FRAGMENTABILITY AND REPRESENTATIONS OF FLOWS

MICHAEL MEGRELISHVILI

Abstract. Our aim is to study weak* continuous representations of semigroup actions into the duals of "good" (e.g., reflexive and Asplund) Banach spaces. This approach leads to flow analogs of Eberlein and Radon-Nikodym compacta and a new class of functions (*Asplund functions*) which intimately is connected with Asplund representations and includes the class of weakly almost periodic functions. We show that a flow is weakly almost periodic iff it admits sufficiently many reflexive representations. One of the main technical tools in this paper is the concept of *fragmentability* (which actually comes from Namioka and Phelps) and widespreadly used in topological aspects of Banach space theory. We explore fragmentability as "a generalized equicontinuity" of flows. This unified approach allows us to obtain several dynamical applications. We generalize and strengthen some results of Akin-Auslander-Berg, Shtern, Veech-Troallic-Auslander and Hansel-Troallic. We establish that frequently, for linear G-actions, weak and strong topologies coincide on, not necessarily closed, G-minimal subsets. For instance such actions are "orbitwise Kadec".

1. Introduction

Every compact jointly continuous G-flow X admits a faithful weak* continuous Banach representation. More precisely, X is G-embedded into the dual ball $B(V^*)$ as a weak* compact G-subset of some Banach space V, where the group G acts continuously on V by linear isometries. Indeed, this is a standard fact (see Teleman's paper [53], or for a more detailed discussion, the survey [47]) for $V = C(X)$, where one can identify $x \in X$ with the point mass $\delta_x \in C(X)^*$. The geometry of $C(X)$, in general, is bad. For example, a very typical disadvantage here is the norm discontinuity of the dual action of G on $C(X)^*$. One of the results of [36] guarantees (see also Corollary 8.7 below) the norm continuity of the dual action of the group G on V^* provided that V is Asplund. Recall that a Banach space V is *Asplund* iff the dual A^* is separable for every separable Banach subspace A of V.

The following general question arises: how good can a Banach space V be among all possible w*-continuous faithful G-linearizations of X into V^* ? For instance when can V be chosen Asplund or reflexive ? We show that the reflexive case (for second countable X) can be reduced completely to the question if X is a weakly almost periodic (in short: *wap*) flow.

Eberlein compact in the sense of Amir and Lindenstrauss [3] is a compact space which can be embedded into $(V, weak)$ for some Banach space V. It is well known [12] that a compact space X is Eberlein iff it can be embedded into the unit ball $(B(V), weak)$ of some reflexive space V. If X is a weak* compact subset in the dual V^* of an Asplund space V then, following Namioka [44], X is called *Radon-Nikodym compact* (in short: RN). Every reflexive Banach space is Asplund. Hence, every Eberlein compact is RN.

Now introduce map versions of these concepts. Let $f : X \to X$ be a selfmap on a compact space X. Let us say that f is an *Eberlein (Radon-Nikodym) map* if it admits a weak* linearization into certain reflexive (resp.: Asplund) Banach space. That is, there

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exists a reflexive (Asplund) Banach space V and a weak* embedding $X \hookrightarrow B(V^*)$ in such a way that $f : X \to X$ is a restriction of the adjoint $F^* : V^* \to V^*$ of some linear operator $F : V \to V$ which is *non-expansive* ($\|F\| \leq 1$). In this point of view, the space X is Eberlein or RN iff the identity mapping $1_X : X \to X$ is Eberlein or RN, respectively.

Clearly, every metric compact space is Eberlein since it is a compact subset of the Hilbert space l_2. In contrast, even simple maps on metric compacta can be non-Eberlein. For example, the $f(x) = x^2$ map on the closed interval $[0, 1]$ is not Eberlein. The map

$$f : \mathbb{T}^2 \to \mathbb{T}^2, \ \ f([a], [b]) = ([a + b], [b + \sqrt{2}])$$

defined on the torus $X = \mathbb{T}^2$ is not even RN (see Example 7.19).

It is significant that a compact metric *cascade* $(\mathbb{Z}, X)$ is wap (equivalently, is an Eberlein flow, by virtue of Corollary 4.10) iff the generating selfhomeomorphism $f : X \to X$ leads to a *wap Markov operator* $T_f : C(X) \to C(X)$ (see Downarowicz [14] and the references there). The study of wap operators and corresponding cascades goes back to the 60's (K. Jacobs, B. Jamison, M. Rosenblatt, R. Sine, J. Montgomery, E. Thomas and others).

The setting of maps and their linearizations admits a natural generalization in terms of flow linearizations. We introduce below Eberlein and Radon-Nikodym flows and show that a flow is weakly almost periodic in the sense of Ellis-Nerurkar [16] iff it is a subdirect product of Eberlein flows. Investigation of RN flows naturally leads also to a new class of functions which we call *Asplund functions.* Our approach emphasizes more the similarities (rather than the differences) between wap and Asplund functions. We show that a function is wap (Asplund) iff it comes from a matrix coefficient defined by a representation into reflexive (resp.: Asplund) spaces. In both cases our method is based on corresponding dualities and a factorization procedure by Davis, Figiel, Johnson and Pelczynski [12]. In the "Asplund case" the technical part uses a modification (using "Asplund subsets" instead of "weakly compact") which is due to Stegall [52].

Let us briefly describe one of the ideas explored in the present paper. Suppose that X is a subflow (under some action by linear isometries) of a weak* compact dual ball $B(V^*)$ of some Banach space V. One of the important questions in Banach space theory is a relationship between norm and weak* topologies on $X \subset V^*$. In the "absolute case" of the coincidence, we say that X is a *Kadec subset* of V^*. In such cases, X, as a flow, is equicontinuous. Conversely, every compact metric equicontinuous flow X admits a flow representation in such a way that X becomes a norm compact subflow (and hence a Kadec subset) of a suitable $B(V^*)$. In general, as an attempt to measure "the level of equicontinuity", we can ask how close can two natural topologies on X inherited from V^* be. A more concrete and flexible enough question is: for what flow representations is the natural mapping $1_X : (X, weak^*) \to (X, norm)$ *fragmented* in the sense of [45, 30]. The latter means that every nonempty subset of X admits relatively weak* open nonempty subsets with arbitrarily small diameters.

The great advantage of Asplund spaces is the (weak*, norm)-fragmentability of bounded subsets in their duals [45, 44]. Many modern investigations in Banach spaces concern Asplund spaces, the notion of fragmentability and closely related Radon-Nikodym property (see [9, 44, 13, 17, 6] and the references therein). In [36, 38] we study some dynamical applications of fragmentability. In the present paper we examine further developments exploring some ideas more familiar in the topological aspects of Banach space theory.

For the convenience of the reader we have tried to make the exposition self-contained.

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2. Preliminaries

The closure and the interior operators in topological spaces will be denoted by cl and int, respectively. If A is a subset in a Banach space then $sp(A)$ is the linear span of A. Let μ be a uniform structure on a set X. Its induced topology on X will be denoted by $top(\mu)$. A uniformity μ on a topological space (X, τ) is said to be *compatible* if $top(\mu) = \tau$. A (left) *flow* (S, X) consists of a topologized semigroup S and a (left) action $\pi : S \times X \to X$ on a topological space X. We reserve the symbol G for the case when S is a group. As usual we write simply sx instead of $\pi(s, x) = \tilde{s}(x) = \tilde{x}(s)$. "Action" means that always $s_1(s_2 x) = (s_1 s_2)x$. If S is a monoid, we assume that the identity e of S acts as the identity transformation of X. Every $x \in X$ defines an orbit map $\tilde{x} : S \to X, s \mapsto sx$. Say that a topologized semigroup S is: (a) left (right) topological; (b) semitopological; (c) topological if the multiplication function $S \times S \to S$ is left (right) continuous, separately continuous, or jointly continuous, respectively. Let S be a semitopological semigroup. A left flow (S, X) is said to be a *semitopological flow* if the action is separately continuous.

A *right flow* (X, S) can be defined analogously. If S^{opp} is the *opposite semigroup* of S with the same topology then (X, S) can be treated as a left flow (S^{opp}, X) (and vice versa).

If not stated otherwise the flows below are assumed to be semitopological. "Compact" will mean compact and Hausdorff.

Let $h : S_1 \to S_2$ be a semigroup homomorphism, S_1 act on X_1 and S_2 on X_2. A map $f : X_1 \to X_2$ is said to be *h-equivariant* if $f(sx) = h(s)f(x)$ for every $(s, x) \in S_1 \times X_1$. For $S_1 = S_2$ with $h = 1_S$, we say *S-map*. The map $h : S_1 \to S_2$ is an *antihomomorphism* iff $h : S_1 \to S_2^{opp}$ (the same assignement) is a homomorphism.

An *S-compactification* of (S, X) is a continuous S-map $\alpha : X \to Y$ with a dense range (*S-compactification map*) into a compact S-flow Y. A (jointly continuous) flow (S, X) is said to be (resp.:*joint continuously*) *compactifiable* if there exists an S-compactification $\gamma : X \to Y$ into a (jointly continuous) S-flow Y such that γ is a topological embedding. Following Junghenn [31] we define a *bicompactification* $m = (h, \alpha) : (S_1, X_1) \rightrightarrows (S_2, X_2)$ as a pair $h : S_1 \to S_2, \alpha : X_1 \to X_2$, where (S_2, X_2) is a semitopological flow with compact S_2 and X_2, the map h is a continuous homomorphism and α is a continuous h-equivariant map with a dense range.

Let V be a Banach space with the dual V^*. Set

$$B(V) = \{v \in V : ||v|| \leq 1\} \text{ and } \Theta(V) = \{\sigma \in L(V, V) : ||\sigma|| \leq 1\}.$$

In most cases we endow the sets $B(V)$, $B^* = B(V^*)$ and $\Theta(V)$ with weak, weak* and weak operator topologies, respectively. Sometimes we use the subscripts "w" and "w*". The subscript "s" will mean the strong operator topology. The pairs $(\Theta(V)_w, B(V)_w)$ and $(B(V^*)_{w^*}, \Theta(V)_w)$ are semitopological flows. The $\Theta(V)_s^{opp}$-flow $B(V^*)_{w^*}$ is jointly continuous as it follows directly from Fact 2.2. It induces a right action of the isometry group $Is(V) = \{g \in Aut(V) : ||g|| = 1\}$. Alternatively, we have a left action defined by

$$Is(V) \times B^* \to B^*, \quad (gf)(v) = f(g^{-1}v)$$

Hence, $(Is(V)_s, B^*_{w^*})$ is a well defined jointly continuous action (see also Remark 3.4).

The Banach algebra of all continuous real valued bounded functions on a topological space X will be denoted by $C(X)$. The same set with the pointwise topology (*p-topology*) is denoted by $C_p(X)$. Let X be a (left) S-flow then it induces the antihomomorphism $h : S \to C(X)$ and the corresponding (right) action $C(X) \times S \to C(X)$ where $(fs)(x) = f(sx)$. In the case of a topological group $S = G$, we can define a homomorphism and a *left action* by $(gf)(x) = f(g^{-1}x)$. While the translations are continuous, the orbit maps $\tilde{f} : S \to C(X)$ are not necessarily (even weakly) continuous. Denote by $RUC_S(X)$ the set of all functions $f \in C(X)$ such that the orbit map $\tilde{f}$ is norm continuous. If we require only weak continuity, then we get the definition of *weakly right uniformly continuous functions* (see [8]). Denote the corresponding set by $WRUC_S(X)$.

The proof of the following fact is straightforward.

Fact 2.1. *If X is compact, then $\pi : S \times X \to X$ is jointly continuous iff $C(X) = RUC_S(X)$.*

For general (separately continuous) action π, the set $RUC_S(X)$ is an S-invariant Banach subalgebra and the corresponding Gelfand compactification $u_R : X \to X^R$ is a universal (maximal) jointly continuous S-compactification of X. If $X = S$ with the left regular action of S, then we simply write $RUC(S)$. If $S = G$ is a topological group, then $RUC(G)$ is the set of all usual right uniformly continuous functions. The algebra of all left uniformly continuous functions (defined for the right regular action of S on S)) will be denoted by $LUC(S)$.

The classical Gelfand-Naimark 1-1 correspondence between Banach subalgebras of $C(X)$ and the compactifications of X can be extended to the category of jointly continuous S-flows using Banach S-subalgebras of $RUC_S(X)$ (like the well-known results for topological group actions (see J. de Vries [57])). One of the ways to verify this is to use the following fact which is a key idea of Teleman's above-mentioned result, as well as in the paper of Uspenskij [55].

Fact 2.2. *Let V be a Banach space. Suppose that a topologized semigroup S acts on V from the right by linear non-expansive operators. The following are equivalent:*
(i) $V \times S \to V$ is norm jointly continuous.
(ii) The induced affine action $S \times B^*_{w^*} \to B^*_{w^*}$ is jointly continuous.

Proof. The dual action defines an injective antihomomorphism of $\Theta(V)$ into $C(B^*, B^*)$. Now observe that the strong operator topology on $\Theta(V)$ coincides with the compact open topology inherited from $C(B^*, B^*)$. □

Recall the definition of weakly almost periodic functions and some relevant facts.

Definition 2.3. Let S be a semitopological semigroup and X be an S-flow.

