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George F. McNulty
University of South Carolina - Columbia, mcnulty@math.sc.edu

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FRAGMENTS OF FIRST ORDER LOGIC, I: UNIVERSAL HORN LOGIC

GEORGE F. McNULTY

§0. **Introduction.** Let L be any finitary language. By restricting our attention to the universal Horn sentences of L and appealing to a semantical notion of logical consequence, we can formulate the universal Horn logic of L. The present paper provides some theorems about universal Horn logic that serve to distinguish it from the full first order predicate logic. Universal Horn equivalence between structures is characterized in two ways, one resembling Kochen's ultralimit theorem. A sharp version of Beth's definability theorem is established for universal Horn logic by means of a reduced product construction. The notion of a consistency property is relativized to universal Horn logic and the corresponding model existence theorem is proven. Using the model existence theorem another proof of the definability result is presented. The relativized consistency properties also suggest a syntactical notion of proof that lies entirely within the universal Horn logic. Finally, a decision problem in universal Horn logic is discussed. It is shown that the set of universal Horn sentences preserved under the formation of homomorphic images (or direct factors) is not recursive, provided the language has at least two unary function symbols or at least one function symbol of rank more than one.

This paper begins with a discussion of how algebraic relations between structures can be used to obtain fragments of a given logic. Only two such fragments seem to be under current investigation: equational logic and universal Horn logic. Other fragments which seem interesting are pointed out. Next there is a survey of theorems concerning universal Horn logic. Some results here are only mentioned in passing, while others find application later in the paper. A new proof for a fundamental theorem of A. I. Mal'cev is provided. This theorem is used several times to obtain some of the new results mentioned above. Open problems are collected in the last section.

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§1. **The construction of logics based on preservation theorems.** For the most part our notation is standard and follows that in the book [5] of Chang and Keisler fairly closely. The reader is directed to this book for all unexplained notions and terminology.

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The ideas central to this section are most easily illustrated by an example. Let L be an ordinary first order predicate logic with equality. Suppose R is a relation between L-structures (L-structures are called models for L in [5].) An L-sentence φ is said to be ***preserved under R*** provided $\mathfrak{B} \models \varphi$ whenever $\mathfrak{A} \models \varphi$ and $\mathfrak{A} \, R \, \mathfrak{B}$. A preservation theorem for L has the following form where Δ is a set of L-sentences and R is a relation between L-structures: An L-sentence φ is preserved under R iff φ is logically equivalent to some sentence in Δ.

If R is a relation with an algebraic motivation and Δ admits a relatively simple syntactical description, the preservation theorem is especially interesting. Of course, the whole notion of preservation theorems extends to relations among structures which are not binary, for example the relation that $\mathfrak{B}$ is isomorphic to the direct product of a system $\langle \mathfrak{A}_i : i \in I \rangle$ of structures.

Historically the first preservation theorem seems to be the following result:

THEOREM. *Let L be a language without relation symbols. An L-sentence φ is preserved under the formation of substructures, homomorphic images, and direct products iff φ is logically equivalent to the universal closure of a conjunction of equations between terms.*

This theorem can be obtained from a result Birkhoff proved in 1935 [3] by a simple application of the compactness theorem. We use this preservation theorem to illustrate how such theorems in general specify a fragment.

We take as the sentences of L_{eq} the universal closures of equations between L-terms, which are sometimes called identities. The semantical notions of truth and logical consequence are inherited from L. L_{eq} is an equational logic. Now observe that L is very much stronger than L_{eq} in the sense that the selection of mathematical notions expressible by sets of L-sentences is much richer than the selection expressible by sets of L_{eq}-sentences. Nevertheless concepts like group, ring, lattice, Boolean algebra, rational vector space, and even the notion of set can be naturally formulated in the appropriate equational logics. One of the major tasks in the model theory of L is to devise means for constructing models of sets of L-sentences on the basis of structures already on hand. In L_{eq} this task is greatly simplified by the inherent preservation properties. This analysis seems to indicate that L_{eq} is semantically poorer than L but algebraically richer. L_{eq} has a simpler, less diverse syntax than L.

The first problem arising in equational logic is to provide a sound and adequate notion of proof that involves only equations. This was done by Birkhoff [3] in 1935. The syntactical notions of proof for equational logic are so sparse that it is difficult to find any deep theorems concerning equational proof theory. Some have been found. In recent years equational logic has experienced vigorous growth. The reader interested in pursuing this topic should consult Tarski's paper [33] which is a survey of results in equational logic prior to 1968.

There are two essential parameters in the illustration provided by equational logic. They are the initial logic L, in this case ordinary first order logic, and the preservation theorem invoked. Our primary concern in the paper leaves L unchanged but we use the preservation theorem for the formation of substructures and arbitrary nontrivial (reduced) products. Before passing to this case we remark that there are many other interesting possibilities. Most of the logics investigated in recent years result from enriching the power of expression of first order logic in various ways, $L_{\omega_1\omega}$ and $L(Q)$ are just two examples. In passing to richer languages the task of constructing models becomes more difficult. By considering fragments of these richer languages specified by preservation theorems it seems possible to regain some of the lost devices for constructing models and yet retain some of the richness of expression. For example, the fragment of $L_{\omega_1\omega}$ which is preserved under ultraproducts is richer than first order logic and yet possesses a compactness theorem. Another intriguing possibility is to develop the theory of "regular" logics. These are the logics specified with respect to first order logic by preservation theorems for relations between structures regular in the sense of Lindström [17], cf. Makkai [19]. Galvin's paper [10] contains many interesting theorems concerning Horn logic, that fragment of first order logic preserved under reduced products. Galvin's results indicate this fragment is very close to first order logic.

§2. A brief survey of unversal Horn logic. Throughout the remainder of this paper, except for the places indicated, L will be a fixed, though arbitrary, first order language. A formula φ of L is a *basic Horn formula* provided φ is a disjunction of formulas at most one of which is atomic and all the remaining are negations of atomic formulas. A basic Horn formula is *strict* if exactly one of its disjuncts is atomic. φ is a (*strict*) *Horn sentence* just in case φ is an L-sentence in prenex normal form the matrix of which is a conjunction of (strict) basic Horn formulas. Strict universal Horn sentences are sometimes called *quasi-identities.*

Let K be a class of L-structures. We use the following terminology:

SK — the class of all isomorphic images of substructures of members of K,

PK — the class of all isomorphic images of direct products of arbitrary systems of members of K,

P^*K — the class of all isomorphic images of direct products of nonempty systems of members of K,

P_RK — the class of all isomorphic images of reduced products of arbitrary systems of members of K,

P_R^*K — the class of all isomorphic images of proper reduced products of nonempty systems of members of K,

P_uK — the class of all isomorphic images of ultraproducts of arbitrary systems of members of K,

P_sK — the class of all isomorphic images of subdirect products of nonempty systems of members of K,

LK — the class of all isomorphic images of direct limits of directed systems of members of K.

