbb{R}^n\setminus B(x,\,|x|/2)} \frac{1}{|x-y|^{2+sp-\alpha p}}\, dy \lesssim |x|^{\alpha p - sp},$$

by $2 + \alpha p - sp > -1$ (due to $s - 3/p < \alpha$), we have

$$\|u\|^p_{W^{s,p}(\Omega)} \lesssim 1 + \int_0^1 \int_0^{x_1^2} \frac{1}{|x|^{-\alpha p+sp}}\, dx_2\, dx_1 \lesssim 1 + \int_0^1 x_1^{2+\alpha p - sp}\, dx_1 \lesssim 1.$$

Assume that u can be extended continuously as a function $\widetilde{u} \in W^{s,p}(\mathbb{R}^2)$. Then by the imbedding of $W^{s,p}(\mathbb{R}^2)$ into the space of Hölder continuous functions or order $s - 2/p$, we know that

$$\sup_{x,\,y\in\Omega, x\neq y} \frac{|u(x) - u(y)|}{|x-y|^{s-2/p}} \lesssim \|u\|_{W^{s,p}(\Omega)}.$$

However, it is easy to see that

$$\frac{|u(x) - u(x/2)|}{|x - x/2|^{s-2/p}} \sim |x|^{\alpha - s + 2/p} \to \infty$$

as $x \to 0$, which is a contradiction. So Ω is not a $W^{s,p}$-extension domain.

The notation used in what follows is standard. We denote by C a positive constant which is independent of the main parameters, but which may vary from line to line. Constants with subscripts, such as C_0, do not change in different occurrences. The symbol $A \lesssim B$ or $B \gtrsim A$ means that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \sim B$. For any locally integrable function u and measurable set X, we denote by $\fint_X u$ the average of f on X, namely, $\fint_X f \equiv \frac{1}{|X|}\int_X f\,dx$. For a set Ω and $x \in \mathbb{R}^n$, we use $d(x, \Omega)$ to denote $\inf_{z\in\Omega} |x - z|$, the distance from x to Ω.

2. Proofs of Theorems 1.1 and 1.2

Obviously, it is easy to see that (ii)⇒(iii) and (iv)⇒(v). It suffices to prove (i)⇒(ii), (ii)⇒(iv), (iii)⇒(v), and (v)⇒(i) . Without loss of generality, we assume that $\operatorname{diam}\Omega = \sup_{x,\, y\in\Omega} |x - y| \ge 2$.

(i)⇒(ii). We first observe that $|\overline{\Omega} \setminus \Omega| = 0$; see [14, Lemma 9], but we also give the argument here. Indeed, for every $x \in \overline{\Omega} \setminus \Omega$ and $r \in (0, 1]$, take $x_j \in \Omega$ such that $x_j \to x$ as $j \to \infty$. We have

$$|B(x, r) \cap \Omega| = \lim_{j\to\infty} |B(x_j, r) \cap \Omega| \ge C_1 r^n,$$

and hence

$$\limsup_{r\to 0} \frac{|B(x, r) \cap (\overline{\Omega} \setminus \Omega)|}{|B(x, r)|} \le 1 - C_1 C(n) < 1.$$

Thus x is not a Lebesgue point of $\chi_{\overline{\Omega}\setminus\Omega}$. By the Lebesgue differential theorem, the set of non-Lebesgue points, and hence $\overline{\Omega} \setminus \Omega$, has measure 0. Therefore, without loss of generality, we may assume that $\overline{\Omega} \ne \mathbb{R}^n$. Let $U \equiv \mathbb{R}^n \setminus \overline{\Omega}$. Then U is an open set and hence enjoys the following Whitney covering; see, for example, [14, Lemma 7].

Lemma 2.1. *There exist a family $\{B(x_i, r_i)\}_{i\in I}$ of countable balls and a constant $M \ge 1$ such that*

1. *$r_i = d(x_i, \Omega)/10$ for all $i \in I$, and the family of balls $\{B(x_i, r_i/5)\}_{i\in I}$ is a maximal family of pairwise disjoint balls;*
2. *$U = \bigcup_{i\in I} B(x_i, r_i) = \bigcup_{i\in I} B(x_i, 5r_i)$;*
3. *if $x \in B(x_i, 5r_i)$ for some $i \in I$, then $5r_i < d(x, \Omega) < 15r_i$;*
4. *for each $i \in I$, there is $x_i^* \in \Omega$ such that $d(x_i, x_i^*) < 15r_i$;*
5. *$\sum_{i\in I} \chi_{B(x_i,5r_i)}(x) \le M$ for all $x \in U$.*

Associated to this covering, there exists a partition of unity (see for example [14, Lemma 8]). That is, there exists a family of smooth functions $\{\varphi_i\}_{i\in I}$ such that

1. $\operatorname{supp}\varphi_i \subset 2B(x_i, r_i)$ for all $i \in I$;
2. $\varphi_i(x) \ge 1/M$ for all $x \in B(x_i, r_i)$ and all $i \in I$;
3. there exists a constant $L > 0$ such that for all $i \in I$, $|\nabla\varphi_i| \le L/r_i$;
4. $\sum_{i\in I} \varphi_i = \chi_\Omega$.

Let J be the collection of all $i \in I$ such that $r_i \le 1$, and set $V \equiv \{x \in \mathbb{R}^n : d(x, \Omega) \le 8\}$. For each $u \in W^{s,\,p}(\Omega)$, we define

$$\widetilde{E}u(x) \equiv \begin{cases} u(x), & x \in \Omega,\\ 0, & x \in \overline{\Omega} \setminus \Omega,\\ \sum_{i\in J} \varphi_i(x) m_u(B(x_i^*, r_i) \cap \Omega), & x \in U, \end{cases} \tag{2.1}$$

where x_i^* is as in Lemma 2.1 and $m_u(X)$ denotes the *median value* of u on set X defined by

$$m_u(X) \equiv \max\left\{a \in \mathbb{R},\, |\{x \in X :\ u(x) < a\}| \le \frac{|X|}{2}\right\}. \tag{2.2}$$

Moreover, let Ψ be a Lipschitz function on $\mathbb{R}^n$ such that $\Psi = 1$ on Ω, $\Psi = 0$ on $\mathbb{R}^n \setminus V$ and $0 \le \Psi \le 1$ on $V \setminus \Omega$. Set

$$Eu \equiv \Psi \widetilde{E}u. \tag{2.3}$$

We point out that the extension operator $\widetilde{E}$ in (2.1), and hence E in (2.3), is an improvement of the construction of [14, Proof of Theorem 6], where, instead of the median value $m_u(B(x_i^*, r_i) \cap \Omega)$, they use the mean value $⨍_{B(x_i^*, r_i)\cap\Omega} u(z)\,dz$. But we do need the median value to handle the case $p \in (0, 1)$ since it has the important property (2.6).

