FRACTIONAL WHITE NOISE CALCULUS AND APPLICATIONS TO FINANCE

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Abstract

The purpose of this paper is to develop a fractional white noise calculus and to apply this to markets modeled by (Wick-) Itô type of stochastic differential equations driven by fractional Brownian motion $B_H(t)$; $1/2 < H < 1$.

We show that if we can use an Itô type of stochastic integration with respect to $B_H(t)$ (as developed in [4]), then the corresponding *Itô fractional Black & Scholes market has no arbitrage*, contrary to the situation when the Stratonovich type of integration is used. Moreover, we prove that our Itô fractional Black & Scholes market is complete and we compute explicitly the price and replicating portfolio of a European option in market. The results are compared to the classical results based on standard Brownian motion $B(t)$.

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1 Introduction

Recall that if $0 < H < 1$ then the *fractional Brownian motion with Hurst parameter* H is the Gaussian process $B_H(t)$; $t \geq 0$ with mean $\mathrm{E}(B_H(t)) = 0$ and covariance

$$\mathrm{E}\left[B_H(t)B_H(s)\right] = \frac{1}{2}\left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\}$$

for all $s, t \geq 0$. Here E denotes the expectation with respect to the probability law for $B_H = B_H(t, \omega)$. For simplicity we assume $B_H(0) = 0$.

If $H = \frac{1}{2}$, then $B_H(t)$ coincides with the standard Brownian motion $B(t)$. If $H > \frac{1}{2}$ then $B_H(t)$ has a *long range dependence*, in the sense that if we put

$$r(n) = \mathrm{cov}\left(B_H(1), \left(B_H(n+1) - B_H(n)\right)\right)$$

then

$$\sum_{n=1}^{\infty} r(n) = \infty\,.$$

For any $H \in (0,1)$ the process $B_H(t)$ is self-similar in the sense that $B_H(\alpha t)$ has the same law as $\alpha^H B_H(t)$ for any $\alpha > 0$.

Because of these properties $B_H(t)$ with Hurst parameter $H \in (\frac{1}{2}, 1)$ has been suggested as a useful tool in many applications, including finance [15]. However, after a stochastic integration theory for fractional Brownian motions was developed [3], [5], [13], it was discovered [19] that mathematical markets based on $B_H(t)$ could have arbitrage. This was the case even for the fractional analogue of the basic Black & Scholes market (see [20] for a short proof). In view of this the process $B_H(t)$ was by many no longer considered promising for mathematical modeling in finance.

However, as pointed out in [4], the first B_H-integration theory was based on using ordinary products and led to an integral which we will denote by

$$\int_a^b f(t, \omega)\delta B_H(t)\,.$$

These integrals do not have expectation zero and we will call them *fractional Stratonovich* integrals, in analogy with the situation for standard Brownian motion. (See [9]).

In this paper we will concentrate on the B_H-integral considered in [4], denoted by

$$\int_a^b f(t, \omega)dB_H(t)\,.$$

These integrals have expectation zero. Since they are based on using Wick products rather than ordinary products, we call them *fractional Itô integrals*, again with reference to the corresponding situation for standard Brownian motion [9].

The relation between these two integrals and properties of them are investigated in [4].

A different kind of stochastic calculus with respect to fractional Brownian motion, based on the Gross-Sobolev derivative on the Wiener space, has been developed in [5].

The purpose of this paper is to develop a white noise calculus based on $B_H(t)$, $\frac{1}{2} < H < 1$, and then use this to prove that the corresponding Itô type fractional Black & Scholes market has no arbitrage. Moreover, we prove that this market is complete and we compute explicitly the price and the replicating portfolio of a European call option in this market. Then we compare the results to the classical results based on standard Brownian motion $B_H(t)$.

Let us point out that even in the standard Brownian motion case, we will get arbitrage if we use Stratonovich integral in the definition of self-financing portfolios. Here is an example essentially due to Shiryaev [20]. Let $\rho > 0$. Let $A_t = e^{\rho t}$ and $X_t = e^{\rho t + B_t}$ be the bond price and stock price, respectively, at time t (see Section 5). Then (A_t, X_t) constitutes a Black & Scholes market, namely,

$$\begin{cases} dA_t = \rho A_t dt \\ dX_t = X_t(\rho dt + \delta B_t)\,, \end{cases}$$

where δ denotes the Stratonovich type of differential. Take the portfolio $\theta(t) = (u_t, v_t)$, where

$$u_t = 1 - e^{2B_t}\,, \quad \text{and} \quad v_t = 2(e^{B_t} - 1)\,.$$

Then

$$X_t^\theta = u_t A_t + v_t X_t = e^{\rho t}\left[e^{B_t} - 1\right]^2\,.$$

It is easy to check that

$$dX_t^\theta = u_t dA_t + v_t \delta X_t\,.$$

So θ is self-financing if one replaces the Itô integral by the Stratonovich integral in the definition of self-financing [17]. It is now easy to see that $X_0^\theta = 0$, $X_t^\theta \geq 0$ for $t > 0$, and $\mathrm{E} X_t^\theta > 0$ for $t > 0$.

Thus the essential point in the problem of arbitrage is not just a problem of martingale or non-martingale. The use of Itô type or Stratonovich type integrals also play a role. We hope that our result will bring new interest in using fractional Brownian motions in mathematical finance.

Our paper is organized as follows: In Section 2 we summarize the results from [4] and prove a new result that we will need. In Section 3 we set up a fractional white noise calculus based on $B_H(t)$. In Section 4 we introduce differentiation and prove a generalized Clark-Ocone formula for fractional Brownian motion. Then in Section 5 we apply our theory to study markets modeled by Itô type stochastic differential equations driven by $B_H(t)$.

2 Background

In this section we summarize the results from [4] that we will need.

Fix a Hurst constant H, $\frac{1}{2} < H < 1$. Define

$$\phi(s,t) = H(2H-1)|s-t|^{2H-2}; \quad s,t \in \mathbb{R}\,. \tag{2.1}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be measurable. Then we say that $f \in L^2_\phi(\mathbb{R})$ if

$$|f|^2_\phi := \int_\mathbb{R}\int_\mathbb{R} f(s)f(t)\phi(s,t)dsdt < \infty\,. \tag{2.2}$$

If we equip $L^2_\phi(\mathbb{R})$ with the inner product

$$(f,g)_\phi := \int_\mathbb{R}\int_\mathbb{R} f(s)g(t)\phi(s,t)dsdt\,; \quad f,g \in L^2_\phi(\mathbb{R}) \tag{2.3}$$

then $L^2_\phi(\mathbb{R})$ becomes a separable Hilbert space. In fact, we have

Lemma 2.1 *Let*

$$\Gamma_\phi f(u) = c_H \int_u^\infty (t-u)^{H-3/2} f(t)dt \tag{2.4}$$

where

$$c_H = \sqrt{\frac{H(2H-1)\Gamma\left(\frac{3}{2}-H\right)}{\Gamma\left(H-\frac{1}{2}\right)\Gamma(2-2H)}}\,,$$

and Γ *denotes the gamma function. Then* Γ_ϕ *is an isometry from* $L^2_\phi(\mathbb{R})$ *to* $L^2(\mathbb{R})$.

Proof By a limiting argument, we may assume that f and g are continuous with compact support. By definition,

$$\begin{aligned}
(\Gamma_\phi(f)\,,\Gamma_\phi(g))_{L^2(\mathbb{R})} &= c_H^2 \int_\mathbb{R} \left\{ \int_u^\infty (s-u)^{H-3/2} f(s)ds \int_u^\infty (t-u)^{H-3/2} g(t)dt \right\} du \\
&= c_H^2 \int_{\mathbb{R}^2} f(s)g(t) \left\{ \int_{-\infty}^{s\wedge t} (s-u)^{H-3/2}(t-u)^{H-3/2} du \right\} dsdt \\
&= \int_\mathbb{R}\int_\mathbb{R} f(s)g(t)\phi(s,t)dsdt = (f,g)_\phi\,,
\end{aligned}$$

where we have used the identity

$$c_H^2 \int_{-\infty}^{s\wedge t} (s-u)^{H-3/2}(t-u)^{H-3/2} du = \phi(s,t)\,.$$

(See for example [6], p.404, for a proof of this identity.) □

If $f \in L^2_\phi(\mathbb{R})$ (deterministic) one can define $\int_{\mathbb{R}} f(t)dB_H(t) = \int_{\mathbb{R}} f(t)\delta B_H(t)$ in the usual way by first considering simple integrands

$$f_m(t) = \sum_i a_i^{(m)} \chi_{[t_i,t_{i+1})}(t)\,,$$

setting

$$\int_{\mathbb{R}} f_m(t)dB_H(t) = \sum_i a_i^{(m)} \left(B_H(t_{i+1}) - B_H(t_i)\right) \tag{2.5}$$

and defining

$$\int_{\mathbb{R}} f(t)dB_H(t) = \lim_{m\to\infty} \int_{\mathbb{R}} f_m(t)dB_H(t)\,. \tag{2.6}$$

The limit exists in $L^2_\phi(\mathbb{R})$ because of the isometry

$$\mathrm{E}\left(\int_{\mathbb{R}} f_m(t)dB_H(t)\right)^2 = |f_m|^2_\phi\,. \tag{2.7}$$

For $f \in L^2_\phi(\mathbb{R})$ define

$$\mathcal{E}(f) = \exp\left(\int_{\mathbb{R}} f dB_H - \frac{1}{2}|f|^2_\phi\right)\,. \tag{2.8}$$

Then we have ([4], Theorem 3.1):

$$\textbf{The linear span of } \left\{\mathcal{E}(f);\ f \in L^2_\phi(\mathbb{R})\right\} \textbf{ is dense in } L^2(\mu_\phi), \tag{2.9}$$

where μ_ϕ is the probability law of B_H (see also next section).

3 Fractional White Noise Calculus

In this section we show how to adapt the traditional white noise calculus (see e.g. [7] or [10]) to the fractional white noise case. As before we fix a Hurst constant $H \in (\frac{1}{2}, 1)$ and we let ϕ, $L^2_\phi(\mathbb{R})$, $|\cdot|_\phi$ and $(\cdot,\cdot)_\phi$ be as in (2.1)-(2.3).