The class K of L-structures is called a *universal Horn class* iff K is the class of all models of some set of universal Horn sentences. Strict universal Horn classes are sometimes called quasi-varieties. The first goal of this section is to characterize the smallest universal Horn class containing a given class. The following theorems are essentially gathered from the literature.

THEOREM 1 (J. C. C. MCKINSEY [27]). *Let φ be a universal L-sentence. φ is preserved under proper (nonempty) direct products iff φ is logically equivalent to a universal Horn sentence.*

This is probably the second preservation theorem to be found. We remark that if the direct product of the empty system of L-structures were considered in the above theorem then φ would be equivalent with a quasi-identity. This modification can be made in most of the theorems below.

THEOREM 2 (CHANG AND MOREL [6]). *If φ is a Horn sentence, then φ is preserved under the formation of proper reduced products.*

The fundamental facts about reduced products can be found in Frayne, Morel, and Scott [7] or Chang and Keisler [5]. We tacitly use some of these facts below.

Theorem 2 together with the Łos-Tarski preservation theorem for the formation of substructures allows us to write down the following well-known version of Theorem 1.

THEOREM 1′. *An L-sentence φ is preserved under the formation of substructures and proper reduced products iff φ is logically equivalent with a universal Horn sentence.*

A. I. Mal′cev was apparently the first to characterize the least universal Horn class containing a given class. See [23, p. 215].

THEOREM 3 (MAL′CEV). *Let K be a class of L-structures. SP_RK is the class of all models of the set of quasi-identities true in K.*

The proof given in Mal′cev [23] is an argument depending on Corollary 2.15 from Frayne, Morel, and Scott [7]. The same argument yields

THEOREM 3′. *Let K be a class of L-structures. SP_R^*K is the least universal Horn class containing K.*

At the beginning of the next section we state an easy corollary of Theorem 3′ and show how it may be proved directly. The proof given can easily be elaborated into a proof of Theorem 3′ different from Mal′cev's. Other ways to characterize the least universal Horn class containing a given class have also been found.

THEOREM 4. *Let K be a class of L-structures,*

(a) (*Grätzer and Lakser* [13]) *SP^*P_uK is the least universal Horn class containing K.*

(b) (*Fujiwara* [9]) *LSP^*K is the least universal Horn class containing K.*

(c) (*Kashiwagi* [15]) *$SP_uP^*K = SP_sP_uK = LSP^*K$.*

Free algebras play an important role in equational logic. Their impact in universal Horn logic promises to be considerable. Mal′cev in [25] showed that every universal Horn class is equipped with free structures. Indeed, his result is somewhat more general. Tabata [32] has achieved an interesting extension of Mal′cev's result.

A. Selman in [29] established a completeness theorem for quasi-identity logic. Independently D. Kelly has shown me a completeness theorem for universal Horn logic.

In [2] Baldwin and Lachlan have explored categoricity in power for universal Horn logic. Working independently Abakumov, Palyutin, Taitslin, and Shishmarev in [1] arrived at similar results. Steven Givant in [11] describes (up to definitional equivalence) those universal Horn classes which can be categorical in infinite powers.

The well-known result of Boone and Novikov on the existence of a finitely presented group with a recursively unsolvable word problem can be construed quite naturally as a theorem about the universal Horn theory of groups. [4] is a good reference of related results.

Various classes of structures arising in mathematics turn out to be universal Horn classes. For example see Herrmann and Poguntke [14] and Makkai and McNulty [21]. Chapter V of Mal'cev's book [23] is devoted to our exposition on quasi-varieties.

§3. Equivalence of structures for universal Horn logic. L_{uH} is the sublogic of L the sentences of which are exactly the universal Horn sentences of L. L_{uH} is specified by the preservation theorem for substructures and proper reduced direct products. Two L-structures $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *universal Horn equivalent*, in symbols $\mathfrak{A} \equiv_{uH} \mathfrak{B}$, iff $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same universal Horn sentences. If K is a class of L-structures $\mathrm{Th}_{uH}K$ denotes the universal Horn theory of K. Hence $\mathrm{Th}_{uH}K = \{\varphi : \varphi$ is a universal Horn sentence and $K \models \varphi\}$. In this section we take up the problem of characterizing universal Horn equivalence algebraically. This problem corresponds to that for L of finding an algebraic characterization of elementary equivalence. The latter problem has been solved in three ways: the well-known Ehrenfeucht-Fraisse games, the ultralimit theorem of Kochen, and the Keisler-Shelah ultrapower theorem. We establish an analogue of Kochen's theorem and provide a very simple counterexample to the most obvious analogue to the ultrapower theorem. We do not know an Ehrenfeucht–Fraisse type game for universal Horn equivalence.

The first step is to draw a Frayne style corollary to Theorem 3′.

COROLLARY 5. *If $\mathfrak{A}$ and $\mathfrak{B}$ are L-structures such that every universal Horn sentence true in $\mathfrak{B}$ is also true in $\mathfrak{A}$, then $\mathfrak{A}$ is embeddable in a proper reduced power of $\mathfrak{B}$.*

REMARK. Since this corollary is central to the section we provide a direct proof of it which specifies more closely the reduced power and the embedding involved.

PROOF OF COROLLARY 5. To simplify notation we assume that $A \times B \cap L$ is empty. $L(A \times B)$ is the language obtained from L by adjoining each element of $A \times B$ as a new constant symbol. If $\mathscr{C}$ is an L-structure and $f: A \times B \rightarrow C$, then $(\mathscr{C}, f)$ is the $L(A \times B)$-structure which uses each new constant $(a, b) \in A \times B$ as a name for $f(a, b)$. ρ_A is the projection of $A \times B$ onto A.

We adopt the following convention: if θ is any $L(A \times B)$-formula, then θ^* is the L-formula obtained from θ by replacing the constants from $A \times B$ by variables so that the same variable replaces both (a, b) and (a', b') iff $a = a'$, and moreover no variable used to replace a constant occurs in θ. (This can be made precise by well ordering $A \times B$.)

Let

$$I = \{\varphi : \varphi \text{ is a conjunction of atomic sentences of } L(A \times B) \text{ and } (\mathfrak{A}, \rho_A) \models \varphi\}.$$

I is the index set of our reduced power. Now observe that $\varphi \in I$ implies $\mathfrak{A} \models \exists \bar{x}\varphi^*$, where every variable occurring in φ^* occurs in the sequence of variables $\bar{x}$. $\exists \bar{x}\varphi^*$ is logically equivalent to the negation of a universal Horn sentence. Consequently $\mathfrak{B} \models \exists \bar{x}\varphi^*$. This observation permits us to pick, for each $\varphi \in I$, a function $f_\varphi : A \times B \to B$ and a function $g_\varphi : A \to B$ such that

(i) $g_\varphi \circ \rho_A = f_\varphi$ for each $\varphi \in I$,

(ii) $(\mathfrak{B}, f_\varphi) \models \varphi$ for each $\varphi \in I$, and

(iii) every function g from a finite subset of A into B can be extended to $\bar{g} : A \to B$ with $\bar{g} \circ \rho_A = f_\varphi$ for some $\varphi \in I$.