We are going to show that Eu gives the desired extension of u into $W^{s,p}(\mathbb{R}^n)$. Obviously, $Eu = u$ on Ω. To see $\|Eu\|_{W^{s,p}(\mathbb{R}^n)} \lesssim \|u\|_{W^{s,p}(\Omega)}$, it suffices to prove that $\widetilde{E}u \in W^{s,p}(V)$ and

$$\|\widetilde{E}u\|_{W^{s,p}(V)} \lesssim \|u\|_{W^{s,p}(\Omega)}. \tag{2.4}$$

Indeed, assume that (2.4) holds for the moment. From this and $0 \le \Psi \le 1$, it follows that

$$\|Eu\|_{L^p(\mathbb{R}^n)} \le \|\widetilde{E}u\|_{L^p(V)} \lesssim \|u\|_{L^p(\Omega)}.$$

Moreover, by Fubini's theorem, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|Eu(x) - Eu(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \\
&\quad = \int_V \int_V \frac{|\Psi(x)\widetilde{E}u(x) - \Psi(y)\widetilde{E}u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \\
&\quad \lesssim \int_V \left(\int_{V\setminus B(x,1)} \frac{|\widetilde{E}u(x)|^p + |\widetilde{E}u(y)|^p}{|x-y|^{n+sp}}\,dy\right)dx \\
&\qquad + \int_V \left(\int_{V\cap B(x,1)} \frac{|\Psi(x)\widetilde{E}u(x) - \Psi(y)\widetilde{E}u(y)|^p}{|x-y|^{n+sp}}\,dy\right)dx \\
&\quad \equiv L_1 + L_2.
\end{aligned}$$

By noticing that $\int_{\mathbb{R}^n\setminus B(x,1)} \frac{1}{|x-y|^{n+sp}}\,dx \lesssim 1$ and $\int_{\mathbb{R}^n\setminus B(y,1)} \frac{1}{|x-y|^{n+sp}}\,dy \lesssim 1$ and by Fubini's theorem, we have

$$\begin{aligned}
L_1 &\le \int_V |\widetilde{E}u(x)|^p \left(\int_{V\setminus B(x,1)} \frac{1}{|x-y|^{n+sp}}\,dy\right)dx \\
&\qquad + \int_V |\widetilde{E}u(y)|^p \left(\int_{V\setminus B(y,1)} \frac{1}{|x-y|^{n+sp}}\,dx\right)dy \\
&\lesssim \|\widetilde{E}u\|^p_{L^p(V)}.
\end{aligned}$$

Since $\int_{B(x,1)} |x-y|^{-n-sp+p}\,dy \lesssim 1$, $0 \le \Psi \le 1$ and Ψ is a Lipschitz function, we also have

$$\begin{aligned}
L_2 &\lesssim \int_V \left(\int_{V\cap B(x,1)} \frac{|\widetilde{E}u(x) - \widetilde{E}u(y)|^p}{|x-y|^{n+sp}}\,dy\right)dx \\
&\qquad + \int_V \left(\int_{V\cap B(x,1)} \frac{|\Psi(x) - \Psi(y)|^p|\widetilde{E}u(x)|^p}{|x-y|^{n+sp}}\,dy\right)dx
\end{aligned}$$

$$\lesssim \|\widetilde{E}u\|^p_{W^{s,\,p}(V)} + \int_V |\widetilde{E}u(x)|^p \left(\int_{B(x,\,1)} \frac{1}{|x-y|^{n+sp-p}}\,dy\right)\,dx$$
$$\lesssim \|\widetilde{E}u\|^p_{W^{s,\,p}(V)}.$$

Thus $Eu \in W^{s,\,p}(\mathbb{R}^n)$ and $\|Eu\|_{W^{s,\,p}(\mathbb{R}^n)} \lesssim \|\widetilde{E}u\|_{W^{s,\,p}(\Omega)}$. So if (2.4) holds, we will have $Eu \in W^{s,\,p}(\mathbb{R}^n)$ and $\|Eu\|_{W^{s,\,p}(\mathbb{R}^n)} \lesssim \|u\|_{W^{s,\,p}(\Omega)}$.

To prove (2.4), we need the following important properties of the median value, which were essentially proved in [5, Lemma 2.4] and [10, (2.4)] (see also [20, (5.9)]).

Lemma 2.2. *For every $\delta \in (0,\,1]$ and $u \in L^\delta_{\mathrm{loc}}(\Omega)$, we have*

$$u(x) = \lim_{r\to 0} m_u(B(x,\,r)\cap\Omega) \tag{2.5}$$

for almost all $x \in \Omega$. Moreover, for every ball B with its center in Ω and each $c \in \mathbb{R}$,

$$|m_u(B\cap\Omega) - c| \le \left\{2 \fint_{B\cap\Omega} |u(w)-c|^\delta\,dw\right\}^{1/\delta}. \tag{2.6}$$

Proof. Observe that for all $0 < r < \operatorname{dist}(x,\,\partial\Omega)$, $B(x,\,r)\cap\Omega = B(x,\,r)$, and hence $m_u(B(x,\,r)\cap\Omega) = m_u(B(x,\,r))$. So the first conclusion (2.5) was exactly the one proved in [5, Lemma 2.2]. The second conclusion (2.6) was essentially proved in [10, (2.4)] (see also [20, (5.9)]). For convenience, we write the details here.

Let B be an arbitrary ball centered at Ω, and let $c \in \mathbb{R}$. We first claim that

$$m_u(B\cap\Omega) - c = m_{u-c}(B\cap\Omega) \text{ and } |m_u(B\cap\Omega)| \le m_{|u|}(B\cap\Omega).$$

Indeed, observe that

$$|\{x \in B\cap\Omega, u(x) < m_u(B\cap\Omega)\}| \le |B\cap\Omega|/2$$

implies that

$$|\{x \in B\cap\Omega, u(x) - c < m_u(B\cap\Omega) - c\}| \le |B\cap\Omega|/2.$$

By the definition of $m_{u-c}(B\cap\Omega)$, we have $m_u(B\cap\Omega) - c \le m_{u-c}(B\cap\Omega)$. Similarly,

$$|\{x \in B\cap\Omega, u(x) - c < m_{u-c}(B\cap\Omega)\}| \le |B\cap\Omega|/2$$

implies that

$$|\{x \in B\cap\Omega, u(x) < m_{u-c}(B\cap\Omega) + c\}| \le |B\cap\Omega|/2.$$

So by the definition of $m_u(B\cap\Omega)$, we have $m_{u-c}(B\cap\Omega) + c \le m_u(B\cap\Omega)$. Therefore, we have $m_u(B\cap\Omega) - c = m_{u-c}(B\cap\Omega)$. Moreover, if $m_u(B\cap\Omega) \ge 0$, then

$$|\{x \in B\cap\Omega, |u(x)| < m_u(B\cap\Omega)\}| \le |\{x \in B\cap\Omega, u(x) < m_u(B\cap\Omega)\}| \le |B\cap\Omega|/2,$$

and hence, by the definition of $m_{|u|}(B\cap\Omega)$, implies that $m_u(B\cap\Omega) \le m_{|u|}(B\cap\Omega)$. If $m_u(B\cap\Omega) < 0$, for every $0 < a < |m_u(B\cap\Omega)|$, we have

$$\begin{aligned}|\{x \in B\cap\Omega, |u(x)| < a\}| &\le |\{x \in B\cap\Omega, u(x) > -a\}|\\ &= |B\cap\Omega| - |\{x \in B\cap\Omega, u(x) \le -a\}|.\end{aligned}$$

Observe that

$$|\{x \in B\cap\Omega, u(x) \le -a\}| \ge |B\cap\Omega|/2;$$

otherwise, $-a \le m_u(B\cap\Omega)$, which is a contradiction. We obtain

$$|\{x \in B\cap\Omega, |u(x)| < a\}| \le |B\cap\Omega|/2,$$

which implies that $a \le m_{|u|}(B\cap\Omega)$, and hence $|m_u(B\cap\Omega)| \le m_{|u|}(B\cap\Omega)$.