(i) A function $f \in C(X)$ is said to be *weakly almost periodic*, (*wap*, in short) if the orbit $fS = \{fs : s \in S\}$ is relatively weakly compact in $C(X)$. Write $f \in WAP_S(X)$.
(ii) We say that X is S-wap, or, (S, X) is wap (otation: $X \in [wap]^S$) if $WAP_S(X)$ separates points and closed subsets of X.
(iii) We say that S is *wap* (and write: $S \in [wap]$) if the regular left action (S, S) is wap.

This general form of definition (i) can be found in the work of Junghenn [31]. For the left action (S, S) we get the classical notion of wap functions on S (see Eberlein [15] and de-Leeuw Glicksberg [34]). We use the notation $WAP(S)$ instead of $WAP_S(S)$.

Replacing 'weakly compact" in Definition 2.3 by "norm compact" we get the definitions of *almost periodic* functions and corresponding S-algebras $AP_S(X), AP(S)$.

Grothendieck's criteria [22] for relative weak compactness leads to the following assertion.

Fact 2.4. *(Grothendieck's DLP) A function $f \in C(X)$ defined on some S-flow is wap iff the following Double Limit Property is satisfied:*
(DLP) For every pair of sequences $s_m \in S$ and $x_n \in X$

$$\lim_m \lim_n f(s_n x_m) = \lim_n \lim_m f(s_n x_m)$$

holds whenever both of the limits exist.

Recall also the following very useful fact.

Fact 2.5. *(Grothendieck's Lemma) Let X be a compact space. Then a bounded subset A of $C(X)$ is w-compact iff A is p-compact.*

The set $WAP_S(X)$ is a Banach S-subalgebra in $C(X)$. This is mentioned in [31]. The proof can be done using Fact 2.5 and the Eberlein-Smulian theorem.

This implies that our general Definition 2.3(ii), for compact X, is equivalent to the definition of wap flows in the sense of Ellis-Nerurkar [16]. Gelfand's compactification $u_W : X \to X^W$ induced by the algebra $WAP_S(X)$ is the *universal wap compactification* of X (see for details [31, Theorem 3.1]). In particular, for the left regular action (S, S) we get the *universal wap semigroup compactification* $u_W : S \to S^W$. It is important to note that in this case S^W is a compact semitopological semigroup and enjoys the corresponding universality property. By our definitions, the flow X (or, the semigroup S) is wap iff u_W is a topological embedding.

Ellis semigroup $E(S,X)$ (or, simply: $E(X)$) for compact X is the pointwise closure of the set of all s-translations $\{\tilde{s} : X \to X : s \in S\}$ in the compact semigroup X^X. Denote by $\lambda : S \to E(X), \lambda(s) = \tilde{s}$ the corresponding natural homomorphism. In general, $E(X)$ is only *right topological*, that is, only the right translations $E(X) \to E(X)$, $s \mapsto sp$ are necessarily continuous.

Fact 2.6. [16, 31] *For a compact S-flow X the following are equivalent:*

1. *X is wap (that is, $C(X) = WAP_S(X)$).*
2. *Each element of $E(S,X)$ is continuous (quasiequicontinuous in terms of* [8]*).*
3. *The pair $(E(X), X)$ is a semitopological flow with the compact semitopological semigroup $E(X)$.*
4. *There exists a bicompactication $(h, \alpha) : (S,X) \rightrightarrows (P,Y)$ such that $\alpha : X \to Y$ is an embedding and $h : S \to P$ is a semigroup compactification into a compact semitopological semigroup P.*

Proof. The principal implication (1) $\Rightarrow$ (2) directly follows from Proposition 4.3 and Remarks 4.4(b).

For (4) $\Rightarrow$ (1) it suffices to show that $C(Y) = WAP_P(Y)$. Let $f \in C(Y)$ then the P-orbit fP is bounded. Since P is compact then fP is pointwise compact in $C(Y)$. By Fact 2.5, fP is even w-compact. Thus, $f \in WAP_P(Y)$.

Other implications are trivial. □

For compact X, Definition 2.3(ii) agrees with the item (1) in Fact 2.6, as it easily follows by Stone-Weierstrass theorem.

If $X \in [wap]^S$ then $Y \in [wap]^P$ for every subsemigroup P of S and every P-subflow Y of X. Moreover, $[wap]^S$ is closed under *subdirect products* (subspaces of products). The class of compact S-wap flows is closed also under quotients.

Fact 2.7. (i) *$WAP_S(X) \subset WRUC_S(X)$ for every semitopological S-flow X. Hence, $WAP(S) \subset WRUC(S)$ for every semitopological semigroup S.*

(ii) *If G is a semitopological group then $WAP_G(X) \subset RUC_G(X)$. In particular, $WAP(G) \subset RUC(G)$ holds.*

Proof. (i) The orbit map $\tilde{f} : S \to C(X)$ is clearly p-continuous. If $f \in WAP_S(X)$ then $cl_w(fS)$ is weakly compact. Hence, $(fS, w) = (fS, p)$. Therefore $\tilde{f}$ is also weakly continuous.

(ii) Follows from Theorem 2.5 and Remark 2.6 (a) of [38]. It can be seen easily also as a corollary of Theorem 8.5 below. □

The inclusion $WAP(G) \subset RUC(G)$ is well known (see for example [24] or [8, Theorem 4.10]). Another proof of the inclusion $WAP_G(X) \subset RUC_G(X)$ can be derived also by results of [28].

For every reflexive Banach space V the semigroup $\Theta(V)$ is a weakly compact semitopological semigroup [34]. Observe that for every vector $v \in V$ with norm 1, the orbit $\Theta(V)v$ of v coincides with $B(V)$. This guarantees the converse: if $\Theta(V)_w$ is compact then B_w is compact , and, hence, V is necessarily reflexive. For every reflexive V, the flows $(\Theta(V), B(V))$ and $(\Theta(V)^{opp}, B(V^*))$ are semitopological and (bi)compact. Hence, wap by Fact 2.6.

One of our applications below (see section 8) provides a simple proof of the following important theorem of Lawson [32] which in itself is a generalization of Ellis theorem.

Fact 2.8. *(Ellis-Lawson's Joint Continuity Theorem).*
Let G be a subgroup of a compact semitopological monoid S. Suppose that (S, X) is a semitopological flow with compact X. Then the action $G \times X \to X$ is jointly continuous and G is a topological group.

A (not necessarily compact) G-flow X is said to be *minimal* if every orbit Gx is dense in X. Equicontinuous compact flows are the simplest one in Topological Dynamics. Every equicontinuous compact flow is wap. The converse is true for every minimal compact wap G-flow X [54, 4]. Below we show (Theorem 6.10) that the compactness assumption is superfluous here. That is, every minimal wap (and even, RN-approximable), not necessarily compact, G-flow is equicontinuous.

3. Banach representations and matrix coefficients

Let V be a Banach space with the canonical duality $<,>: V \times V^* \to \mathbb{R}$. If a semigroup S acts from the right on V (equivalently: if we have an antihomomorphism $S \to L(V, V)$) then it induces a left action of S on the dual V^* such that $< vs, \psi >=< v, s\psi >$ for every $v \in V$ and $\psi \in V^*$.

Definition 3.1. A (non-expansive) *V-representation* of a flow (S, X) is an equivariant pair

$$(h, \alpha) : (S, X) \rightrightarrows (\Theta(V)^{opp}, B^*)$$

where $h : S \to \Theta(V)^{opp}$ is a weak continuos homomorphism (equivalently: antihomomorphism $S \to \Theta(V)$) and $\alpha : X \to B^*$ is weak* continuous and equivariant, that is $\alpha(sx) = h(s)\alpha(x)$.

We say that a representation is *strongly continuous* if $h : S \to \Theta(V)_s$ is continuous. *Topologically faithful* (or, simply: *faithful*) will mean that $\alpha : X \to (B^*, w^*)$ is a topological embedding.

Let $\mathcal{K} \subset BAN$ be a subclass of Banach spaces. We say that a flow (S, X) is:

(a) *$\mathcal{K}$-representable* if there exists a *faithful* V-representation of (S, X) for some $V \in \mathcal{K}$.
(b) *$\mathcal{K}$-approximable* if there exists a system (h_i, α_i) of representations of (S, X) in V_i separating points and closed subsets in X with $V_i \in \mathcal{K}$ (equivalently, if X is a subdirect product of $\mathcal{K}$-representable S-flows).
(c) *Eberlein* if it is REFL-representable.
(d) *Radon-Nikodym* (in short: RN) if it is ASP-representable.
(e) RN-*approximable* if it is ASP-approximable.

In this definition REFL and ASP mean the classes of all reflexive and Asplund spaces respectively. Since REFL $\subset$ ASP, every Eberlein flow is RN. If S is a trivial monoid and X is compact, then the definitions (c) and (d) give exactly the classical notions of Eberlein and RN compacta mentioned in the introduction.

Remark 3.2. Sometimes weak continuous (anti)homomorphisms automatically are strongly continuous. This happens for instance if either: (a) S is an arbitrary semitopological group and V is reflexive; (b) S is a locally compact Hausdorff topological group; or (c) S is a topological group metrizable by a complete metric. The first assertion follows from [38, Theorem 2.8] (or, from Corollary 8.2 below). For the last two assertions see [8]. For some other results of the nature "weak implies strong" see also [24, 25, 33, 36, 38].

The following standard fact (see for example [53]) states actually that every jointly continuous action on compact spaces admits a faithful Banach representation.

Lemma 3.3. *Let (S, X) be a jointly continuous semigroup action on a compact X. Then there exists a Banach space V and a faithful strongly continuous representation (h, α) of (S, X) into the jointly continuous affine action $(\Theta(V)^{opp}_s, B(V^*)_{w^*})$.*

Proof. Take $V = C(X)$ and define the antihomomorphism $h : S \to \Theta(V)$ induced by the natural right action $C(X) \times S \to C(X)$. This action is norm continuous by Fact 2.1 because $RUC_S(X) = C(X)$. Thus, h is strongly continuous by Fact 2.2. Finally define the natural weak* embedding $\alpha : X \to (B(C(X)^*)$ identifying each $x \in X$ with the point mass $\delta_x \in B(C(X)^*)$. □

For every weakly continuous antihomomorphism $h : S \to L(V, V)$ and every chosen pair of vectors $v \in V$ and $\psi \in V^*$, there exists a canonically associated *(generalized) matrix coefficient* $m_{v,\psi} : S \to \mathbb{R}, \quad s \mapsto < vs, \psi > = < v, s\psi >$

$$\begin{array}{ccc} S & \xrightarrow{\ m_{v,\psi}\ } & \mathbb{R} \\ {\scriptstyle h}\big\downarrow & & \big\uparrow{\scriptstyle \psi} \\ L(V,V) & \xrightarrow{\ \tilde{v}\ } & V \end{array}$$

Remark 3.4. In many important cases we can use homomorphisms instead of antihomomorphisms. Indeed, if S is a topological group (or, a semigroup with a continuous involution), then we can define a *homomorphism* $h^* : S \to L(V, V), s \mapsto h(s^{-1})$ and redefine the function $m_{v,\psi}$ by $s \mapsto < s^{-1}v, \psi >$.

It is natural to expect that matrix coefficients reflect good properties of flow representations (see, for example, [47]). We recall two well-known facts. The first example is the case of Hilbert representations. If $h : G \to Is(H)$ is a group representation into Hilbert space H and $\psi = v$, then the corresponding map $g \mapsto < g^{-1}v, v >$ is a *positive definite function* on G. The converse is also true: every continuous positive definite function comes from some continuous Hilbert representation. Every positive definite function is wap (see [11]).

The second example comes from Eberlein [15] (see also [8, Examples 1.2.f]). If V is reflexive, then every bounded V-representation (h, α) and arbitrary pair (v, ψ) lead to a weakly almost periodic function $m_{v,\psi}$ on S. This follows easily by the (weak) continuity of the natural operators defined by the following rule. For every fixed $\psi \in V^*$ $(v \in V)$ define *introversion type operators* by

$$L_\psi : V \to C(S) \text{ and } R_v : V^* \to C(S), \text{where } L_\psi(v) = R_v(\psi) = m_{v,\psi}.$$

We say that a vector $v \in V$ is strongly (weakly) continuous if the corresponding orbit map $\tilde{v} : S \to V, \tilde{v}(s) = vs$, defined through $h : S \to \Theta(V)$, is strongly (weakly) continuous.

Fact 3.5. *Let $h : S \to L(V, V)$ be a weakly continuous antihomomorphism with the norm bounded range. Then*

(1) *$L_\psi : V \to C(S)$ (and $R_v : V^* \to C(S)$) are linear bounded S-operators between right (left) S-actions.*

(2) *If* ψ *(resp.:* $v \in V$*) is norm continuous, then* $m_{v,\psi}$ *is left (resp.: right) uniformly continuous on* S*.*
(3) *If* V *is reflexive, then* $m_{v,\psi} \in WAP(S)$*.*

Proof. (1) Is straightforward.

(2) Since $h(S)$ is norm bounded, $sup\{||vt|| : t \in S\} = c < \infty$. Let ψ be a norm continuous vector. In order to establish that $m_{v,\psi} \in LUC(S)$, observe that

$$|m_{v,\psi}(ts) - m_{v,\psi}(ts_0)| = |< vts, \psi > - < vts_0, \psi > | =$$

$$|< vt, s\psi > - < vt, s_0\psi > | \leq ||vt|| \cdot ||s\psi - s_0\psi|| \leq c \cdot ||s\psi - s_0\psi||.$$

Similar verification is valid for the second case.

(3) If the orbit vS is relatively weakly compact in V (e.g., if V is reflexive), then the same is true for $L_\psi(vS) = m_{v,\psi}S$ in $C(S)$. Thus $m_{v,\psi}$ is wap.