This last condition is assured since I contains sufficiently many validities. For example, if $\{(a_0, b_0), \cdots, (a_{n-1}, b_{n-1})\}$ is the finite function to be extended, then we can take φ to be

$$(a_0, b_0) = (a_0, b_0) \wedge \cdots \wedge (a_{n-1}, b_{n-1}) = (a_{n-1}, b_{n-1}),$$

and require that $f_\varphi(a_i, b_i) = b_i$ for $0 \leq i < n$, while choosing other values of f_φ so that (i) is fulfilled.

Now let $J_\varphi = \{\sigma : (\mathfrak{B}, f_\sigma) \models \varphi \text{ and } \sigma \in I\}$ for each $\varphi \in I$. Evidently $F = \{B : J_\varphi \subseteq B \subseteq I \text{ for some } \varphi \in I\}$ is a filter. Finally, we define $h : A \to B^I/F$ such that

$$h(a) = (g_\sigma(a) : \sigma \in I)/F$$

for each $a \in A$. It remains to show that h embeds $\mathfrak{A}$ into $\mathfrak{B}^I/F$.

Let ψ be an atomic L-formula. For the sake of convenience we assume that only one variable occurs in ψ. Let $a \in A$, $b \in B$, and observe that the following statements are equivalent on the basis of the definitions involved.

(1) $\mathfrak{B}^I/F \models \psi(h(a))$,

(2) $\mathfrak{B}^I/F \models \psi((g_\sigma(a) : \sigma \in I)/F)$,

(3) $\{\sigma : \mathfrak{B} \models \psi(g_\sigma(a))\} \in F$,

(4) $\{\sigma : (\mathfrak{B}, f_\sigma) \models \psi((a, b))\} \in F$.

Now (4) implies that for some $\varphi \in I$

$$J_\varphi \subseteq \{\sigma : (\mathfrak{B}, f_\sigma) \models \psi((a, b))\}.$$

Fix such a $\varphi \in I$. Consequently for all $\sigma \in I$ if $(\mathfrak{B}, f_\sigma) \models \varphi$, then $(\mathfrak{B}, f_\sigma) \models \psi((a, b))$. But this means that $(\mathfrak{B}, f_\sigma) \models \varphi \to \psi((a, b))$ for every $\sigma \in I$. According to (iii) we obtain

$$\mathfrak{B} \models \forall \bar{x}[\varphi \to \psi((a, b))]^*$$

where every variable occurring in $\varphi \to \psi((a, b))$ occurs in $\bar{x}$. Now $\forall \bar{x}[\varphi \to \psi((a, b))]^*$ is a universal Horn L-sentence. Therefore

$$\mathfrak{A} \models \forall \bar{x}[\varphi \to \psi((a, b))]^*.$$

Hence

$$(\mathfrak{A}, \rho_A) \models \varphi \rightarrow ((a, b)).$$

Since $\varphi \in I$, this means that $(\mathfrak{A}, \rho_A) \models \psi((a, b))$. Hence (4) implies

(5) $\mathfrak{A} \models \psi(a)$.

But (5) clearly implies $(\mathfrak{A}, \rho_A) \models \psi((a, b))$ which puts $\psi((a, b)) \in I$. So $J_{\psi((a,b))} \in F$. Thus (5) implies (4). Finally we conclude that (1) and (5) are equivalent and hence h is an embedding. This completes the proof of the corollary.

It is not very difficult to elaborate the proof above to the point where a similar proof of Theorem 3′ emerges. Care is needed to formulate the appropriate extension of (iii).

DEFINITION. $\mathfrak{A}$ is a *reduced limit* of $\mathfrak{B}$ provided $\mathfrak{A}$ is the direct limit of some system of reduced powers of $\mathfrak{B}$ directed by a system of embeddings.

This definition is like the definition of ultralimits, the chief difference being that here the canonical embeddings play no special role.

THEOREM 6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be any two L-structures. $\mathfrak{A} \equiv_{uH} \mathfrak{B}$ iff some reduced limit of $\mathfrak{A}$ is isomorphic with some reduced limit of $\mathfrak{B}$.*

PROOF. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ have isomorphic reduced limits, $\mathfrak{A}^*$ and $\mathfrak{B}^*$ respectively. Let φ be a universal Horn sentence true in $\mathfrak{A}$. Then φ is true in every reduced power of $\mathfrak{A}$ and, since universal sentences are preserved under direct limits, φ must be true in $\mathfrak{A}^*$ and hence in $\mathfrak{B}^*$. Since $\mathfrak{B}$ can be embedded in $\mathfrak{B}^*$ we know that $\mathfrak{B} \models \varphi$. By a symmetric argument we conclude that $\mathfrak{A} \equiv_{uH} \mathfrak{B}$.

The converse can be established by a straightforward "alternating chain" construction with the help of Corollary 5.

It is tempting to think that two L-structures should be universal Horn equivalent iff they have isomorphic reduced powers. The next simple example shows that this is false.

EXAMPLE. Let $\mathfrak{A} = (\omega, <)$ and $\mathfrak{B} = (\omega + 1, <)$. It is easy to see that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same universal sentences (and so the same universal Horn sentences). However

$$\mathfrak{A} \models \forall x \exists y [x < y]$$

whereas

$$\mathfrak{B} \models \exists x \forall y [\neg(x < y)].$$

Both of these sentences are Horn sentences and each is logically equivalent with the negation of the other. So $\forall x \exists y [x < y]$ is true in every reduced power of $\mathfrak{A}$ while $\exists x \forall y [\neg(x < y)]$ is true in every reduced power of $\mathfrak{B}$. So $\mathfrak{A}$ and $\mathfrak{B}$ cannot have isomorphic reduced powers.

The next theorem characterizes universal Horn equivalence in terms of reduced powers and universal equivalence, which is the best characterization by quantifier complexity to hope for, according to the example.

THEOREM 7. *$\mathfrak{A} \equiv_{uH} \mathfrak{B}$ iff some reduced power of $\mathfrak{A}$ is universally equivalent with some reduced power of $\mathfrak{B}$.*

PROOF. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ have universally equivalent reduced powers. Since $\mathfrak{A}$ is universal Horn equivalent with each of its reduced powers and $\mathfrak{B}$ with each of its reduced powers, it follows that $\mathfrak{A}$ and $\mathfrak{B}$ are universal Horn equivalent.