The above claim leads to that

$$|m_u(B\cap\Omega) - c| = |m_{u-c}(B\cap\Omega)| \le m_{|u-c|}(B\cap\Omega).$$

By this, (2.6) is reduced to

$$m_{|u-c|}(B\cap\Omega) \le \left\{2 \fint_{B\cap\Omega} |u(w)-c|^\delta\, d\mu(w)\right\}^{1/\delta}. \tag{2.7}$$

To see (2.7), set $\sigma \equiv \fint_{B\cap\Omega}|u(w)-c|^\delta\, dw$. By Chebyshev's inequality, for every $a > 2$, we have

$$\begin{aligned}\left|\left\{w\in B\cap\Omega:\ |u(w)-c| \ge (a\sigma)^{1/\delta}\right\}\right| &= \left|\left\{w\in B\cap\Omega:\ |u(w)-c|^\delta \ge a\sigma\right\}\right|\\ &\le (a\sigma)^{-1}\int_{B\cap\Omega}|u(w)-c|^\delta\, dw\\ &< \frac{|B\cap\Omega|}{2}.\end{aligned}$$

This yields that

$$\left|\left\{w\in B\cap\Omega:\ |u(w)-c| < (a\sigma)^{1/\delta}\right\}\right| > \frac{|B\cap\Omega|}{2},$$

and hence, by the definition of $m_{|u-c|}(B\cap\Omega)$, we have $m_{|u-c|}(B\cap\Omega) \le (a\sigma)^{1/\delta}$. Letting $a\to 2$, we obtain (2.7) and hence prove (2.6). This finishes the proof of Lemma 2.2. $\square$

We return to the proof of (2.4). First, we show that $\|\widetilde{E}u\|_{L^p(V)} \lesssim \|u\|_{L^p(\Omega)}$. For $x \in V\setminus\overline{\Omega}$, denote by I_x the collection of $i\in I$ such that $x\in B(x_i,\, 2r_i)$. Then by Lemma 2.1,

$$\sharp I_x \le M, \tag{2.8}$$

and for $i \in I_x$, by (1.3),

$$\begin{aligned}&B(x_i^*,\, r_i) \subset B(x,\, 5d(x,\,\Omega)),\\ &|B(x_i^*,\, r_i)\cap\Omega| \sim |B(x,\, 5d(x,\,\Omega))\cap\Omega| \sim |B(x,\, 5d(x,\,\Omega))|.\end{aligned} \tag{2.9}$$

Notice that if $i\in I\setminus J$, then $r_i \ge 1$ and hence $d(z,\,\Omega) > 8r_i \ge 8$ for all $z \in B(x_i,\, 2r_i)$; that is, $B(x_i,\, 2r_i)\cap\Omega = \emptyset$. Thus $I_x \subset J$ and $\sum_{i\in I_x}\varphi_i(x) = 1$. Take $\delta \equiv \min\{1/2,\, p/2\}$. By Lemma 2.2, (2.8) and (2.9), we have

$$\begin{aligned}\widetilde{E}u(x) &\le \sum_{i\in I_x}\varphi_i(x)|m_u(B(x_i^*,\, r_i)\cap\Omega)|\\ &\lesssim \sum_{i\in I_x}\varphi_i(x)\left(\fint_{B(x_i^*,\, r_i)\cap\Omega}|u(z)|^\delta\, dz\right)^{1/\delta}\\ &\lesssim \left(\fint_{B(x,\, 5d(x,\,\Omega))}|u(z)|^\delta\chi_\Omega(z)\, dz\right)^{1/\delta}\\ &\lesssim \mathcal{M}_\delta(u\chi_\Omega)(x),\end{aligned}$$

where and in what follows,

$$\mathcal{M}_\delta(g)(x) \equiv \sup_{B(x,\,r)} \left(\fint_{B(x,\,r)} |g(z)|^\delta \, dz \right)^{1/\delta} = [\mathcal{M}(|g|^\delta)]^{1/\delta},$$

and $\mathcal{M}$ is the Hardy-Littlewood operator. By the $L^{p/\delta}$-boundedness of $\mathcal{M}$, we obtain $\|\widetilde{E}u\|_{L^p(V)} \lesssim \|u\|_{L^p(\Omega)}$.

Moreover, we write

$$\begin{aligned}
&\int_V \int_V \frac{|\widetilde{E}u(x) - \widetilde{E}u(y)|^p}{|x-y|^{n+sp}} \, dx\, dy \\
&= \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dx\, dy + 2 \int_{V\setminus\Omega} \int_\Omega \frac{|\widetilde{E}u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dy\, dx \\
&\quad + \int_{V\setminus\Omega} \int_{V\setminus\Omega} \frac{|\widetilde{E}u(x) - \widetilde{E}u(y)|^p}{|x-y|^{n+sp}} \, dx\, dy \\
&\equiv H_1 + H_2 + H_3.
\end{aligned}$$

Obviously, $H_1 \le \|u\|^p_{W^{s,\,p}(\Omega)}$. For $x \in V\setminus\Omega$ and $y \in \Omega$, by $\sum_{i\in I_x} \varphi_i(x) = 1$, Lemma 2.2, (2.8) and (2.9), we obtain

$$\begin{aligned}
|\widetilde{E}u(x) - u(y)| &\le \sum_{i\in I_x} \varphi_i(x)\, |m_u(B(x_i^*,\, r_i)\cap\Omega) - u(y)| \\
&\le \sum_{i\in I_x} \varphi_i(x) \left(\fint_{B(x_i^*,\, r_i)\cap\Omega} |u(z)-u(y)|^\delta \, dz \right)^{1/\delta} \\
&\lesssim \left(\fint_{B(x,\, 5d(x,\,\Omega))\cap\Omega} |u(z)-u(y)|^\delta \, dz \right)^{1/\delta}.
\end{aligned}$$

For $y \in \Omega$ and $z \in B(x,\, 5d(x,\,\Omega)) \cap \Omega$, since $|x-y| \ge d(x,\,\Omega)$, we always have

$$|z-y| \le |z-x| + |x-y| \le 5d(x,\,\Omega) + |x-y| \lesssim |x-y|.$$

Hence

$$\begin{aligned}
\frac{|\widetilde{E}u(x) - u(y)|}{|x-y|^{n/p+s}} &\lesssim \left(\fint_{B(x,\,5d(x,\,\Omega))\cap\Omega} \frac{|u(z)-u(y)|^\delta}{|z-y|^{n\delta/p+s\delta}} \, dz \right)^{1/\delta} \\
&\lesssim \mathcal{M}_\delta \left(\frac{|u(\cdot)-u(y)|}{|\cdot - y|^{n/p+s}} \chi_\Omega(\cdot) \right)(x),
\end{aligned}$$

which together with the $L^{p/\delta}$-boundedness of Hardy-Littlewood maximal operator implies that

$$\begin{aligned}
H_2 &\lesssim \int_{V\setminus\Omega} \int_\Omega \left[\mathcal{M}_\delta \left(\frac{|u(\cdot)-u(y)|}{|\cdot - y|^{n/p+s}} \chi_\Omega(\cdot) \right)(x) \right]^p dy\, dx \\
&\lesssim \int_\Omega \int_{\mathbb{R}^n} \left[\mathcal{M}_\delta \left(\frac{|u(\cdot)-u(y)|}{|\cdot - y|^{n/p+s}} \chi_\Omega(\cdot) \right)(x) \right]^p dx\, dy \\
&\lesssim \int_\Omega \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n/p+s}} \chi_\Omega(x) \, dx\, dy \\
&\lesssim \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dx\, dy \lesssim \|u\|^p_{W^{s,\,p}(\Omega)}.
\end{aligned}$$