□

Fact 3.6. *Let* $(h, \alpha) : (S, X) \rightrightarrows (\Theta(V)^{opp}, B^*)$ *be an equivariant pair with weak* continuous* α *but without no continuity assumptions on* h*.*

(i) *The map* $T : V \to C(X), v \mapsto T(v)$*, where* $T(v) : X \to \mathbb{R}$ *is defined by*

$$T(v)(x) =< v, \alpha(x) >$$

is a linear S*-operator (between right* S*-actions) with* $||T|| \leq 1$*.*

(ii) $T(v_0) \in RUC_S(X)$ *for every strongly continuous vector* v_0 *in* V*. Hence, if* h *is strongly continuous then* $T(V) \subset RUC_S(X)$*.*

(iii) *If* V *is reflexive, then* $T(V) \subset WAP_S(X)$*.*

Proof. (i) is straightforward.

(ii) Observe that $||\alpha(x)|| \leq 1$ for every $x \in X$. We get

$$||T(v_0)s - T(v_0)s_0|| = sup\{| < v_0s - v_0s_0, \alpha(x) > | : x \in X\} \leq$$

$$\leq ||v_0s - v_0s_0|| \cdot ||\alpha(x)|| \leq ||v_0s - v_0s_0||.$$

This implies that $T(v_0) \in RUC_S(X)$.

(iii) If V is reflexive, the orbit vS is relatively weakly compact for each $v \in V$. By the (weak) continuity of the S-operator T, the same is true for the orbit of $T(v)$ in $C(X)$. Therefore we get $T(v) \in WAP_S(X)$. □

Proposition 3.7. *For every* S*-flow* X *the following are equivalent:*

(1) $f \in RUC_S(X)$*.*
(2) *There exist: a Banach space* V*, a strongly continuous antihomomorphism* $h : S \to \Theta(V)$*, a weak* continuous equivariant map* $\alpha : X \to B^*$*, and a vector* $v \in V$ *such that* $f(x) =< v, \alpha(x) >$ *(that is* $f = T(v)$*.*

Proof. (1) $\Longrightarrow$ (2) The function f belongs to an S-invariant Banach subalgebra $\mathcal{A}$ of $RUC_S(X)$. The right action of S on $V := \mathcal{A}$ is jointly continuous. Then by Fact 2.2, corresponding left action of S on the dual ball (B^*, w^*) is jointly continuous. Then the naturally associated map $\alpha : X \to B^*$ and the vector $v := f$ satisfy the desired property.

(1) $\Longleftarrow$ (2) Immediate by Fact 3.6 (ii). □

Proposition 3.8. *For every semitopological monoid* S *the following are equivalent:*

(1) $f \in RUC(S)$*.*
(2) *There exist: a Banach space* V*, a strongly continuous antihomomorphism* $h : S \to \Theta(V)$*, and a pair of vectors* $v \in V$ *and* $\psi \in V^*$ *such that* $f = m_{v,\psi}$*.*

Proof. (1) $\Longrightarrow$ (2) Consider the Gelfand compactification $u_R : S \to S^R$ defined by $RUC(S) = C(S^R)$. Then the action $S \times S^R \to S^R$ is jointly continuous by Fact 2.1. Now define: $V := C(S^R)$, corresponding strongly continuous $h : S \to \Theta(V)$ (induced by the right action of S on $C(S^R)$), $v := f \in V$ and $\psi = u_R(e) \in V^*$.

(1) $\Longleftarrow$ (2) Immediate by Fact 3.5.2. □

As we already have seen a right uniformly continuous function can be represented as a matrix coefficient $m_{v,\psi}$ of some strongly continuous *Banach representation*. We mentioned also the well known case of Hilbert representations. A positive definite function on a topological group G is exactly a matrix coefficients of some unitary representation. One of our aims is to understand the role of matrix coefficients for intermediate cases of reflexive and Asplund representations. We show that wap functions are exactly the *reflexive matrix coefficients*. In the "Asplund case" this approach leads to a definition of *Asplund functions* introduced in Section 7.

4. Reflexive representations of flows

Definition 4.1. A (bounded) *duality* is a separately continuous (resp., bounded) mapping $<,>: Y \times X \to \mathbb{R}$. We say that the duality is *right strict* if the corresponding continuous map $q_X : X \to C_p(Y), \quad q_X(x) =< y, x >$ is a topological embedding (e.g., an injection if X is compact).

The "left" version can be defined analogously. Then "strict" will mean left and right strict simultaneously.

Let a semigroup S act on X and Y by the following actions:

$$\pi_X : S \times X \to X, \quad \pi_Y : Y \times S \to Y.$$

The duality is an *S-duality* (or, *S-invariant*) if $< ys, x >=< y, sx >$.

Consider two typical examples:

1. "Canonical reflexive duality": $rB \times B^* \to \mathbb{R}$ with compact spaces rB and B^* (under weak topologies) is defined for every reflexive V, a positive number $r > 0$ and an antihomomorphism $h : S \to \Theta(V)$. In particular, we can choose the natural action of $S = \Theta(V)^{opp}$. Observe that $\Theta(V)^{opp} = \Theta(V^*)$ (compare Proposition 4.3).
2. Let $K \subset V$ be a weakly compact S-invariant subset in a Banach space V with respect to some antihomomorphism $h : S \to \Theta(V)$. Then $K \times B(V^*) \to \mathbb{R}$ is a left strict S-duality.

Lemma 4.2. *Let $<,>: Y \times X \to \mathbb{R}$ be an S-duality.*

1. *$<,>$ is left strict iff a net y_i converges to y in Y exactly when $< y_i, x >\to< y, x >$ in $\mathbb{R}$ for every $x \in X$. Similarly, $<,>$ is right strict iff a net x_i converges to x in X exactly when $< y, x_i >\to< y, x >$ in $\mathbb{R}$ for every $y \in Y$.*
2. *Let $<,>$ be a left (right) strict S-duality. Then all s-translations $\tilde{s} : Y \to Y$ (resp., $\tilde{s} : X \to X$) are continuous.*
3. *Let $<,>$ be a strict S-duality. Then π_Y is separately continuous iff π_X is separately continuous.*
4. *Let $<,>: Y \times X \to \mathbb{R}$ be a left strict S-duality. Then it can be reduced canonically to the naturally associated strict S-duality $<,>_q: Y \times X_q \to \mathbb{R}$. If π_Y is separately continuous then π_{X_q} is also separately continuous.*

Proof. (1) Follows from the net characterization of the product topology.

(2) We have to show that every s-translation $\pi_Y^s : Y \to Y$ is continuous (the case of π_X^s is similar). Let $y_i \to y$. In order to show that $sy_i \to sy$, it suffices by (1) to check that $< y_i s, x > \to < ys, x >$ for each $x \in X$. Or, equivalently, we have to show that $< y_i, sx > \to < y, sx >$. The latter follows from the assumption $y_i \to y$ and the separate continuity of $<,>$.

(3) Similar to the proof of (2).

(4) Consider the canonical continuous map $q_X : X \to C_p(Y)$ and the corresponding range $X_q = q_X(X) \subset C_p(Y)$. Define

$$<,>_q : Y \times X_q \to \mathbb{R}, \ < y, q_X(x) >_q := < y, x > .$$

It is easy to show that this is a well-defined strict duality. Moreover, the action of S on X induces the natural action of S on X_q such that $q : X \to X_q$ is S-equivariant and $<,>_q$ is S-invariant. Apply (3) to $<,>_q$. If π_Y is separately continuous then π_{X_q} is also separately continuous by virtue of (3).

□

Proposition 4.3. *Let X and Y be Hausdorff S-flows such that X is compact and every orbit closure $cl(yS)$ in Y is compact. Assume that $<,>: Y \times X \to \mathbb{R}$ is a strict S-duality. Then the Ellis semigroup $E(S,X) = E(X)$ is semitopological and there exist separately continuous actions $E(X) \times X \to X$ on $Y \times E(X) \to Y$ extending the original actions of S and such that $<,>: Y \times X \to \mathbb{R}$ becomes an $E(X)$-duality .*

Proof. By Lemma 4.2 (assertions (2) and (3)) we need only to check that there exists an action of $E(X)$ on Y which extends the original right action of S on Y in such a way that the duality $<,>$ becomes an $E(X)$-invariant.

Denote by $\lambda : S \to E(S,X)$ the canonical semigroup compactification. We follow the idea of [16, Proposition II.2]. Let $p \in E(X)$. Choose arbitrarily a net $s_i \in S$ such that $\lambda(s_i)$ converges to p in $E(X)$. Let $y \in Y$. Using the compactness of $cl(yS)$, one can pick a subnet t_j of s_i such that yt_j converges to some $z \in Y$. Define $yp := z$. Then for every $x \in X$ we have

$$< z, x > = < \lim(yt_j), x > = \lim < yt_j, x > = \lim < y, t_j x > =$$

$$= < y, \lim(t_j x) > = < y, \lim \lambda(t_j) x > = < y, px > .$$

The element $< y, px >$ does not depend on the choice of subnets in the definition of z. Since the duality $Y \times X \to \mathbb{R}$ is (left) strict, we can conclude that such z is uniquely determined. Thus, yp is well-defined. These computations show also that $< yp, x > = < y, px >$ because each of them is the limit of $< yt_j, x > = < y, t_j x >$. Since the given duality is strict it follows that the function $Y \times E(X) \to Y, \quad (y,p) \mapsto yp$ is a right action which extends the given action of S on Y. Indeed, we can choose for $p := \lambda(s)$ the constant net $s_i = s$ in the above definition.

□

Remarks 4.4. (a) If Y is compact then it is easy to see that the Ellis semigroups $E(S,X)$ and $E(Y,S)$ are antiisomorphic as semitopological semigroups.

(b) Proposition 4.3 provides a proof of the crucial implication (1) $\Longrightarrow$ (2) of Fact 2.6. Indeed, endow the S-flow $Y := C(X) = WAP_S(X)$ with the pointwise topology and apply Proposition 4.3 to the natural S-pair $Y \times X \to \mathbb{R}$.

(c) As a corollary we get that the semitopological flow $(E(X), X)$ of Proposition 4.3 is wap. In fact this flow is even Eberlein as it follows by Theorem 4.5 below.

(d) Proposition 4.3 implies that a compact S-flow X is S-wap iff $B^* = B(C(X)^*)$ is S-wap. Indeed, by Fact 2.6 it suffices to show that the action of the Ellis semigroup

$E(S,X)$ on B^* is separately continuous. This follows from Lemma 4.2.3 because by Fact 2.7 (i) the action of $E(S,X)$ on $B(C(X))_w$ is separately continuous and $B(C(X))_w \times B^*_{w^*} \to \mathbb{R}$ is a strict $E(S,X)$-duality. Therefore if X is a compact S-wap then we can conclude that the S-subflow $P(X) \subset B^*$ of all probability measures is wap, too. This fact was established earlier by Glasner [19]. Below we show (Theorem 4.11) that $P(X)$ is S-Eberlein if X is S-Eberlein.

Now we prove that all bounded S-dualities $Y \times X \to \mathbb{R}$ come as restrictions of canonical reflexive S-dualities. This fact seems to be interesting even in a purely topological context (that is, for trivial S) which has been proved by Krivine and Maurey [27] for metrizable compacta X and Y. In the proof we provide a modification for flows of the well-known construction of Davis, Figiel, Johnson and Pelczynski [12].

Theorem 4.5. *Let $\nu : Y \times X \to \mathbb{R}$ be a bounded strict S-duality, with semitopological compact S-spaces X and Y. Then there exist: a reflexive Banach space V, a positive number $r > 0$, a weakly continuous antihomomorphism $h : S \to \Theta(V)$, and weak embeddings $\gamma_1 : X \to B^*$ and $\gamma_2 : Y \to rB$ such that the following diagram is commutative.*

$$\begin{array}{ccc} Y \times X & \longrightarrow & \mathbb{R} \\ \gamma_2 \downarrow \quad \downarrow \gamma_1 & & \downarrow id \\ rB \times B^* & \longrightarrow & \mathbb{R} \end{array}$$

If the action $S \times X \to X$ is jointly continuous then we can suppose that $h : S \to \Theta(V)$ is strongly continuous.

Proof. By Proposition 4.3 we can suppose that S is a compact semitopological semigroup. Adjoining the isolated identity e, one can assume even that S is a monoid and $ex = x$. The map $q_Y : Y \to (C(X), w)$ is a topological embedding by Grothendieck's Lemma. We will identify $q_Y(Y)$ and Y. Denote by E the Banach subspace of $C(X)$ topologically generated by Y. That is, $E = cl(sp(Y))$.