To establish the converse suppose $\mathrm{Th}_{uH}\mathfrak{A} = \mathrm{Th}_{uH}\mathfrak{B}$. We define $G = \{\Gamma : \Gamma \cup \mathrm{Th}_{uH}\mathfrak{A}$ is consistent and Γ is a set of existential sentences$\}$. In view of the compactness theorem G is a set of finite character. According to the Teichmüller–Tukey lemma, let Δ be a maximal member of G. Hence, $\Delta \cup \mathrm{Th}_{uH}\mathfrak{A} = \Delta \cup \mathrm{Th}_{uH}\mathfrak{B}$ is consistent. Let $\mathscr{C}$ be a model of $\Delta \cup \mathrm{Th}_{uH}\mathfrak{A}$. According to Corollary 5, $\mathscr{C}$ can be embedded into some reduced power of $\mathfrak{A}$ and also into some reduced power of $\mathfrak{B}$. By the maximal nature of Δ these two reduced powers must be universally equivalent. So the theorem is proved.

§4. Definability in universal Horn logic. In this section we will prove a version of Beth's definability theorem. We exploit the well-known connections between Beth's theorem, Craig's interpolation theorem, and Robinson's joint consistency theorem. Throughout this section L_0 and L_1 denote two fixed languages with the *same* function symbols.

THEOREM 8 (JOINT CONSISTENCY). *Let Γ_0 and Γ_1 be any two sets of universal Horn sentences in the languages L_0 and L_1 respectively. $\Gamma_0 \cup \Gamma_1$ is inconsistent iff there is a universal Horn sentence θ in $L_0 \cap L_1$ with $\Gamma_0 \models \theta$ and $\{\theta\} \cup \Gamma_1$ inconsistent.*

PROOF. Suppose that $\Gamma_0 \cup \Gamma_1$ is inconsistent. Let K_i be the class of reducts to $L_0 \cap L_1$ of all models of Γ_i, for $i = 0, 1$. Clearly $K_0 \cap K_1$ is empty. Both K_0 and K_1 are closed under the formation of reduced products. K_0 and K_1 are also closed under the formation of substructures since $L_0 \cap L_1$ includes all of the functions symbols in $L_0 \cup L_1$. Consequently $SP_R^*K_0 = K_0$ and $SP_R^*K_1 = K_1$. By Theorem 3′, K_0 and K_1 are universal Horn classes of $L_0 \cap L_1$-structures. Let Δ_i be the set of universal Horn sentences true in K_i, for $i = 0, 1$. So $\Delta_0 \cup \Delta_1$ is inconsistent. The conclusion follows immediately from compactness.

THEOREM 9 (INTERPOLATION). *Let φ and ψ be universal Horn sentences such that $\varphi \models \neg\psi$. There is a universal Horn sentence θ with $\varphi \models \theta$ and $\theta \models \neg\psi$ such that each relation symbol occurring in θ occurs in both φ and ψ.*

Let $\mathscr{F}$ be a set of relation symbols not occurring in the language L. Let Σ be a set of universal Horn sentences of $L \cup \mathscr{F}$. Recall that Σ is said to be an *implicit definition* of $\mathscr{F}$ provided the reduct to L of each model of Σ has exactly one expansion to a model of Σ. Moreover, the formula $\varphi(x_0, \cdots, x_{n-1})$ is an *explicit definition* of the n-ary relation symbol P with respect to Σ if

$$\Sigma \models \forall x_0, \cdots, x_{n-1}[P(x_0, \cdots, x_{n-1}) \Leftrightarrow \varphi(x_0, \cdots, x_{n-1})].$$

THEOREM 10 (DEFINABILITY). *Let $\mathscr{F}$ be any set of relation symbols not occurring in the language L and let Σ be any set of universal Horn sentences for $L \cup \mathscr{F}$. Then Σ implicitly defines $\mathscr{F}$ iff for each relation symbol $P \in \mathscr{F}$ there is a universal Horn L-formula φ such that φ is an explicit definition of P with respect to Σ.*

The details of how Theorems 9 and 10 may be derived from Theorem 8 can be found in Chang and Keisler [5, pp. 87–88]. In this proof of Theorem 10 it is important to notice that Theorem 9 be used in place of Craig's interpolation theorem.

It is should be pointed out that the restrictions concerning function symbols in Theorem 9 and their consequent lack in the formulation of the definability theorem are essential. In fact, Harvey Friedman in [8] has recently shown that the quantifier complexity of explicit definitions for function symbols cannot be bounded even when the implicit definition is a set of four identities. He suggested the following example to me which serves to illustrate that Theorem 10 cannot be expanded to include function symbols. Friedman's constructions in [8] are much more subtle.

EXAMPLE. Consider a language L with a binary function symbol $+$, a unary function symbol f, and two constants 0 and a. Let Σ be the following set of equations: $\{\forall x[x+0 \approx 0+x],\ \forall x[x+0 \approx x],\ f(0) \approx a\}$. Evidently $\{0, a\}$ is implicity defined by Σ. Suppose $\theta(x)$ is a formula of $L \sim \{0, a\}$ such that $\Sigma \models \forall x[\theta(x) \Leftrightarrow x \approx a]$, i.e. $\theta(x)$ explicitly defines a with respect to Σ. Let $A = \{\bar{0}, \bar{a}\}$, $\bar{f}(\bar{a}) = \bar{a}$, $\bar{f}(\bar{0}) = \bar{a}$, $\bar{0} \mathbin{\bar{+}} \bar{a} = \bar{a}$, $\bar{a} + \bar{0} = \bar{a}$, $\bar{0} + \bar{0} = \bar{0}$, and $\bar{a} + \bar{a} = \bar{a}$. Then $\mathfrak{A} = \langle A, \bar{+}, \bar{f}, \bar{0}, \bar{a}\rangle$ is a model of Σ. Let $B = \{\bar{0}, \bar{a}, 0'\}$, $f'(\bar{0}) = \bar{a}$, $f'(\bar{a}) = \bar{a}$, $f'(0') = \bar{0}$, $\bar{0} +' 0' = \bar{0}$, $0' +' \bar{0} = \bar{0}$, $0' +' 0' = 0'$, and $c +' d = c \mathbin{\bar{+}} d$ for all $c, d \in A$. Then $\mathfrak{B} = \langle B, +', f', 0', \bar{0}\rangle$ is a model of Σ. Now $\langle A, \bar{+}, \bar{f}, \bar{0}\rangle$ is a subalgebra of $\langle B, +', f', \bar{0}\rangle$. But observe that $\langle B, +', f', \bar{0}\rangle \models \theta(\bar{0})$. If $\theta(\bar{0})$ where a universal sentence, then $\langle A, \bar{+}, \bar{f}, \bar{0}\rangle \models \theta(\bar{0})$. But $\mathfrak{A} = \theta(c)$ iff $c = \bar{a}$. Since $\bar{a} \neq \bar{0}$ any explicit definition of a cannot be universal, much less universal Horn.

