![HOKKAIDO UNIVERSITY]

Title	Framed curves in the Euclidean space
Author(s)	Honda, Shun'ichi; Takahashi, Masatomo
Citation	Advances in geometry, 16(3), 265-276 https://doi.org/10.1515/advgeom-2015-0035
Issue Date	2016-07
Doc URL	http://hdl.handle.net/2115/66356
Type	article
File Information	[Advances in Geometry] Framed curves in the Euclidean space.pdf

Instructions for use

Shun'ichi Honda* and Masatomo Takahashi

Framed curves in the Euclidean space

Abstract: A framed curve in the Euclidean space is a curve with a moving frame. It is a generalization not only of regular curves with linear independent condition, but also of Legendre curves in the unit tangent bundle. We define smooth functions for a framed curve, called the curvature of the framed curve, similarly to the curvature of a regular curve and of a Legendre curve. Framed curves may have singularities. The curvature of the framed curve is quite useful to analyse the framed curves and their singularities. In fact, we give the existence and the uniqueness for the framed curves by using their curvature. As applications, we consider a contact between framed curves, and give a relationship between projections of framed space curves and Legendre curves.

Keywords: Framed curve, existence, uniqueness, curvature of a framed curve, congruent as framed curves.

Communicated by: K. Ono

1 Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space equipped with the inner product $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{n} a_i b_i$, where $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\boldsymbol{b} = (b_1, \dots, b_n)$. Let $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n-1} \in \mathbb{R}^n$ be vectors $\boldsymbol{a}_i = (a_{i1}, \dots, a_{in})$ for $i = 1, \dots, n-1$. We define the vector product

$$\boldsymbol{a}_1 \times \dots \times \boldsymbol{a}_{n-1} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n-11} & \cdots & a_{n-1n} \\ e_1 & \cdots & e_n \end{vmatrix} = \sum_{i=1}^{n} \det(\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n-1}, e_i)e_i,$$

where $e_1, \dots, e_n$ are the canonical basis vectors of $\mathbb{R}^n$. Then $(\boldsymbol{a}_1 \times \dots \times \boldsymbol{a}_{n-1}) \cdot \boldsymbol{a}_i = 0$ for $i = 1, \dots, n-1$. Note that for the case of $n = 3$,

$$\boldsymbol{a}_1 \times \boldsymbol{a}_2 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ e_1 & e_2 & e_3 \end{vmatrix} = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}.$$

The set

$$\Delta_{n-1} = \{\boldsymbol{v} = (v_1, \dots, v_{n-1}) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n \mid v_i \cdot v_j = \delta_{ij}, i, j = 1, \dots, n-1\}$$
$$= \{\boldsymbol{v} = (v_1, \dots, v_{n-1}) \in S^{n-1} \times \dots \times S^{n-1} \mid v_i \cdot v_j = 0, i \neq j, i, j = 1, \dots, n-1\}$$

is an $n(n-1)/2$-dimensional smooth manifold. If $\boldsymbol{v} = (v_1, \dots, v_{n-1}) \in \Delta_{n-1}$, we define a unit vector $\boldsymbol{\mu} = v_1 \times \dots \times v_{n-1}$ of $\mathbb{R}^n$. It follows that $(\boldsymbol{v}, \boldsymbol{\mu}) \in \Delta_n$ and $\det(\boldsymbol{v}, \boldsymbol{\mu}) = 1$.

A framed curve in the Euclidean space is a curve with a moving frame. It is a generalization not only of regular curves with linear independent condition, but also of Legendre curves in the unit tangent bundle.

Definition 1.1. We say that $(\gamma, \boldsymbol{v}) : I \to \mathbb{R}^n \times \Delta_{n-1}$ is a *framed curve* if $\dot{\gamma}(t) \cdot v_i(t) = 0$ for all $t \in I$ and $i = 1, \dots, n-1$. We also say that $\gamma : I \to \mathbb{R}^n$ is a *framed curve* (or a *framed base curve*) if there exists $\boldsymbol{v} : I \to \Delta_{n-1}$ such that $(\gamma, \boldsymbol{v})$ is a framed curve.

*Corresponding author: Shun'ichi Honda:** Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan, email: s-honda@math.sci.hokudai.ac.jp
Masatomo Takahashi: Muroran Institute of Technology, Muroran 050-8585, Japan, email: masatomo@mmm.muroran-it.ac.jp