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$ and let $\Omega = \mathcal{S}'(\mathbb{R})$ be the dual of $\mathcal{S}(\mathbb{R})$, *i.e.* Ω is the space of *tempered distributions* ω on $\mathbb{R}$. The map

$$f \to \exp(-\frac{1}{2}|f|^2_\phi)\,; \quad f \in \mathcal{S}(\mathbb{R})$$

is positive definite on $\mathcal{S}(\mathbb{R})$, so by the Bochner-Minlos theorem there exists a probability measure μ_ϕ on Ω such that

$$\int_\Omega e^{i\langle\omega,f\rangle} d\mu_\phi(\omega) = e^{-\frac{1}{2}|f|^2_\phi} \quad \text{for all} \quad f \in \mathcal{S}(\mathbb{R})\,, \tag{3.1}$$

where $\langle \omega, f\rangle$ denotes the usual pairing between $\omega \in \mathcal{S}'(\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$. It follows from (3.1) that

$$\mathbb{E}\left[\langle \cdot, f\rangle\right] = 0 \quad \text{and} \quad \mathbb{E}\left[\langle \cdot, f\rangle^2\right] = |f|_\phi^2 . \tag{3.2}$$

Using this we see that we may define

$$\tilde{B}_H(t) = \tilde{B}_H(t,\omega) = \langle \omega, \chi_{[0,t]}(\cdot)\rangle \tag{3.3}$$

as an element of $L^2(\mu_\phi)$ for each $t \in \mathbb{R}$, where

$$\chi_{[0,t]}(s) = \begin{cases} 1 & \text{if } 0 \le s \le t \\ -1 & \text{if } t \le s \le 0 \\ 0 & \text{otherwise}. \end{cases}$$

By Kolmogorov's continuity theorem $\tilde{B}_H(t)$ has a t-continuous version which we will denote by $B_H(t)$. From (3.2) we see that $B_H(t)$ is a Gaussian process with

$$\mathbb{E}_{\mu_\phi}\left[B_H(t)\right] = 0 \quad \text{and} \quad \mathbb{E}_{\mu_\phi}\left[B_H(s)B_H(t)\right] = \frac{1}{2}\left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\} . \tag{3.4}$$

It follows that $B_H(t)$ is a *fractional Brownian motion.* Moreover, if $f \in L^2_\phi(\mathbb{R})$ then by approximating by step functions we see from (2.5)-(2.6) that

$$\langle \omega, f\rangle = \int_{\mathbb{R}} f(t)dB_H(t,\omega) . \tag{3.5}$$

In the following we let

$$h_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2/2}\right) ; \quad n = 0,1,2,\cdots \tag{3.6}$$

be the *Hermite polynomials.*

Lemma 3.1 *There is an orthonormal basis $\{e_i\}_{i=1}^\infty$ of $L^2_\phi(\mathbb{R})$ such that for any $t \in \mathbb{R}$ such that*

$$\left|\int_{\mathbb{R}} e_n(s)\phi(s,t)ds\right| < C_t n^{\frac{2}{3}} \tag{3.7}$$

Proof Define the Hermite functions as in [10] (see also [22])

$$\tilde{h}_n(x) = \pi^{-1/4}((n-1)!)^{-1/2}h_{n-1}(\sqrt{2}x)e^{-\frac{x^2}{2}}, \quad n = 1,2,\cdots$$

Then from [22], $\left\{\tilde{h}_n(x),\ n = 1,2,\cdots\right\}$ is an orthonormal basis of $L^2(\mathbb{R})$ and

$$|\tilde{h}_n(x)| \le \begin{cases} Cn^{-\frac{1}{12}} & \text{when } |x| \le 2\sqrt{n} \\ Ce^{-\gamma x^2} & \text{when } |x| > 2\sqrt{n}, \end{cases}$$

where γ and C are certain positive constants, independent of n. (See for example, [22], p.26, Lemma 1.5.1. See also [21]). Set

$$e_n(u) = c_H \int_u^\infty (t-u)^{H-3/2}\tilde{h}_n(t)dt\,. \tag{3.8}$$

Then by Lemma 2.1, $\{e_n\,,\ n = 1, 2, \cdots\}$ is an orthonormal basis of $L^2_\phi(\mathbb{R})$. We have also

$$\begin{aligned}\int_{\mathbb{R}} e_i(s)\phi(s,t)ds &= \int_{\mathbb{R}}\left\{\int_s^\infty (v-s)^{H-3/2}\tilde{h}_n(v)dv\right\}\phi(s,t)ds \\ &= \int_{\mathbb{R}}\left\{\int_{-\infty}^v (v-s)^{H-3/2}\phi(s,t)ds\right\}\tilde{h}_n(v)dv\,.\end{aligned}$$

It is easy to verify that there is a positive constant C_t such that

$$\begin{aligned}\int_{-\infty}^v (v-s)^{H-3/2}\phi(s,t)ds &\le C_t|v|^{3H-\frac{5}{2}} \\ &\le C_t\sqrt{|v|} \quad as \quad |v|\to\infty\,.\end{aligned}$$

(This integral is bounded when v is bounded.) Therefore we have

$$\begin{aligned}\left|\int_{\mathbb{R}} e_i(s)\phi(s,t)ds\right| &\le \int_{\mathbb{R}} C_t|v|^{1/2}|\tilde{h}_n(v)|dv \\ &\le C_t\int_{|v|\le 2\sqrt{n}} |v|^{1/2}n^{-\frac{1}{12}}dv + C_t\int_{|v|>2\sqrt{n}} |v|^{1/2}e^{-\gamma|v|^2}dv \\ &\le C_t n^{\frac{2}{3}}\,.\end{aligned}$$

This proves the lemma. □

From now on we let $\{e_n\}_{n=1}^\infty$ be the orthonormal basis of $L^2_\phi(\mathbb{R})$ defined in (3.8). Then the e_i's are smooth. Moreover, we see that

$$t \to \int_{\mathbb{R}} e_i(s)\phi(s,t)ds \quad \text{is continuous for each } i. \tag{3.9}$$

Let $\mathcal{I} = (\mathbb{N}_0^{\mathbb{N}})_c$ denote the set of all (finite) multi-indices $\alpha = (\alpha_1, \cdots, \alpha_m)$ of nonnegative integers, ($\mathbb{N}$ is the set of natural numbers and $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$). Then if $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathcal{I}$ we put

$$\mathcal{H}_\alpha(\omega) := h_{\alpha_1}(\langle\omega\,, e_1\rangle)\cdots h_{\alpha_m}(\langle\omega\,, e_m\rangle)\,. \tag{3.10}$$

In particular, if we let $\varepsilon^{(i)} := (0, \cdots, 0, 1, 0, \cdots, 0)$ denote the i'th unit vector, then by (3.5) we get

$$\mathcal{H}_{\varepsilon^{(i)}}(\omega) := h_1(\langle\omega\,, e_i\rangle) = \langle\omega, e_i\rangle = \int_{\mathbb{R}} e_i(t)dB_H(t)\,. \tag{3.11}$$

The following result is a fractional Wiener-Itô chaos expansion theorem:

Theorem 3.2 *[4]* *Let $F \in L^2(\mu_\phi)$. Then there exist constants $c_\alpha \in \mathbb{R}$, $\alpha \in \mathcal{I}$, such that*

$$F(\omega) = \sum_{\alpha\in\mathcal{I}} c_\alpha \mathcal{H}_\alpha(\omega) \quad (\text{ convergence in } \; L^2(\mu_\phi)) . \tag{3.12}$$

Moreover,

$$\|F\|^2_{L^2(\mu_\phi)} = \sum_{\alpha\in\mathcal{I}} \alpha! c_\alpha^2 \tag{3.13}$$

where $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_m!$ if $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathcal{I}$.

Proof This result may be regarded as a reformulation of Theorem 6.9 in [4], where such a chaos expansion is proved in terms of iterated Itô fractional integrals. (See Theorem 3.21).

A direct proof is the following:

Let $\mathcal{E}$ be as defined in (2.8). Then if $a_k \in \mathbb{R}$, $k = 1, 2, \cdots$, we have

$$\begin{aligned} \mathcal{E}(a_k e_k) &= \exp\left(a_k\langle\omega, e_k\rangle - \frac{1}{2}a_k^2\right) \\ &= \sum_{n=0}^{\infty} \frac{a_k^n}{n!} h_n(\langle\omega, e_k\rangle) . \end{aligned} \tag{3.14}$$

(See e.g. [10], Appendix C for more information about the Hermite polynomials).

It follows that if

$$f = \sum_{k=1}^{\infty} a_k e_k \in L^2_\phi(\mathbb{R})$$

then

$$\begin{aligned} \mathcal{E}(f) &= \exp\left(\sum_{k=1}^{\infty} a_k\langle\omega, e_k\rangle - \frac{1}{2}\sum_{k=1}^{\infty} a_k^2\right) \\ &= \lim_{N\to\infty} \prod_{k=1}^{\infty} \left(\sum_{n=0}^{N} \frac{a_k^n}{n!} h_n(\langle\omega, e_k\rangle)\right) \\ &= \lim_{N\to\infty} \sum_{\alpha\in\mathcal{I}^{(N)}} \prod_{k=1}^{\infty} \frac{a_k^{\alpha_k}}{\alpha_k!} h_{\alpha_k}(\langle\omega, e_k\rangle) \\ &= \lim_{N\to\infty} \sum_{\alpha\in\mathcal{I}^{(N)}} c_\alpha \mathcal{H}_\alpha(\omega) \quad (\text{limit in } \; L^2_\phi(\mu_\phi)) , \end{aligned} \tag{3.15}$$

where $\mathcal{I}^{(N)}$ denotes the set of all multi-indices $\alpha = (\alpha_1, \cdots, \alpha_m)$ of nonnegative integers with $\alpha_i \leq N$ and we have put

$$c_\alpha = \prod_{k=1}^{\infty} \frac{a_k^{\alpha_k}}{\alpha_k!} \quad \text{if} \quad \alpha = (\alpha_1, \cdots, \alpha_m) .$$

If we combine (2.9) with (3.15) we obtain that the linear span of $\{\mathcal{H}_\alpha; \alpha \in \mathcal{I}\}$ is dense in $L^2(\mu_\phi)$.

