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Research Article

Fractional Subequation Method for Cahn-Hilliard and Klein-Gordon Equations

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The fractional subequation method is applied to solve Cahn-Hilliard and Klein-Gordon equations of fractional order. The accuracy and efficiency of the scheme are discussed for these illustrative examples.

1. Introduction

Fractional calculus deals with fractional integrals and derivatives of any order [1–8]. Numbers of very interesting and novel applications of fractional partial differential equations (FPDEs) in physics, chemistry, engineering, finance, biology, hydrology, signal processing, viscoelastic materials, fractional variational principles, and so forth, developed mainly in the last few decades [1–15], have led recently to an intensive effort to find accurate and stable numerical methods that are also straightforward to be implemented.

Also, the exact solutions of most of the FPDEs cannot be found easily; thus analytical and numerical methods must be used. Some of the numerical methods for solving fractional differential equations (FDE) and FPDEs were discussed in (see [7, 16–23] and the references therein).

By taking into account the results from [24], a new direct method titled fractional subequation method to search for explicit solutions of FPDEs was proposed [25]. We notice that the method relies on the homogeneous balance principle [26], Jumarie's modified Riemann-Liouville derivative [27, 28], and the symbolic computation. With the help of this method, some exact solutions of nonlinear time fractional biological population model as well as the $(4 + 1)$-dimensional space-time fractional Fokas equation were reported [25]. Recently, the improved fractional subequation method was proposed, and it was used to solve the following two FPDEs in fluid mechanics [29].

In this paper, we suggest the fractional subequation method and utilize this method to solve the following two FPDEs.

(a) The space-time fractional Cahn-Hilliard equation in the form

$$D_t^{\alpha}u - \gamma D_x^{\alpha}u - 6u\left(D_x^{\alpha}u\right)^2 - \left(3u^2 - 1\right)D_x^{2\alpha}u + D_x^{4\alpha}u = 0, \quad (1)$$

where $0 < \alpha \leq 1$ and u are the functions of (x, t). For the case corresponding to $\alpha = 1$, this equation is related with a number of interesting physical phenomena like the spinodal decomposition, phase separation, and phase ordering dynamics. On the other hand it becomes important in material sciences [30, 31]. However we notice that this equation is very difficult to be solved and several articles investigated it (see, e.g., [32] and the references therein).

(b) The nonlinear fractional Klein-Gordon equation [33] with quadratic nonlinearity reads as

$$D_{tt}^{2\alpha}u - D_{xx}^{2\alpha}u + \gamma u - \beta u^2 = 0, \quad \gamma, \beta \neq 0. \tag{2}$$

We notice that the nonlinear fractional Klein-Gordon equation describes many types of nonlinearities. On the other hand the Klein-Gordon equation plays a significant role in several real world applications, for example, the solid state physics, nonlinear optics, and quantum field theory.

The paper suggests a fractional subequation method to find the exact analytical solutions of nonlinear fractional partial differential equations with the Jumarie's modified Riemann-Liouville derivative of order α which is defined as [27]

$$D_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^x (x-\xi)^{-\alpha-1} \left[f(\xi) - f(0)\right], & \\ \qquad\qquad\qquad\qquad\qquad \alpha < 0, & \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-\xi)^{-\alpha} \left[f(\xi) - f(0)\right], & \\ \qquad\qquad\qquad\qquad\qquad 0 < \alpha < 1, & \\ \left[f^{(\alpha-n)}(x)\right]^{(n)}, \qquad n \leq \alpha < n+1, \ n \geq 1. & \end{cases} \tag{3}$$

As pointed out by Kolwankar and Gangal [34], even though the variable t is taking all real positive values the actual evolution takes place only for values of t in the fractal set C. We take $\chi(t) = 1$ which is a flag function. We conclude that, from the viewpoint of the Kolwankar-Gangal's local fractional derivative, the parameter α is the fractal dimension of time. Thus, the approximate solution is generated by some distribute function defined over the fractal sets in some closed interval $[0, 1]$. They are continuous but not differentiable functions with respect to t.

The organization of the manuscript is as follows. In Section 2, we briefly explain the fractional subequation method for solving fractional partial differential equations. In Section 3, we extend the application of the proposed method to two nonlinear equations. Finally, Section 4 is devoted to our conclusions.

2. The Method

The fundamental ingredients of the fractional subequation method for solving fractional partial differential equations are described in [29]. The starting point is to consider a given nonlinear fractional partial differential equation in $u(x,t)$

$$p\left(u, u_x, u_t, D_t^\alpha u, D_x^\alpha u, \ldots\right) = 0, \quad 0 < \alpha < 1, \tag{4}$$

where $D_t^\alpha u$ and $D_x^\alpha u$ are Jumarie's modified Riemann-Liouville derivatives of u, $u = u(x,t)$ is an unknown function, and P is a polynomial in u and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