Consider the right action $C(X) \times S \to C(X)$, $(fs)(x) := f(sx)$. Then every s-translation $\tilde{s} : C(X) \to C(X)$ is a contractive linear operator. The orbit map $\tilde{y} : S \to C(X)$ is p-continuous for every $y \in Y \subset C(X)$. By our assumption S is compact. Therefore, the orbit yS is bounded p-compact, and, hence w-compact by Grothendieck's Lemma. Since p-topology coincides with the w-topology on yS, it follows that $\tilde{y} : S \to C(X)$ is w-continuous. Then the same is true for every $u \in E = cl(sp(Y))$ (as it follows, for example, from [8, Proposition 6.1.2]). By the Hahn-Banach Theorem, the weak topology of E is the same as its relative weak topology as a subset of $C(X)$. Therefore we get that $((E,w),S)$ is a semitopological flow. Consider the convex hull $co(-Y \cup Y) = W$. By the Krein-Smulian Theorem, W is relatively weakly compact in E. Since W is also convex and symmetric, we can apply a factorization procedure of [12]. For each natural n, set $U_n = 2^n W + 2^{-n} B(E)$. Let $\| \ \|_n$ be the Minkowski's functional of the set U_n. That is, $\|v\|_n = \inf \{\lambda > 0 \mid v \in \lambda U_n\}$. Then $\| \ \|_n$ is a norm on E equivalent to the given norm of E. For $v \in E$, let

$$N(v) := \left(\sum_{n=1}^{\infty} \|v\|_n^2 \right)^{1/2} \quad \text{and} \quad V := \{v \in E \mid N(v) < \infty\}.$$

Denote by $j : V \to E$ the inclusion map. Then:

(1) (V, N) is a reflexive Banach space, $j : V \to E$ is a continuous linear injection and $Y \subset W \subset B(V)$.

(2) The restriction of $j : V \to E$ on each bounded subset A of V induces a homeomorphism of A and $j(A)$ in the weak topologies.

By our construction W and $B(E)$ are S-invariant. Thus we get

(3) V is an S-subset of E and $N(vs) \leq N(v)$ for every $v \in V$ and every $s \in S$.

(4) For every $v \in V$, the orbit map $\tilde{v} : S \to V, \ \ \tilde{v}(s) = vs$ is weakly continuous.

Indeed, by (3), the orbit $\tilde{v}(S) = vS$ is N-bounded in V. Our assertion follows from (2) (for $A = vS$), taking into account that $\tilde{v} : S \to E$ is weakly continuous.

By (3), for every $s \in S$, the translation map $\tilde{s} : V \to V, v \mapsto vs$ is a linear contraction of (V, N). Therefore, we get the antihomomorphism $h : S \to \Theta(V), \quad h(s) = \tilde{s}$.

Now, directly from (4), we obtain the following assertion.

(5) $h : S \to \Theta(V)$ is a w-continuous monoid antihomomorphism.

By (1) and (2) the natural inclusion map $\gamma_2 : Y \to B = B(V)$ is a topological (weak) embedding. Define the weak star embedding $\gamma_1 : X \to V^*$ by $\gamma_1(x)(v) = j(v)(x) =< v, x >$. Clearly, $\gamma_1(x) = (i^* \circ j^*)(x)$, where $i^* : C(X)^* \to E^*$ is the adjoint of the inclusion $i : E \to C(X)$. In particular, we get that $\gamma_1(X)$ is a bounded subset of V^*. It is evident that γ_1 is S-equivariant and weak* (=weak) continuous. On the other hand, $< \gamma_2(y), \gamma_1(x) >=< y, x >$. Since the original duality is strict, we obtain that γ_1 is injective and hence a topological (weak) embedding. As we already mentioned $\gamma_1(X)$ is norm bounded in V^*. Therefore, $\gamma_1(X) \subset rB^*$ for some $r > 0$. By renorming V (defining the new norm as $||v||_{new} := rN(v)$ and observing that $||vs||_{new} \leq ||v||_{new}$ for every $s \in S$), we can suppose without restricting of generality that in fact $\gamma_1(X) \subset B^*$ and $\gamma_2(Y) \subset rB$.

If $S \times X \to X$ is jointly continuous, then the action $C(X) \times S \to C(X)$ is jointy continuous with respect to the norm. By the definition of the Banach space (V, N), it is straightforward to show that all orbit maps $\tilde{v} : S \to V$ are N-norm continuous (recall that each $\| \ \|_n$ is equivalent to the norm of E). This will guarantee that $h : S \to \Theta(V)_s$ is continuous. □

Now we can prove the representation theorem.

Theorem 4.6. *(***WAP Representation Theorem***) Let X be a semitopological S-flow. The following conditions are equivalent:*

(i) *$f : X \to \mathbb{R}$ is weakly almost periodic.*

(ii) *There exist: a representation (h, α) of (S, X) into reflexive V with a weak continuous antihomomorphism $h : S \to \Theta(V)$, weak continuous $\alpha : X \to B^*$, and a vector $v \in V$ such that $f(x) =< v, \alpha(x) >$.*

(iii) *As in (ii) but with no continuity assumptions on h.*

If either: a) $S = G$ is a semitopological group; or b) X is compact and the action $S \times X \to X$ is jointly continuous, then in (ii) we can suppose that h is strongly continuous.

Proof. (i) $\Longrightarrow$ (ii) We can suppose that S is a monoid. For the desired representation of $f \in WAP_S(X)$ by some reflexive V, choose a left strict bounded duality

$$<,>: K_f \times D_f \to \mathbb{R}$$

where $K_f = (cl_w(fS), w)$ and $D_f = (B(WAP_S(X)^*), w^*)$. The weak and pointwise topologies coincide on K_f. Therefore the action of S on K_f is separately continuous. Note, however, that the action of S on D_f is not necessarily separately continuous. By Lemma 4.2 we can pass to the naturally associated strict separately continuous S-duality $<,>_q$: $K_f \times (D_f)_q \to \mathbb{R}$. Lemma 4.2.3 guarantees that the action of S on $(D_f)_q$ is also separately continuous. Denote by $t : X \to (D_f)_q$ the composition of two natural maps $X \to D_f$ and $D_f \to (D_f)_q$. Now, by Theorem 4.5, there exist: a reflexive Banach space

V and a weakly continuous antihomomorphism $h : S \to \Theta(V)$ such that S-duality $<,>_q$ equivariantly can be realized as a part of a reflexive duality $rB \times B^* \to \mathbb{R}$.

$$\begin{array}{ccc} K_f \times (D_f)_q & \longrightarrow & \mathbb{R} \\ \gamma_2 \big\downarrow \quad \big\downarrow \gamma_1 & & \big\downarrow id \\ rB \times B^* & \longrightarrow & \mathbb{R} \end{array}$$

Define $\alpha : X \to B^*$ by $\alpha(x) = \gamma_1(t(x))$ and pick $v := \gamma_2(f)$. Then $f(x) =< v, \alpha(x) >$, as desired.

(ii) $\Longrightarrow$ (iii) Is trivial.

(iii) $\Longrightarrow$ (i) Is immediate by Fact 3.6 (iii).

If S is a semitopological group, then every weakly continuous reflexive (anti)representation is automatically strongly continuous as we mentioned in Section 3 (see Remark 3.2). This proves the case "a)". In the second case "b)", we can apply directly Theorem 4.5. $\square$

Now we easily obtain one of our main results.

Theorem 4.7. *An S-flow X is wap iff X is REFL-approximable.*

Proof. If X is REFL-approximable then X is wap by Fact 3.6 (iii).

The nontrivial part follows from Theorem 4.6 because if X has sufficiently many wap functions, then (S, X) has sufficiently many reflexive representations. $\square$

Corollary 4.8. *Every wap flow (S, X) is RN-approximable.*

It is well known that a countable product of Eberlein (RN) compacta is again Eberlein (resp.: RN). We show that the same is true for flows.

Lemma 4.9. *The classes of Eberlein and RN S-flows are closed under countable products.*

Proof. Let X_n be a sequence of Eberlein (or, RN) S-flows. By the definition there exists a sequence of reflexive (Asplund) representations

$$(h_n, \alpha_n) : (S, X_n) \rightrightarrows (\Theta(V_n)^{opp}, B(V_n^*)).$$

We can suppose that each X_n is compact and $\alpha_n(X_n) \subset 2^{-n} B(V_n^*)$. Turn to the l_2-sum of representations. That is, consider

$$(h, \alpha) : (S, X) \rightrightarrows (\Theta(V)^{opp}, B(V^*))$$

where $V := (\sum_n V_n)_{l_2}$, $h(s)(v) = \sum_n h(s)(v_n)$ for every $v = \sum_n v_n$, and $\alpha(x) = \sum_n \alpha_n(x_n)$ for every $x = (x_1, x_2, \cdots) \in \prod_n X_n$. It is easy to show that $\alpha(x) \in B(V^*)$, α is weak* continuous and injective (hence, a topological embedding). Now use the fact that the l_2-sum of reflexive (Asplund) spaces is again reflexive (Asplund) [17]. $\square$

Corollary 4.10. (i) *Every second countable wap flow is Eberlein.*
(ii) *Every second countable RN-approximable flow is an RN flow.*

Proof. Assertion (ii) is immediate by Lemma 4.9. For (i), we need also Theorem 4.7. $\square$

The following theorem provides, in particular, a flow generalization of a result by Amir-Lindenstrauss [3] which states that if X is an Eberlein compact then $B^* = (B(C(X)^*), w^*)$ is Eberlein, too.

Theorem 4.11. *Let X be a compact semitopological S-flow. The following are equivalent:*

(i) *X is S-Eberlein.*

(ii) *There exists a Banach space E, a homomorphism $h : S \to \Theta(V)$ (no continuity assumptions on h), and an S-embedding $\alpha : X \to (V, w)$.*

(iii) *There exists a compact space* Y *and a (right) strict* S*-duality* $Y \times X \to \mathbb{R}$.
(iv) *There exists a sequence of* S*-invariant weakly compact subsets* $K_n \subset C(X)$ *such that* $\cup_{n \in \mathbb{N}} K_n$ *separates the points of* X.
(v) *There exists an* S*-invariant weakly compact subset* M *in* $C(X)$ *such that* $cl(sp(M)) = C(X)$.
(vi) B^* *is* S*-Eberlein.*
(vii) $P(X)$ *is* S*-Eberlein.*

Proof. (i) $\Longrightarrow$ (ii) By the definition there exists a faithful reflexive V-representation. That is, we can choose a weakly continuous homomorphism $h : S \to \Theta(V)^{opp} = \Theta(V^*)$ and an equivariant embedding $\alpha : X \to B(V^*)$. It suffices to choose $E := V^*$.

(ii) $\Longrightarrow$ (iii) By our assumption X is S-embedded into (E, w). Define the right strict S-duality $Y \times X \to \mathbb{R}$ as a restriction of the canonical duality where $Y := (B(E^*), w^*)$.

(iii) $\Longrightarrow$ (iv) Use the "right version" of Lemma 4.2.4. Then our right strict $Y \times X \to \mathbb{R}$ duality induces the strict S-duality $Y_q \times X \to \mathbb{R}$. We can suppose in addition that this duality is bounded. Now define simply $K_n := Y_q \subset C(X)$ for each n and use Fact 2.5.

(iv) $\Longrightarrow$ (v) Look at K_n as an S-subflow of $(C(X), w)$. We can suppose that $K_n \subset B(C(X))$. Following a method of Rosenthal [49], consider S-invariant set M_n consisting of the constant function equal to 1 on X and of all products of functions $f_1 \cdot f_2 \cdots f_n$ where $f_i \in (\cup_{m=1}^{n} K_m) \cup \{1\}$. By Fact 2.5 and the Eberlein-Smulian theorem, it is easy to see that each M_n is weakly compact. Then $M := \cup_{n \in \mathbb{N}} 2^{-n} M_n$ is also S-invariant and weakly compact in $C(X)$. By the Stone-Weierstrass theorem, $sp(M)$ is dense in $C(X)$.

(v) $\Longrightarrow$ (vi) We can suppose that $M \subset B(C(X))$. The corresponding left strict S-duality $M \times B^* \to [-1, 1]$ is also right strict because $cl(sp(M)) = C(X)$. Now we can apply Theorem 4.5.

(vi) $\Longrightarrow$ (vii) and (vii) $\Longrightarrow$ (i) are trivial because $P(X)$ is an S-subflow of of B^* and X can be treated is an S-subflow of $P(X)$.

□

5. Reflexive representations of (semi)groups

Now we examine a particular but important case of the flows (S, S), left regular actions of a semitopological semigroup S on itself by multiplication. Every compact semitopological semigroup is wap. In general, S is wap iff the universal semitopological compactification $S \to S^W$ is an embedding iff S is a subsemigroup of a compact semitopological semigroup.

Every locally compact Hausdorff topological group G is wap being a subsemigroup of its one-point compactification (which clearly is a compact semitopological semigroup). Moreover, it is well known that such G is even unitarily representable because it can be embedded into the unitary group $Is(H)_s$ of the Hilbert space $H = L_2(G, m_{Haar})$, where m_{Haar} is the Haar measure on G.

It is also easy to show that every *non-Archimedean* (having a local base of open subgroups) topological group is unitarily representable (and, hence wap). Distinguishing unitarily and reflexive representability (and answering a question of Shtern [51]), we show in [39] that the additive group of $L_4[0, 1]$ is wap but not unitarily representable. The proof is based on Grothendieck's double limit property for wap functions. It is still an open question if every abelian Hausdorff topological group (e.g., the additive group of a Banach space) is wap.

Not every topological (even Polish) group is wap. Indeed, the group $G = \mathrm{Homeo}_+[0, 1]$ of all orientation preserving selfhomeomorphisms of the closed interval is not wap [37]. In fact we show that every wap function on such G is necessarily constant (conjectured by Pestov). As a corollary this implies that the universal semitopological compactification G^W

of G is trivial (answering a question of Ruppert [50]) and every weakly continuous bounded representation $h : G \to Aut(V)$ into a reflexive space V is trivial. This example also shows (answering a question of Milnes [42]) that there exists a nonprecompact Hausdorff topological group G such that $WAP(G) = AP(G)$.

Turn again to the WAP Representation theorem. It implies that every wap function comes from a reflexive matrix coefficient.