To estimate H_3 for the given x, we split $V \setminus \Omega$ into two parts:

$$X_1(x) \equiv \{y \in V \setminus \Omega : \ |x-y| \ge \frac{1}{2} \max\{d(x, \Omega), d(y, \Omega)\}\}$$

and

$$X_2(x) \equiv \{y \in V \setminus \Omega : \ |x-y| < \frac{1}{2} \max\{d(x, \Omega), d(y, \Omega)\}\}.$$

Write

$$\begin{aligned} H_3 &= \int_{V\setminus\Omega} \int_{X_1(x)} \frac{|\widetilde{E}u(x) - \widetilde{E}u(y)|^p}{|x-y|^{n+sp}} \, dy \, dx + \int_{V\setminus\Omega} \int_{X_2(x)} \frac{|\widetilde{E}u(x) - \widetilde{E}u(y)|^p}{|x-y|^{n+sp}} \, dy \, dx \\ &\equiv H_{3,1} + H_{3,2}. \end{aligned}$$

If $x \in V \setminus \Omega$ and $y \in X_1(x)$, by

$$\sum_{i \in I_x} \varphi_i(x) = 1 = \sum_{i \in I_y} \varphi_i(y)$$

we have

$$\widetilde{E}u(x) - \widetilde{E}u(y) = \sum_{i \in I_x} \sum_{j \in I_y} \varphi_i(x)\varphi_j(y)[m_u(B(x_i^*, r_i) \cap \Omega) - m_u(B(x_j^*, r_j) \cap \Omega)].$$

Applying Lemma 2.2 twice, by (2.9) we have

$$\begin{aligned} &|m_u(B(x_i^*, r_i) \cap \Omega) - m_u(B(x_j^*, r_j) \cap \Omega)| \\ &\lesssim \left(\fint_{B(x_i^*, r_i)\cap\Omega} \fint_{B(x_j^*, r_j)\cap\Omega} |u(z) - u(w)|^\delta \, dz \, dw \right)^{1/\delta} \\ &\lesssim \left(\fint_{B(y, 5d(x, \Omega))\cap\Omega} \fint_{B(x, 5d(y, \Omega))\cap\Omega} |u(z) - u(w)|^\delta \, dz \, dw \right)^{1/\delta}. \end{aligned}$$

Observe that for all $z \in B(x, 5d(x, \Omega)) \cap \Omega$ and $w \in B(y, 5d(y, \Omega)) \cap \Omega$, since $|x-y| \ge \frac{1}{2} \max\{d(x, \Omega), d(y, \Omega)\}$,

$$|z-w| \le |x-y| + 5d(x, \Omega) + 5d(y, \Omega) \lesssim |x-y|.$$

This together with (2.8) leads to the fact that

$$\begin{aligned} \frac{|\widetilde{E}u(x) - \widetilde{E}u(y)|}{|x-y|^{n/p+s}} &\lesssim \left(\fint_{B(x, 5d(x, \Omega))\cap\Omega} \fint_{B(y, 5d(y, \Omega))\cap\Omega} \frac{|u(z) - u(w)|^\delta}{|z-w|^{n\delta/p+s\delta}} \, dz \, dw \right)^{1/\delta} \\ &\lesssim (\mathcal{M}_\delta \times \mathcal{M}_\delta)(F)(x, y), \end{aligned}$$

where

$$F(z, w) \equiv \frac{|u(z) - u(w)|}{|w-z|^{n/p+s}} \chi_\Omega(z) \chi_\Omega(w),$$

and $(\mathcal{M}_\delta \times \mathcal{M}_\delta)(F)(x, y)$ denotes the iterated Hardy-Littlewood maximal function of F. That is, first for a given w, taking the maximal function of $F(z, w)$ with respect to the variable z and evaluating at x, we get $\mathcal{M}_\delta(F(\cdot, w)(x)$. Then for a given x, taking maximal function of $\mathcal{M}_\delta(F(\cdot, w)(x)$ with respect to w and evaluating at y, we obtain $(\mathcal{M}_\delta \times \mathcal{M}_\delta)(F)(x, y)$.

By the $L^{p/\delta}$-boundedness of Hardy-Littlewood operator, we obtain

$$\begin{aligned} H_{3,\,1} &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [(\mathcal{M}_\delta \times \mathcal{M}_\delta)(F)(x,\,y)]^p\,dx\,dy \\ &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\chi_\Omega(x)\chi_\Omega(y)\,dx\,dy \\ &\lesssim \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dy\,dx \\ &\lesssim \|u\|^p_{W^{s,\,p}(\Omega)}. \end{aligned}$$

If $x \in V \setminus \Omega$ and $y \in X_2(x)$, noticing that $\sum_{i\in I_x\cup I_y}[\varphi_i(x)-\varphi_i(y)] = 0$, by Lemma 2.1, we arrive at

$$\begin{aligned} &|\widetilde{E}u(x) - \widetilde{E}u(y)| \\ &\quad= \left|\sum_{i\in I_x\cup I_y}[\varphi_i(x)-\varphi_i(y)]m_u(B(x_i^*,\,r_i)\cap\Omega)\right| \\ &\quad= \left|\sum_{i\in I_x\cup I_y}[\varphi_i(x)-\varphi_i(y)]\left[m_u(B(x_i^*,\,r_i)\cap\Omega) - m_u(B(x,\,5d(x,\,\Omega))\cap\Omega)\right]\right| \\ &\quad\lesssim \sum_{i\in I_x\cup I_y}\frac{|x-y|}{r_i}\left|m_u(B(x_i^*,\,r_i)\cap\Omega) - m_u(B(x,\,5d(x,\,\Omega))\cap\Omega)\right|. \end{aligned}$$

Applying Lemma 2.2 twice with $\delta < p$ and the Hölder inequality, we have

$$\begin{aligned} &|m_u(B(x_i^*,\,r_i)\cap\Omega) - m_u(B(x,\,5d(x,\,\Omega))\cap\Omega)| \\ &\quad\le \left\{ ⨍_{B(x_i^*,\,r_i)\cap\Omega} ⨍_{B(x,\,5d(x,\,\Omega))\cap\Omega} |u(z)-u(w)|^\delta\,dw\,dz \right\}^{1/\delta} \\ &\quad\le \left\{ ⨍_{B(x_i^*,\,r_i)\cap\Omega} \left[⨍_{B(x,\,5d(x,\,\Omega))\cap\Omega} |u(z)-u(w)|^p\,dw \right]^{\delta/p} dz \right\}^{1/\delta}. \end{aligned}$$

For all $i \in I_x \cup I_y$, we claim that $r_i \sim d(x,\,\Omega)$,

$$B(x_i^*,\,r_i) \subset B(x,\,20d(x,\,\Omega)) \text{ and } |B(x_i^*,\,r_i)| \sim |B(x,\,20d(x,\,\Omega))\cap\Omega|.$$

If $i \in I_x$, this follows from (2.9). To see this for $i \in I_y$, observe that

$$\frac{1}{3}d(y,\,\Omega) \le d(x,\,\Omega) \le 3d(y,\,\Omega). \tag{2.10}$$

Indeed, taking $\overline{y} \in \overline{\Omega}$ so that $|y-\overline{y}| = d(y,\,\Omega)$, by $|x-y| \le \frac{1}{2}\max\{d(x,\,\Omega),\,d(y,\,\Omega)\}$ we have

$$\begin{aligned} d(x,\,\Omega) &\le |x-\overline{y}| \le |x-y| + |y-\overline{y}| \\ &\le \frac{1}{2}d(x,\,\Omega) + \frac{1}{2}d(y,\,\Omega) + d(y,\,\Omega) \\ &= \frac{1}{2}d(x,\,\Omega) + \frac{3}{2}d(y,\,\Omega), \end{aligned}$$

which implies that $d(x,\Omega)\le 3d(y,\Omega)$. Similarly, we have $d(y,\Omega)\le 3d(x,\Omega)$. Thus $B(y,5d(y,\Omega))\subset B(x,20d(x,\Omega))$, and by (1.3),

$$|B(y,5d(y,\Omega))\cap\Omega|\sim|B(x,20d(x,\Omega))\cap\Omega|.$$

So, for all $i\in I_y$, by Lemma 2.1 and (1.3) we have

$$\begin{aligned}&B(x_i^*,r_i)\subset B(y,5d(y,\Omega))\subset B(x,20d(x,\Omega)),\\ &|B(x_i^*,r_i)\cap\Omega|\sim|B(y,5d(y,\Omega))\cap\Omega|\sim|B(x,20d(x,\Omega))\cap\Omega|,\end{aligned}$$

as desired in the above claim.