To specify u explicitly, we use in this paper the proposal in four basic steps proposed in [29, 35]; namely, we reduce, by using the traveling wave transformation, the given nonlinear FPDE to a nonlinear fractional differential equation (FDE). After that we assume that the reduced equation obtained previously admits the following solution

$$u(\xi) = \sum_{i=0}^{n} a_i \varphi^i, \tag{5}$$

where a_i $(i = 0, 1, \ldots, n-1, n)$ are constants to be found, n denotes a positive integer determined by balancing the highest order derivatives with the highest nonlinear terms in (4) or the modified one (see [35] for more details), and the new variable $\varphi = \varphi(\xi)$ fulfilling the fractional Riccati equation:

$$D_\xi^\alpha \varphi = \sigma + \varphi^2, \quad 0 < \alpha \leq 1. \tag{6}$$

The next step is to substitute (5) along with (6) into the modified version of the equation and to use the properties of Jumarie's modified Riemann-Liouville derivative, in order to get a polynomial in $\varphi(\xi)$. Requesting all coefficients of φ^k $(k = 0, 1, 2, \ldots)$ to be zero, we end up to a set of overdetermined nonlinear algebraic equations for c, k, a_i $(i = 0, 1, \ldots, n-1, n)$.

Finally, assuming that c, k, a_i $(i = 0, 1, \ldots, n-1, n)$ are obtained by solving the algebraic equations in the previous step, and substituting these constants and the solutions of (6) into (5), we get the explicit solutions of (4).

3. Main Results

In this section, we apply the method presented in Section 2 for solving the FPDEs (1) and (2), respectively.

Example 1. We consider the space-time fractional Cahn-Hilliard equation as

$$D_t^\alpha u - \gamma D_x^\alpha u - 6u\left(D_x^\alpha u\right)^2 - \left(3u^2 - 1\right) D_x^{2\alpha} u + D_x^{4\alpha} u = 0. \tag{7}$$

Making use of the travelling wave transformation

$$u = u(\xi), \quad \xi = kx + ct. \tag{8}$$

Equation (7) is reduced into a nonlinear FDE easy to solve, namely,

$$\begin{aligned} c^\alpha D_\xi^\alpha u - \gamma k^\alpha D_\xi^\alpha u - 6u\left(k^\alpha D_\xi^\alpha u\right)^2 - \left(3u^2 - 1\right) k^{2\alpha} D_\xi^{2\alpha} u & \\ + k^{4\alpha} D_\xi^{4\alpha} u = 0. & \end{aligned} \tag{9}$$

Next we suppose that (9) has a solution in the form given below

$$u = \sum_{i=0}^{n} a_i \varphi^i, \tag{10}$$

where φ obeys the subequation (6).

By balancing the highest order derivative terms and nonlinear terms in (9), gives the value of $n = 1$, we substitute (10), along with (6), into (9), and then setting the coefficients of φ^j $(j = 0, 1, \ldots, 5)$ to zero, we finally end up with a system of algebraic equations, namely,

$$\begin{gathered}
a_1\sigma c^{\alpha} - 6a_0a_1^2\sigma^2k^{2\alpha} - a_1\gamma\sigma k^{\alpha} = 0,\\
-6a_1^3\sigma^2k^{2\alpha} - 16a_1\sigma^2k^{4\alpha} - 6a_0^2a_1\sigma k^{2\alpha} + 2a_1\sigma k^{2\alpha} = 0,\\
a_1c^{\alpha} - 24a_0a_1^2\sigma k^{2\alpha} - a_1\gamma k^{\alpha} = 0,\\
-18a_1^3\sigma k^{2\alpha} - 40a_1\sigma k^{4\alpha} - 6a_0^2a_1k^{2\alpha} + 2a_1k^{2\alpha} = 0,\\
-18a_0a_1^2k^{2\alpha} = 0,\\
-12a_1^3k^{2\alpha} - 24a_1k^{4\alpha} = 0.
\end{gathered} \tag{11}$$

Solving the set of algebraic equations yields

$$a_0 = 0, \qquad a_1 = \pm i\sqrt{2k^{2\alpha}}, \tag{12}$$

where $k^{2\alpha} = 1/2\sigma$, $c^{\alpha} = k^{\alpha}$ and σ denotes an arbitrary constant.

By using (8)–(12) after some tedious calculations, the exact solutions of (7), namely, generalized hyperbolic function solutions (see [24] for their definitions) and generalized trigonometric function solutions are obtained as

$$u = \begin{cases}
\pm i\sqrt{2k^{2\alpha}}\left(\sqrt{-\sigma}\tanh_{\alpha}\left(\sqrt{-\sigma}\left(kx+ct\right)\right)\right), & \sigma < 0,\\
\pm i\sqrt{2k^{2\alpha}}\left(\sqrt{-\sigma}\coth_{\alpha}\left(\sqrt{-\sigma}\left(kx+ct\right)\right)\right), & \sigma < 0,\\
\pm\sqrt{2k^{2\alpha}}\left(\sqrt{\sigma}\tan_{\alpha}\left(\sqrt{\sigma}\left(kx+ct\right)\right)\right), & \sigma > 0,\\
\pm i\sqrt{2k^{2\alpha}}\left(\sqrt{\sigma}\cot_{\alpha}\left(\sqrt{\sigma}\left(kx+ct\right)\right)\right), & \sigma > 0.
\end{cases} \tag{13}$$

We stress on the fact that when $\alpha \to 1$ these obtained exact solutions give the ones of the standard form equation of the space-time fractional Cahn-Hilliard equation (7).