It remains to prove that

$$\mathrm{E}_{\mu_\phi}[\mathcal{H}_\alpha \mathcal{H}_\beta] = 0 \quad \text{if} \quad \alpha \neq \beta \tag{3.16}$$

and

$$\mathrm{E}_{\mu_\phi}\left[\mathcal{H}_\alpha^2\right] = \alpha! \tag{3.17}$$

To this end note that from (3.1) it follows that

$$\mathrm{E}_{\mu_\phi}[f(\langle\omega, e_1\rangle, \cdots, \langle\omega, e_m\rangle)] = \int_{\mathbb{R}^m} f(x) d\lambda_m(x) \tag{3.18}$$

for all $f \in L^1(\lambda_m)$, where λ_m is the normal distribution on $\mathbb{R}^m$, *i.e.*

$$d\lambda_m(x) = (2\pi)^{-m/2} e^{-\frac{1}{2}|x|^2} dx_1 \cdots dx_m; \quad x = (x_1, \cdots, x_m) \in \mathbb{R}^m .$$

Therefore, if $\alpha = (\alpha_1, \cdots . \alpha_m)$, $\beta = (\beta_1, \cdots . \beta_m)$ we have

$$\begin{aligned}
\mathrm{E}_{\mu_\phi}[\mathcal{H}_\alpha \mathcal{H}_\beta] &= \mathrm{E}_{\mu_\phi}\left[\prod_{k=1}^m h_{\alpha_k}(\langle\omega, e_k\rangle) h_{\beta_k}(\langle\omega, e_k\rangle)\right] \\
&= \int_{\mathbb{R}^m} \prod_{k=1}^m h_{\alpha_k}(x_k) h_{\beta_k}(x_k) d\lambda_m(x_1, \cdots, x_m) \\
&= \prod_{k=1}^m \int_{\mathbb{R}} h_{\alpha_k}(x_k) h_{\beta_k}(x_k) d\lambda_m(x_k) \\
&= \prod_{k=1}^m \delta_{\alpha_k, \beta_k} \alpha_k! = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \alpha! & \text{if } \alpha = \beta \end{cases}
\end{aligned}$$

where we have used the following orthogonality relation for Hermite polynomials:

$$\int_{\mathbb{R}} h_i(x) h_j(x) e^{-\frac{1}{2}x^2} dx = \delta_{ij} \sqrt{2\pi} j! .$$

□

Example 3.3 Note that by orthogonality of the family $\{\mathcal{H}_\alpha\}_{\alpha \in \mathcal{I}}$ in $L^2(\mu_\phi)$ we have that the coefficients c_α in the expansion (3.12) of F are given by

$$c_\alpha = \frac{1}{\alpha!} \mathrm{E}_{\mu_\phi}[F \mathcal{H}_\alpha] . \tag{3.19}$$

Choose $f \in L_\phi^2(\mathbb{R})$ and put $F(\omega) = \langle\omega, f\rangle = \int_{\mathbb{R}} f(s) dB_H(s)$. Then F is Gaussian and by (3.16) and (2.7) we deduce that

$$\begin{aligned}
\mathrm{E}_{\mu_\phi}[F \mathcal{H}_{\varepsilon^{(i)}}] &= \mathrm{E}_{\mu_\phi}[\langle\omega, f\rangle \langle\omega, e_i\rangle] \\
&= (f, e_i)_\phi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e_i(v) \phi(u, v) du dv .
\end{aligned}$$

Moreover,

$$\mathrm{E}_{\mu_\phi}[F\mathcal{H}_{\varepsilon^{(i)}}] = 0 \quad \text{if} \quad |\alpha| > 1 .$$

We conclude that we have the expansion

$$\int_{\mathbb{R}} f(s)dB_H(s) = \sum_{i=1}^{\infty}(f, e_i)_\phi \mathcal{H}_{\varepsilon^{(i)}}(\omega) ; \quad f \in L^2_\phi(\mathbb{R}) . \tag{3.20}$$

In particular, for fractional Brownian motion we get, by choosing $f = \chi_{[0,t]}$,

$$B_H(t) = \sum_{i=1}^{\infty} \left[\int_0^t \left(\int_{-\infty}^{\infty} e_i(v)\phi(u,v)dv \right) du \right] \mathcal{H}_{\varepsilon^{(i)}}(\omega) . \tag{3.21}$$

We proceed to define the fractional Hida test function and distribution spaces (compare with Definition 2.3.2 in [10]):

Definition 3.4 a) *(The fractional Hida test function spaces) Define* $(\mathcal{S})_H$ *to be the set of all*

$$\psi(\omega) = \sum_{\alpha \in \mathcal{I}} a_\alpha \mathcal{H}_\alpha(\omega) \in L^2(\mu_\phi) \quad \textit{such that}$$

$$\|\psi\|^2_{H,k} := \sum_{\alpha \in \mathcal{I}} \alpha! a^2_\alpha (2\mathbb{N})^{k\alpha} < \infty \quad \text{for all} \quad k \in \mathbb{N} , \tag{3.22}$$

where

$$(2\mathbb{N})^\gamma = \prod_j (2j)^{\gamma_j} \quad \text{if} \quad \gamma = (\gamma_1, \cdots, \gamma_m) \in \mathcal{I} .$$

b) *(The fractional Hida distribution spaces) Define* $(\mathcal{S})^*_H$ *to be the set of all formal expansions*

$$G(\omega) = \sum_{\beta \in \mathcal{I}} b_\beta \mathcal{H}_\beta(\omega)$$

such that

$$\|\psi\|^2_{H,-q} := \sum_{\beta \in \mathcal{I}} \beta! b^2_\beta (2\mathbb{N})^{-q\beta} < \infty \quad \text{for some} \quad q \in \mathbb{N} . \tag{3.23}$$

We equip $(\mathcal{S})_H$ with the projective topology and $(\mathcal{S})^*_H$ with the inductive topology. Then $(\mathcal{S})^*_H$ can be identified with the dual of $(\mathcal{S})_H$ and the action of $G \in (\mathcal{S})^*_H$ on $\psi \in (\mathcal{S})_H$ is given by

$$\langle\langle G, \psi \rangle\rangle := \langle G, \psi \rangle_{(\mathcal{S})_H, (\mathcal{S})^*_H} := \sum_{\alpha \in \mathcal{I}} \alpha! a_\alpha b_\alpha . \tag{3.24}$$

In particular, if G belongs to $L^2(\mu_\phi) \subset (\mathcal{S})^*_H$ and $\psi \in (\mathcal{S})_H \subset L^2(\mu_\phi)$ then

$$\langle\langle G, \psi \rangle\rangle = \mathrm{E}_{\mu_\phi}[G\psi] = (G, \psi)_{L^2(\mu_\phi)} .$$

We can in a natural way define $(\mathcal{S})^*_H$-valued integrals as follows:

Definition 3.5 *Suppose $Z : \mathbb{R} \to (\mathcal{S})^*_H$ is a given function with property that*

$$\langle\langle Z(t), \psi(t) \rangle\rangle \in L^2(\mu_\phi) \quad \textit{for all } \psi \in (\mathcal{S})_H . \tag{3.25}$$

Then $\int_{\mathbb{R}} Z(t)dt$ is defined to be the unique element of $(\mathcal{S})^*_H$ such that

$$\langle\langle \int_{\mathbb{R}} Z(t)dt, \psi(t) \rangle\rangle = \int_{\mathbb{R}} \langle\langle Z(t), \psi(t) \rangle\rangle dt \quad \textit{for all} \quad \psi \in (\mathcal{S})_H . \tag{3.26}$$

Just as in [10], Proposition 8.1, one can show that (3.26) defines $\int_{\mathbb{R}} Z(t)dt$ as an element of $(\mathcal{S})^*_H$.

If (3.25) holds, then we say that $Z(t)$ is *integrable* in $(\mathcal{S})^*_H$.

Example 3.6 The *fractional white noise* $W_H(t)$ at time t is defined by

$$W_H(t) = \sum_{i=1}^{\infty} \left[\int_{\mathbb{R}} e_i(v) \phi(t, v) dv \right] \mathcal{H}_{\varepsilon^{(i)}}(\omega) . \tag{3.27}$$

We see that for $q > 13/9$ we have

$$\begin{aligned} \|W_H(t)\|^2_{H,-q} &= \sum_{i=1}^{\infty} \varepsilon^{(i)}! \left[\int_{\mathbb{R}} e_i(v) \phi(t, v) dv \right]^2 (2\mathbb{N})^{-q\varepsilon^{(i)}} \\ &= \sum_{i=1}^{\infty} \left[\int_{\mathbb{R}} e_i(v) \phi(t, v) dv \right]^2 (2i)^{-q} < \infty \end{aligned}$$

by (3.7). Hence $W_H(t) \in (\mathcal{S})^*_H$ for all t. Moreover, by (3.9) it follows that $t \to W_H(t)$ is a continuous function from $\mathbb{R}$ into $(\mathcal{S})^*_H$. Hence $W_H(t)$ is integrable in $(\mathcal{S})^*_H$ for $0 \le s \le t$ and

$$\int_0^t W_H(s) ds = \sum_{i=1}^{\infty} \left[\int_0^t \left(\int_{\mathbb{R}} e_i(v) \phi(u, v) dv \right) du \right] \mathcal{H}_{\varepsilon^{(i)}}(\omega) = B_H(t) \tag{3.28}$$

by Example 3.2. Therefore $t \to B_H(t)$ is differentiable in $(\mathcal{S})^*_H$ and

$$\frac{d}{dt} B_H(t) = W_H(t) \quad \text{in} \quad (\mathcal{S})^*_H . \tag{3.29}$$

This justifies the name *fractional white noise* for $W_H(t)$.

Definition 3.7 *Let*

$$F(\omega) = \sum_{\alpha \in \mathcal{I}} a_\alpha \mathcal{H}_\alpha(\omega) \quad \text{and} \quad G(\omega) = \sum_{\beta \in \mathcal{I}} b_\beta \mathcal{H}_\beta(\omega)$$

be two members of $(\mathcal{S})^*_H$. Then we define the Wick product $F \diamond G$ of F and G by

$$(F \diamond G)(\omega) = \sum_{\alpha, \beta \in \mathcal{I}} a_\alpha b_\beta \mathcal{H}_{\alpha+\beta}(\omega) = \sum_{\gamma \in \mathcal{I}} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) \mathcal{H}_\gamma(\omega) . \tag{3.30}$$

Just as for the usual white noise theory one can now prove (see [10], Lemma 2.4.4)

Lemma 3.8 **a)** $F,\, G \in (\mathcal{S})^*_H \Longrightarrow F \diamond G \in (\mathcal{S})^*_H$
b) $\psi,\, \eta \in (\mathcal{S})_H \Longrightarrow \psi \diamond \eta \in (\mathcal{S})_H$

Example 3.9 Let $f,\, g \in L^2_\phi(\mathbb{R})$. Then by (3.10)

$$\begin{aligned}
&\left(\int_{\mathbb{R}} f dB_H\right) \diamond \left(\int_{\mathbb{R}} g dB_H\right) = \left(\sum_{i=1}^\infty (f, e_i)_\phi \mathcal{H}_{\varepsilon^{(i)}}\right) \diamond \left(\sum_{j=1}^\infty (g, e_j)_\phi \mathcal{H}_{\varepsilon^{(j)}}\right) \\
&= \sum_{i,j=1}^\infty (f, e_i)_\phi (g, e_j)_\phi \mathcal{H}_{\varepsilon^{(i)}+\varepsilon^{(j)}} \\
&= \sum_{\substack{i,j=1 \\ i\neq j}}^\infty (f, e_i)_\phi (g, e_j)_\phi \langle \omega, e_i\rangle\langle \omega, e_j\rangle + \sum_{i=1}^\infty (f, e_i)_\phi (g, e_i)_\phi \left(\langle \omega, e_i\rangle^2 - 1\right) \\
&= \left(\sum_{i=1}^\infty (f, e_i)_\phi \langle \omega, e_i\rangle\right)\left(\sum_{j=1}^\infty (g, e_j)_\phi \langle \omega, e_j\rangle\right) + \sum_{i=1}^\infty (f, e_i)_\phi (g, e_i)_\phi\,.
\end{aligned}$$