Moreover, by (2.10), for all $z,w\in B(x,20d(x,\Omega))\cap\Omega$, we have $|z-w|\lesssim d(x,\Omega)$. This together with $\sharp I_x+\sharp I_y\lesssim 1$ (by Lemma 2.1) yields that

$$\begin{aligned}&|\widetilde{E}u(x)-\widetilde{E}u(y)|\\ &\le\frac{|x-y|}{d(x,\Omega)}\left\{\fint_{B(x,20d(x,\Omega))\cap\Omega}\left[\fint_{B(x,5d(x,\Omega))\cap\Omega}|u(z)-u(w)|^p\,dw\right]^{\delta/p}dz\right\}^{1/\delta}\\ &\le\frac{|x-y|}{d(x,\Omega)^{1-s}}\left\{\fint_{B(x,20d(x,\Omega))\cap\Omega}\left[\int_{B(x,5d(x,\Omega))\cap\Omega}\frac{|u(z)-u(w)|^p}{|z-w|^{n+sp}}dw\right]^{\delta/p}dz\right\}^{1/\delta}\\ &\le\frac{|x-y|}{d(x,\Omega)^{1-s}}\mathcal{M}_\delta\left(\left(\int_\Omega\frac{|u(\cdot)-u(w)|^p}{|\cdot-w|^{n+sp}}\,dw\right)^{1/p}\chi_\Omega(\cdot)\right)(x).\end{aligned}$$

This implies that

$$\begin{aligned}H_{3,2}\lesssim&\int_{V\setminus\Omega}\left(\int_{X_2(x)}\frac{|x-y|^{p-n-sp}}{d(x,\Omega)^{p-sp}}\,dy\right)\\ &\times\left[\mathcal{M}_\delta\left(\left(\int_\Omega\frac{|u(\cdot)-u(w)|^p}{|\cdot-w|^{n+sp}}\,dw\right)^{1/p}\chi_\Omega(\cdot)\right)(x)\right]^p dx.\end{aligned}$$

Since (2.10) implies that $X_2(x)\subset B(x,15d(x,\Omega))$, we have

$$\int_{X_2(x)}\frac{|x-y|^{p-n-sp}}{d(x,\Omega)^{p-sp}}\,dy\lesssim\int_{B(x,15d(x,\Omega))}\frac{|x-y|^{p-n-sp}}{d(x,\Omega)^{p-sp}}\,dy\lesssim 1.$$

This together with the $L^p(\mathbb{R}^n)$-boundedness of $\mathcal{M}_\delta$ yields that

$$\begin{aligned}H_{3,2}&\lesssim\int_{\mathbb{R}^n}\left[\mathcal{M}_\delta\left(\left(\int_\Omega\frac{|u(\cdot)-u(w)|^p}{|\cdot-w|^{n+sp}}\,dw\right)^{1/p}\chi_\Omega(\cdot)\right)(x)\right]^p dx\\ &\lesssim\int_{\mathbb{R}^n}\left[\left(\int_\Omega\frac{|u(x)-u(w)|^p}{|x-w|^{n+sp}}\,dw\right)^{1/p}\chi_\Omega(x)\right]^p dx\\ &\lesssim\int_\Omega\int_\Omega\frac{|u(x)-u(w)|^p}{|x-w|^{n+sp}}\,dw\,dx\\ &\lesssim\|u\|^p_{W^{s,p}(\Omega)}.\end{aligned}$$

Now we arrive at

$$H_3\le H_{3,1}+H_{3,2}\lesssim\|u\|^p_{W^{s,p}(\Omega)}.$$

Combining the estimates of H_1, H_2 and H_3 yields (2.4). This gives (iii).

(ii)⇒(iv). If $sp \neq n$ and $p \geq 1$, this is well known; see, for example, [21, Theorems 6.5 and 8.2].

Case $p \in (0, 1)$. For $u \in \dot{W}^{s,\,p}(\mathbb{R}^n)$, set

$$g(x) \equiv \sup_{j\in\mathbb{Z}} 2^{js} \left(⨍_{B(x,\,2^{-j})} |u(z) - m_u(B(x,\,2^{-j}))|^p\,dz \right)^{1/p}.$$

For each j, by Lemma 2.2 and $|x - z| \leq 2^{-j}$ for all $z \in B(x,\,2^{-j})$, we have

$$\begin{aligned}
&⨍_{B(x,\,2^{-j})} |u(z) - m_u(B(x,\,2^{-j}))|^p\,dz \\
&\lesssim ⨍_{B(x,\,2^{-j})} |u(z) - u(x)|^p\,dz + |u(x) - m_u(B(x,\,2^{-j}))|^p \\
&\lesssim ⨍_{B(x,\,2^{-j})} |u(z) - u(x)|^p\,dz \\
&\lesssim 2^{-jsp} \int_{B(x,\,2^{-j})} \frac{|u(z) - u(x)|^p}{2^{-j(n+sp)}}\,dz \\
&\lesssim 2^{-jsp} \int_{B(x,\,2^{-j})} \frac{|u(z) - u(x)|^p}{|z - x|^{n+sp}}\,dz.
\end{aligned}$$

Hence

$$g(x) \lesssim \sup_{j\in\mathbb{Z}} \left(\int_{B(x,\,2^{-j})} \frac{|u(z) - u(x)|^p}{|x - z|^{n+sp}}\,dz \right)^{1/p} \lesssim \left(\int_{\mathbb{R}^n} \frac{|u(z) - u(x)|^p}{|x - z|^{n+sp}}\,dz \right)^{1/p},$$

which implies that $\|g\|_{L^p(\mathbb{R}^n)} \lesssim \|u\|_{W^{s,\,p}(\mathbb{R}^n)}$. For each ball $B(0,\,2^{-k})$, we have

$$\begin{aligned}
\|u\|_{L^{np/(n-sp)}(B(0,\,2^{-k}))} &\leq \|u - m_u(B(\cdot,\,2^{-k}))\|_{L^{np/(n-sp)}(B(0,\,2^{-k}))} \\
&\quad + \sup_{x\in B(0,\,2^{-k})} |m_u(B(x,\,2^{-k}))||B(0,\,2^{-k})|^{(n-sp)/np}.
\end{aligned}$$

Notice that by Lemma 2.2, for each $x \in B(0,\,2^{-k})$,

$$\begin{aligned}
&|m_u(B(x,\,2^{-k}))||B(0,\,2^{-k})|^{(n-sp)/np} \\
&\leq |B(0,\,2^{-k})|^{(n-sp)/np} \left(⨍_{B(x,\,2^{-k})} |u(z)|^p\,dz \right)^{1/p} \\
&\lesssim \|u\|_{L^p(\mathbb{R}^n)} |B(0,\,2^{-k})|^{-s/n}.
\end{aligned}$$