§5. Consistency properties for universal Horn logic. Consistency properties find their origins in Henkin's proof of the completeness theorem for first order logic. They were first explicitly formulated for first order logic in Smullyan [30] (see also [31]). Their most fruitful applications so far have been to infinitary logic, see Makkai [19] and Keisler [16]. Here we propose a straightforward modification of the notion appropriate to the fragment in which we are interested: universal Horn logic. In order to demonstrate the usefulness of this notion we establish a model existence theorem for universal Horn consistency properties and employ it to obtain the joint consistency result of the previous section; we also observe that logical axioms and rules of inference can be obtained from the notion of consistency property and we give some indication of how to prove that these axioms and rules of inference completely characterize logical consequence. This provides another proof of the result of Selman [29] and David Kelly.

Let L be a language and C be a nonempty set of new constant symbols. Let M be the language obtained from L by adjoining C.

DEFINITION. S is a *universal Horn consistency property* if and only if S is a collection of sets of universal Horn M-sentences such that S is of finite character and for all $s \in S$,

(1) $\varphi \notin s$ or $\neg\varphi \notin s$ for every atomic M-sentence φ.

(2) If θ is a basic Horn M-sentence, $\neg\varphi$ is a disjunct, and $\theta, \varphi \in s$, then $s \cup \{\theta'\} \in S$ where θ' is obtained from θ by deleting the disjunct $\neg\varphi$.

(3) If $\varphi \wedge \psi \in s$, then $s \cup \{\varphi\} \in S$ and $s \cup \{\psi\} \in S$.

(4) If $\forall x\varphi(x) \in s$, then $s \cup \{\varphi(\tau)\} \in S$ for every M-term τ in which no variable occurs.

(5) For every M-term τ in which no variables occur, $s \cup \{\tau \approx \tau\} \in S$.

(6) If σ, τ, and ρ are terms in which no variables occur and $\sigma \approx \tau$, $\sigma \approx \rho \in s$, then $s \cup \{\rho \approx \tau\} \in S$.

(7) If σ and τ are terms in which no variables occur and φ is an atomic M-sentence such that $\sigma \approx \tau$, $\varphi \in s$, then $s \cup \{\varphi'\} \in S$ where φ' is obtained from φ by replacing any one occurrence of τ by σ.

Our definition of universal Horn consistency properties differs from the more usual treatment of ordinary consistency properties (see Keisler [16]) in condition (2) and in the treatment of equations. Condition (2) reflects the fact that disjunctions are permitted only in basic Horn formulas. In usual formulations it is enough to state condition (6) only for the new constants from C and to state condition (7) for the case when $\sigma \in C$. In connection with condition (7) this is still possible. On the other hand condition (6) turns out to be inadequate, in this weaker form, for the proof of the model existence theorem. This fact is more clearly revealed after the second proof of the joint consistency theorem.

THEOREM 11 (MODEL EXISTENCE). *If S is a universal Horn consistency property and $s \in S$, then s has a model.*

PROOF. Let $E = \{s' : s \cup s' \in S\}$. E is a set of finite character since S is. Let s_ω be a maximal member of E, which must exist according to the Teichmüller-Tukey lemma. Observe that $s \subseteq s_\omega$. Therefore it is enough to construct a model of s_ω which is what we will do. Let Te be the set of M-terms in which no variables occur. For $\sigma, \tau \in Te$ define $\sigma \sim \tau$ iff $\sigma \approx \tau \in s_\omega$. Conditions (5) and (6) and the maximality of s_ω insure that $\sim$ is an equivalence relation on Te. Let $A = Te/\sim$, the set of equivalence classes. If $\sigma \in Te$ let $\bar\sigma$ denote the $\sim$ equivalence class of σ. We define the structure $\mathfrak{A}$ with universe A in the following way. Let f be a function symbol, say of rank r, and let $\bar\sigma_0, \cdots, \bar\sigma_r \in A$. $f^{\mathfrak{A}}(\bar\sigma_0, \cdots, \bar\sigma_{r-1}) = \bar\sigma_r$ iff $f\sigma_0 \cdots \sigma_{r-1} \approx \sigma_r \in s_\omega$. To show that $f^{\mathfrak{A}}$ is a well-defined function condition (7) is essential. By a simple argument using induction over terms it follows that $\sigma^{\mathfrak{A}} = \bar\sigma$ for every $\sigma \in Te$. Hence $\mathfrak{A} \models \sigma \approx \tau$ iff $\sigma \approx \tau \in s_\omega$ for all $\sigma, \tau \in Te$. Now let R be an n-ary relation symbol and let $\sigma_0, \cdots, \sigma_{n-1} \in Te$. Define $R^{\mathfrak{A}}(\bar\sigma_0, \cdots, \bar\sigma_{n-1})$ iff $R(\sigma_0, \cdots, \sigma_{n-1}) \in s_\omega$. $R^{\mathfrak{A}}$ is well defined according to condition (7) and the maximality of s_ω. The resulting structure $\mathfrak{A}$ is our desired model. So if φ is an atomic M-sentence, then $\mathfrak{A} \models \varphi$ iff $\varphi \in s_\omega$.

Claim. $\mathfrak{A} \models s_\omega$.

PROOF. Proceed by induction on $\theta \in s_\omega$.

Initial step. θ is a basic Horn sentence. If θ is atomic we have already essentially observed that $\mathfrak{A} = \theta$. If θ is $\neg\varphi$ and φ is atomic, then by condition (1) $\varphi \notin s_\omega$ and so $\mathfrak{A} \not\models \varphi$. Now suppose φ is atomic and $\neg\varphi$ is a disjunct of θ. If $\mathfrak{A} \models \neg\varphi$, then $\mathfrak{A} \models \theta$. Otherwise by condition (2) and the maximality of s_ω, $\theta' \in s_\omega$, where θ' is obtained from θ by omitting $\neg\varphi$ as a disjunct. In this way all disjuncts of θ of the form $\neg\varphi$ where $\mathfrak{A} \models \varphi$ can be deleted. By condition (1) some disjuncts remain and $\mathfrak{A}$ is a model of at least one of them. So $\mathfrak{A} \models \theta$.

Inductive step. This is a straightforward use of conditions (3) and (4) and the maximality of s_ω.

With the claim the proof is complete.