We conclude that

$$\left(\int_{\mathbb{R}} f dB_H\right) \diamond \left(\int_{\mathbb{R}} g dB_H\right) = \left(\int_{\mathbb{R}} f dB_H\right) \cdot \left(\int_{\mathbb{R}} g dB_H\right) - (f, g)_\phi\,. \tag{3.31}$$

Example 3.10 If $X \in (\mathcal{S})^*_H$ then we define its Wick powers $X^{\diamond n}$ by

$$X^{\diamond n} = X \diamond X \diamond \cdots \diamond X \quad \text{(n factors)} \tag{3.32}$$

and we define its Wick exponential $\exp^\diamond(X)$ by

$$\exp^\diamond(X) = \sum_{n=0}^\infty \frac{1}{n!} X^{\diamond n} \tag{3.33}$$

provided that the series converges in $(\mathcal{S})^*_H$. Note that by definition of the Wick product we have

$$\langle \omega, e_k\rangle^{\diamond n} = (\mathcal{H}_{\varepsilon^{(k)}})^{\diamond n} = \mathcal{H}_{n\varepsilon^{(k)}} = h_n(\langle \omega, e_k\rangle)\,. \tag{3.34}$$

Therefore, if $c_k \in \mathbb{R}$ we get

$$\begin{aligned}
\exp^\diamond(c_k\langle \omega, e_k\rangle) &= \sum_{n=0}^\infty \frac{c_k^n}{n!} \langle \omega, e_k\rangle^{\diamond n} \\
&= \sum_{n=0}^\infty \frac{c_k^n}{n!} h_n(\langle \omega, e_k\rangle) \\
&= \exp\left(c_k\langle \omega, e_k\rangle - \frac{1}{2} c_k^2\right)\,,
\end{aligned}$$

by the generating property of Hermite polynomials. More generally, if $f \in L^2_\phi(\mathbb{R})$ we get

$$\begin{aligned}
\exp^\diamond(\langle \omega, f\rangle) &= \exp^\diamond\left(\sum_k (f, e_k)_\phi \langle \omega, e_k\rangle\right) \\
&= \prod_k \diamond \exp^\diamond\left((f, e_k)_\phi \langle \omega, e_k\rangle\right) \\
&= \prod_k \exp^\diamond\left((f, e_k)_\phi \langle \omega, e_k\rangle\right) \\
&= \prod_k \exp\left((f, e_k)_\phi \langle \omega, e_k\rangle - \frac{1}{2}(f, e_k)^2_\phi\right) \\
&= \exp\left(\sum_k (f, e_k)_\phi \langle \omega, e_k\rangle - \frac{1}{2}\sum_k (f, e_k)^2_\phi\right) \\
&= \exp\left(\langle \omega, f\rangle - \frac{1}{2}|f|^2_\phi\right). \qquad (3.35)
\end{aligned}$$

Thus

$$\exp^\diamond(\langle \omega, f\rangle) = \mathcal{E}(f) \quad \text{for all} \quad f \in L^2_\phi(\mathbb{R}). \qquad (3.36)$$

More generally, if $g : \mathbb{C} \to \mathbb{C}$ is an entire function ($\mathbb{C}$ is the set of complex numbers) with the power series expansion

$$g(z_1, \cdots, z_n) = \sum_\alpha c_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n} =: \sum_\alpha c_\alpha z^\alpha$$

(where we have put $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ if $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathcal{I}$), then we define, for $X = (X_1, \cdots, X_n) \in ((\mathcal{S})^*_H)^n$,

$$g^\diamond(X_1, \cdots, X_n) = \sum_\alpha c_\alpha X^{\diamond\alpha}. \qquad (3.37)$$

It is useful to note that with this notation we in fact have

$$\mathcal{H}_\alpha(\omega) = \langle \omega, e_1\rangle^{\diamond\alpha_1} \diamond \cdots \diamond \langle \omega, e_n\rangle^{\diamond\alpha_n} \qquad (3.38)$$

if $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathcal{I}$. Or, if we define

$$\xi_k(\omega) = \langle \omega, e_k\rangle, \quad \xi = (\xi_1, \xi_2, \cdots) \qquad (3.39)$$

then

$$\mathcal{H}_\alpha(\omega) = \xi^{\diamond\alpha}. \qquad (3.40)$$

We now proceed to define a generalized fractional stochastic integral of Itô type.

Definition 3.11 *Suppose $Y : \mathbb{R} \to (\mathcal{S})^*_H$ is a given function such that $Y(t) \diamond W_H(t)$ is integrable in $(\mathcal{S})^*_H$. Then we define its fractional stochastic integral of Itô type, $\int_\mathbb{R} Y(t)dB_H(t)$, by*

$$\int_\mathbb{R} Y(t)dB_H(t) := \int_\mathbb{R} Y(t) \diamond W_H(t)dt. \qquad (3.41)$$

Example 3.12 Suppose

$$Y(t) = \sum_{i=1}^{n} F_i(\omega)\chi_{[t_i,t_{i+1})}(t), \quad \text{where} \quad F_i \in (\mathcal{S})_H^* .$$

Then by (3.28) we see that

$$\int_{\mathbb{R}} Y(t)dB_H(t) = \sum_{i=1}^{n} F_i(\omega) \diamond (B_H(t_{i+1}) - B_H(t_i)) .$$

Hence our definition (3.41) is an extension of the fractional Itô integral introduced in [4], (3.17).

In the classical Brownian motion case (3.41) represents a generalization of the Hitsuda-Skorohod integral. See [14], [2] and [10], Theorem 2.5.9. Note that - as mentioned in the introduction - in general the fractional Itô integral $\int Y(t)dB_H(t)$ differs from the fractional Stratonovich integral $\int Y(t)\delta B_H(t)$, which up to now has been the type of fractional integral most commonly studied.

There is an Itô formula for the Itô fractional stochastic integral (see [4], Theorem 4.1) and as in the case of the classical Brownian motion one can use this to compute such integrals. But in some cases it is easier to work directly within the Wick calculus in $(\mathcal{S})_H^*$. The method is similar to the one presented in e.g. [10], Chapter 3. We illustrate it by means of two simple examples:

Example 3.13

$$\begin{aligned}\int_0^t B_H(s)dB_H(s) &= \int_0^t B_H(s) \diamond W_H(s)ds \\ &= \int_0^t B_H(s) \diamond \frac{d}{ds}B_H(s)ds = \frac{1}{2}B_H^{\diamond 2}(t) \\ &= \frac{1}{2}B_H^2(t) - \frac{1}{2}t^{2H}, \end{aligned} \tag{3.42}$$

where we have used (3.41), (3.29), standard Wick calculus, (3.31) and finally the fact that

$$\int_0^t \int_0^t \phi(u,v)dudv = t^{2H} . \tag{3.43}$$

Example 3.14 *(Geometric fractional Brownian motion)* Consider the fractional stochastic differential equation

$$dX(t) = \mu X(t)dt + \sigma X(t)dB_H(t)\,; \quad X(0) = x > 0\,, \tag{3.44}$$

where x, μ and σ are constants. We rewrite this as the following equation in $(\mathcal{S})_H^*$:

$$\frac{dX(t)}{dt} = \mu X(t) + \sigma X(t) \diamond W_H(t)$$

or

$$\frac{dX(t)}{dt} = (\mu + \sigma W_H(t)) \diamond X(t) \,.$$

Using Wick calculus we see that the solution of this equation is

$$\begin{aligned} X(t) &= x \exp^{\diamond}\left(\mu t + \sigma \int_0^t W_H(s)ds\right) \\ &= x \exp^{\diamond}\left(\mu t + \sigma B_H(t)\right) \,. \end{aligned} \tag{3.45}$$

By (3.36) and (3.42) this can be written

$$X(t) = x \exp\left(\sigma B_H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right) \,. \tag{3.46}$$

Note that

$$\mathrm{E}_{\mu_\phi}\left[X(t)\right] = x e^{\mu t} \,. \tag{3.47}$$

As in [10], we define the translation operator as follows: Let $\omega_0 \in \mathcal{S}'(\mathrm{R})$. For $F \in (\mathcal{S})_H$, we define

$$T_{\omega_0} F(\omega) = F(\omega + \omega_0) \,; \quad \omega \in \mathcal{S}'(\mathrm{R}) \,.$$

It is easy to verify as in the proof of Theorem 2.10.1 that $f \to T_{\omega_0} f$ is a continuous linear map from $(\mathcal{S})_H$ to $(\mathcal{S})_H$. We then define the adjoint translation operator $T_{\omega_0}^*$ from $(\mathcal{S})_H^*$ to $(\mathcal{S})_H^*$ by

$$\langle\langle T_{\omega_0}^* X \,, F \rangle\rangle = \langle\langle X \,, T_{\omega_0} F \rangle\rangle \,, \quad X \in (\mathcal{S})_H^* \,, \; F \in (\mathcal{S})_H \,.$$

Lemma 3.15 *Let $\omega_0 \in L_\phi^2(\mathrm{R})$ and define $\tilde{\omega}_0(t) = \int_{\mathrm{R}} \omega_0(u)\phi(t,u)du$. Then*

$$T_{\tilde{\omega}_0}^* X = X \diamond \exp^{\diamond}\left(\langle \omega \,, \omega_0 \rangle\right) \,. \tag{3.48}$$

Proof By a density argument it suffices to show

$$\langle\langle T_{\omega_0}^* X \,, F \rangle\rangle = \langle\langle X \diamond \exp^{\diamond}\left(\langle \omega, \omega_0 \rangle\right) \,, F \rangle\rangle$$

for

$$X = \exp\left(\langle \omega, g \rangle - \frac{1}{2}|g|_\phi^2\right) \,, \quad F = \exp\left(\langle \omega, f \rangle - \frac{1}{2}|f|_\phi^2\right)$$

where $f, g \in L_\phi^2(\mathrm{R})$ and $\omega_0 \in \mathcal{S}(\mathrm{R})$. In this case, we have by definition

$$\begin{aligned} \langle\langle T_{\tilde{\omega}_0}^* X, F \rangle\rangle &= \langle\langle X, F(\omega + \tilde{\omega}_0) \rangle\rangle \\ &= e^{\langle \tilde{\omega}_0, f \rangle} \langle\langle X, F(\omega) \rangle\rangle \\ &= e^{\langle \omega_0, f \rangle_\phi - \langle f, g \rangle_\phi} \,. \end{aligned}$$

On the other hand,

$$\begin{aligned}\langle\!\langle X \diamond \exp^{\diamond}(\langle \omega , \omega_0 \rangle) , F \rangle\!\rangle &= \langle\!\langle e^{\langle \omega, g+\omega_0\rangle - \frac{1}{2}|g+\omega_0|_\phi^2} , F\rangle\!\rangle \\ &= e^{\langle g+\omega_0 , f\rangle_\phi} .\end{aligned}$$

This shows the lemma. □

With the white noise machinery established one can now verify that the proof of the Benth-Gjessing version of the Girsanov formula, as presented in Corollary 2.10.5 in [10], applies to give the following fractional version:

Theorem 3.16 *(Fractional Girsanov formula I)*

Let $\psi \in L^p(\mu_\phi)$ for some $p > 1$ and let $\gamma \in L^2_\phi(\mathrm{R}) \subset \mathcal{S}'(\mathrm{R})$. Let $\tilde{\gamma}$ be defined by $\tilde{\gamma}(t) = \int_{\mathrm{R}} \phi(t,s)\gamma(s)ds$. Then the map $\omega \to \psi(\omega + \tilde{\gamma})$ belongs to $L^\rho(\mu_\phi)$ for all $\rho < p$ and

$$\int_{\mathcal{S}'(\mathrm{R})} \psi(\omega+\tilde{\gamma})d\mu_\phi(\omega) = \int_{\mathcal{S}'(\mathrm{R})} \psi(\omega)\cdot \exp^{\diamond}(\langle \omega,\gamma\rangle)d\mu_\phi(\omega) . \tag{3.49}$$

Proof By Lemma 3.15, we have

$$\langle\!\langle X , T_{\tilde{\gamma}}\psi \rangle\!\rangle = \langle\!\langle X \diamond \exp^{\diamond}(\langle \omega , \gamma \rangle) , \psi \rangle\!\rangle . \tag{3.50}$$

Let $X = 1$. We see that the left hand of (3.50) is $\int_{\mathcal{S}'(\mathrm{R})} \psi(\omega+\tilde{\gamma})d\mu_\phi(\omega)$ and the right hand side of (3.50) is $\int_{\mathcal{S}'(\mathrm{R})} \psi(\omega)\cdot\exp^{\diamond}(\langle\omega,\gamma\rangle)d\mu_\phi(\omega)$. This completes the proof of this theorem. □

Corollary 3.17 *Let $g : \mathrm{R} \to \mathrm{R}$ be bounded and let $\gamma \in L^2_\phi(\mathrm{R})$. Then, with $\mathcal{E}(\cdot)$ as in (2.8),*

$$\mathrm{E}_{\mu_\phi}\left[g(B_H(t) + \int_0^t \tilde{\gamma}(s)ds)\right] = \mathrm{E}_{\mu_\phi}\left[g(B_H(t))\mathcal{E}(\gamma)\right] . \tag{3.51}$$

Proof Define $\psi(\omega) = g(\langle \omega, \chi_{[0,t]}\rangle) = g(B_H(t))$. Then

$$\psi(\omega+\tilde{\gamma}) = g(\langle \omega+\tilde{\gamma}, \chi_{[0,t]}\rangle) = g(B_H(t) + \int_0^t \tilde{\gamma}(s)ds)$$

so the result follows from (3.36) and Theorem 3.16. □

The following Girsanov theorem will be used in Section 5.

Theorem 3.18 *(Fractional Girsanov formula II) Let $T > 0$ and let γ be a continuous function with* supp $\gamma \subset [0,T]$. *Let K be a function with* supp $K \subset [0,T]$ *and such that*

$$(K,f)_\phi = (\gamma, f)_{L^2(\mathrm{R})} , \quad \forall \ f \in (\mathcal{S}) , \ \mathrm{supp}\, f \subset [0,T] , \tag{3.52}$$

i.e.

$$\int_{\mathrm{R}} K(s)\phi(s,t)ds = \gamma(t) , \quad 0 \le t \le T . \tag{3.53}$$

Define a probability measure $\mu_{\phi,\gamma}$ on the σ-algebra $\mathcal{F}_T^{(H)}$ generated by $\{B_H(s); s \le T\}$ by

$$\frac{d\mu_{\phi,\gamma}}{d\mu_\phi} = \exp^\diamond \{-\langle \omega, K\rangle\} \tag{3.54}$$

Then $\hat{B}_H(t) = B_H(t) + \int_0^t \gamma_s ds$, $0 \le t \le T$, is a fractional Brownian motion under $\mu_{\phi,\gamma}$.

Proof It suffices to show that for any $G(\omega) = \exp\{\langle \omega, f\rangle\}$ with $f \in \mathcal{S}(\mathrm{R})$, supp $f \subset [0, T]$ we have

$$\mathrm{E}_{\mu_{\phi,\gamma}} \{G(\omega+\gamma)\} = \mathrm{E}_{\mu_\phi} \{G(\omega+\gamma) \exp^\diamond [-\langle \omega, K\rangle]\} = \mathrm{E}_{\mu_\phi} G(\omega) .$$

But in this case

$$\begin{aligned}
\mathrm{E}_{\mu_\phi} \{G(\omega+\gamma) \exp^\diamond [-\langle \omega, K\rangle]\} &= \mathrm{E}_{\mu_\phi} \exp\left\{\langle \omega+\gamma\,, f\rangle - \langle \omega, K\rangle - \frac{1}{2}|K|_\phi^2\right\} \\
&= \mathrm{E}_{\mu_\phi} \exp\left\{\langle \omega\,, f-K\rangle + (\gamma, f)_{L^2(\mathrm{R})} - \frac{1}{2}|K|_\phi^2\right\} \\
&= \exp\left\{\frac{1}{2}|f-K|_\phi^2 + (\gamma, f)_{L^2(\mathrm{R})} - \frac{1}{2}|K|_\phi^2\right\} \\
&= \exp\left\{\frac{1}{2}|K|_\phi^2 - \langle K, f\rangle_\phi + \frac{1}{2}|f|_\phi^2 + (\gamma, f)_{L^2(\mathrm{R})} - \frac{1}{2}|K|_\phi^2\right\} \\
&= e^{\frac{1}{2}|f|_\phi^2} = \mathrm{E}_{\mu_\phi} e^{\langle \omega, f\rangle} = \mathrm{E}_{\mu_\phi} G(\omega) .
\end{aligned}$$

□

Remark 3.19 *Since $B_H(t)$ is not a martingale, unlike in the standard Brownian motion case, the restriction of $\frac{d\mu_{\phi,\gamma}}{d\mu_\phi}$ to $\mathcal{F}_t^{(H)}$, $0 < t < T$ is in general* **not** *given by $\exp^\diamond \left\{-\langle \omega, \chi_{[0,t]} K\rangle\right\}$.*

Remark 3.20 *In [16], a special case of (3.51) was obtained. In Section 5, Theorem 3.18 will be used.*

We end this section by giving an alternative Wiener-Itô chaos expansion theorem in terms of iterated integrals:

Let $\hat{L}_\phi^2(\mathrm{R}^n)$ denote the set of functions $f(x_1, \cdots, x_n)$ on R^n which are symmetric with respect to its n variables and which satisfies

$$\|f\|_{L_\phi^2(\mathrm{R}^n)}^2 := (f, f)_{L_\phi^2(\mathrm{R}^n)} < \infty ,$$

where

$$(f, g)_{L_\phi^2(\mathrm{R}^n)} := \int_{\mathrm{R}^n \times \mathrm{R}^n} f(u_1, \cdots, u_n) g(v_1, \cdots, v_n) \phi(u_1, v_1) \cdots \phi(u_n, v_n) du_1 \cdots du_n dv_1 \cdots dv_n \tag{3.55}$$

Thus one can define (see [4], Theorem 6.7) the iterated integral

$$I_n(f_n) := \int_{\mathrm{R}^n} f dB_H(s)^{\otimes n} := n! \int_{s_1 < \cdots < s_n} f(s_1, \cdots, s_n) dB_H(s_1) \cdots dB_H(s_n) . \tag{3.56}$$

For $n = 0$ and $f = f_0$ constant we set $I_0(f_0) = f_0$ and $\|f_0\|_{L_\phi^2(\mathrm{R}^0)}^2 = f_0^2$.

Theorem 3.21 *[4] Let $F \in L^2(\mu_\phi)$. Then there exist $f_n \in \hat{L}^2_\phi(\mathbb{R}^n)$ for $n = 0, 1, 2, \cdots$ such that*

$$F(\omega) = \sum_{n=0}^{\infty} I_n(f_n)\,. \tag{3.57}$$

Moreover,

$$\|F\|^2_{L^2(\mu_\phi)} = \sum_{n=0}^{\infty} n!\|f_n\|^2_{L^2_\phi(\mathbb{R}^n)}\,. \tag{3.58}$$

Proof For a direct proof see [4], Theorem 6.9. The result can be also deduced from our Theorem 3.1 by using the identity

$$\mathcal{H}_\alpha(\omega) = \int_{\mathbb{R}^n} e_1^{\otimes\alpha_1}\hat{\otimes}\cdots\hat{\otimes}e_n^{\otimes\alpha_n} dB_H^{\otimes n} \tag{3.59}$$

if $\alpha = (\alpha_1, \cdots, \alpha_n)$, where $\hat{\otimes}$ denotes symmetrized tensor product. (See the proof of Theorem 6.7 in [4]). □

4 Differentiation. A Fractional Clark-Ocone Theorem

Now that the basic fractional white noise theory is established, we can proceed as in [1] to define stochastic gradient and prove a generalized Clark-Ocone formula in the fractional case.

Definition 4.1 *Let $F : \mathcal{S}'(\mathbb{R}) \to \mathbb{R}$ be a given function and let $\gamma \in \mathcal{S}'(\mathbb{R})$. We say that F has a directional derivative in the direction γ if*

$$D_\gamma F(\omega) := \lim_{\varepsilon\to 0} \frac{F(\omega + \varepsilon\gamma) - F(\omega)}{\varepsilon}$$

exists in $(\mathcal{S})^*_H$. If this is the case, we call $D_\gamma F$ the directional derivative of F in the direction γ.

Example 4.2 If $F(\omega) = \langle\omega, f\rangle = \int_{\mathbb{R}} f(t)dB_H(t)$, for some $f \in \mathcal{S}(\mathbb{R})$ and $\gamma \in L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ then

$$\begin{aligned} D_\gamma F(\omega) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} [\langle\omega + \varepsilon\gamma, f\rangle - \langle\omega, f\rangle] \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} [\langle\varepsilon\gamma, f\rangle] = \langle\gamma, f\rangle = \int_{\mathbb{R}} f(t)\gamma(t)dt\,. \end{aligned}$$

Definition 4.3 *We say that $F : \mathcal{S}'(\mathbb{R}) \to \mathbb{R}$ is differentiable if there exists a map $\Psi : \mathbb{R} \to (\mathcal{S})^*_H$ such that*

$$\Psi(t)\gamma(t) = \Psi(t,\omega)\gamma(t) \quad \textit{is } (\mathcal{S})^*_H - \textit{ integrable}$$

and

$$D_\gamma F(\omega) = \int_{\mathbb{R}} \Psi(t,\omega)\gamma(t)dt \quad \text{for all} \quad \gamma \in L^2(\mathbb{R}).$$

In this case we put

$$D_t F(\omega) := \frac{dF}{d\omega}(t,\omega) := \Psi(t,\omega)$$

and we call $D_t F(\omega) = \frac{dF}{d\omega}(t,\omega)$ *the stochastic gradient (or the Hida / Malliavin derivative) of* F *at* t.