Example 2. The next step is to investigate the fractional nonlinear Klein-Gordon equation in the following form:

$$D_{tt}^{2\alpha}u - D_{xx}^{2\alpha}u + \gamma u - \beta u^2 = 0. \tag{14}$$

To solve (14), we perform the traveling wave transformation

$$u = u(\xi), \quad \xi = kx + ct; \tag{15}$$

therefore (14) is reduced to the following nonlinear fractional ODE, namely,

$$c^{2\alpha}D_{\xi}^{2\alpha}u - k^{2\alpha}D^{2\alpha}u + \gamma u - \beta u^2 = 0. \tag{16}$$

Next, we assume that (16) admits a solution in the form

$$u = \sum_{i=0}^{n} a_i\varphi^i. \tag{17}$$

At this stage we apply the same technique as in the case of the previous example. Namely, by balancing the highest order derivative terms and nonlinear terms in (16), then substituting (17), with $n = 2$, with (6) into (16), we finally obtain the corresponding system of algebraic equations as

$$\begin{gathered}
-a_0^2\beta + a_0\gamma + 2a_2\sigma^2c^{2\alpha} - 2a_2\sigma^2k^{2\alpha} = 0,\\
a_1\gamma - 2a_0a_1\beta + 2a_1\sigma c^{2\alpha} - 2a_1\sigma k^{2\alpha} = 0,\\
-a_1^2\beta + a_2\gamma - 2a_0a_2\beta + 8a_2\sigma c^{2\alpha} - 8a_2\sigma k^{2\alpha} = 0,\\
-2a_1a_2\beta + 2a_1c^{2\alpha} - 2a_1k^{2\alpha} = 0,\\
-a_2^2\beta + 6a_2c^{2\alpha} - 6a_2k^{2\alpha} = 0.
\end{gathered} \tag{18}$$

After using the Mathematica to solve (18) the following solutions are reported:

$$\begin{gathered}
a_0 = \frac{\gamma + 8\sigma c^{2\alpha} - 8\sigma k^{2\alpha}}{2\beta}, \qquad a_1 = 0,\\
a_2 = -\frac{6\left(k^{2\alpha} - c^{2\alpha}\right)}{\beta},
\end{gathered} \tag{19}$$

where σ denotes an arbitrary constant. Finally, from (15)–(19) we obtain the following generalized hyperbolic function solutions, generalized trigonometric function solutions, and the rational solution of (14) as

$$u = \begin{cases}
\dfrac{\gamma+8\sigma c^{2\alpha}-8\sigma k^{2\alpha}}{2\beta} + \dfrac{6\sigma\left(k^{2\alpha}-c^{2\alpha}\right)}{\beta} \\
\quad\times\left(\tanh_{\alpha}^2\left(\sqrt{-\sigma}\xi\right)\right), & \sigma < 0,\\
\dfrac{\gamma+8\sigma c^{2\alpha}-8\sigma k^{2\alpha}}{2\beta} + \dfrac{6\sigma\left(k^{2\alpha}-c^{2\alpha}\right)}{\beta} \\
\quad\times\left(\coth_{\alpha}^2\left(\sqrt{-\sigma}\xi\right)\right), & \sigma < 0,\\
\dfrac{\gamma+8\sigma c^{2\alpha}-8\sigma k^{2\alpha}}{2\beta} - \dfrac{6\sigma\left(k^{2\alpha}-c^{2\alpha}\right)}{\beta} \\
\quad\times\left(\tan_{\alpha}\left(\sqrt{\sigma}\xi\right)\right), & \sigma > 0,\\
\dfrac{\gamma+8\sigma c^{2\alpha}-8\sigma k^{2\alpha}}{2\beta} - \dfrac{6\sigma\left(k^{2\alpha}-c^{2\alpha}\right)}{\beta} \\
\quad\times\left(\cot_{\alpha}\left(\sqrt{\sigma}\xi\right)\right), & \sigma > 0,\\
\dfrac{\gamma}{2\beta} - \dfrac{6\left(k^{2\alpha}-c^{2\alpha}\right)\Gamma^2(1+\alpha)}{\beta(\xi^{\alpha}+\omega)^2}, & \sigma = 0,
\end{cases} \tag{20}$$

where $\xi = kx + ct$.

As $\alpha \to 1$ (20) the results obtained above become the ones of (14).

4. Conclusions

In this paper, a fractional subequation method is used to construct the exact analytical solutions of the space-time fractional Cahn-Hilliard (1) and the fractional nonlinear Klein-Gordon equation (2). These solutions include the generalized hyperbolic function solutions, generalized trigonometric function solutions, and rational function solutions, which may be very useful to further understand the mechanisms of the complicated nonlinear physical phenomena and FPDEs. Also, this method help us to find all exact solutions of the Fan subequations involving all possible parameters, it is concise and efficient. *Mathematica* has been used for computations and programming in this work.

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