We define smooth functions for a framed curve similarly to the curvature of a regular curve and of a Legendre curve. Let $(\gamma, \mathbf{v}) : I \to \mathbb{R}^n \times \Delta_{n-1}$ be a framed curve. We define $\boldsymbol{\mu} : I \to S^{n-1}$ by $\boldsymbol{\mu}(t) = v_1(t) \times \cdots \times v_{n-1}(t)$. By definition, $(\mathbf{v}(t), \boldsymbol{\mu}(t)) \in \Delta_n$ for each $t \in I$ and we call $\{\mathbf{v}(t), \boldsymbol{\mu}(t)\}$ a *moving frame along the framed base curve* $\gamma(t)$. Then we have the Frenet–Serret type formula

$$\begin{pmatrix} \dot{\mathbf{v}}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = A(t) \begin{pmatrix} \mathbf{v}(t) \\ \boldsymbol{\mu}(t) \end{pmatrix},$$

where $A(t) = (\alpha_{ij}(t)) \in \mathfrak{o}(n)$ for $i, j = 1, \ldots, n$, and $\mathfrak{o}(n)$ is the set of alternative matrices. Moreover, there exists a smooth mapping $\alpha : I \to \mathbb{R}$ such that

$$\dot{\gamma}(t) = \alpha(t)\boldsymbol{\mu}(t).$$

We call the functions $(\alpha_{ij}(t), \alpha(t))$ the *curvature of the framed curve* (with respect to the parameter t). Clearly, t_0 is a singular point of γ if and only if $\alpha(t_0) = 0$. The curvature of the framed curve is quite useful to analyse the framed curves and singularities, see Theorems 1.3 and 1.4.

Definition 1.2. Let $(\gamma, \mathbf{v})$ and $(\tilde{\gamma}, \tilde{\mathbf{v}}) : I \to \mathbb{R}^n \times \Delta_{n-1}$ be framed curves. We say that $(\gamma, \mathbf{v})$ and $(\tilde{\gamma}, \tilde{\mathbf{v}})$ are (positive) *congruent as framed curves* if there exists a matrix $X \in \mathrm{SO}(n)$ and a constant vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$\tilde{\gamma}(t) = X(\gamma(t)) + \mathbf{x}, \quad \tilde{\mathbf{v}}(t) = X(\mathbf{v}(t))$$

for all $t \in I$, where $\mathrm{SO}(n)$ is the set of special orthogonal matrices.

The main results are the following (for $n = 2$, see [6]).

Theorem 1.3 (The Existence Theorem). *Let $(\alpha_{ij}, \alpha) : I \to \mathfrak{o}(n) \times \mathbb{R}$ be a smooth mapping. There exists a framed curve $(\gamma, \mathbf{v}) : I \to \mathbb{R}^n \times \Delta_{n-1}$ whose associated curvature is (α_{ij}, α).*

Theorem 1.4 (The Uniqueness Theorem). *Let $(\gamma, \mathbf{v})$ and $(\tilde{\gamma}, \tilde{\mathbf{v}}) : I \to \mathbb{R}^n \times \Delta_{n-1}$ be framed curves whose curvatures (α_{ij}, α) and $(\widetilde{\alpha_{ij}}, \tilde{\alpha})$ coincide. Then $(\gamma, \mathbf{v})$ and $(\tilde{\gamma}, \tilde{\mathbf{v}})$ are congruent as framed curves.*

We shall prove these theorems in §2. We consider properties of the curvature of framed curves and concentrate in §3 on the case $n = 3$ of framed curves in $\mathbb{R}^3$. We consider contact between framed curves, and give a relationship between projections of framed space curves and Legendre curves. Moreover, we give the arc-length parameter of framed immersions. In §4, we give examples of framed curves in $\mathbb{R}^3 \times \Delta_2$.

All maps and manifolds considered here are differentiable of class C^∞.

2 Proofs for the existence and uniqueness theorems

First we prove the existence theorem by using the theorem of existence and uniqueness for a system of linear ordinary differential equations.

Proof of Theorem 1.3. Choose any fixed value $t = t_0$ of the parameter. We consider the initial value problem

$$\frac{d}{dt}F(t) = A(t)F(t), \quad F(t_0) = I_n,$$

where $F(t) \in M(n)$, $A(t) = (\alpha_{ij}(t)) \in \mathfrak{o}(n)$ for $i, j = 1, \ldots, n$; here $M(n)$ is the set of $n \times n$ matrices and I_n is the identity matrix. By the existence and the uniqueness of the solution of a system of linear ordinary differential equations, there exists a solution $F(t)$. Since $A(t) \in \mathfrak{o}(n)$,

$$\frac{d}{dt}({}^tF(t)F(t)) = \left(\frac{d}{dt}{}^tF(t)\right)F(t) + {}^tF(t)\left(\frac{d}{dt}F(t)\right) = {}^tF(t)({}^tA(t) + A(t))F(t) = O.$$