Example 4.4 Let $F(\omega) = \langle \omega, f\rangle$ with $f \in \mathcal{S}(\mathbb{R})$. Then by Example 4.2 F is differentiable and its stochastic gradient is

$$D_t F(\omega) = f(t) \quad \text{for} \quad a.a.t, \omega.$$

Just as in [1], Lemma 3.6 we now get

Lemma 4.5 *(The chain rule I) Let* $P(y) = \sum_\alpha c_\alpha y^\alpha$ *be a polynomial in* n *variables* $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$. *Choose* $f_i \in \mathcal{S}(\mathbb{R})$ *and put* $Y = (Y_1, \cdots, Y_n)$ *with*

$$Y_i(\omega) = \langle \omega, f_i\rangle = \int_{\mathbb{R}} f_i(t)dB_H(t)\,; \quad 1 \le u \le n.$$

Then both $P(Y)$ *and* $P^\diamond(Y)$ *are differentiable and*

$$D_t P(Y) = \sum_{i=1}^n \frac{\partial P}{\partial x_i}(Y_1, \cdots, Y_n) f_i(t) = \sum_\alpha \sum_i \alpha_i Y^{\alpha - \varepsilon^{(i)}} f_i(t)$$

and

$$D_t P^\diamond(Y) = \sum_{i=1}^n \frac{\partial P^\diamond}{\partial x_i}(Y_1, \cdots, Y_n) f_i(t) = \sum_\alpha \sum_i \alpha_i Y^{\diamond(\alpha - \varepsilon^{(i)})} f_i(t)$$

Similarly, if we define $Y^{(t)} = (Y_1^{(t)}, \cdots, Y_n^{(t)})$ with

$$Y_i^{(t)}(\omega) = \int_0^t f_i(s)dB_H(s) = \int_{\mathbb{R}} f_i(s)\chi_{[0,t]}(s)dB_H(s)\,; \quad 1 \le i \le n$$

then we obtain, as in [1], Lemma 3.7:

Lemma 4.6 *(Chain rule II)*

$$\frac{d}{dt} P^\diamond(Y^{(t)}) = \sum_{j=1}^n f_j(t) \left(\frac{\partial P}{\partial x_j}\right)^\diamond (Y^{(t)}) \diamond W_H(t).$$

We want to extend the differentiation operator to a space of random variables containing $L^2(\mu_\phi)$. A convenient pair of spaces to work with is the following:

Definition 4.7 *([18], [1])*

(i) Let $k \in \mathrm{N}$. We say that a function

$$\psi(\omega) = \sum_{n=0}^{\infty} \int_{\mathrm{R}^n} f_n dB_H^{\otimes n}(t) \in L^2(\mu_\phi) ; \quad f_n \in \hat{L}_\phi^2(\mathrm{R}^n)$$

belongs to the space $\mathcal{G}_k = \mathcal{G}_k(\mu_\phi)$ if

$$\|\psi\|_{\mathcal{G}_k}^2 := \sum_{n=0}^{\infty} n! \|f_n\|_{L_\phi^2(\mathrm{R}^n)}^2 e^{2kn} < \infty . \tag{4.1}$$

We define

$$\mathcal{G} = \mathcal{G}(\mu_\phi) = \cap_{k=1}^{\infty} \mathcal{G}_k(\mu_\phi) \tag{4.2}$$

and equip $\mathcal{G}$ with the projective topology.

(ii) Let $q \in \mathrm{N}$. We say that a formal expansion

$$G = \sum_{n=0}^{\infty} \int_{\mathrm{R}^n} g_n dB_H^{\otimes n}(t) ; \quad g_n \in \hat{L}_\phi^2(\mathrm{R}^n)$$

belongs to the space $\mathcal{G}_{-q} = \mathcal{G}_{-q}(\mu_\phi)$ if

$$\|G\|_{\mathcal{G}_q}^2 = \sum_{n=0}^{\infty} n! \|g_n\|_{L_\phi^2(\mathrm{R}^n)}^2 e^{-2qn} < \infty . \tag{4.3}$$

We define

$$\mathcal{G}^* = \mathcal{G}^*(\mu_\phi) = \cup_{q \in \mathrm{N}} \mathcal{G}_{-q}(\mu_\phi) \tag{4.4}$$

and equip $\mathcal{G}^*$ with the inductive topology. Then $\mathcal{G}^*$ is the dual of $\mathcal{G}$. And the action of $G \in \mathcal{G}^*$ on $\psi \in \mathcal{G}$ is given by

$$\langle\langle G, \psi \rangle\rangle = \sum_{n=0}^{\infty} n! (g_n, f_n)_{L_\phi^2(\mathrm{R}^n)} . \tag{4.5}$$

Remark 4.8 *Note that by Theorem 3.21 we have*

$$(\mathcal{S})_H \subset \mathcal{G}(\mu_\phi) \subset L^2(\mu_\phi) = \left(L^2(\mu_\phi)\right)^* \subset \mathcal{G}^*(\mu_\phi) \subset (\mathcal{S})_H^* .$$

Let $\mathcal{F}_t^{(H)}$ be the σ-algebra generated by $B_H(s, \cdot)$; $s \le t$. The following operator is useful:

Definition 4.9 a) *Let $G = \sum_{n=0}^{\infty} \int_{\mathrm{R}^n} g_n(s) dB_H^{\otimes n}(s) \in \mathcal{G}^*$. Then we define the quasi-conditional expectation of G with respect to $\mathcal{F}_t^{(H)}$ by*

$$\tilde{\mathrm{E}}_{\mu_\phi}\left[G | \mathcal{F}_t^{(H)}\right] = \sum_{n=0}^{\infty} \int_{\mathrm{R}^n} g_n(s) \cdot \chi_{0 \le s \le t} dB_H^{\otimes n}(s) . \tag{4.6}$$

b) *We say that $G \in \mathcal{G}^*$ is $\mathcal{F}_t^{(H)}$-measurable if*

$$\tilde{\mathrm{E}}_{\mu_\phi}\left[G | \mathcal{F}_t^{(H)}\right] = G .$$

Remark 4.10 *The quasi-conditional expectation* $\tilde{\mathrm{E}}$ *is different from the ordinary conditional expectation. For example, it is easy to check that* $\tilde{\mathrm{E}}\left[B_H(t)|\mathcal{F}_s^{(H)}\right] = B_H(s)$ *for* $0 \le s \le t$. *But the computation of* $\mathrm{E}\left[B_H(t)|\mathcal{F}_s^{(H)}\right]$ *is much more complicated. See for example [6].*

As in [1], Lemma 2.8 we can get

Lemma 4.11 *a)* $F \in \mathcal{G}_r \Longrightarrow \|\tilde{\mathrm{E}}_{\mu_\phi}\left[F|\mathcal{F}_t^{(H)}\right]\|_{\mathcal{G}_r} \le \|F\|_{\mathcal{G}_r}$.

b) $F \in \mathcal{G}^* \Longrightarrow \tilde{\mathrm{E}}_{\mu_\phi}\left[F|\mathcal{F}_t^{(H)}\right] \in \mathcal{G}^*$.

c) $F, G \in \mathcal{G}^* \Longrightarrow \tilde{\mathrm{E}}_{\mu_\phi}\left[F \diamond G|\mathcal{F}_t^{(H)}\right] = \tilde{\mathrm{E}}_{\mu_\phi}\left[F|\mathcal{F}_t^{(H)}\right] \diamond \tilde{\mathrm{E}}_{\mu_\phi}\left[G|\mathcal{F}_t^{(H)}\right]$.

Motivated by Lemma 4.5 we now make the following definition:

Definition 4.12 *Let* $F = \sum_\alpha c_\alpha \mathcal{H}_\alpha(\omega) \in \mathcal{G}^*$. *Then we define the stochastic gradient of* F *at* t *by*

$$\begin{aligned} D_t F(\omega) &= \frac{dF}{d\omega}(t,\omega) := \sum_\alpha c_\alpha \sum_i \alpha_i \mathcal{H}_{\alpha-\varepsilon^{(i)}} e_i(t) \\ &= \sum_\beta \left(\sum_i c_{\beta+\varepsilon^{(i)}} (\beta_i + 1) e_i(t) \right) \mathcal{H}_\beta(\omega). \end{aligned} \tag{4.7}$$

Lemma 4.13 *(A fractional Clark-Ocone formula for polynomials) Let* $F(\omega)$ *be* $\mathcal{F}_T^{(H)}$*-measurable and suppose* $F(\omega) = P^\diamond(Y)$ *for some polynomial* $P(y) = \sum_\alpha c_\alpha y^\alpha$, *where* $Y = (Y_1, \cdots, Y_n)$ *with* $Y_j = \langle \omega, f_j \rangle$ *as in Lemma 4.5 ,* $1 \le j \le n$. *Then*

$$F(\omega) = P^\diamond(Y^{(T)}), \quad \text{where} \quad Y^{(T)} = \langle \omega, f_j \chi_{[0,T]} \rangle \tag{4.8}$$

and

$$F(\omega) = \mathrm{E}_{\mu_\phi}[F] + \int_0^T \tilde{\mathrm{E}}_{\mu_\phi}\left[D_t F|\mathcal{F}_t^{(H)}\right] dB_H(t). \tag{4.9}$$

Proof The proof of Lemma 3.8 in [1] applies, with only conceptual modifications. For completeness we give the details:

Note that

$$\begin{aligned} F(\omega) &= \tilde{\mathrm{E}}_{\mu_\phi}\left[F|\mathcal{F}_T^{(H)}\right] = \sum_\alpha c_\alpha \tilde{\mathrm{E}}_{\mu_\phi}\left[Y|\mathcal{F}_T^{(H)}\right]^{\diamond\alpha} \\ &= \sum_\alpha c_\alpha \left(Y^{(T)}\right)^{\diamond\alpha} = P^\diamond(Y^{(T)}), \end{aligned}$$

where we have used Lemma 4.10 c) and (4.6). Hence by Lemma 4.5 and Lemma 4.6 we get

$$
\begin{aligned}
&\int_0^T \tilde{\mathrm{E}}_{\mu_\phi}\left[D_tF|\mathcal{F}_t^{(H)}\right] dB_H(t) \\
=& \int_0^T \tilde{\mathrm{E}}_{\mu_\phi}\left[\sum_{i=1}^n \left(\frac{\partial P}{\partial y_i}\right)^\diamond (Y) f_i(t)|\mathcal{F}_t^{(H)}\right] dB_H(t) \\
=& \int_0^T \sum_{i=1}^n \left(\frac{\partial P}{\partial y_i}\right)^\diamond (Y^{(t)}) f_i(t) \diamond W_H(t)dt \\
=& \int_0^t \frac{d}{dt} P^\diamond (Y^{(t)})dt \\
=& P^\diamond(Y^{(T)}) - P^\diamond(Y^{(0)}) = F - P^\diamond(Y^{(0)}) \\
=& F - \mathrm{E}_{\mu_\phi}[F]
\end{aligned}
$$

□

The proof of the following is identical to the proof of Lemma 3.10 in [1]:

Lemma 4.14 *a) If $F \in \mathcal{G}^*$ then $D_tF \in \mathcal{G}^*$ for a.a. t.*

b) Suppose F, $F_m \in \mathcal{G}^*$ and $F_n \to F$ in $\mathcal{G}^*$. Then there is a subsequence $\{F_{m_k}\}_{k=1}^\infty$ such that

$$D_tF_{m_k} \to D_t \quad \text{in} \quad \mathcal{G}^* \quad \text{for} \quad \text{a.a.} \quad t$$

We now have all the ingredients for the proof of the following results. We refer to [1], Theorem 3.15 and Theorem 3.11 for proofs, which are similar to our case.