THEOREM 9′ (JOINT CONSISTENCY). Let φ and ψ be universal Horn sentences such that $\{\theta_1, \theta_2\}$ is consistent whenever $\varphi \models \theta_1$, $\psi \models \theta_2$, and θ_1 and θ_2 are universal Horn sentences such that every relation symbol occurring in either θ_1 or θ_2 occurs in both φ and ψ. Then $\{\varphi, \psi\}$ is consistent.

PROOF. We display a consistency property S of which $\{\varphi, \psi\}$ is clearly a member. Let $s \in S$ iff the following three conditions hold:

(i) s is a set of universal Horn M-sentences.

(ii) $s = s_\varphi \cup s_\psi$ where s_φ is a set of sentences involving only those relation symbols occurring in φ and s_ψ is a set of sentences involving only those relation symbols occurring in ψ.

(iii) $\{\theta_1, \theta_2\}$ is consistent whenever $s_\varphi \models \theta_1$ and $s_\psi \models \theta_2$ and θ_1 and θ_2 are universal Horn sentences involving only those relation symbols occurring in both φ and ψ.

S is of finite character according to the compactness theorem. There is no difficulty in establishing conditions (1)–(7). Since $\{\varphi, \psi\} \in S'$ it follows from the model existence theorem that $\{\varphi, \psi\}$ is consistent.

This proof of the joint consistency theorem differs from the proof of the Craig-Lopez-Escobar interpolation theorem for $L_{\omega_1,\omega}$ given in Keisler [16] (see also Makkai [20] and Smullyan [30]), but only to the extent that a universal Horn consistency property was used. As remarked earlier, the statement analogous to Theorem 9′ concerning functional symbols is false. It is now apparent why the variant on the notion of universal Horn consistency property described immediately following the definition of universal Horn consistency property is inadequate. If a model existence theorem were true for that variant, then the function symbol analog to Theorem 9′ could be proved in the same manner. Consequently there is no model existence theorem appropriate for the variant. Indeed, one could construct a collection of sets of universal Horn sentences on the basis of the example following Theorem 10 and the proof just given for Theorem 9′, which would fulfill the variant notion of universal Horn consistency property and yet every nonempty set in the collection would have no model.

The notion of universal Horn consistency property suggests a syntactical characterization of logical consequence which lies entirely within the domain of universal Horn logic. A. Selman in [29] has published such a syntactical notion and ours will be similar to his. In order to simplify the presentation we are going to suppress all the quantifiers of L_{uH}. This is actually harmless since they are all universal and all our sentences are in prenex normal form anyway. Our syntactical notion of proof will be specified by a set of axioms and rules of inference in the usual way.

Axioms for universal Horn logic.

(1) $x \approx x$.

(2) $\neg x \approx y \vee \neg x \approx z \vee z \approx y$.

(3) $\neg x \approx y \vee \neg \varphi \vee \varphi'$ for every atomic formula φ and φ' is obtained from φ by replacing any one occurrence of y in φ by z.

(4) $\varphi \vee \neg \varphi$ for every atomic formula φ.

REMARK. (3) and (4) above are axiom schemata.

Rules of inference for universal Horn logic.

(1) ($\wedge$-elimination) From $\varphi \wedge \psi$ deduce both φ and ψ.

(2) ($\wedge$-introduction) From φ and ψ deduce $\varphi \wedge \psi$.

(3) (Substitution) From φ deduce φ' where φ' is obtained from φ by substituting any term for every occurrence of any variable in φ.

(4) ($\vee$-simplification) Let θ and ψ be any basic Horn formulas such that if φ is an atomic disjunct of ψ, then $\neg\varphi$ is a disjunct of θ. Let θ' be any basic Horn formula such that all the negated atomic disjuncts of ψ are disjuncts of θ' and every disjunct of θ that is not the negation of an atomic disjunct of ψ is a disjunct of θ'. From θ and ψ deduce θ'.

We use $\Sigma \vdash \varphi$ to denote that there is a proof of (the matrix) of φ from (the set of matrices of) Σ on the basis of the axioms and rules of inference just given. In order to prove the completeness theorem for our notion of proof it is convenient to prove a version of the deduction theorem. Recall that every universal Horn formula is a (universal closure of a) conjunction of basic Horn formulas.

THEOREM 12 (DEDUCTION THEOREM). *If $\Sigma \cup \{\varphi\} \vdash \theta$ and φ is an atomic Horn sentence, then $\Sigma \vdash \theta^*$ where θ^* is obtained from θ by inserting $\neg\varphi$ as a disjunct in each conjunct of θ.*

The proof of this theorem is a straightforward induction on proofs. The proof does not depend on the equality axioms (1)–(3) nor on the rule of inference (3) (substitution). All the other rules of inference are used as is the remaining axiom schema (4).

THEOREM 13 (COMPLETENESS). *If $\Sigma \models \varphi$, then $\Sigma \vdash \varphi$.*

PROOF (SKETCH). We suppose that $\Sigma \nvdash \varphi$ and construct a model for Σ in which φ is false. To the language L adjoin a countable set $\{d_0, d_1, d_2, \cdots\}$ of distinct new constant symbols. Suppose that $x_0, x_1, \cdots$ is a one-to-one listing of all the variables. For every L-formula ψ let ψ^* be the formula obtained by substituting d_i for each occurrence in ψ of x_i for every natural number i. By checking the rules of inference and the axioms we conclude that $\Sigma \nvdash \varphi^*$. In view of the rule of inference (2) we assume without loss of generality that φ is a basic Horn formula. Because of the deduction theorem we can further conclude that $\Sigma \cup \{\psi_0^*, \cdots, \psi_{n-1}^*\} \nvdash \theta^*$ where $\neg\psi_0, \cdots, \neg\psi_{n-1}, \theta$ is a list of all the disjuncts of φ. Note that θ^* is atomic or negated atomic. So it is sufficient to construct a model of $\Sigma \cup \{\psi_0^*, \cdots, \psi_{n-1}^*\}$ in which θ^* fails. Let

$$S = \{s : \Sigma \cup \{\psi_0^*, \cdots, \psi_{n-1}^*\} \cup s \nvdash \theta^*\}.$$

It is easy to check that S is a consistency property. S is of finite character since our proofs are finite. Conditions (2)–(7) are almost direct reflections of our axioms or rules of inference. We check (1). Let ρ be an atomic sentence. In view of $\vee$-simplification $\neg\rho \vdash \theta^* \vee \neg\rho$ and $\{\rho, \theta^* \vee \neg\rho\} \vdash \theta^*$. Consequently $\{\rho, \neg\rho\} \vdash \theta^*$ and so for $s \in S$ either $\rho \notin s$ or $\neg\rho \notin s$. Now let s_ω be a maximal member of S and let $\mathfrak{A}$ be the model of s_ω constructed in the proof of the model existence theorem. Recall that $\mathfrak{A} \models \rho$ iff $\rho \in s_\omega$ for every atomic sentence ρ. Since $\theta^* \notin s_\omega$ we are done unless θ^* is $\neg\pi$ for some atomic sentence π. So suppose this is the case. Then by the deduction theorem $s_\omega \cup \{\pi\} \nvdash \theta^*$. By the maximality of s_ω, $\pi \in s_\omega$. Therefore $\mathfrak{A} \models \pi$ and so $\mathfrak{A} \not\models \theta^*$. Finally $\mathfrak{A} \models \Sigma$ but $\mathfrak{A} \not\models \varphi$ and the proof is complete.