It follows that ${}^tF(t)F(t)$ is constant. Thus ${}^tF(t)F(t) = {}^tF(t_0)F(t_0) = I_n$, and $F(t)$ is an orthogonal matrix. Let $F(t) = {}^t(v_1(t), \dots, v_{n-1}(t), \boldsymbol{\mu}(t))$. Since $(d/dt)(\det F(t)) = 0$, we have $\det F(t) = \det F(t_0) = \det I_n = 1$. Then $\boldsymbol{\mu}(t) = v_1(t) \times \cdots \times v_{n-1}(t)$. Next we consider the initial value problem

$$\dot{\gamma}(t) = \alpha(t)\boldsymbol{\mu}(t), \quad \gamma(t_0) = \boldsymbol{x},$$

where $\boldsymbol{x}$ is a point in $\mathbb{R}^n$. By the existence and the uniqueness of the solution of a system of linear ordinary differential equations, there exists a solution $\gamma(t)$. Therefore, there exists a framed curve $(\gamma, \boldsymbol{v}) : I \to \mathbb{R}^n \times \Delta_{n-1}$ whose associated curvature is (α_{ij}, α). □

In order to prove the Uniqueness Theorem (Theorem 1.4), we need two lemmas.

Lemma 2.1. *Let $(\gamma, \boldsymbol{v})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{v}}) : I \to \mathbb{R}^n \times \Delta_{n-1}$ be congruent as framed curves. Then their curvatures coincide.*

Proof. Since $(\gamma, \boldsymbol{v})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{v}})$ are congruent as framed curves, there exist a matrix $X \in SO(n)$ and a constant vector $\boldsymbol{x} \in \mathbb{R}^n$ with the property that

$$\widetilde{\gamma}(t) = X(\gamma(t)) + \boldsymbol{x}, \quad \widetilde{\boldsymbol{v}}(t) = X(\boldsymbol{v}(t))$$

for all $t \in I$. By definition of $\boldsymbol{\mu}$, we have $\widetilde{\boldsymbol{\mu}}(t) = X(\boldsymbol{\mu}(t))$ for all $t \in I$. By a direct calculation, we have

$$\widetilde{\alpha_{ij}}(t) = \dot{\widetilde{v}}_i(t) \cdot \widetilde{v}_j(t) = X(\dot{v}_i(t)) \cdot X(v_j(t)) = \dot{v}_i(t) \cdot v_j(t) = \alpha_{ij}(t),$$

$$\dot{\widetilde{\gamma}}(t) = X(\dot{\gamma}(t)) = X(\alpha(t)\boldsymbol{\mu}(t)) = \alpha(t)X(\boldsymbol{\mu}(t)) = \alpha(t)\widetilde{\boldsymbol{\mu}}(t).$$

Hence we have $\alpha_{ij}(t) = \widetilde{\alpha_{ij}}(t)$ and $\alpha(t) = \widetilde{\alpha}(t)$. □

Lemma 2.2. *Let $(\gamma, \boldsymbol{v})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{v}}) : I \to \mathbb{R}^n \times \Delta_{n-1}$ be framed curves having equal curvature, that is, $(\alpha_{ij}(t), \alpha(t)) = (\widetilde{\alpha_{ij}}(t), \widetilde{\alpha}(t))$ for all $t \in I$. If there exists a parameter $t = t_0$ for which $(\gamma(t_0), \boldsymbol{v}(t_0)) = (\widetilde{\gamma}(t_0), \widetilde{\boldsymbol{v}}(t_0))$, then $(\gamma, \boldsymbol{v})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{v}})$ coincide.*

Proof. Here we put $v_n(t) = \boldsymbol{\mu}(t)$. Define a smooth function $f : I \to \mathbb{R}$ by $f(t) = \sum_{i=1}^n v_i(t) \cdot \widetilde{v}_i(t)$. Since $\alpha_{ij}(t) = \widetilde{\alpha_{ij}}(t)$ and $\alpha_{ij}(t) = -\alpha_{ji}(t)$, we have

$$\dot{f}(t) = \sum_{i=1}^n (\dot{v}_i(t) \cdot \widetilde{v}_i(t) + v_i(t) \cdot \dot{\widetilde{v}}_i(t))$$

$$= \sum_{i=1}^n \left\{ \left(\sum_{j=1}^n \alpha_{ij}(t)v_j(t) \right) \cdot \widetilde{v}_i(t) + v_i(t) \cdot \left(\sum_{j=1}^n \widetilde{\alpha_{ij}}(t)\widetilde{v}_j(t) \right) \right\} = 2 \sum_{i=1}^n \sum_{j=1}^n (\alpha_{ij}(t) + \alpha_{ji}(t))v_i(t) \cdot \widetilde{v}_j(t) = 0.$$