Theorem 4.15 *(A fractional Clark-Ocone Theorem)*

a) Let $G(\omega) \in \mathcal{G}^*$ be $\mathcal{F}_T$-measurable. Then $D_tG \in \mathcal{G}^*$ and $\tilde{\mathrm{E}}_{\mu_\phi}\left[D_tG|\mathcal{F}_t^{(H)}\right] \in \mathcal{G}^*$ for a.a. t. $\tilde{\mathrm{E}}_{\mu_\phi}\left[D_tG|\mathcal{F}_t^{(H)}\right] \diamond W_H(t)$ is integrable in $(\mathcal{S})_H^*$ and

$$G(\omega) = \mathrm{E}_{\mu_\phi}[G] + \int_0^T \tilde{\mathrm{E}}_{\mu_\phi}\left[D_tG|\mathcal{F}_t^{(H)}\right] \diamond W_H(t)dt\,. \tag{4.10}$$

b) Suppose $G(\omega) \in L^2(\mu_\phi)$ is $\mathcal{F}_T$-measurable. Then

$$(t,\omega) \to \tilde{\mathrm{E}}_{\mu_\phi}\left[D_tG|\mathcal{F}_t^{(H)}\right](\omega) \in L^2(\lambda \times \mu)\,,$$

where λ is the Lebesgue measure on $[0,T]$. Moreover,

$$G(\omega) = \mathrm{E}_{\mu_\phi}[G] + \int_0^T \tilde{\mathrm{E}}_{\mu_\phi}\left[D_tG|\mathcal{F}_t^{(H)}\right] dB_H(t)\,. \tag{4.11}$$

We shall call $\tilde{\mathrm{E}}_{\mu_\phi}\left[D_tG|\mathcal{F}_t^{(H)}\right]$ the *fractional Clark derivative* of G in analogy with the classical Brownian motion case. We will use the notation

$$\nabla_t^\phi G = \tilde{\mathrm{E}}_{\mu_\phi}\left[D_tG|\mathcal{F}_t^{(H)}\right]\,.$$

Example 4.16 Let $\xi \in \mathbb{R}$ and let

$$X(t) = \exp\left(i\xi B_H(t) + \frac{1}{2}\xi^2 t^{2H}\right). \tag{4.12}$$

Then from Example 3.14 it follows that

$$X(T) = 1 + i\xi \int_0^T X(s) dB_H(s)$$

Thus we have

$$\nabla_t^\phi X(T) = i\xi X(t).$$

Consequently,

$$\nabla_t^\phi e^{i\xi B_H(T)} = i\xi e^{i\xi B_H(t) + \frac{1}{2}\xi^2\left(t^{2H} - T^{2H}\right)}. \tag{4.13}$$

Let $f \in \mathcal{S}(\mathbb{R})$ and let $\hat{f}$ be the Fourier transform of f, *i.e.*

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx.$$

Then

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\hat{f}(\xi) d\xi.$$

$$f(B_H(T)) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iB_H(T)\xi}\hat{f}(\xi) d\xi.$$

Therefore by (4.13) we obtain

$$\begin{aligned}
\nabla_t^\phi f(B_H(T)) &= \frac{1}{2\pi}\int_{\mathbb{R}} \nabla_t^\phi e^{iB_H(T)\xi}\hat{f}(\xi) d\xi \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} i\xi e^{i\xi B_H(t) - \frac{1}{2}\xi^2\left(T^{2H} - t^{2H}\right)}\hat{f}(\xi) d\xi \\
&= g(B_H(t)),
\end{aligned}$$

where g is the inverse Fourier transform of the product of the following two functions: $\hat{f}(\xi)$ and

$$Q(\xi) = i\xi e^{-\frac{1}{2}\xi^2\left(T^{2H} - t^{2H}\right)}.$$

However, $Q(\xi)$ is the Fourier transform of $\frac{d}{dx}P_{t,T}(x)$, where

$$P_{t,T}(x) = \frac{1}{\sqrt{2\pi(T^{2H} - t^{2H})}} e^{-\frac{x^2}{2(T^{2H} - t^{2H})}} \tag{4.14}$$

which is the heat kernel at time $T^{2H} - t^{2H}$. Thus we have obtained

$$g(x) = \int_{\mathbb{R}} q_{t,T}(x - y) f(y) dy,$$

where $q_{t,T}(x) = \frac{d}{dx}P_{t,T}(x)$.

In general, we can obtain the following

Proposition 4.17 *Let f be a function such that $\mathbb{E}|f(B_H(T))| < \infty$. Then*

$$\nabla_t^\phi f(B_H(T)) = \int_{\mathbb{R}} q_{t,T}(B_H(T) - y) f(y) dy \,, \tag{4.15}$$

where

$$q_{t,T}(x) = \frac{d}{dx} P_{t,T}(x) \quad \text{with} \quad P_{t,T}(x) = \frac{1}{\sqrt{2\pi(T^{2H} - t^{2H})}} e^{-\frac{x^2}{2(T^{2H} - t^{2H})}} \tag{4.16}$$

Remark 4.18 *When $H = 1/2$, (4.15)-(4.16) reduce to known formulas. (See [8]).*

5 Application to Finance

With the Clark-Ocone formula to our disposal, we can now follow the approach in [1], adapted to the *fractional Black & Scholes market.* This market has two investment possibilities:

(i) A *bank account* or a *bond*, where the price $A(t)$ at time t develops according to the equation

$$dA(t) = \rho A(t) dt \,, \quad A(0) = 1 \,; \quad 0 \le t \le T \,, \tag{5.1}$$

where $\rho > 0$ is constant.

(ii) A stock, where the price $X(t)$ at time t satisfies the equation

$$dX(t) = \mu X(t) dt + \sigma X(t) dB_H(t) \,; \quad X(0) = x > 0 \,, \tag{5.2}$$

where μ and $\sigma \neq 0$ are constants, $0 \le t \le T$. By Example 3.13 we know that

$$X(t) = x \exp\left(\sigma B_H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right) \,; \quad t \ge 0 \,. \tag{5.3}$$

A *portfolio* $\theta(t) = \theta(t, \omega) = (u(t), v(t))$ is an $\mathcal{F}_t^{(H)}$-adapted 2-dimensional process giving the number of units $u(t)$, $v(t)$ held at time t of the bond and the stock, respectively. The corresponding *value process* $Z(t) = Z^\theta(t, \omega)$ is given by

$$Z^\theta(t, \omega) = u(t) A(t) + v(t) X(t) \,. \tag{5.4}$$

The portfolio is called admissible if $Z^\theta(t)$ is lower bounded a.s. (t, ω) and -in addition- *self-financing*, in the sense that

$$dZ^\theta(t, \omega) = u(t) dA(t) + v(t) dX(t) \,; \quad t \in [0, T] \,. \tag{5.5}$$

This is the usual mathematical way of expressing that no money is going in or out of the market $(A(t), X(t))$. It may be regarded as the continuous time limit of the corresponding natural discrete time condition

$$Z^\theta(t_{k+1}) = Z^\theta(t_k) + u(t_k) \Delta A(t_k) + v(t_k) \Delta X(t_k) \,,$$

where

$$\Delta A(t_k) = A(t_{k+1}) - A(t_k)\,, \quad \Delta X(t_k) = X(t_{k+1}) - X(t_k)\,.$$

Assume from now on that all portfolios $\theta = (u, v)$ that we consider are admissible. Then by (5.4) we have

$$u(t) = \frac{Z^\theta(t) - v(t)X(t)}{A(t)} \tag{5.6}$$

which substituted into (5.5) gives

$$dZ^\theta(t) = \rho Z^\theta(t)dt + \sigma v(t)X(t)\left[\frac{\mu - \rho}{\sigma}dt + dB_H(t)\right]. \tag{5.7}$$

By the *Girsanov theorem* for fractional Brownian motion (Theorem 3.18) we see that

$$\hat{B}_H(t) := \frac{\mu - \rho}{\sigma}t + B_H(t) \tag{5.8}$$

is a fractional Brownian motion with respect to the measure $\hat{\mu}_\phi$ defined on $\mathcal{F}_T^{(H)}$ by

$$d\hat{\mu}_\phi(\omega) = \exp\left(-\int_0^T K(s)dB_H(s) - \frac{1}{2}|K|_\phi^2\right) d\mu_\phi(\omega)\,, \tag{5.9}$$

where $K(s) = K(T, s)$ is defined by the the following properties: supp $K \subset [0, T]$ and

$$\int_0^T K(T, s)\phi(t, s)ds = \frac{\mu - \rho}{\sigma}\,, \quad \text{for} \quad 0 \le t \le T\,. \tag{5.10}$$

By Appendix, $K(T, s)$ is given explicitly by

$$K(T, s) = \frac{(\mu - \rho)}{2\sigma H(2H-1)\Gamma(2H-1)\Gamma(2-2H)\cos(\pi(H-1/2))}(Ts - s^2)^{1/2-H}\,. \tag{5.11}$$

In terms of $\hat{B}_H(t)$ the equation (5.7) can be written

$$dZ^\theta(t) = \rho Z^\theta(t)dt + \sigma v(t)X(t)d\hat{B}_H(t)\,. \tag{5.12}$$

Multiplying by $e^{-\rho t}$ and integrating we get

$$e^{-\rho t}Z^\theta(t) = e^{-\rho t}Z^{\theta(t),z} = z + \int_0^t e^{-\rho s}\sigma v(s)X(s)d\hat{B}_H(s)\,, \quad 0 \le t \le T\,, \tag{5.13}$$

where $z = Z^\theta(0)$ (constant) is the initial fortune.