It should be reiterated here that both A. Selman and David Kelly obtained similar complete syntactic notions of proof for universal Horn logic. Though our system seems to be somewhat simpler than the one given by Selman in [29] we present it here more as an application of consistency properties and the model existence theorem.

§6. The decision problem for universal Horn sentences preserved under homomorphisms. The last section suggests that function symbols present the only source of any complexity or power of expression that universal Horn logic might possess. In this section we will be concerned with the terms available in L. Since we wish to discuss decision problems in a meaningful setting we will assume that L and all the usual sets associated with it, e.g. the set of all variables, the set of all L-terms, the set of all universal Horn L-formulas, are recursive. The reader interested in a detailed structure for L under these stipulations should consult [28]. In [18] R. Lyndon showed that the set of all L-sentences preserved under homomorphisms is not recursive, assuming L is provided with a relation symbol of rank at least two. We add to Lyndon's result by showing that the set of universal Horn L-sentences preserved under homomorphisms is not recursive, provided L has a function symbol of rank at least two or at least two unary function symbols. Our major tool for establishing this result will be sets of terms satisfying the subterm condition. We refer to [28] for all necessary details and proofs, but display below the definitions and theorems from [28] that we will use.

DEFINITION. A set Δ of terms *satisfies the subterm condition* provided no variable is a member of Δ and if $\varphi, \psi \in \Delta$ and θ is a nonvariable subterm of φ, then no substitution instance of θ is a substitution instance of φ unless $\theta = \varphi = \psi$.

DEFINITION. The language L is *nontrivial* iff L has at least two unary function symbols or some function symbol of rank at least two.

THEOREM 14 (SEE THEOREM 2.9 IN [28]). *If L is nontrivial, then there is an infinite set Δ of L-terms in a single variable such that Δ satisfies the subterm condition.*

The usefulness of sets of terms satisfying the subterm condition in building models is revealed in the following definition and theorem.

DEFINITION. Let L_0 and L_1 be two languages and let δ be a map from the function symbols of L_0 into terms of L_1. We define in_δ, a map from terms of L_0 into the terms of L_1, by the following recursion:

(i) $\mathrm{in}_\delta x = x$ for every variable x,

(ii) $\mathrm{in}_\delta(f\theta_0 \cdots \theta_{r-1}) = \delta_f[\mathrm{in}_\delta\theta_0, \cdots, \mathrm{in}_\delta\theta_{r-1}]$ where δ_f is the image of f under δ and $\delta_f[\mathrm{in}_\delta\theta_0, \cdots, \mathrm{in}_\delta\theta_{r-1}]$ is obtained from δ_f by substituting $\mathrm{in}_\delta\theta_i$ for x_i, for each $i < r$.

THEOREM 15 (SEE THEOREM 2.5 IN [28]). *Let Δ be any set of L_1-terms in a single variable which satisfies the subterm condition. Let L_0 be a language whose only nonlogical symbols are the unary function symbols f_δ, one for each $\delta \in \Delta$.*

For any infinite L_0-structure $\mathfrak{A}$ there is an L_1-structure $\mathfrak{B}$ with the same universe as $\mathfrak{A}$ such that $(\mathrm{in}_\delta \theta)^{\mathfrak{B}} = \theta^{\mathfrak{A}}$ for every L_0-term θ. (Here $\theta^{\mathfrak{A}}$ is the unary function which interprets θ in $\mathfrak{A}$.)

THEOREM 16 (MAL'CEV [24]). *In a language with two unary function symbols f and g there is a finite set M of identities in f, g, and the variable x such that $\{\xi : M \models \xi$ and ξ is an identity in f, g, and $x\}$ is not recursive.*

One more theorem is necessary before we can accomplish our goal.

THEOREM 17 (SEE LEMMA 3.8 IN [28]). *For any infinite L-structure $\mathfrak{A}$ there is an infinite L-structure $\mathfrak{B}$ such that for any two L-terms φ and ψ in just the variable x*

(i) *If $\mathfrak{A} \models \forall x[\varphi \approx \psi]$, then $\mathfrak{B} \models \forall x[\varphi \approx \psi]$.*

(ii) *If $\mathfrak{A} \not\models \forall x[\varphi \approx \psi]$, then there are one-to-one functions a, b, and c from B into B with the range of b disjoint from the range of c so that $\varphi^{\mathfrak{B}}(a_i) = b_i$ and $\psi^{\mathfrak{B}}(a_i) = c_i$, for each $i \in B$.*

As a matter of notation we use $\theta[\eta]$ to denote the result of substituting η for every variable in θ, whenever θ, η are terms.

THEOREM 18. *If L is a nontrivial (recursive) language, then $\{\varphi : \varphi$ is a universal Horn L-sentence and φ is preserved under the formation of homomorphic images$\}$ is not recursive.*

PROOF. Let f and g be two distinct unary function symbols not in L and let M, the finite set of identities specified by Theorem 16, be written in terms of f, g, and x. Let δ be a one-to-one map from $\{f, g, 2, 3, 4\}$ into the set of L-terms in the variable x whose range satisfies the subterm condition. Let φ, ψ be any two terms in f, g and x. Define

$$\begin{aligned} B(\varphi, \psi) = \mathrm{in}_\delta M \cup \{&\forall xy(\delta_2[\mathrm{in}_\delta \varphi[\delta_3]] \approx \delta_2[\mathrm{in}_\delta \varphi[\delta_3[y]]]), \\ &\forall x(\delta_2[\mathrm{in}_\delta \psi[\delta_3]] \approx x), \forall x(\delta_4[\delta_4] \approx x), \\ &\forall xy(\neg \delta_4 \approx x \vee x \approx y)\}. \end{aligned}$$

So $B(\varphi, \psi)$ is always a finite set of universal Horn L-sentences. We will treat $B(\varphi, \psi)$ as if it were itself a universal Horn sentence. Now suppose $M \models \forall x(\varphi \approx \psi)$. Then $\mathrm{in}_\delta M \models \forall x(\mathrm{in}_\delta \varphi \approx \mathrm{in}_\delta \psi)$ by an easy semantical argument. So $B(\varphi, \psi) \models \forall xy(x \approx y)$ by the way in which $B(\varphi, \psi)$ was defined. Consequently $B(\varphi, \psi)$ has only one-element models and hence is preserved under all homomorphisms.