It follows that f is constant. Moreover, $\boldsymbol{v}(t_0) = \widetilde{\boldsymbol{v}}(t_0)$, so $\boldsymbol{\mu}(t_0) = \widetilde{\boldsymbol{\mu}}(t_0)$. Hence $f(t_0) = n$ and the function f is constant with value n. By the Cauchy–Schwarz inequality, we have

$$v_i(t) \cdot \widetilde{v}_i(t) \le |v_i(t)||\widetilde{v}_i(t)| = 1,$$

for each $i = 1, \dots, n$. If one of these inequalities were strict, the value of $f(t)$ would be less than n. It follows that these inequalities are equalities, and we have $v_i(t) \cdot \widetilde{v}_i(t) = 1$ for all $t \in I$ and $i = 1, \dots, n$. Then we have

$$|v_i(t) - \widetilde{v}_i(t)|^2 = v_i(t) \cdot v_i(t) - 2v_i(t) \cdot \widetilde{v}_i(t) + \widetilde{v}_i(t) \cdot \widetilde{v}_i(t) = 0.$$

Hence $v_i(t) = \widetilde{v}_i(t)$ for all $t \in I$ and $i = 1, \dots, n$. Since $\dot{\gamma}(t) = \alpha(t)\boldsymbol{\mu}(t)$, $\dot{\widetilde{\gamma}}(t) = \widetilde{\alpha}(t)\widetilde{\boldsymbol{\mu}}(t)$ and with the assumption $\alpha(t) = \widetilde{\alpha}(t)$ we obtain $(d/dt)(\gamma(t) - \widetilde{\gamma}(t)) = 0$. It follows that $\gamma(t) - \widetilde{\gamma}(t)$ is constant. By the condition $\gamma(t_0) = \widetilde{\gamma}(t_0)$, we have $\gamma(t) = \widetilde{\gamma}(t)$ for all $t \in I$. □

Proof of Theorem 1.4. Choose any fixed value $t = t_0$ of the parameter. By using a matrix $X \in SO(n)$ and a constant vector $\boldsymbol{x} \in \mathbb{R}^n$, we can assume that $\widetilde{\gamma}(t_0) = X(\gamma(t_0)) + \boldsymbol{x}$ and $\widetilde{\boldsymbol{v}}(t_0) = X(\boldsymbol{v}(t_0))$. By Lemma 2.1, the curvatures of the framed curves $(\gamma(t), \boldsymbol{v}(t))$ and $(X(\gamma(t)) + \boldsymbol{x}, X(\boldsymbol{v}(t)))$ coincide. By Lemma 2.2, we have

$$\widetilde{\gamma}(t) = X(\gamma(t)) + \boldsymbol{x}, \quad \widetilde{\boldsymbol{v}}(t) = X(\boldsymbol{v}(t))$$

for all $t \in I$. It follows that $(\gamma, \boldsymbol{v})$ and $(\widetilde{\gamma}, \widetilde{\boldsymbol{v}})$ are congruent as framed curves. □

Remark 2.3. The Uniqueness Theorem 1.4 can be proved also by using the theorem of uniqueness of the solution of a system of ordinary differential equations.

3 Framed curves in $\mathbb{R}^3 \times \Delta_2$

In this section, we focus on space curves. One can extend the results to higher dimensional curves. However, it is rather tedious; we concentrate on the case of $n = 3$.

We fix the following notation throughout this section. Let $(\gamma, v_1, v_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve and $\boldsymbol{\mu}(t) = v_1(t) \times v_2(t)$. The Frenet–Serret type formula is given by

$$
\begin{pmatrix} \dot{v}_1(t) \\ \dot{v}_2(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) & m(t) \\ -\ell(t) & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{pmatrix} \begin{pmatrix} v_1(t) \\ v_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix},
$$

where $\ell(t) = \dot{v}_1(t) \cdot v_2(t)$, $m(t) = \dot{v}_1(t) \cdot \boldsymbol{\mu}(t)$ and $n(t) = \dot{v}_2(t) \cdot \boldsymbol{\mu}(t)$. Moreover, there exists a smooth mapping $\alpha : I \to \mathbb{R}$ such that

$$
\dot{\gamma}(t) = \alpha(t)\boldsymbol{\mu}(t).
$$

Example 3.1. Typical examples of framed curves are regular curves with linear independent conditions. Let $\gamma : I \to \mathbb{R}^3$ be a regular curve with linear independent condition, namely, $\dot{\gamma}(t)$ and $\ddot{\gamma}(t)$ are linear independent for all $t \in I$. If we take $v_1(t) = \boldsymbol{n}(t)$ and $v_2(t) = \boldsymbol{b}(t)$, then $(\gamma, v_1, v_2) : I \to \mathbb{R}^3 \times \Delta_2$ is a framed curve. Note that $\boldsymbol{\mu}(t) = v_1(t) \times v_2(t) = \boldsymbol{t}(t)$. Here

$$
\boldsymbol{t}(t) = \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}, \quad \boldsymbol{n}(t) = \frac{(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \times \dot{\gamma}(t)}{|(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \times \dot{\gamma}(t)|}, \quad \boldsymbol{b}(t) = \frac{\dot{\gamma}(t) \times \ddot{\gamma}(t)}{|\dot{\gamma}(t) \times \ddot{\gamma}(t)|}.
$$

We give a relationship between regular curves and framed curves.