Moreover, for almost all $x \in B(0,\,2^{-k})$, we claim that

$$|u(x) - m_u(B(x,\,2^{-k}))| \lesssim \left(\int_{B(0,\,2^{-k+2})} \frac{[g(z)]^{p/2}}{|z - x|^{n-sp/2}}\,dz \right)^{2/p}. \tag{2.11}$$

Assume that this holds for the moment. Denoting by $I_{sp/2}$ the Riesz potential of order $sp/2$ and by its boundedness from $L^2(\mathbb{R}^n)$ to $L^{2n/(n-sp)}(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\|u - m_u(B(\cdot,\,2^{-k}))\|_{L^{np/(n-sp)}(B(0,\,2^{-k}))} \\
&\lesssim \|I_{sp/2}(g^{p/2})\|^{2/p}_{L^{2n/(n-sp)}(\mathbb{R}^n)} \lesssim \|g^{p/2}\|^{2/p}_{L^2(\mathbb{R}^n)} \\
&\lesssim \|g\|_{L^p(\mathbb{R}^n)} \lesssim \|u\|_{W^{s,\,p}(\mathbb{R}^n)}.
\end{aligned}$$

Thus,

$$\|u\|_{L^{np/(n-sp)}(B(0,\,2^{-k}))} \lesssim \|u\|_{W^{s,\,p}(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} |B(0,\,2^{-k})|^{-s/n},$$

which, when $k \to \infty$, implies that $\|u\|_{L^{np/(n-sp)}(\mathbb{R}^n)} \lesssim \|u\|_{W^{s,\,p}(\mathbb{R}^n)}$.

To see (2.11), for almost all $x \in B$, we have $u(x) = \lim_{j\to\infty} m_u(B(x,\,2^{-j}))$ by Lemma 2.2 with $\Omega = \mathbb{R}^n$, and hence

$$\begin{aligned} &|u(x) - m_u(B(x,\,2^{-k}))| \\ &\quad = \left|\sum_{j\ge k} [m_u(B(x,\,2^{-j-1})) - m_u(B(x,\,2^{-j}))]\right| \\ &\quad \le \sum_{j\ge k} |m_u(B(x,\,2^{-j-1})) - m_u(B(x,\,2^{-j}))|. \end{aligned}$$

Choose a ball $B(x_j,\,2^{-j-2}) \subset B(x,\,2^{-j}) \setminus B(x,\,2^{-j-1})$. For each $z \in B(x_j,\,2^{-j-2})$, we have $B(x,\,2^{-j}) \subset B(z,\,2^{-j+2})$, and hence, by Lemma 2.2 again,

$$\begin{aligned} &|m_u(B(x,\,2^{-j})) - m_u(B(x,\,2^{-j-1}))| \\ &\quad \lesssim |m_u(B(x,\,2^{-j})) - m_u(B(z,\,2^{-j+2}))| \\ &\qquad + |m_u(B(z,\,2^{-j-4})) - m_u(B(x,\,2^{-j+2}))| \\ &\quad \lesssim \sum_{\ell=j-1}^{j} \left(⨍_{B(x,\,2^{-\ell})} |u(w) - m_u(B(z,\,2^{-j+2}))|^p \,dw \right)^{1/p} \\ &\quad \lesssim \left(⨍_{B(z,\,2^{-j+2})} |u(w) - m_u(B(z,\,2^{-j+2}))|^p \,dw \right)^{1/p} \\ &\quad \lesssim 2^{-js} g(z), \end{aligned}$$

which implies that

$$|m_u(B(x,\,2^{-j})) - m_u(B(x,\,2^{-j-1}))| \lesssim 2^{-js} \left(⨍_{B(x_j,\,2^{-j-2})} [g(z)]^{p/2} \,dz \right)^{p/2}.$$

Observing that $|z - x| \sim 2^{-j}$ for all $z \in B(x_j,\,2^{-j-2})$ and $\{B(x_j,\,2^{-j-2})\}_{j\ge k}$ are pairwise disjoint, by the Cauchy-Schwarz inequality we arrive at

$$\begin{aligned} |u(x) - m_u(B(x,\,2^{-k}))| &\lesssim \sum_{j\ge k} 2^{-js} \left(⨍_{B(x_j,\,2^{-j-2})} [g(z)]^{p/2} \,dz \right)^{p/2} \\ &\lesssim 2^{-ks} \left(\sum_{j\ge k} ⨍_{B(x_j,\,2^{-j-2})} [g(z)]^{p/2} \,dz \right)^{p/2} \\ &\lesssim \left(\int_{B(0,\,2^{-k+2})} \frac{[g(z)]^{p/2}}{|z-x|^{n-sp/2}} \,dz \right)^{p/2}. \end{aligned}$$

This gives (2.11).

Case $sp = n$. This relies on the following result.

Lemma 2.3. *Let* $s \in (0, 1)$. *Then there exist positive constants* $C_4, C > 0$ *such that for all balls* $B \subset \mathbb{R}^n$ *and* $u \in W^{s,n/s}(8B)$,

$$\int_B \exp\left(C_4 \frac{|u(x) - u_B|}{\|u\|_{W^{s,n/s}(8B)}}\right)^{n/(n-s)} dx \le C|B|. \tag{2.12}$$

Now assume that Ω is a $W^{s,n/s}$-extension domain and let $u \in W^{s,p}(\Omega)$. Then u has an extension $\widetilde{u} \in W^{s,p}(\mathbb{R}^n)$ with $\|u\|_{W^{s,p}(\mathbb{R}^n)} \le C_3\|u\|_{W^{s,p}(\Omega)}$. For each ball $B = B(x, r)$ with $x \in \Omega$ and $r \in (0, 1]$, by Lemma 2.3, we have that (2.12) holds for $\widetilde{u}$. Then $\|\widetilde{u}\|_{\dot{W}^{s,n/s}(8B)} \le \|\widetilde{u}\|_{W^{s,n/s}(\mathbb{R}^n)} \le C\|u\|_{W^{s,n/s}(\Omega)}$ yields that

$$\inf_{c\in\mathbb{R}} \int_{B\cap\Omega} \exp\left(C_3 \frac{|u(x) - c|}{C\|u\|_{W^{s,n/s}(\Omega)}}\right)^{n/(n-s)} dx \lesssim |B|.$$

This gives (iv).

Proof of Lemma 2.3. Assume that $B \equiv B(x_0, 2^{-k_0})$ for some $x_0 \in \mathbb{R}^n$ and $k_0 \in \mathbb{Z}$. Let $u \in \dot{W}^{s,p}(4B)$ and take

$$g(x) \equiv \sup_{j \ge k_0 - 2} 2^{js} \fint_{B(x,2^{-j})} |u(z) - u_{B(x,2^{-j})}|\, dz$$

for all $x \in 2B$ and $g(x) = 0$ otherwise. Then $\|g\|_{L^{n/s}(8B)} \lesssim \|u\|_{W^{s,n/s}(8B)}$, which follows from

$$\begin{aligned} g(x) &\lesssim \sup_{j \ge k_0 - 2} \left(\int_{B(x,2^{-j})} \frac{|u(z) - u(x)|^{n/s}}{|x - z|^{2n}}\, dz\right)^{n/s} \\ &\lesssim \left(\int_{8B} \frac{|u(z) - u(x)|^{n/s}}{|x - z|^{2n}}\, dz\right)^{n/s}. \end{aligned}$$

Moreover, for almost all $x \in B$, we have

$$|u(x) - u_B| \lesssim \int_{B(x_0, 2^{-k_0+2})} \frac{|g(y)|}{|y - x|^{n-s}}\, dy. \tag{2.13}$$

Indeed, by an argument similar to but easier than the case $p \in (0, 1)$, for every Lebesgue point $x \in B$ of u,

$$|u(x) - u_{B(x,2^{-k_0-1})}| \lesssim \int_{B(x_0,2^{-k_0+2})} \frac{|g(y)|}{|y - x|^{n-s}}\, dy.$$

Similarly,

$$|u_B - u_{B(x,2^{-k_0-1})}| \lesssim \fint_B |u(z) - u_B|\, dz \lesssim \int_{B(x_0,2^{-k_0+2})} \frac{|\mathcal{M}(g)(y)|}{|y - x|^{n-s}}\, dy.$$

This gives (2.13).