Theorem 5.1. *For every semitopological monoid S the function $f : S \to \mathbb{R}$ is wap iff f is a matrix coefficient of a weak continuous antihomomorphism $S \to \Theta(V)$ for a reflexive V. That is, there exist $v \in V$ and $\psi \in V^*$ such that $f(s) =< vs, \psi >$.*

Proof. Apply Theorem 4.6 to the flow (S, S). Then for $f \in WAP(S)$ there exists a reflexive V and a representation $h : S \to \Theta(V)$, $\alpha : S \to B(V^*)$ such that $f(s) =< v, \alpha(s) >$ for a suitable $v \in V$. Denote by e the identity of S. Then $f = m_{v,\psi}$ where $\psi = \alpha(e)$. □

If we wish to get a *homomorphism*, just consider $h : S \to \Theta(V)^{opp} = \Theta(V^*)$.

It is also easy now to establish the following result first established by Shtern [51] (see also [38]).

Fact 5.2. *The following are equivalent:*

(i) *A semitopological semigroup S is wap (equivalently, S is embedded into a compact semitopological monoid).*

(ii) *There exists a reflexive space E such that S is embedded (as a semitopological subsemigroup) into $\Theta(E)_w$.*

Proof. (i) $\Longrightarrow$ (ii) We can suppose that S is a monoid. Consider $X := S^W$ the universal semitopological compactification of S. Then the corresponding universal map $u_W : S \to S^W$ is a topological embedding by (i) and hence, the action (S, S^W) is *left strict*. That is, there is no strictly coarser topology on S under which S is a semitopological semigroup and S^W is still a semitopological S-flow. By Theorem 4.6 there exists a separating family (h_i, α_i) of reflexive V_i-representations $(i \in I)$ of (S, S^W). Then the l_2-sum of these representations defined on the Banach space $V := (\sum_i V_{i \in I})_{l_2}$ will induce a weakly continuous antihomomorphism $h : S \to \Theta(V)$. Since the original action is left strict, it is easy to show that h must be a topological embedding. Define $E := V^*$. It is clear that the antihomomorphism h defines the desired homomorphism $h :\to \Theta(V)^{opp} = \Theta(V^*) = \Theta(E)$.
(ii) $\Longrightarrow$ (i) It is well known [34] that $\Theta(V)_w$ is a compact semitopological semigroup for every reflexive V. □

By [38] (or, Corollary 8.3 below), $Is(V)_s = Is(V)_w$ for every reflexive V. Therefore we obtain the following result.

Fact 5.3. [38] *Let G be a topological group. The following are equivalent:*

(i) *G is wap.*

(ii) *G is a topological subgroup of the group $Is(V)_s$ (endowed with the strong operator topology) of all linear isometries for a suitable reflexive V.*

Remarks 5.4. (i) Theorem 4.7 implies that every wap S-flow X is compactifiable. Moreover, if $S = G$ is a semitopological group and $\alpha : X \to Y$ is a corresponding faithful wap G-compactification (which exists by Theorem 4.7) then the action $G \times Y \to Y$ is jointly continuous. This follows from Fact 2.8. Therefore every noncompactifiable in a joint continuous way G-space provides an example of a non-wap flow. Such examples can be found even for jointly continuous group actions of Polish topological groups G on Polish spaces X (see [35, 40]).

(ii) It is well known (as noted for example in Arhangelskij [5] or Namioka-Wheeler [46]) that a compact space is Eberlein iff it can be included into some right strict duality. Theorem 4.5 provides "a flow version".
(iii) Theorems 4.5 , 4.6 and 5.1 admit also "almost periodic versions", replacing: separately continuous S-dualities by jointly continuous, weakly compact by norm compact and weak* continuous α by norm continuous.

6. Fragmentability and flows

Definition 6.1. Let (X, τ) be a topological space and μ be a uniformity on the set X. Then X is *(τ, μ)-fragmented* if for each nonempty $A \subset X$ and each $\varepsilon \in \mu$ there exists a τ-open subset O of X such that $O \cap A \neq \emptyset$ and $O \cap A$ is ε-small.

This definition (for metrics) is explicitly defined by Jayne and Rogers [30] and implicitly it appears even earlier in Namioka-Phelps [45] (see also [26]). There are several generalizations: for covers (Bouziad [10]), for functions [29, 36]. Similar concepts are studied in many contexts: *cliquish* (Thielman 1953), *huskable* (in French, *epluchable*) (Godefroy 1977).

The works [36, 38] are devoted to a systematic study of the fragmentability concept in the context of (semi)group actions and topological dynamics.

Namioka's famous *joint continuity theorem* implies that every weakly compact subset of a Banach space is norm-fragmented [44]. We need the following generalization for locally convex spaces (V, μ) where μ denotes the usual additive uniform structure on V.

Lemma 6.2. *Every relatively weakly compact $X \subset V$ in a l.c.s. V is $(weak, \mu)$-fragmented.*

Proof. See [36, Proposition 3.5]. □

We will use the following useful observation.

Fact 6.3. *Let (X, τ) be a Baire space and d a pseudometric on the set X. If X is (τ, d)-fragmented, then $1_X : (X, \tau) \to (X, d)$ is continuous at the points of a dense G_δ subset D of X.*

Proof. Easily follows using the standard Baire arguments. See for example the proof of Lemma 1.1 in [44] or [36, Lemma 3.2 (d)]. □

The following characterization of Asplund spaces (which is a result of many works) in terms of fragmentability is very important in our setting.

Fact 6.4. *A Banach space V is Asplund iff every bounded subset $A \subset V^*$ of the dual V^* is (weak*,norm)-fragmented.*

Standard examples of Asplund spaces include: reflexive spaces and $c_0(\Gamma)$ spaces. Let K be compact. Then $C(K) \in ASP$ iff K is *scattered* (that is, every nonempty subspace of K contains an isolated point).

Let μ be a uniformity on an S-flow X. We say:

a) $z \in X$ is *a point of equicontinuity* (or, a *Lyapunov stable*) (denote $z \in Equic_\mu(S, X)$ or, simply, $z \in Equic$) if there exists a compatible uniformity μ such that for all $\varepsilon > 0$ there exists a neighborhood $U(z)$ of z such that $(sx, sz) \in \varepsilon$ for every $(x, s) \in U \times S$.
b) (S, X) is *(almost) μ-equicontinuous* if (resp.: $X = cl(Equic)$) $X = Equic$.
c) (S, X) is *uniformly μ-equicontinuous* if for every $\varepsilon \in \mu$ there exists $\delta \in \mu$ such that $(sx, sy) \in \varepsilon$ for every $(x, y) \in \delta$ and every $s \in S$.
d) A point $z \in X$ is the point of *local μ-equicontinuity* in the sense of Glasner and Weiss [21] if $z \in Equic_\mu(S, cl(Sz))$ (we do not require that X be compact). If this

condition holds for every point in X, we say that (X, μ) is locally equicontinuous and write $X \in$ LE.

e) (S, X) is (almost, locally) equicontinuous if X is (resp.: almost, locally) μ-equicontinuous with respect to some compatible uniformity μ on X.

f) $(S, (X, \mu))$ is *not sensitive* (see for example [20] and the references there) if for every $\varepsilon \in \mu$ there exists a non-empty open subset O of X such that $(sx, sy) \in \varepsilon$ for all $x, y \in O$ and $s \in S$.

Theorem 6.5. *(i) Every RN (e.g., Eberlein) Baire flow (S, X) is almost equicontinuous.*
(ii) Every RN-approximable (e.g., wap) Polish S-flow X is almost equicontinuous.

Proof. (i) There exists an Asplund representation $h : S \to \Theta(V)_w$, $\alpha : X \to B(V^*)$. Then according to Fact 6.4, $f(X)$ is (weak*, norm)-fragmented. The action of $\Theta(V)^{opp}$ on $(B(V^*), norm)$ is obviously uniformly equicontinuous. Every point $z \in X$ of continuity of the map $1_X : (X, w^*) \to (X, norm)$ is a point of equicontinuity in the S-flow (X, w^*). Fact 6.3 guarantees that such points are dense in X. Therefore (S, X) is almost equicontinuous.

(ii) Follows from (i) because every RN-approximable second countable flow is RN (Corollary 4.10). □

As a conclusion of the part (ii) and Corollary 4.8 we get the following known result.

Corollary 6.6. *(Akin-Auslander-Berg [1]) Let G be a topological group and X a metrizable compact G-flow. Assume that the G-flow X is wap. Then X is almost equicontinuous.*

The following definition is an important tool for our purposes.

Definition 6.7. Let (X, τ) be an S-flow and μ a uniformity on the set X such that $\tau \subset top(\mu)$. We say that the flow (X, τ) is *μ-equifragmented* if X is (τ, μ)-fragmented, the action of S on X is uniformly μ-equicontinuous and for some uniformity $\xi \subset \mu$ we have $top(\xi) = \tau$.

We collect here some useful stability conditions for equifragmentability.

Lemma 6.8. (i) *The class of equifragmented flows is preserved under subflows.*

(ii) *Equifragmentability is preserved under products. More precisely, if X_i is μ_i-eqifragmented then the product $\prod X_i$ of S-flows is $\prod \mu_i$-equifragmented.*

(iii) *For every Asplund space V the flow $(\Theta(V)^{opp}, (B(V^*), w^*))$ is $\mu_{\| \ \|}$-equifragmented where $\mu_{\| \ \|}$ is the norm uniformity of V^*.*

(iv) $(\Theta(V), (B(V), w))$ *is $\mu_{\| \ \|}$-equifragmented for every reflexive V.*

(v) *Every RN-approximable (e.g., wap) flow (S, X) is equifragmented.*

(vi) *If a compact flow X is equifragmented then X is not sensitive. Therefore, every RN-approximable S-flow X is not sensitive.*

Proof. The assertion (i) is trivial, (ii) and (vi) are straightforward. For (iii) and (iv), we can use Fact 6.4 and Lemma 6.2, respectively. In order to establish (v), combine (i), (ii) and (iii). □

Let a group G act on a topological space X. We say that:

a) a point $z \in X$ is *transitive* (write: $z \in Trans$) if $cl(Gz) = X$. If $Trans \neq \emptyset$, then, as usual, X is called transitive.

b) a point $z \in X$ is *quasitransitive* (write: $z \in qTrans$) if $int(cl(Gz)) \neq \emptyset$.

c) X is *quasiminimal* if $X = qTrans$.

d) X is *minimal* if $cl(Gz) = X$ for all $z \in X$.

Let (X, τ) be a topological space and μ be a uniformity on X such that $\tau \subset top(\mu)$. We say that a subset $K \subset X$ is (τ, μ)-*Kadec* if $\tau|_K = top(\mu)|_K$. Denote by $Cont(\tau, \mu)$ the subset of all points of continuity of the identity map $1_X : (X, \tau) \to (X, \mu)$. Clearly, $Cont(\tau, \mu)$ is an example of a (τ, μ)-Kadec set.

Theorem 6.9. *Let a topologized group G act on a topological space (X, τ) by homeomorphisms. If this action is μ-equifragmented (with respect to $\xi \subset \mu$ such that $top(\xi) = \tau$) then:*

(i) *$qTrans \subset Cont(\tau, \mu) \subset Equic_\mu(G, X)$. In particular, every point of quasitransitivity of X is a point of ξ-equicontinuity.*

(ii) *If a G-subflow Y is quasiminimal (e.g., 1-orbit subset $Y = Gz$) then Y is a (τ, μ)-Kadec set. Hence $\mu|_Y$ is an compatible uniformity on Y and Y is a uniformly $\mu|_Y$-equicontinuous G-flow.*

Proof. (i) Let $z \in qTrans$. We have to show that for every $\varepsilon \in \mu$ there exists a τ-neighborhood $O(z)$ of z such that O is ε-small. Choose $\delta \in \mu$ such that $(gy_1, gy_2) \in \varepsilon$ for every $(y_1, y_2) \in \delta$ and $g \in G$. Since $z \in qTrans$, the set $A := int(cl(Gz))$ is non-void. Since X is (τ, μ)-fragmentable, we can pick a *non-void* τ-open subset W of X such that $W \subset A$ and W is δ-small in X. Clearly, $W \cap Gz \neq \emptyset$. One can choose $g_0 \in G$ such that $g_0 z \in W \cap Gz$. Denote by O the open subset $g_0^{-1}W$ of (X, τ). Then O is a τ-neighborhood of z and is ε-small. This proves the inclusion $qTrans \subset Cont(\tau, \mu)$. The second inclusion $Cont(\tau, \mu) \subset Equic_\mu(G, X)$ is trivial because X is uniformly μ-equicontinuous.

(ii) By the quasiminimality of Y, $qTrans(Y) = Y$. Therefore, the assertion (i) implies that $\tau|_Y = top(\mu)|_Y$. □

Theorem 6.10. *Let G be a semitopological group and X be an RN-approximable semitopological G-flow. Then $X \in LE$ and every G-quasiminimal subspace (for instance, every orbit) of X is equicontinuous.*

Proof. By Lemma 6.8 (v), X is μ-equifragmented. For every fixed $z \in X$ consider the S-subflow $Y := cl(Gz)$. Clearly, z is a point of quasitransitivity of Y. Then we can apply Theorem 6.9 to (G, Y) and conclude that z is a point of local equicontinuity of Y (and hence of X). □

Corollary 6.11. *("Generalized Veech-Troallic-Auslander Theorem")*
Every wap (not necessarily compact or metrizable) G-flow X is LE and every G-quasiminimal subflow of X is equicontinuous.