Proposition 3.2. *With notation as in Example* 3.1, *the relationships between the curvature of the framed curve* $(\ell(t), m(t), n(t), \alpha(t))$, *and the curvature* $\kappa(t)$ *and torsion* $\tau(t)$ *of* γ *are given by*

$$
|\alpha(t)|\kappa(t) = \sqrt{m^2(t) + n^2(t)}, \tag{1}
$$

$$
\alpha(t)(m^2(t) + n^2(t))\tau(t) = m(t)\dot{n}(t) - \dot{m}(t)n(t) + (m^2(t) + n^2(t))\ell(t). \tag{2}
$$

Proof. By a direct calculation, we have

$$
\dot{\gamma}(t) = \alpha(t)\boldsymbol{\mu}(t), \tag{3}
$$

$$
\ddot{\gamma}(t) = \dot{\alpha}(t)\boldsymbol{\mu}(t) - \alpha(t)m(t)v_1(t) - \alpha(t)n(t)v_2(t), \tag{4}
$$

$$
\dddot{\gamma}(t) = (\ddot{\alpha}(t) - \alpha(t)m^2(t) - \alpha(t)n^2(t))\boldsymbol{\mu}(t) - (2\dot{\alpha}(t)m(t) + \alpha(t)\dot{m}(t) - \alpha(t)n(t)\ell(t))v_1(t) \tag{5}
$$
$$
- (2\dot{\alpha}(t)n(t) + \alpha(t)\dot{n}(t) + \alpha(t)m(t)\ell(t))v_2(t).
$$

It follows that

$$
|\dot{\gamma}(t)| = |\alpha(t)|
$$

$$
|\dot{\gamma}(t) \times \ddot{\gamma}(t)| = \alpha^2(t)\sqrt{m^2(t) + n^2(t)}
$$

$$
\det(\dot{\gamma}(t), \ddot{\gamma}(t), \dddot{\gamma}(t)) = \alpha^3(t)(m(t)\dot{n}(t) - \dot{m}(t)n(t) + (m^2(t) + n^2(t))\ell(t))
$$

Therefore, the curvature $\kappa(t)$ and the torsion $\tau(t)$ are given by

$$
\kappa(t) = \frac{|\dot{\gamma}(t) \times \ddot{\gamma}(t)|}{|\dot{\gamma}(t)|^3} = \frac{\sqrt{m^2(t) + n^2(t)}}{|\alpha(t)|}
$$

and

$$
\tau(t) = \frac{\det(\dot{\gamma}(t), \ddot{\gamma}(t), \dddot{\gamma}(t))}{|\dot{\gamma}(t) \times \ddot{\gamma}(t)|^2} = \frac{m(t)\dot{n}(t) - \dot{m}(t)n(t) + (m^2(t) + n^2(t))\ell(t)}{\alpha(t)(m^2(t) + n^2(t))}. \qquad \square
$$

Let $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ be a space curve germ and write $\gamma(t) = (x(t), y(t), z(t))$. It can be shown that, if γ is not infinitely flat, namely, if either $x(t)$, $y(t)$ or $z(t)$ does not belong to $\mathfrak{m}_1^\infty$ (the ideal of infinitely flat function germs), then γ is a framed base curve. Suppose without loss of generality that $x(t)$ does not belong to $\mathfrak{m}_1^\infty$ and that order $x(t) \leq \min\{\text{order } y(t), \text{order } z(t)\}$. Then there exist smooth function germs $a(t)$ and $b(t)$ such that $\dot{y}(t) = a(t)\dot{x}(t)$ and $\dot{z}(t) = b(t)\dot{x}(t)$. Thus if we take

$$v_1(t) = \frac{1}{\sqrt{1 + a^2(t)}}(-a(t), 1, 0),$$

$$v_2(t) = \frac{1}{\sqrt{(1 + a^2(t))(1 + a^2(t) + b^2(t))}}(-b(t), -a(t)b(t), 1 + a^2(t)),$$

then (γ, v_1, v_2) is a framed curve. Note that

$$\boldsymbol{\mu}(t) = v_1(t) \times v_2(t) = \frac{1}{\sqrt{1 + a^2(t) + b^2(t)}}(1, a(t), b(t)).$$

On the other hand, constant maps are also framed base curves, which do not satisfy the above sufficient condition. In particular an analytic curve germ is always a framed base curve, because if it is infinitely flat, then it is constant.