Applying [9, Lemma 7.2], for all $q \ge n/s$, we obtain

$$\|u - u_B\|_{L^q(B)} \le q^{1-s/n+1/q} |B(0, 1)|^{1-s/n} |B|^{1/q} \|\mathcal{M}(g)\|_{L^{n/s}(4B)},$$

which, together with the $L^{n/s}(\mathbb{R}^n)$-boundedness of $\mathcal{M}$ and the fact that

$$\|g\|^q_{L^{n/s}(8B)} \lesssim \|u\|_{W^{s,n/s}(8B)},$$

implies that

$$\fint_B |u(z)-u_B|^q\,dz \lesssim q^{1+(n-s)/(nq)}|B(0,1)|^{nq/(n-s)}\|u\|_{W^{s,n/s}(8B)}.$$

Thus when $q\ge n/s-1$, we have

$$\fint_B |u(z)-u_B|^{qn/(n-s)}\,dz \lesssim \frac{nq}{n-s}\left(|B(0,1)|\frac{nq}{n-s}\|u\|^{n/(n-s)}_{W^{s,n/s}(8B)}\right)^q.$$

Taking $\sigma > [e|B(0,1)|n/(n-s)]^{(n-s)/n}$, this yields that

$$\begin{aligned}\fint_B \sum_{j\ge\lfloor n/s\rfloor}\frac{1}{j!}\left(\frac{|u(x)-u_B|}{\sigma\|u\|_{W^{s,n/s}(8B)}}\right)^{jn/(n-s)} dx\\ \lesssim \sum_{j\ge 1}\left(\frac{n|B(0,1)|}{(n-s)\sigma^{n/(n-s)}}\right)^j\frac{j^j}{(j-1)!}\\ \lesssim 1.\end{aligned}$$

Notice that by Hölder's inequality, we have

$$\begin{aligned}\fint_B \sum_{j=0}^{\lfloor n/s\rfloor}\frac{1}{j!}\left(\frac{|u(x)-u_B|}{\sigma\|u\|_{W^{s,n/s}(8B)}}\right)^{jn/(n-s)} dx\\ \lesssim \sum_{j=0}^{\lfloor n/s\rfloor}\left(\fint_B \frac{|u(x)-u_B|^{n/s}}{\|u\|^{n/s}_{W^{s,n/s}(8B)}}\,dx\right)^{(n-s)/(j-s)}\\ \lesssim 1.\end{aligned}$$

This gives (2.12) and thus finishes the proof of Lemma 2.3. □

(iii)⇒(v). This case follows from the arguments that are exactly the same as in the case (ii)⇒(iv).

(v)⇒(i). We need the following estimates.

Lemma 2.4. *Let $s\in(0,1)$ and $p\in(0,\infty)$. For $z\in\Omega$ and $0<t<r\le 1$, define a function u on Ω by setting*

$$u(x)\equiv\begin{cases}1, & \text{if } x\in B(z,t)\cap\Omega,\\ \dfrac{r-|x-z|}{r-t}, & \text{if } x\in (B(z,r)\setminus B(z,t))\cap\Omega,\\ 0, & \text{if } x\in\Omega\setminus B(z,r).\end{cases}\tag{2.14}$$

Then there exists a constant $C>0$ independent of z, t, r such that

$$\|u\|_{W^{s,p}(\Omega)}\le C\frac{|B(z,r)\cap\Omega|^{1/p}}{(r-t)^s}.\tag{2.15}$$

The proof of Lemma 2.4 will be given later. With the help of Lemma 2.4, we consider the following cases: *Case $sp<n$*, *Case $sp=n$* and *Case $sp>n$*, separately.

Case $sp<n$. Take arbitrary $z\in\Omega$ and $r\in(0,1]$. Notice that there always exists a unique $b\in(0,1)$ such that

$$|B(z,br)\cap\Omega| = \frac12|B(z,r)\cap\Omega|.$$

We claim that there exists a constant $C > 0$, independent of x and r, such that

$$r - br \le C|B(z, r) \cap \Omega|^{1/n}. \tag{2.16}$$

Indeed, since Ω is a $W^{s,p}$-imbedding domain, we know that $\|u\|_{L^{np/(n-sp)}(\Omega)} \lesssim \|u\|_{W^{s,p}(\Omega)}$, and hence by Lemma 2.4 and $\|u\|_{L^{np/(n-sp)}(\Omega)} \ge |B(z, br) \cap \Omega|^{(n-sp)/(np)}$, we further have

$$|B(z, br) \cap \Omega|^{(n-sp)/(np)} \lesssim \frac{|B(z, r) \cap \Omega|^{1/p}}{(r - br)^s},$$

which yields $r - br \lesssim |B(z, r) \cap \Omega|^{1/n}$. This gives (2.16). Moreover, let $b_0 = 1$ and $b_j \in (0, 1)$ for $j \in \mathbb{N}$ such that

$$|B(z, b_j r) \cap \Omega| = 2^{-1}|B(z, b_{j-1}r) \cap \Omega| = 2^{-j}|B(z, r) \cap \Omega|. \tag{2.17}$$

Then $b_j \to 0$ as $j \to \infty$, and hence

$$\begin{aligned} r &= \sum_{j\in\mathbb{N}} (b_{j-1}r - b_j r) \\ &\lesssim \sum_{j\in\mathbb{N}} |B(z, b_{j-1}r) \cap \Omega|^{1/n} \\ &\lesssim \sum_{j\in\mathbb{N}} 2^{-j/n}|B(z, r) \cap \Omega|^{1/n} \\ &\le |B(z, r) \cap \Omega|^{1/n}, \end{aligned}$$

as desired.