Proof. By Corollary 4.8 every wap flow is RN-approximable. □

Remark 6.12. Troallic [54] and also Auslander [4] proved that every minimal compact wap G-flow X is equicontinuous. Previously such a result was established for compact Eberlein (in our terminology) G-flows by Veech [56].

Combining Theorem 4.7 and Corollary 6.11 for general G-flows, we can draw the following diagram

$$\text{Eberlein} \subset \text{WAP} = \text{REFL}_{app} \subset \text{RN}_{app} \subset \text{LE}$$

Consider the case of $S = \mathbb{Z}$ and metrizable compact cascades. The fact that WAP $\neq$ LE is discussed in [21]. The authors constructed (see main example in [21, page 350]) a transitive cascade $(\mathbb{Z}, X)$ such that X is in LE but not wap and every point of transitivity is *recurrent*. If we do not require the last assumption then there exists an elementary example distinguishing wap and RN (and , hence also wap and LE). Namely, the two-point compactification X of $\mathbb{Z}$ with the natural action of $\mathbb{Z}$ on X is transitive and contains two fixed points. Fact 6.13 implies that such (Z, X) can not be wap. On the other hand, X is clearly scattered. Therefore (Z, X) is RN by Proposition 7.15 below.

Fact 6.13. *Let X be a wap transitive compact G-flow. Then X contains a unique minimal compact subflow.*

Proof. Let $E = E(G, X)$ be the Ellis (semitopological) semigroup. By [16, Proposition II.5] this semigroup contains a unique minimal ideal K which is closed in E. It follows by transitivity that $Et_0 = X$ for some $t_0 \in X$. Then the unique minimal compact subset of X is Kt_0. □

If the group action (G, X) is RN then there exists a compatible uniformity μ on X (the precompact uniformity of the corresponding weak star G-embedding of X into $B(V^*)$ with Asplund V) such that X is not sensitive (see Lemma 6.8 (vi)). Another observation comes from Theorem 6.10. It implies that every RN-approximable 1-orbit group action is equicontinuos. This provides an easy way producing examples of G-flows which fail to be RN. Roughly speaking, RN G-flow cannot be "too chaotic" or "too massive".

Let G be a topological group. Consider the natural action (call it a "Δ-action")

$$\pi_\Delta : (G \times G) \times G \to G, \quad (s, t)x = sxt^{-1}.$$

It actually coincides with the coset G-space action $(G \times G, (G \times G)/H)$, where $H = \Delta := \{(g, g) : g \in G\}$.

Lemma 6.14. *Let G be a topological group such that $(G \times G, G, \pi_\Delta)$ is an RN-approximable (e.g., wap) flow. Then G satisfies SIN (small invariant neighborhoods).*

Proof. The given (1-orbit) Δ-action is μ-equicontinuous, by Theorem 6.10, with respect to some compatible uniformity μ on G. Let $U(e)$ be an arbitrary neighborhood of the identity in G. Choose $\varepsilon \in \mu$ such that the neighborhood $\varepsilon(e) = \{x \in G : (e, x) \in \varepsilon\}$ is contained in $U(e)$. By the μ-equicontinuity of the Δ-action at the point e one can choose a neighborhood $O(e)$ such that sOt^{-1} is ε-small for all $(s, t) \in G \times G$. Then $gOg^{-1} \subset \varepsilon(e) \subset U(e)$ for every $g \in G$. This is equivalent to the condition $G \in SIN$. □

Now we can strengthen a result of Hansel and Troallic. Let G be a topological group. Following [23] we say that a function $f \in C(G)$ is *strictly wap* (notation: $f \in sWAP(G)$) if GfG is relatively w-compact in $C(G)$. Denote by $[swap]$ the class of groups such that $sWAP(G)$ separates points and and closed subsets of G. Denote by $[WS]$ the class of groups for which every wap function is strictly wap. Clearly, $[wap] \cap [WS] \subset [swap]$.

Proposition 6.15. *Let $G \in [swap]$ then $G \in SIN$.*

Proof. First observe that $G \in [swap]$ iff the Δ-action $(G \times G, G)$ is wap and hence RN-approximable. Now Lemma 6.14 finishes the proof. □

Corollary 6.16. *(Hansel-Troallic* [23]*) Let $G \in [wap] \cap [WS]$. Then $G \in SIN$.*

7. Asplund functions and representations

Recall that a Radon-Nikodym compact space [43] is a compact subset in (V^*, w^*) for an Asplund space V. We introduce a generalization for flows. Our approach synthesizes some ideas from [56, 52, 44, 17].

The following definition goes back to Stegall [52] and Namioka [44].

Definition 7.1. Let M be a nonempty bounded subset of a Banach space V. Say that M is an *Asplund set* in V if for every countable $C \subset M$ the pseudometric space (V^*, ρ_C) is separable, where

$$\rho_C(\xi, \eta) = sup\{| < c, \xi > - < c, \eta > | : \ c \in C\}.$$

Generalizing slightly this definition, let's say that M is an *Asplund set for* $K \subset V^*$ if the pseudometric subspace (K, ρ_C) is separable for every countable $C \subset M$.

We need the following lemma of Namioka in the form presented by Fabian.

Lemma 7.2. [17, Lemma 1.5.3] *Let X be a compact space (canonically embedded into $C(X)^*$) and let $M \subset C(X)$ be a bounded subset. Assume that (X, ρ_M) is separable. Then the pseudometric space $(C(X)^*, \rho_M)$ is also separable.*

Corollary 7.3. *$M \subset C(X)$ is an Asplund set for compact X iff M is an Asplund set for $C(X)^*$.*

Remark 7.4. The family of Asplund sets in V has nice properties being stable under taking subsets, finite unions, closures, linear continuous images, finite linear combinations, etc. Note also that if M_1 and M_2 are Asplund sets in $C(X)$ for a compact X, then the subset $M_1 \cdot M_2$ is also Asplund. For these and some other results we refer to [17].

We say that a *bounded* duality $Y \times X \to \mathbb{R}$ is an *Asplund duality* if $q_Y(Y)$ is an Asplund subset of $C(X)$. Conversely, the subset $M \subset C(X)$ is an Asplund set iff the corresponding duality $M \times X \to \mathbb{R}$ is an Asplund duality where M is endowed with the pointwise topology inherited from $C_p(X)$.

The following Lemma is a reformulation of a result of Namioka [44, Theorem 3.4].

Lemma 7.5. *Let V be a Banach space and K be a compact subspace in the dual ball $B^*_{w^*}$. Suppose that K is (weak*, norm)-fragmented in V^*. Then $B = B(V)$ is an Asplund set for K and $B \times K \to \mathbb{R}$ is an Asplund duality for every topology τ on B such that $\psi : B \to \mathbb{R}$ is τ-continuous for every $\psi \in K$.*

By Fact 6.4 and Lemma 7.5, the strict duality $rB_w \times B^*_{w^*} \to \mathbb{R}$ is an Asplund duality (call it: *canonical Asplund duality*) for every Asplund space V (and $r > 0$).

Definition 7.6. Let X be a compact S-flow with a separately continuous left action. We say that X is *w-admissible* if $C(X) = WRUC_S(X)$.

This happens if either: a) (S, X) is jointly continuous (then by Fact 2.1 we have even $C(X) = RUC_S(X)$); b) (S, X) is wap (by Fact 2.7); or c) S is a k-space (use Fact 2.5).

Theorem 7.7. *Let X be a compact w-admissible S-flow. Every Asplund strict S-duality $Y \times X \to \mathbb{R}$ is an S-restriction of a canonical Asplund S-duality with respect to a weakly continuous antihomomorphism $h : S \to \Theta(V)$. More precisely, there exist: a suitable Asplund space V, a positive number $r > 0$ and equivariant maps: $\gamma_1 : X \to B^*_{w^*}$ and $\gamma_2 : Y \to B$ such that the following diagram is commutative*

$$\begin{array}{ccc} Y \times X & \longrightarrow & \mathbb{R} \\ \gamma_2 \downarrow \quad \downarrow \gamma_1 & & \downarrow id \\ rB \times B^* & \longrightarrow & \mathbb{R} \end{array}$$

where we require that γ_1 is a topological embedding and γ_2 is an injective map.

If the action $S \times X \to X$ is jointly continuous, then we can suppose that $h : S \to \Theta(V)$ is strongly continuous.

Proof. Consider the natural continuous injective map $q_Y : Y \to C_p(X)$ and denote by K the subset $q_Y(Y) \subset C(X)$. Then K is an Asplund subset in $C(X)$. Then K is an Asplund subset also in the Banach subspace $E = cl(sp(K))$ of $C(X)$. Following the method of [52] and, especially, [17, Section 1.4], one can modify the proof of Theorem 4.5 using the factorization procedure for Asplund S-sets (instead of weakly compact sets). We define a sequence $\|\cdot\|_n$ of norms on E each of them equivalent to the original norm. Namely, for every natural n consider Minkowski's functional of the set

$$P_n := 2^n co(-K \cup K) + 2^{-n} B(E).$$

It is important that the subset $\cap_{n\in\mathbb{N}} P_n$ is Asplund. Moreover, by [17, Theorem 1.4.4] we get a linear injective continuous mapping $j : V \to E$ where V is an Asplund space. Since K is an S-invariant subset of $C(X)$, the same is true for E, $B(E)$ and P_n. Therefore, every norm $||\cdot||_n$ is S-nonexpansive. Then it follows by the construction that the corresponding norm N on the Banach space (V, N) is also S-nonexpansive.

Define $\gamma_2 : Y \to B(V)$ as a natural S-inclusion (of sets). Since X is w-admissible we have $WRUC_S(X) = C(X)$. This guarantees that every orbit map $\tilde{z} : S \to C(X)$ is weakly continuous. Hence the action of S on $(E, weak)$ is separately continuous.

On the other hand, by [17, Theorem 1.4.4], the adjoint map $j^* : E^* \to V^*$ has the norm dense range. It follows that for every bounded subset A of V, the weak topology of V and the weak topology of E, considering of A as a subset of E and $C(X)$, are the same. In particular, this implies that every orbit map $\tilde{v} : S \to (V, w)$ is weakly continuous. Thus, the antihomomorphism $h : S \to \Theta(V)$ is weakly continuous.

We get also that the dual (left) action of S on V^* is weak* separately continuous. The natural S-inclusion $j : V \to C(X)$ is a linear continuous S-map. The adjoint $j^* : C(X)^* \to V^*$ is a weak*-weak* continuous S-operator. Denote by γ_1 the restriction of this map on $X \subset C(X)^*$. Clearly, $< y, x >=< \gamma_2(y), \gamma_1(x) >$. Then γ_1 is injective (and hence a topological embedding) because the original duality is (right) strict.

If (S, X) is a jointly continuous flow then, like Theorem 4.5, we can prove that h is strongly continuous, too.

□

It is well known (see [44, 48]) that, similarly to the "Eberlein case", a compact space X is RN iff the unit ball $B^* \subset C(X)^*$ (and hence P(X)) is RN. The following result provides, in particular, a generalization for flows.

Theorem 7.8. *Let (S, X) be a compact w-admissible (e.g., jointly continuous) flow. The following conditions are equivalent:*

(i) *The flow (S, X) is RN.*
(ii) *There exists a representation (h, α) of (S, X) into a Banach space V such that the antihomomorphism $h : S \to \Theta(V)$ is weakly continuous, $\alpha : X \to V^*$ is a bounded weak* embedding, and $\alpha(X)$ is (weak*, norm)-fragmented.*
(iii) *There exists a representation (h, α) of (S, X) into a Banach space V such that $h : S \to \Theta(V)$ is an antihomomorphism (no continuity assumptions on h), $\alpha : X \to V^*$ is a bounded weak* embedding, and $\alpha(X)$ is (weak*, norm)-fragmented.*
(iv) *There exists a (right) strict Asplund S-duality $Y \times X \to \mathbb{R}$.*
(v) *There exists a bounded S-invariant subset $M \subset C(X)$ such that M separates points of X and M is an Asplund set for X (equivalently, for $(C(X)^*)$).*
(vi) *There exists an S-invariant Asplund set Q in $C(X)$ such that cl(sp(Q))=C(X).*
(vii) *$(S, B(C(X)^*)$ is RN.*
(viii) *$(S, P(X))$ is RN.*

If (S, X) is jointly continuous, then in the assertions (i), (ii), (vii), (viii) we can suppose in addition that the corresponding $h : S \to \Theta(V)$ is strongly continuous.

Proof. (i) $\Longrightarrow$ (ii) By definition of RN, there exists a faithful Asplund V-representation. By Fact 6.4, $\alpha(X)$ is $(w^*, norm)$-fragmented in V^*.

(iii) $\Longrightarrow$ (iv) The ball $B(V)$ is S-invariant and separates points of $\alpha(X)$. It follows from Lemma 7.5 that the right strict S-duality

$$B(V) \times X \to [-1,1], \ (v,x) \mapsto < v, \alpha(x) >$$

is an Asplund duality. In order to get a strict duality, pass to the associated reduced duality $<,>_q$: $B_q \times X \to [-1,1]$ (using the "dual version" of Lemma 4.2.4). Clearly, $<,>_q$ is also an Asplund duality.

(iv) $\Longrightarrow$ (v) Take $M := q_Y(Y) \subset C(X)$ (and use Corollary 7.3).