Let $(\gamma, v_1, v_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve with the curvature of the framed curve (ℓ, m, n, α). By (3), (4) and (5) in the proof of Proposition 3.2, we have the following Taylor expansion of γ:

$$\gamma(t) = \gamma(t_0) + (t - t_0)\alpha(t_0)\boldsymbol{\mu}(t_0) + \frac{(t - t_0)^2}{2}(\dot{\alpha}(t_0)\boldsymbol{\mu}(t_0) - \alpha(t_0)m(t_0)v_1(t_0) - \alpha(t_0)n(t_0)v_2(t_0))$$

$$+ \frac{(t - t_0)^3}{3!}\Big((\ddot{\alpha}(t_0) - \alpha(t_0)m^2(t_0) - \alpha(t_0)n^2(t_0))\boldsymbol{\mu}(t_0) - (2\dot{\alpha}(t_0)m(t_0) + \alpha(t_0)\dot{m}(t_0)$$

$$- \alpha(t_0)n(t_0)\ell(t_0))v_1(t_0) - (2\dot{\alpha}(t_0)n(t_0) + \alpha(t_0)\dot{n}(t_0) + \alpha(t_0)m(t_0)\ell(t_0))v_2(t_0)\Big) + o(4).$$

If t_0 is a singular point of γ, then we have

$$\gamma(t) = \gamma(t_0) + \frac{(t - t_0)^2}{2}\dot{\alpha}(t_0)\boldsymbol{\mu}(t_0) + \frac{(t - t_0)^3}{3!}(\ddot{\alpha}(t_0)\boldsymbol{\mu}(t_0) - 2\dot{\alpha}(t_0)m(t_0)v_1(t_0) - 2\dot{\alpha}(t_0)n(t_0)v_2(t_0)) + o(4).$$

Let $(\gamma, v_1, v_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve with the curvature of the framed curve (ℓ, m, n, α). For the normal plane of $\gamma(t)$, spanned by $v_1(t)$ and $v_2(t)$, there is some ambient of framed curves similarly to the case of the Bishop frame of a regular space curve (cf. [3]). We define $(\bar{v}_1(t), \bar{v}_2(t)) \in \Delta_2$ by

$$\begin{pmatrix} \bar{v}_1(t) \\ \bar{v}_2(t) \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix}\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix},$$

where $\theta(t)$ is a smooth function. Then $(\gamma, \bar{v}_1, \bar{v}_2) : I \to \mathbb{R}^3 \times \Delta_2$ is also a framed curve and

$$\bar{\boldsymbol{\mu}}(t) = \bar{v}_1(t) \times \bar{v}_2(t) = (\cos\theta(t)v_1(t) - \sin\theta(t)v_2(t)) \times (\sin\theta(t)v_1(t) + \cos\theta(t)v_2(t))$$

$$= v_1(t) \times v_2(t) = \boldsymbol{\mu}(t).$$

By a direct calculation, we have

$$\dot{\bar{v}}_1(t) = (\ell(t) - \dot{\theta}(t))\sin\theta(t)v_1(t) + (\ell(t) - \dot{\theta}(t))\cos\theta(t)v_2(t) + (m(t)\cos\theta(t) - n(t)\sin\theta(t))\boldsymbol{\mu}(t),$$

$$\dot{\bar{v}}_2(t) = -(\ell(t) - \dot{\theta}(t))\cos\theta(t)v_1(t) + (\ell(t) - \dot{\theta}(t))\sin\theta(t)v_2(t) + (m(t)\sin\theta(t) + n(t)\cos\theta(t))\boldsymbol{\mu}(t).$$

If we take a smooth function $\theta : I \to \mathbb{R}$ which satisfies $\dot{\theta}(t) = \ell(t)$, then we call the frame $\{\bar{v}_1(t), \bar{v}_2(t), \boldsymbol{\mu}(t)\}$ an *adapted frame* along the framed base curve $\gamma(t)$. It follows that the Frenet–Serret type formula is given by

$$\begin{pmatrix} \dot{\bar{v}}_1(t) \\ \dot{\bar{v}}_2(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \bar{m}(t) \\ 0 & 0 & \bar{n}(t) \\ -\bar{m}(t) & -\bar{n}(t) & 0 \end{pmatrix}\begin{pmatrix} \bar{v}_1(t) \\ \bar{v}_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix},$$

where $\overline{m}(t)$ and $\overline{n}(t)$ are given by

$$\begin{pmatrix} \overline{m}(t) \\ \overline{n}(t) \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} m(t) \\ n(t) \end{pmatrix}. \tag{6}$$

We now consider framed curves in a plane. Let $(\gamma, v_1, v_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve with the curvature of the framed curve (ℓ, m, n, α). We denote a plane by

$$P(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^3 \mid \boldsymbol{x} \cdot \boldsymbol{v} = c\},$$

where $\boldsymbol{v} \in S^2$ and $c \in \mathbb{R}$. If $\gamma(t) \in P(\boldsymbol{v}, c)$, then we have $\det(\dot{\gamma}(t), \ddot{\gamma}(t), \dddot{\gamma}(t)) = 0$. It follows that

$$\alpha(t)\big(m(t)\dot{n}(t) - \dot{m}(t)n(t) + (m^2(t) + n^2(t))\ell(t)\big) = 0$$

for all $t \in I$. Conversely, we have the following result.