An admissible portfolio θ is called an *arbitrage* for this market $(A(t), X(t))$; $t \in [0, T]$ if

$$Z^\theta(0) \le 0\,, \quad Z^\theta(T) \ge 0 \quad \text{a.s. and} \quad \mu_\phi\left(\{\omega; Z^\theta(T, \omega) > 0\}\right) > 0\,. \tag{5.14}$$

From (5.13) and (5.9) we deduce that *no arbitrage can exist*, because by taking the expectation with respect to $\hat{\mu}_\phi$ we get

$$E_{\hat{\mu}_\phi}\left[e^{-\rho T}Z^\theta(T)\right] = Z^\theta(0)\,. \tag{5.15}$$

The market $(A(t), X(t))$; $t \in [0,T]$ is called *complete* if for every $\mathcal{F}_T^{(H)}$-measurable bounded random variable $F(\omega)$ there exist $z \in \mathbb{R}$ and portfolio $\theta = (u,v)$ such that

$$F(\omega) = Z^{\theta,z}(T,\omega) \quad a.s.\ \ \mu_\phi\,. \tag{5.16}$$

By (5.13) this is the same as requiring that

$$e^{-\rho T}F(\omega) = z + \int_0^T e^{-\rho t}\sigma v(t)X(t)d\hat{B}_H(t)\,. \tag{5.17}$$

If we apply the fractional Clark-Ocone theorem (Theorem 4.13 b)) to $G(\omega) = e^{-\rho T}F(\omega)$ and with $B_H(t)$ replaced by $\hat{B}_H(t)$ we get

$$e^{-\rho T}F(\omega) = E_{\hat{\mu}_\phi}\left[e^{-\rho T}F\right] + \int_0^T \tilde{E}_{\hat{\mu}_\phi}\left[e^{-\rho T}\hat{D}_tF|\mathcal{F}_t^{(H)}\right]d\hat{B}_H(t)\,, \tag{5.18}$$

where $\hat{D}_t$ denotes the stochastic gradient with respect to $\hat{\mu}_\phi$. Note that the σ-algebra $\hat{\mathcal{F}}_t^{(H)}$ generated by $\hat{B}_H(s)$, $s \le t$ is the same as $\mathcal{F}_t^{(H)}$.

Comparing (5.17) and (5.18) we conclude that our market is indeed complete. Moreover, there is a unique initial value

$$z = Z^\theta(0) = E_{\hat{\mu}_\phi}\left[e^{-\rho T}F\right]$$

and a unique portfolio $\theta(t) = (u(t), v(t))$ needed to replicate (hedge) the claim $F(\omega)$. This initial value is called the *price* of the (European) claim F. By an approximation argument we see that the same conclusion holds for any lower bounded $\mathcal{F}_T^{(H)}$-measurable $F(\omega)$ such that $E_{\hat{\mu}_\phi}[F^2] < \infty$.

We summarize what we have proved in the following:

Theorem 5.1 *The fractional Black and Scholes market (5.1)-(5.2) has no arbitrage. It is complete and the price z of a lower bounded $\mathcal{F}_T^{(H)}$-measurable claim $F(\omega) \in L^2(\hat{\mu}_\phi)$ is given by*

$$z = e^{-\rho T}E_{\hat{\mu}_\phi}[F]\,, \tag{5.19}$$

where $\hat{\mu}_\phi$ is defined in (5.9).

Moreover, the corresponding replicating / hedging portfolio $\theta(t) = (u(t), v(t))$ for the claim F is

$$v(t) = e^{-\rho(T-t)}\sigma^{-1}X^{-1}(t)\tilde{E}_{\hat{\mu}_\phi}\left[\hat{D}_tF|\mathcal{F}_t^{(H)}\right] \tag{5.20}$$

and $u(t)$ is determined by (5.6), (5.20) and (5.13).

A claim of special interest is the *European call*, where

$$F(\omega) = (X(T,\omega) - c)^+ \,, \tag{5.21}$$

$c > 0$ being a constant (the exercise price). In this case we get:

Corollary 5.2 *(Fractional Black and Scholes formula)*
The price of the fractional European call (5.21) is

$$z = e^{-\rho T} \int_{\mathbb{R}} \frac{1}{T^H \sqrt{2\pi}} \left(x \exp\left[\sigma y + \rho T - \frac{1}{2}\sigma^2 T^{2H}\right] - c \right)^+ \exp\left[-\frac{y^2}{2T^{2H}}\right] dy \,. \tag{5.22}$$

The corresponding replicating portfolio $\theta(t) = (u(t), v(t))$ is given by (5.6), (5.13), and

$$v(t) = \sigma^{-1} e^{-\rho(T-t)} X^{-1}(t) \kappa(X(t)) \,, \tag{5.23}$$

where

$$\kappa(y) \;=\; \frac{1}{\sqrt{2\pi(T^{2H} - t^{2H})}} \int_{\mathbb{R}} e^{-\frac{(\tilde{y}-z)^2}{2(T^{2H}-t^{2H})}} h(z) dz \,, \quad \text{at} \tag{5.24}$$

$$\tilde{y} = \frac{\log y - \log x - \rho t + \frac{1}{2}\sigma^2 t^{2H}}{\sigma} \,, \tag{5.25}$$

where

$$h(z) \;=\; \sigma \chi_{[c,\infty)} \left(x e^{\sigma z + \rho T - \frac{1}{2}\sigma^2 T^{2H}} \right) \,. \tag{5.26}$$

Here log *denotes the natural logarithm.*

Proof By (5.19), (5.3) and (5.8) we have

$$\begin{aligned}
z &= e^{-\rho T} \mathrm{E}_{\hat{\mu}_\phi} \left[(X(T,\omega) - c)^+ \right] \\
&= e^{-\rho T} \mathrm{E}_{\hat{\mu}_\phi} \left[\left(x \exp\left(\sigma B_H(T) + \mu T - \frac{1}{2}\sigma^2 T^{2H} \right) - c \right)^+ \right] \\
&= e^{-\rho T} \mathrm{E}_{\hat{\mu}_\phi} \left[\left(x \exp\left(\sigma \hat{B}_H(T) + \rho T - \frac{1}{2}\sigma^2 T^{2H} \right) - c \right)^+ \right] \qquad (5.27) \\
&= e^{-\rho T} \mathrm{E}_{\mu_\phi} \left[\left(x \exp\left(\sigma B_H(T) + \rho T - \frac{1}{2}\sigma^2 T^{2H} \right) - c \right)^+ \right] \\
&= e^{-\rho T} \int_{\mathbb{R}} \frac{1}{T^H \sqrt{2\pi}} \left(x \exp\left(\sigma y + \rho T - \frac{1}{2}\sigma^2 T^{2H} \right) - c \right)^+ \exp\left[-\frac{y^2}{2T^{2H}} \right] dy \qquad (5.28)
\end{aligned}$$

which is (5.22).

Now we show (5.23)-(5.26). By (5.17), the definition of the fractional Clark derivative, and (5.21), we obtain

$$v(t) = \sigma^{-1} e^{-\rho(T-t)} X^{-1}(t) \hat{\nabla}_t^\phi F(\omega)\,, \tag{5.29}$$

where $F(\omega) = (X(T) - c)^+$ and $\hat{\nabla}_t^\phi F$ denotes the fractional Clark derivative of F when F is considered as a functional of $\hat{B}_H = B_H + \int_0^\cdot \frac{\mu-\rho}{\sigma} ds$.

However,

$$\begin{aligned} F(\omega) &= (X(T) - c)^+ \\ &= \left(xe^{\sigma \hat{B}_H(T) + \rho T - \frac{1}{2}\sigma^2 T^{2H}} - c\right)^+ \\ &= g(\hat{B}_H(T))\,, \end{aligned}$$

where

$$g(z) = \left(xe^{\sigma z + \rho T - \frac{1}{2}\sigma^2 T^{2H}} - c\right)^+ .$$

Hence by Proposition 4.17, we obtain

$$\hat{\nabla}_t^\phi F = \int_{\mathbb{R}} q_{t,T}(\hat{B}_H(t) - y) g(y) dy\,,$$

where $q_{t,T}(x)$ is given by (4.16). Applying the integration by parts, we have

$$\begin{aligned} \hat{\nabla}_t^\phi F &= -\int_{\mathbb{R}} \frac{\partial P_{t,T}}{\partial y}(\hat{B}_H(t) - y) g(y) dy \\ &= \int_{\mathbb{R}} P_{t,T}(\hat{B}_H(t) - y) g'(y) dy\,. \end{aligned}$$

Noticing that

$$\hat{B}_H(t) = \frac{\log X(t) - \log x - \rho t + \frac{1}{2}\sigma^2 t^{2H}}{\sigma}$$

and $g'(y) = h(y)$, we obtain the corollary. □

Remark 5.3 *One notices that z is independent of μ. One may compare this corollary with the classical results for the classical Brownian motion case, e.g. ([17], p. 274-275, Theorem 12.3.6).*

6 Appendix

It is interesting to obtain the explicit form of the Radon-Nikodým derivative in (5.9), *i.e.* the explicit solution $K(T, s)$ of (5.10). Equation (5.10) was studied in details in for example

[12]. The Lemma 3 of [11] can be applied directly to our equation with $c(\cdot) = \frac{\mu-\rho}{\sigma}$. Thus the solution of (5.10) is given by

$$\begin{aligned} K(T,s) &= -\frac{1}{H(2H-1)d_H}\frac{\mu-\rho}{\sigma}s^{\frac{1}{2}-H} \\ &\quad \frac{d}{ds}\left\{\int_s^T\left[\left(\frac{d}{dw}\int_0^w z^{\frac{1}{2}-H}(w-z)^{\frac{1}{2}-H}dz\right)w^{2H-1}(w-s)^{\frac{1}{2}-H}\right]dw\right\}, \end{aligned} \tag{6.1}$$

where

$$d_H = 2\Gamma^2\left(\frac{3}{2}-H\right)\Gamma(2H-1)\cos(\pi(H-\frac{1}{2}))\,. \tag{6.2}$$

It is well-known that

$$\int_0^w z^{\frac{1}{2}-H}(w-z)^{\frac{1}{2}-H}dz = \frac{\Gamma^2(\frac{3}{2}-H)}{\Gamma(3-2H)}w^{2-2H}\,.$$

So

$$\begin{aligned} \frac{d}{dw}\int_0^w z^{\frac{1}{2}-H}(w-z)^{\frac{1}{2}-H}dz &= \frac{(2-2H)\Gamma^2(\frac{3}{2}-H)}{\Gamma(3-2H)}w^{1-2H} \\ &= \frac{\Gamma^2(\frac{3}{2}-H)}{\Gamma(2-2H)}w^{1-2H}\,. \end{aligned} \tag{6.3}$$

But

$$\int_s^T (w-s)^{\frac{1}{2}-H}dw = \left(\frac{3}{2}-H\right)^{-1}(T-s)^{\frac{3}{2}-H}\,.$$

Hence

$$\frac{d}{ds}\int_s^T w^{2H-1}(w-s)^{\frac{1}{2}-H}w^{1-2H}dw = -(T-s)^{\frac{1}{2}-H}\,. \tag{6.4}$$

By (6.1)-(6.4), we have

$$K(T,s) = \kappa_H\frac{\mu-\rho}{\sigma}s^{\frac{1}{2}-H}(T-s)^{\frac{1}{2}-H}\,, \tag{6.5}$$

where

$$\kappa_H = \frac{1}{2H(2H-1)\Gamma(2-2H)\Gamma(2H-1)\cos(\pi(H-\frac{1}{2}))}\,. \tag{6.6}$$

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