(v) $\Longrightarrow$ (vi) Suppose that a set M satisfies assumptions of (v). As in the proof of Theorem 4.11, produce inductively the sequence of subsets $M_n = M_1 \cdot M_1 \cdots M_1$, where $M_1 = M \cup \{1\}$. Then it is well known that each M_n is again an Asplund set (Remark 7.4). Moreover, even the set $Q = \cup_{n \in \mathbb{N}} 2^{-n} M_n$ is Asplund. On the other hand, by the Stone-Weierstrass theorem $cl(sp(Q)) = C(X)$. By the construction Q is S-invariant.

(vi) $\Longrightarrow$ (vii) Since Q is an Asplund set in $C(X)$ we obtain that $Q \times B^* \to \mathbb{R}$ is a (left strict) Asplund S-duality. This duality actually is right strict (and hence strict) because $cl(sp(Q)) = C(X)$. Now we can conclude that B^* is RN S-flow as it follows directly from Theorem 7.7. The same result guarantees that h is strongly continuous provided that (S,X) is a jointly continuous flow.

Other implications are trivial.

□

Definition 7.9. Let X be a semitopological compact S-flow. Let us say that a function $f \in C(X)$ is *S-Asplund* if the orbit fS is an Asplund set in $C(X)$. Equivalently, if $(fS) \times X \to \mathbb{R}$ is an Asplund duality. More explicitly, taking into account Corollary 7.3, we see that f is Asplund iff for every countable subset (equivalently, separable) $C \subset S$ the pseudometric space (X, ρ_C) is separable, where

$$\rho_C(x,y) = sup\{|f(sx) - f(sy)| : \ s \in C\}.$$

If S is separable then it is equivalent to check the separability of the single semimetric space (X, ρ_S).

Denote by $Asp_S(X)$ the set of all S-Asplund functions on a compact X. The product $F = f_1 f_2$ of two Asplund functions f_1 and f_2 on X is again Asplund. This follows from the inclusion $FG \subset (f_1 G) \cdot (f_2 G)$ taking into account Remark 7.4. It is easy to show that in fact $Asp_S(X)$ is a Banach S-subalgebra of $C(X)$ for every compact S-flow X.

Lemma 7.10. *$WAP_S(X) \subset Asp_S(X)$ for every semitopological compact S-flow X.*

Proof. Every weakly compact subset of a Banach space is an Asplund set (see [17]). In particular, $fS \subset C(X)$ is an Asplund set if fS is relatively weakly compact. □

Theorem 7.11. *(***RN Representation Theorem***) Let X be a compact w-admissible S-flow. The following are equivalent:*

(i) *$f \in Asp_S(X)$.*
(ii) *$(fS) \times X \to \mathbb{R}$ is an Asplund S-duality.*
(iii) *There exist: a representation (h, α) of (S,X) into an Asplund V with a (weakly continuous) antihomomorphism h and a vector $v \in V$ such that $f(x) =< v, \alpha(x) >$.*
(iv) *There exist: a representation (h, α) of (S,X) into a Banach space V with a weak* $\alpha : X \to B(V^*)$ (no continuity assumptions on h) such that $\alpha(X)$ is $(w^*, norm)$-fragmented and there exists a vector $v \in V$ satisfying $f(x) =< v, \alpha(x) >$ for every $x \in X$.*

If (S,X) is jointly continuous, "weakly continuous" can be replaced by "strongly continuous".

Proof. (i) $\iff$ (ii) Is trivial as it was mentioned earlier.

(ii) $\implies$ (iii) Using Lemma 4.2.4 pass to the associated strict S-duality $(fS) \times X_q \to \mathbb{R}$ (which again is Asplund) and apply Theorem 7.7.

(iii) $\implies$ (iv) is trivial by Fact 6.4.

(iv) $\implies$ (i) Observe that the orbit vS is an Asplund set for $\alpha(X)$ by Lemma 7.5. Now observe that the orbit $T(vS) = fS$ of f is an Asplund set for X (and hence for $C(X)^*$ by Corollary 7.3). □

Corollary 7.12. *Let X be a compact w-admissible S-flow. The following are equivalent:*
(i) X is RN-approximable S-flow.
(ii) $C(X) = Asp_S(X)$.

Proof. (i) $\implies$ (ii) Let X be RN-approximable. Then by Theorem 7.11, $Asp_S(X)$ separates points of X. On the other hand, $Asp_S(X)$ is a Banach subalgebra of $C(X)$ containing the constants. Therefore by the Stone-Weierstrass Theorem we have the coincidence $Asp_S(X) = C(X)$.

(ii) $\implies$ (i) The algebra $C(X) = Asp_S(X)$ separates points and closed subsets of X. Hence, by Theorem 7.11 there are sufficiently many Asplund representations of (S, X). □

Proposition 7.13.
(1) *Let X be a compact S-flow and $q : X \to Y$ be an S-quotient. A continuous bounded function $f : Y \to \mathbb{R}$ is S-Asplund iff the composition $F = f \circ q : X \to \mathbb{R}$ is S-Asplund.*
(2) *$F \in Asp(X)$ iff it comes from an RN S-factor. That is, there exist an RN S-flow Y, an S-factor $q : X \to Y$ and a continuous function $f \in C(Y)$ such that $F = f \circ q$.*
(3) *A factor Y of RN-approximable compact w-admissible S-flow X is again RN-approximable and w-admissible. If Y is metrizable then (S, Y) is an RN flow.*

Proof. (1) For every pair x_1, x_2 in X and every countable $C \subset G$ we have

$$\rho_{FC}(x_1, x_2) = sup\{|F(sx_1) - F(sx_2)| : \ s \in C\} =$$
$$= sup\{|f(sq(x_1)) - f(sq(x_2))| : \ s \in C\} = \rho_{fC}(q(x_1), q(x_2)).$$

Thus, $q : (X, \rho_{FC}) \to (Y, \rho_{fC})$ is a surjective "pseudometric-preserving" map. In particular, (X, ρ_{FC}) is separable iff (Y, ρ_{fC}) is separable.

(2) Combine the first assertion, Corollary 7.12 and RN representation Theorem 7.11.

(3) First observe that $f \in WRUC_S(Y)$ iff $F \in WRUC_S(X)$ because $q : X \to Y$ induces the S-inclusion $q^* : C(Y) \hookrightarrow C(X)$ of Banach S-algebras. This implies that Y is also w-admissible. We have to show that Y is RN-approximable. By Theorem 7.11 it is equivalent to check that $C(Y) = Asp_S(Y)$. The latter follows directly from (1).

If, in addition, Y is metrizable, then by Corollary 4.10, (S, Y) is RN. □

For every fixed S, the class of all RN-approximable compact S-flows is closed under subdirect products. Therefore, using a well-known method (see for example [31, 58]) we obtain that for every compact S-flow X there exists a universal RN-approximable compactification $u_A : X \to X^A$ which is a topological embedding iff $Asp_S(X)$ separates points and closed subsets. Indeed, by Corollary 7.12 and Proposition 7.13.1, it is easy to see that $u_A : X \to X^A$ is a compactification of X associated to the algebra $Asp_S(X)$.

Proposition 7.14. *For every compact RN-approximable S-flow X, its Ellis semigroup $E(X)$, as an S-flow, is also RN-approximable.*

Proof. By the definition, $E(X)$ is a an S-subflow of X^X. Hence, $E(X)$ is a subdirect product of RN-approximable S-flows ("X many copies" of the flow X). □

Proposition 7.15. *Every scattered compact jointly continuous S-flow X is RN.*

Proof. It is well known that X is scattered iff $C(X)$ is Asplund. Hence the canonical representation $S \to \Theta(V)_s$, $X \to B(V^*)_{w^*}$ into an Asplund space $V := C(X)$ is the desired. $\square$

Let a semitopological group G act joint continuously on compact X. The following scheme gives some intuitive explanation about the real place of Asplund functions.

$$\begin{array}{lccccc} \text{functions: } & WAP_G(X) & \subset & Asp_G(X) & \subset & C(X) \\ \text{compactifications: } & X^W & \leftarrow & X^A & \leftarrow & X^R = X \\ \text{representations: } & REFL & \subset & ASP & \subset & BAN \end{array}$$

Now define Asplund functions on a semitopological group G via the universal compactification $u_R : G \to G^R$ (we identify G with $u_R(G)$). A continuous bounded function $f : G \to \mathbb{R}$ is said to be an *Asplund function* (and write: $f \in Asp(G)$) if there exists an Asplund function $F : G^R \to \mathbb{R}$ on the G-flow G^R such that $f = F \circ u_R$. It is equivalent to say that the orbit fG is an Asplund set in the Banach space $RUC(G)$. In particular, $Asp(G) \subset RUC(G)$.

Define by $\lambda : C(G^R) \to RUC(G)$ the natural isomorphism of G-algebras induced by the compactification $u_R : G \to G^R$.

Proposition 7.16. *Let G be a semitopological group.*

1. *$Asp(G)$ is a G-invariant Banach subalgebra of $RUC(G)$ canonically G-isomorphic to $Asp_G(G^R)$. More precisely, $Asp(G) = \lambda(Asp_G(G^R))$.*
2. *$WAP(G) \subset Asp(G)$.*
3. *Denote by $u_A : G \to G^A$ the G-compactification induced by the algebra $Asp(G)$. Then $u_A : G \to G^A$ is the universal RN-approximable jointly continuous G-compactification of G. More precisely, for every jointly continuous G-compactification $\alpha : G \to Y$ with RN-approximable compact G-flow Y, there exists a (necessarily unique) G-map $\psi : G^A \to Y$ such that $\psi \circ u_A = \alpha$.*
4. *G^A is a right topological monoid naturally isomorphic to the Ellis semigroup $E(G, G^A)$ and u_A is a right topological semigroup compactification of G.*

Proof. (1) Follows by the definition of $Asp(G)$.

(2) If $f \in WAP(G)$, then Fact 2.7(ii) guarantees that $fG \subset RUC(G)$. Then fG, being a relatively weakly compact in $RUC(G)$, is necessarily an Asplund set (see [17]).

(3) For joint continuity of the action of G on G^A, recall that $Asp(G)$ is a G-invariant subalgebra of $RUC(G)$. Universality follows from the fact that G^A canonically can be identified with $(G^R)^A$ defined for the jointly continuous compact G-flow G^R.

(4) Let $i : G \to E(G^A)$ be the natural homomorphism of G into the Ellis semigroup of the G-flow G^A. Consider the orbit map $\gamma : E(G^A) \to G^A, \gamma(p) = p(u_A(e))$. Clearly, $\gamma(i(g)) = u_A(g)$ for every $g \in G$. Therefore γ is a morphism between two compactifications $i : G \to E(G^A)$ and $u_A : G \to G^A$. It suffices to show that γ is an isomorphism of these transitive G-flows. By [58, D.2] we need the existence of a morphism of compactifications in the reverse direction. We can use Proposition 7.14 which states that $E(G^A)$ is RN-approximable. By the universality property of u_A, there exists a continuous G-map $\nu : G^A \to E(G^A)$ such that $\nu \circ u_A = i$. Hence, ν is the desired morphism between the compactifications. $\square$

The G-algebra $Asp(G)$ is *m-admissible* in the sense of [8] as it follows by Proposition 7.16 and [8, Theorem 3.1.7].

Theorem 7.17. *Let G be a semitopological group. The following are equivalent:*

(i) $f \in Asp(G)$.
(ii) *There exist: an Asplund space V, a strongly continuous antihomomorphism $h : G \to Is(V)$, vectors $v \in V$, and $\psi \in V^*$ such that $f(g) =< v, g\psi >$ (that is, $f = m_{v,\psi}$).*
(iii) *There exist a G-compactification $\alpha : G \to Y$ with a jointly continuous RN G-flow Y and a function $F \in C(Y) = Asp_G(Y)$ such that $f = F \circ \alpha$.*
(iv) *There exist a G-compactification $\alpha : G \to Y$ with a jointly continuous G-flow Y and a function $F \in Asp_G(Y)$ such that $f = F \circ \alpha$.*

Proof. (i) $\Longrightarrow$ (ii) Since the G-action on a compact space $X = G^R$ is jointly continuous, we can use "strongly continuous version" of Theorem 7.11 obtaining strongly continuous antihomomorphism $h : G \to Is(V)$ and a weak* continuous $\alpha_0 : G^R \to B(V^*)$ such that $f(g) =< v, \alpha_0(g) >$. Then $f = m_{v,\psi}$ where $\psi = \alpha_0(e)$ and e is the identity of G.

(ii) $\Longrightarrow$ (iii) Define Y as the weak* closure of $cl_{w^*}(G\psi)$ of the orbit of $G\psi$. By Fact 2.2, Y is a jointly continuous G-flow. Moreover, Y is an RN G-flow (Definition 3.1). Then, $F : Y \to \mathbb{R}$, $F(y) =< v, y >$ is the desired function by Corollary 7.12.

(iii) $\Longrightarrow$ (iv) is trivial.

(iv) $\Longrightarrow$ (i) By our assumption FG is an Asplund set in $C(Y)$. The natural G-embedding (induced by α) of Banach algebras $C(Y) \to RUC(G)$ maps FG onto fG. Therefore, fG is an Asplund set in $RUC(G)$. □

Proposition 7.18. *Let G be a semitopological group and X be a compact minimal G-flow. Then $AP_G(X) = WAP_G(X) = Asp_G(X)$.*

Proof. In general, $AP_G(X) \subset WAP_G(X) \subset Asp_G(X)$ by Lemma 7.10. So we have only to show that in our situation $AP_G(X) \supset Asp_G(X)$ holds. Let $F \in Asp_G(X)$. Then by the universality of the canonical S-quotient $u_A : X \to X^A$, we have $F = f \circ u_A$ for some $f \in Asp_G(X^A)$. Since (G, X^A) is also minimal, (G, X^A) is equicontinuous by Theorem 6.10. Hence, $f \in AP_G(X^A)$. Then, clearly $F \in AP_G(X)$.