Case $sp = n$. Take arbitrary $z \in \Omega$ and $r \in (0, 1]$. Let $b_0 = 1$ and $b_j \in (0, 1)$ for $j \in \mathbb{N}$ such that (2.17) holds. Considering the function u associated to z, $b_1 r$, $b_2 r$ as in (2.15), by Lemma 2.3, and applying (1.4) to the ball $B(z, r)$, we have

$$\inf_{c\in\mathbb{R}} \int_{B(z, r)\cap\Omega} \exp\left(C \frac{|u(x) - c|(b_1 r - b_2 r)^s}{|B(x, b_1 r) \cap \Omega|^{s/n}} \right)^{n/(n-s)} dx \le C_2 r^n.$$

Observe that, for each $c \in \mathbb{R}$, $|u - c| \ge 1/2$ either on $(B(z, r) \setminus B(z, b_1 r)) \cap \Omega$ or on $B(z, b_2 r) \cap \Omega$. By (2.17), we have

$$|B(z, b_1 r) \cap \Omega| \exp\left(\frac{(b_1 r - b_2 r)^{sn/(n-s)}}{|B(x, b_1 r) \cap \Omega|^{s/(n-s)}} \right) \lesssim r^n,$$

which implies that

$$b_1 r - b_2 r \le |B(x, b_1 r) \cap \Omega|^{1/n} \left[\log\left(\frac{C r^n}{|B(z, b_1 r) \cap \Omega|} \right) \right]^{(n-s)/sn}.$$

Similar inequalities also hold for $b_j r - b_{j+1} r$ with $j \ge 2$. This leads to

$$\begin{aligned} b_1 r &= \sum_{j\in\mathbb{N}} (b_j r - b_{j+1} r) \le \sum_{j\in\mathbb{N}} |B(x, b_j r) \cap \Omega|^{1/n} \left[\log\left(\frac{C(b_{j-1}r)^n}{|B(z, b_j r) \cap \Omega|} \right) \right]^{(n-s)/sn} \\ &\le \sum_{j\in\mathbb{N}} 2^{-j/n}|B(x, r) \cap \Omega|^{1/n} \left[\log\left(2^j \frac{C r^n}{|B(z, r) \cap \Omega|} \right) \right]^{(n-s)/sn} \\ &\lesssim |B(x, r) \cap \Omega|^{1/n} \left[\log\left(\frac{C r^n}{|B(z, r) \cap \Omega|} \right) \right]^{(n-s)/sn}. \end{aligned}$$

If $b_1 \ge 1/10$, observing that $t(\log \frac{1}{t})^{(n-s)/s} \ge 1$ implies that $t \gtrsim 1$, we have $|B(x, r) \cap \Omega|^{1/n} \gtrsim r$, as desired. If $b_1 \le 1/10$, we choose $R = 2r/5$ and a point $y \in B(z, r) \cap \Omega$ such that $|y - z| = b_1 r + R/2$. Then $B(z, b_1 r) \subset B(y, R) \subset B(z, r)$ but $B(y, R/2) \cap B(z, b_1 r) = \emptyset$. Therefore, if $|B(y, \widetilde{b}_1 R) \cap \Omega| = \frac{1}{2}|B(y, R) \cap \Omega|$, then by $|B(z, b_1 r) \cap \Omega| \ge \frac{1}{2}|B(y, R) \cap \Omega|$ we have $\widetilde{b}_1 \ge 1/2$. Applying the result when $b_1 \ge 1/10$, we conclude that $|B(y, R) \cap \Omega| \gtrsim R^n$, which implies that $|B(z, r) \cap \Omega| \gtrsim r^n$, as desired.

Case $sp > n$. For $z \in \Omega$ and $r \in (0, 1]$, take $t \in (0, r/4)$, and for such z, r, t, set u as in (2.14). Then for all $x, y \in \Omega$, by Lemma 2.4 and $r/2 \le r - t \le r$, we have

$$|u(x) - u(y)| \le C\|u\|_{W^{s,p}(\Omega)}|x - y|^{s-n/p} \lesssim \frac{|B(z, r) \cap \Omega|^{1/p}}{r^s}|x - y|^{s-n/p}.$$

In particular, let $x \in B(z, t) \cap \Omega$ and $y \in (B(z, r + r/2) \cap \Omega) \setminus B(z, r)$. Then $|x - z| \le r/4$, $r \le |y - z| \le 3r/2$, and hence $r/2 \le |x - y| \le 2r$. Therefore, $r^n \lesssim |B(z, r) \cap \Omega|$, as desired.

Proof of Lemma 2.4. Obviously, $\|u\|_{L^p(\Omega)} \lesssim |B(z, r) \cap \Omega|^{1/p}$. It then suffices to prove that

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dx\, dy \lesssim \frac{|B(z, r) \cap \Omega|}{(r - t)^{sp}}.$$

To this end, observing $u = 0$ on $\Omega \setminus B(z, r)$, we write

$$\begin{aligned}\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dx\, dy &= 2\int_{B(z,r)\cap\Omega} \int_{\Omega\setminus B(z,r)} \frac{|u(x)|^p}{|x - y|^{n+sp}}\, dx\, dy \\ &\quad + \int_{B(z,r)\cap\Omega} \int_{B(z,r)\cap\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dx\, dy \\ &\equiv H_1 + H_2.\end{aligned}$$

For each $x \in B(z, r) \cap \Omega$, we have $\Omega \setminus B(z, r) \subset \mathbb{R}^n \setminus B(x, r - |x - z|)$, and hence

$$\int_{\Omega\setminus B(z,r)} \frac{1}{|x - y|^{n+sp}}\, dy \le \int_{\mathbb{R}^n\setminus B(x,r-|x-z|)} \frac{1}{|x - y|^{n+sp}}\, dy \le \frac{1}{(r - |x - z|)^{sp}}.$$

Thus

$$\begin{aligned}H_1 &\le \int_{(B(z,r)\setminus B(z,t))\cap\Omega} \left(\frac{r - |x - z|}{r - t}\right)^p \frac{1}{(r - |x - z|)^{sp}}\, dx \\ &\quad + \int_{B(z,t)\cap\Omega} \frac{1}{(r - |x - z|)^{sp}}\, dx \\ &\lesssim \frac{|B(z, r) \cap \Omega|}{(r - t)^{sp}}.\end{aligned}$$

Write

$$\begin{aligned}H_2 &= \int_{B(z,r)\cap\Omega} \int_{B(x,r-t)\cap\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dx\, dy \\ &\quad + \int_{B(z,r)\cap\Omega} \int_{(B(z,r)\setminus B(x,r-t))\cap\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dx\, dy \\ &\equiv H_{2,1} + H_{2,2}.\end{aligned}$$

Since $|\nabla u| \le 1/(r-t)$, for all $x,\, y \in \Omega$, we have

$$|u(x) - u(y)| \le \frac{1}{r-t}||x-z| - |y-z|| \le \frac{1}{r-t}|x-y|.$$

Then by $\int_{B(x,\, r-t)} \frac{1}{|x-y|^{n+sp-p}}\, dy \lesssim (r-t)^{p-sp}$ we obtain

$$\begin{aligned} H_{2,\,1} &\le \int_{B(z,\, r)\cap\Omega} \int_{B(x,\, r-t)} \frac{1}{(r-t)^p} \frac{1}{|x-y|^{n+sp-p}}\, dy\, dx \\ &\lesssim \int_{B(z,\, r)\cap\Omega} \frac{(r-t)^{p-sp}}{(r-t)^p}\, dx \\ &\lesssim |B(z,\, r)\cap\Omega| \frac{1}{(r-t)^{sp}}. \end{aligned}$$

Observing $0 \le u \le 1$ and $\int_{\mathbb{R}^n \setminus B(x,\, r-t)} \frac{1}{|x-y|^{n+sp}}\, dy \lesssim (r-t)^{-sp}$, we also have

$$\begin{aligned} H_{2,\,2} &\lesssim \int_{B(z,\, r)\cap\Omega} \int_{\mathbb{R}^n \setminus B(x,\, r-t)} \frac{1}{|x-y|^{n+sp}}\, dy\, dx \\ &\lesssim \int_{B(z,\, r)\cap\Omega} \frac{1}{(r-t)^{sp}}\, dx \\ &\lesssim |B(z,\, r)\cap\Omega| \frac{1}{(r-t)^{sp}}. \end{aligned}$$

So $H_2 \lesssim \frac{\Omega|}{(r-t)^{sp}}$, as desired. This finishes the proof of Lemma 2.4. □

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