Proposition 3.3. *Let* $(\gamma, v_1, v_2) : I \to \mathbb{R}^3 \times \Delta_2$ *be a framed curve with curvature* (ℓ, m, n, α).

(1) *If* $\alpha(t) = 0$ *for all* $t \in I$, *then* $\gamma(t)$ *is a point.*

(2) *If* $m(t) = n(t) = 0$ *for all* $t \in I$, *then* $\gamma(t)$ *is a part of a straight line.*

(3) *If* $m(t)\dot{n}(t) - \dot{m}(t)n(t) + (m^2(t) + n^2(t))\ell(t) = 0$ *and* $m^2(t) + n^2(t) \neq 0$ *for all* $t \in I$, *then there exist a vector* $\boldsymbol{v} \in S^2$ *and a constant* $c \in \mathbb{R}$ *such that* $\gamma(t) \in P(\boldsymbol{v}, c)$.

Proof. (1) By $\dot{\gamma}(t) = \alpha(t)\boldsymbol{\mu}(t) = 0$ for all $t \in I$, $\gamma(t)$ is a point.

(2) By the Frenet–Serret type formula, $\dot{\boldsymbol{\mu}}(t) = 0$ for all $t \in I$ and hence $\dot{\gamma}(t) = \alpha(t)\boldsymbol{\mu}(t) = \alpha(t)\boldsymbol{v}$, where $\boldsymbol{v} \in S^2$ is a constant vector. Then there exists a constant vector $\boldsymbol{x}$ such that $\gamma(t) = (\int \alpha(t)dt)\boldsymbol{v} + \boldsymbol{x}$. It follows that $\gamma(t)$ is a part of a straight line.

(3) We take an adapted frame $\{\overline{v}_1(t), \overline{v}_2(t), \boldsymbol{\mu}(t)\}$ along the framed base curve $\gamma(t)$. By (6) and a direct calculation, we have

$$\overline{m}(t)\dot{\overline{n}}(t) - \dot{\overline{m}}(t)\overline{n}(t) = m(t)\dot{n}(t) - \dot{m}(t)n(t) + (m^2(t) + n^2(t))\ell(t) = 0$$

and

$$\overline{m}^2(t) + \overline{n}^2(t) = m^2(t) + n^2(t) \neq 0$$

for all $t \in I$. It follows that $\overline{m}(t)$ and $\overline{n}(t)$ are linear dependent on I (cf. [4; 9; 10]). Thus, there exists a non-zero constant vector (c_1, c_2) such that $c_1\overline{m}(t) + c_2\overline{n}(t) = 0$ for all $t \in I$. Then $\tilde{\boldsymbol{v}} = c_1\overline{v}_1(t) + c_2\overline{v}_2(t)$ is a non-zero constant vector. Let $\boldsymbol{v} = \tilde{\boldsymbol{v}}/\sqrt{c_1^2 + c_2^2}$. Since $\dot{\gamma}(t) \cdot \boldsymbol{v} = \alpha(t)\boldsymbol{\mu}(t) \cdot \boldsymbol{v} = 0$ for all $t \in I$, there exists a constant $c \in \mathbb{R}$ such that $\gamma(t) \in P(\boldsymbol{v}, c)$. □

Remark 3.4. If $(\gamma, v_1, v_2) : I \to \mathbb{R}^3 \times \Delta_2$ is an analytic framed curve, then $\overline{m}(t)$ and $\overline{n}(t)$ are also analytic functions. Hence if $m(t)\dot{n}(t) - \dot{m}(t)n(t) + (m^2(t) + n^2(t))\ell(t) = 0$ for all $t \in I$, then $\overline{m}(t)$ and $\overline{n}(t)$ are linear dependent on I (cf. [4; 10]). It follows that there exist a vector $\boldsymbol{v} \in S^2$ and a constant $c \in \mathbb{R}$ such that $\gamma(t) \in P(\boldsymbol{v}, c)$.

We also define a Legendre curve on a plane.

Definition 3.5. We say that $(\gamma, v) : I \to \mathbb{R}^3 \times S^2$ is a *Legendre curve on the plane* $P(\boldsymbol{v}, c)$ if $\gamma(t) \cdot \boldsymbol{v} = c$, $\dot{\gamma}(t) \cdot v(t) = 0$ and $v(t) \cdot \boldsymbol{v} = 0$ for all $t \in I$.