□

Examples 7.19. (1) As it was mentioned above, the two-point compactification of $\mathbb{Z}$, as a cascade, is RN but not wap. Similarly the two-point compactification of $\mathbb{R}$, as an $\mathbb{R}$-flow, is RN but not wap.

(2) Define $f : \mathbb{Z} \to \mathbb{R}$ by $f(z) = 1$ iff z is a positive integer and $f(z) = 0$ otherwise. Then $f \in Asp(\mathbb{Z}) \setminus WAP(\mathbb{Z})$.

Indeed, $f \in Asp(\mathbb{Z})$ by Theorem 7.17 because f comes from the two-point compactification Y of $\mathbb{Z}$ which is RN $\mathbb{Z}$-flow by (1). On the other hand, $f \notin WAP(\mathbb{Z})$ because f does not satisfy Grothendieck's DLP (see Fact 2.4). Choose $s_n = n$ and $x_m = -m$. Then $\lim_m \lim_n f(n - m) = 1 \neq 0 = \lim_n \lim_m f(n - m)$.

(3) As in (2), it is easy to show that $f \in Asp(\mathbb{R}) \setminus WAP(\mathbb{R})$ for the functions $f(x) = \frac{x}{1+|x|}$ and $f(x) = arcsinx$.

(4) The cascade $(\mathbb{Z}, [0, 1])$ generated by the $f(x) = x^2$ map is not wap. Indeed, it contains, as a subflow, the two-point compactification of Z. Take, for example, the $\mathbb{Z}$-orbit of the point $x = \frac{1}{2}$. Together with the endpoints $\{0\}$ and $\{1\}$, we get the closure of this orbit.

(5) Let X be a minimal compact jointly continuous G-flow which is not equicontinuous. Then $C(X) \neq Asp_G(X)$ (X is not RN-approximable) and $RUC(G) \neq Asp(G)$.

Indeed, by Theorem 6.10, X is not RN-approximable. Theorem 7.12 guarantees that $C(X) \neq Asp_G(X)$. Another proof of the same fact follows from the equality $AP_G(X) = Asp_G(X)$ (Proposition 7.18).

Now we check that $RUC(G) \neq Asp(G)$. Fix $f \in C(X) \setminus Asp_G(X)$ and a point $z \in X$. Since z is a point of transitivity of X, there exists a continuous onto G-map $q : G^R \to X$ such that $q(u_R(g)) = gz$ for every $g \in G$. Define $F : G^R \to \mathbb{R}$ as the composition $f \circ q$. Then $F \notin Asp_G(G^R)$ by Proposition 7.13. Thus the restriction $F|_G(g) = f(gz)$ of F on G satisfies $F|_G \in RUC(G) \setminus Asp(G)$.

As a concrete example consider the cascade on the two-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ generated by the selfhomeomorphism (see [7, Example 5.1.7])

$$\sigma_\theta : \mathbb{T}^2 \to \mathbb{T}^2, \ \ \sigma_\theta([a],[b]) = ([a+b],[a+\theta])$$

where θ is a given irrational number. Then the corresponding flow $(\mathbb{Z}, \mathbb{T}^2, \pi_\theta)$ is minimal but not equicontinuous. The minimality one can check by results of Furstenberg [18]. In particular, the cascade $(\mathbb{Z}, \mathbb{T}^2, \pi_\theta)$ is not RN. As an another corollary, $Asp(\mathbb{Z}) \neq RUC(\mathbb{Z}) = C(\mathbb{Z})$ and $\mathbb{Z}^R$ is not RN-approximable.

Remarks 7.20. (i) A result from [41] states that a topological group G is precompact iff $WAP(G) = RUC(G)$, previously obtained in [2] for monothetic groups. Is it true the same assuming $Asp(G) = RUC(G)$?

(ii) Namioka and Phelps [45] proved a generalized Ryll-Nardzewski fixed-point theorem for S-flows which are weak star compact convex subsets in the dual of an Asplund space. Hence, this situation is a particular case of RN flows. It is interesting to analyze possible applications for amenability context as well as for decomposition theorems.

8. Kadec property: "when does weak imply strong ?"

Recall that a Banach space V has the *Kadec property* if the weak and norm topologies coincide on the unit (or some other) sphere of V. Let us say that a subset X of a locally convex space (l.c.s.) (V, τ) is a *Kadec subset* (*light subset* in [38]) if the weak topology coincides with the strong topology. *Light* linear subgroups $G \leq Aut(V)$ (with respect to the weak and strong *operator topologies*) can be defined Analogously. Clearly, if G is *orbitwise Kadec* on V that is, all orbits Gv are light in V, then G is necessarily light. The simplest examples are the spheres (orbits of the unitary group $Is(H)$) in Hilbert spaces H.

The following results show that linear actions frequently are "orbitwise Kadec".

Theorem 8.1. *Let a subgroup $G \leq Aut(V)$ be equicontinuous, X be a bounded, $(weak, \mu)$-fragmented G-invariant subset of an l.c.s. V with the natural uniformity μ. Then every, not necessarily closed, quasiminimal G-subspace (e.g., the orbits) Y of X is a Kadec subset.*

Proof. The equicontinuity of the subgroup $G \leq Aut(V)$ implies that the action of G on the bounded subspace $X \subset V$ is uniformly μ-equicontinuous with respect to the natural uniformity μ on V. Since X is $(weak, \mu)$-fragmented we get that in fact the G-flow X is (weak, $\mu|_X$)-equifragmented. Therefore we can apply Theorem 6.9. □

We say that an l.c.s. V is *boundedly fragmented* (write: $V \in BF$) if every bounded subset $X \subset V$ is (weak, μ)-fragmented, where μ, as above, is the natural uniformity of V.

Corollary 8.2. *Let $V \in BF$. Then every equicontinuous $G \leq Aut(V)$ is a light subgroup and every orbit Gv is a light subset in V.*

The class BF is large (see the relevant references in [38]) and includes among others: Banach spaces with PCP (point of continuity property), semireflexive l.c.s., Frechet spaces with the Radon-Nikodym Property.

Corollary 8.3. [38] *Let V be a Banach space with PCP. Then any bounded subgroup G of $Aut(V)$ (e.g., $Is(V)$) is light.*

Now, combining Corollary 8.3 and Theorem 4.7, one can obtain a transparent proof of Fact 2.8.

Ellis-Lawson's Joint Continuity Theorem: Let G be a subgroup of a compact semitopological monoid S. Suppose that (S, X) is a semitopological flow with compact X. Then the action $G \times X \to X$ is jointly continuous and G is a topological group.

Proof. (S, X) is wap by Fact 2.6. Therefore by Theorem 4.7 there exists an approximating family (h_i, α_i) of reflexive V_i-representations. It suffices to prove the theorem for the canonical wap $\Theta(V)^{opp}$-flow B^*. Let G be a subgroup of $\Theta(V)^{opp}$. Then by Corollary 8.3 the strong operator topology on G coincides with the weak topology. In particular, G is a topological group. Moreover, by Fact 2.2 the action of G on (B^*, w^*) is jointly continuous. □

Recall that X is a *Namioka space* if for every compact space Y and a separately continuous map $\gamma : Y \times X \to \mathbb{R}$ there exists a dense subset $P \subset X$ such that γ is jointly continuous at every $(y, p) \in Y \times P$. A topological space is said to be Čhech-*complete* if it can be represented as a G_δ-subset of a compact space. Every Čhech-complete (e.g., locally compact or Polish) space is a Namioka space.

Proposition 8.4. *Let G be a semitopological group, X be a semitopological G-flow and $f \in C(X)$. Suppose that $(cl_w(fG), weak)$ be a Namioka space. Then fG is light in $C(X)$.*

Proof. As in the proof of Lemma 6.2, it is easy to show that if a bounded subset of a Banach space is a Namioka space under the weak topology then it is (weak-norm)-fragmented. Therefore we can complete the proof by Theorem 8.1. □

Theorem 8.5. *Let G be a semitopological group. Then for every semitopological G-flow X and every $f \in WAP(X)$ the pointwise and norm topologies coincide on the orbit fG. In particular, $WAP_G(X) \subset RUC_G(X)$.*

Proof. Proposition 8.4 guarantees that fG is (weak, norm)-Kadec. On the other hand, the weak and pointwise topologies coincide on the weak compact set $cl_w(fG)$. □

Now we turn to the weak* version of the lightness concept. Let (V, τ) be an l.c.s. with its strong dual (V^*, τ^*). Denote by μ^* the corresponding uniformity on V^*. Let's say that a subset A of V^* is *weak* light* if weak* and strong topologies coincide on A. If G is a subgroup of $Aut(V^*)$, then the weak* (resp., strong*) topology on G is the weakest topology which makes all orbit maps $\{\tilde{\psi} : G \to V^* : \ \psi \in V^*\}$ weak* (resp., strong) continuous.

Following [36] we say that an l.c.s. V is a *Namioka-Phelps space* ($V \in NP$) if every equicontinuous subset $X \subset V^*$ is (w^*, μ^*)-fragmented. The class NP is closed under subspaces, products and l.c. sums and includes: Asplund Banach spaces, semireflexive l.c.s. and Nuclear l.c.s.

Theorem 8.6. *Suppose that V is an NP space, $G \leq Aut(V)$ is an equicontinuous subgroup, and $X \subset V^*$ is an equicontinuous G-invariant subset.*

(i) *If (X, w^*) is a quasiminimal (e.g., 1-orbit) G-subset, then X is weak* light.*
(ii) *The weak* and strong* operator topologies coincide on G.*

Proof. (i) The strong topology on the dual space (V^*, μ^*) is the topology of bounded convergence. Since G is an equicontinuous subgroup of $Aut(V)$, it is easy to show that the dual action of G on V^* is also equicontinuous. On the other hand, $X \subset V^*$ is (w^*, μ^*)-fragmented as it follows by the definition of NP spaces. We obtain in fact that G-flow X is (weak*, $\mu^*|_X$)-equifragmented. Now use once again Theorem 6.9.

(ii) Directly follows from (i) because every G-orbit in V^* is an equicontinuous subset. □

The last result is useful in the context of continuity of *dual actions* (for more information see [36] and the references there). More precisely, let V be an l.c.s. and $h : G \to Aut(V)$ be a homomorphism such that $h(G)$ is an equicontinuous subgroup of $Aut(V)$ and the action $G \times V \to V$ is jointly continuous. Then we can ask: is the dual action

$$\pi^* : G \times (V^*, \mu^*) \to (V^*, \mu^*), \ (gf)(v) = f(g^{-1}v)$$

also jointly continuous ?

Since $h(G)$ is equicontinuous, clearly (G, V^*) is equicontinuous with respect to the dual action π^*. Therefore it is equivalent to ask if the orbit maps $\tilde{f} : G \to (V^*, \mu^*)$ are continuous for all $f \in V^*$. Since $\tilde{f} : G \to (V^*, w^*)$ is continuous, it suffices to show that the orbits Gf are (weak*, strong)-Kadec subsets of V^*. This fact follows directly from Theorem 8.6 provided that $V \in NP$. Hence we obtain the following result.

Corollary 8.7. *Let $V \in NP$ (e.g., Asplund Banach space) and $\pi : G \times V \to V$ be a linear jointly continuous equicontinuous action. Then the dual action $\pi^* : G \times V^* \to V^*$ is also jointly continuous.*

Remark 8.8. Corollary 8.7 can be derived also from [36, section 6]. If V is an Asplund Banach space then we can drop the condition about equicontinuity (in fact, boundedness) as it follows by [36, Corollary 6.9].

Proposition 8.9. *Let V be an Asplund Banach space, G be a semitopological group, and $h : G \to Aut(V)$ be a bounded weakly continuous antihomomorphism. Assume that $v \in V$ and $\psi \in V^*$ are some fixed vectors. Then the corresponding matrix coefficient $m_{v,\psi} : G \to \mathbb{R}$ is left uniformly continuous. Moreover, if the vector v is norm-continuous, then $m_{v,\psi}$, in addition, is right uniformly continuous.*

Proof. The antihomomorphism h sends G into a norm bounded subgroup of $Aut(V)$. Therefore by Fact 3.5 it suffices to show that ψ is a norm G-continuous vector. Since $h : G \to Aut(V)$ is weak continuous, the orbit map $\tilde{\psi} : G \to V^*$ is weak star continuous. Since V is Asplund (and hence NP), Theorem 8.6 implies that the weak star and norm topologies coincide on the orbit $G\psi$. Then $\tilde{\psi} : G \to V^*$ is even norm continuous. □

Corollary 8.10. *For every semitopological group G*

$$WAP(G) \subset Asp(G) \subset LUC(G) \cap RUC(G)$$

holds.

Proof. The inclusion $WAP(G) \subset Asp(G)$ is a part of Proposition 7.16. Let $f \in Asp(G)$. By Theorem 7.17 the function f coincides with a matrix coefficient $m_{v,\psi}$ for a suitable strongly continuous antihomomorphism $h : G \to Is(V)_s$. Now we can apply Proposition 8.9 to $f = m_{v,\psi}$.

□

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Bar-Ilan University, Israel
E-mail address: megereli@math.biu.ac.il
URL: http://www.math.biu.ac.il/~megereli