Now suppose $M \not\models \forall x(\varphi \approx \psi)$. Then M has an infinite model $\mathfrak{A}$ in which $\forall x(\varphi \approx \psi)$ fails. On the basis of Theorem 17, M has an infinite model $\mathfrak{B}$ such that there are one-to-one functions a, b, and c from B into B such that $\varphi^{\mathfrak{B}}(a_i) = b_i$ and $\psi^{\mathfrak{B}}(a_i) = c_i$ for each $i \in B$, and such that b and c have disjoint ranges. Let h, k, and l be three entirely new unary function symbols. Let d be any function from B into B such that

$$d(b_i) = d(b_j) \quad \text{for any } i, j \in B,$$

$$d(c_i) = i \qquad \text{for any } i \in B.$$

Let e be any involution of B without fixed points. Then $(\mathfrak{B}, d, a, e)$ is a model of

$$M \cup \{\forall xy(h\varphi[kx] \approx h\varphi[ky], \forall x(h\psi[kx] \approx x),$$

$$\forall x(llx \approx x), \forall xy(\neg lx \approx x \vee x = y)\}.$$

Similarly if e' is the identity function on B, then $(\mathfrak{B}, d, a, e')$ is a model of

$$M \cup \{\forall xy(h\varphi[kx] \approx h\varphi[ky]), \forall x(h\psi[kx] \approx x), \forall x(lx \approx x)\}.$$

Apply Theorem 15 to these two expansions of $\mathfrak{B}$ to obtain two infinite L-structures $\mathscr{C}$ and $\mathscr{C}'$ so that $\mathscr{C} \models B(\varphi, \psi)$ and

$$\mathscr{C}' \models [B(\varphi, \psi) \sim \{\forall xy(\neg\delta_4 \approx x \vee x \approx y)\}].$$

Now,

$$\mathscr{C} \times \mathscr{C}' \models [B(\varphi, \psi) \sim \{\forall xy(\neg\delta_4 \approx x \vee x \approx y)\}]$$

since this set is a set of universal Horn sentences. On the other hand $\delta_4^{\mathscr{C}}$ is an involution of C without any fixed points while $\delta_4^{\mathscr{C}'}$ is the identity function on C'. Consequently, $\delta_4^{\mathscr{C} \times \mathscr{C}'}$ is an involution of $C \times C'$ without fixed points. But this means $\mathscr{C} \times \mathscr{C}' \models B(\varphi, \psi)$ while $\mathscr{C}' \not\models B(\varphi, \psi)$. Since $\mathscr{C}'$ is a homomorphic image of $\mathscr{C} \times \mathscr{C}'$ it follows that $B(\varphi, \psi)$ is not preserved under the formation of homomorphic images. Thus we have shown

$$M \models \forall x(\varphi \approx \psi) \quad \text{iff} \quad B(\varphi, \psi)$$

is preserved under the formation of homomorphic images iff $B(\varphi, \psi)$ is preserved under the relation of forming direct factors. According to Theorem 16 this is sufficient to establish the theorem.

COROLLARY 19. *If L is a nontrivial (recursive) language, then $\{\varphi : \varphi$ is a universal Horn L-sentence and φ is preserved under the formation of direct factors$\}$ is not recursive.*

A set of sentences is *satisfiable* provided it has a model. The following theorem is closely related to a result of Ralph McKenzie, see especially item (4) in the introduction to McKenzie's paper [26].

THEOREM 20. *Let L be a nontrivial language. The following sets are not recursive:*

(1) $\{\varphi : \varphi$ *is a satisfiable universal Horn L-sentence*$\}$.

(2) $\{\varphi : \varphi$ *is a valid existential Horn L-sentence*$\}$.

PROOF. As in the proof of Theorem 18 let f, g, h, k, and l be unary operation symbols not occurring in L and let M, the finite set of identities specified by Theorem 16, be written in terms of f, g, and the variable x. Let $\{\delta_f, \delta_g, \delta_2, \delta_3, \delta_4\}$ be a set of five L-terms which satisfies the subterm conditions. Let φ and ψ be any two terms in f, g, and the variable x and this time define

$$B(\varphi, \psi) = \text{in}_\delta M \cup \{\forall xy(\delta_2[\text{in}_\delta\varphi[\delta_3]] \approx [\text{in}_\delta\varphi[\delta_3[y]]]),$$

$$\forall x(\delta_2[\text{in}_\delta\psi[\delta_3]] \approx x), \forall x[\neg\delta_4 \approx x]\}.$$

Evidently $B(\varphi, \psi)$ is equivalent to a universal Horn sentence, so we will again ignore the slight difference. Also $\neg B(\varphi, \psi)$ is logically equivalent to an existential Horn sentence and we treat it as such. It is an important though obvious point that the universal and existential sentences we have in mind can be obtained from $B(\varphi, \psi)$ in a recursive way. Now proceeding just as in the proof of Theorem 18 suppose that $M \models \forall x[\varphi \approx \psi]$. Then

$$B(\varphi, \psi) \models \forall xy[x \approx y] \wedge \forall x[\neg \delta_4 \approx x]$$

and so $B(\varphi, \psi)$ is not satisfiable and $\neg B(\varphi, \psi)$ is valid. Now if $M \not\models \forall x[\varphi \approx \psi]$ then the structure $\mathscr{C}$ described in the proof of Theorem 18 is a model of $B(\varphi, \psi)$. Hence $B(\varphi, \psi)$ is satisfiable and $\neg B(\varphi, \psi)$ is not valid. In view of Theorem 16, this completes theorem proof.

While McKenzie's proof holds only if L has a function symbol of rank at least 2, under this provision he does prove (Corollary 5.2 in [26]) that the set of universal Horn sentences which have finite models with more than one element is recursively inseparable from the set of universal Horn sentences without models with more than 'one element.

§7. Open problems.

1. Find an Ehrehfeucht–Fraisse type characterization for universal Horn equivalence between structures.
2. Provide examples to show that the various definability theorems of Svenonius, Kueker, and Chang–Makkai (see § 5.3 in Chang and Keisler [5]) fail for universal Horn theories when universal Horn defining formulas are required.
3. Explore the connections between the canonical structures provided by the model existence theorem and the free structures discussed in Grätzer [12], Mal'cev [25], and Tabata [32].
4. Develop the theory of logics specified by regular relations as described in §1 above.
5. Characterize those sentences of $L_{\omega_1\omega}$ preserved (Łos style) under the formation of ultraproducts. Develop the logic specified by that preservation theorem.

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UNIVERSITY OF SOUTH CAROLINA
COLUMBIA, SOUTH CAROLINA 29208