Proposition 3.6. (1) *If* $(\gamma, v) : I \to \mathbb{R}^3 \times S^2$ *is a Legendre curve on the plane* $P(\boldsymbol{v}, c)$, *then* $(\gamma, \boldsymbol{v}, v) : I \to \mathbb{R}^3 \times \Delta_2$ *is a framed curve with* $\ell(t) = m(t) = 0$ *for all* $t \in I$. *Conversely, if* $(\gamma, v_1, v_2) : I \to \mathbb{R}^3 \times \Delta_2$ *is a framed curve with* $\ell(t) = m(t) = 0$ *for all* $t \in I$, *then there exist a constant vector* $\boldsymbol{v} \in S^2$ *and a constant* $c \in \mathbb{R}$ *such that* $(\gamma, v_2) : I \to \mathbb{R}^3 \times S^2$ *is a Legendre curve on the plane* $P(\boldsymbol{v}, c)$.

(2) *If* $(\gamma, v) : I \to \mathbb{R}^3 \times S^2$ *is a Legendre curve on the plane* $P(\boldsymbol{v}, c)$, *then* $(\gamma, v, \boldsymbol{v}) : I \to \mathbb{R}^3 \times \Delta_2$ *is a framed curve with* $\ell(t) = n(t) = 0$ *for all* $t \in I$. *Conversely, if* $(\gamma, v_1, v_2) : I \to \mathbb{R}^3 \times \Delta_2$ *is a framed curve with* $\ell(t) = n(t) = 0$ *for all* $t \in I$, *then there exist a constant vector* $\boldsymbol{v} \in S^2$ *and a constant* $c \in \mathbb{R}$ *such that* $(\gamma, v_1) : I \to \mathbb{R}^3 \times S^2$ *is a Legendre curve on the plane* $P(\boldsymbol{v}, c)$.

Proof. (1) By definition, we have $\dot{\gamma}(t) \cdot \boldsymbol{v} = 0$ and $(\boldsymbol{v}, \nu(t)) \in \Delta_2$. Since $\boldsymbol{v}$ is a constant, we have $\ell(t) = m(t) = 0$ for all $t \in I$. Conversely, by the Frenet–Serret type formula, $\boldsymbol{v} = \nu_1(t) \in S^2$ is a constant vector. Moreover, since $\dot{\gamma}(t) \cdot \boldsymbol{v} = \alpha(t)\boldsymbol{\mu}(t) \cdot \nu_1(t) = 0$ for all $t \in I$, there exists a constant $c \in \mathbb{R}$ such that $\gamma(t) \cdot \boldsymbol{v} = c$. It follows that $(\gamma, \nu_2) : I \to \mathbb{R}^3 \times S^2$ is a Legendre curve on the plane $P(\boldsymbol{v}, c)$.

Assertion (2) can be proved similarly.   □

3.1 Contact between framed curves

In this subsection, we discuss contact between framed curves. Let

$$(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta_2; t \mapsto (\gamma(t), \nu_1(t), \nu_2(t)) \text{ and } (\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : \widetilde{I} \to \mathbb{R}^3 \times \Delta_2; u \mapsto (\widetilde{\gamma}(u), \widetilde{\nu}_1(u), \widetilde{\nu}_2(u))$$

be framed curves and let k be a natural number. We say that (γ, ν_1, ν_2) *and* $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ *have k-th order contact at $t = t_0, u = u_0$* if

$$(\gamma, \nu_1, \nu_2)(t_0) = (\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)(u_0), \quad \frac{d}{dt}(\gamma, \nu_1, \nu_2)(t_0) = \frac{d}{du}(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)(u_0), \quad \dots,$$

$$\frac{d^{k-1}}{dt^{k-1}}(\gamma, \nu_1, \nu_2)(t_0) = \frac{d^{k-1}}{du^{k-1}}(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)(u_0), \quad \frac{d^k}{dt^k}(\gamma, \nu_1, \nu_2)(t_0) \neq \frac{d^k}{du^k}(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)(u_0)$$

(cf. [6; 8]). Moreover, we say that (γ, ν_1, ν_2) *and* $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ *have at least k-th order contact at $t = t_0, u = u_0$* if

$$(\gamma, \nu_1, \nu_2)(t_0) = (\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)(u_0), \quad \frac{d}{dt}(\gamma, \nu_1, \nu_2)(t_0) = \frac{d}{du}(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)(u_0), \quad \dots,$$

$$\frac{d^{k-1}}{dt^{k-1}}(\gamma, \nu_1, \nu_2)(t_0) = \frac{d^{k-1}}{du^{k-1}}(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)(u_0).$$

In general, we may assume that (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ have at least first order contact at any point $t = t_0, u = u_0$, up to congruence as framed curves. We denote the curvatures of the framed curves $(\gamma(t), \nu_1(t), \nu_2(t))$ by $\mathcal{F}(t) = (\ell(t), m(t), n(t), \alpha(t))$ and that of $(\widetilde{\gamma}(u), \widetilde{\nu}_1(u), \widetilde{\nu}_2(u))$ by $\widetilde{\mathcal{F}}(u) = (\widetilde{\ell}(u), \widetilde{m}(u), \widetilde{n}(u), \widetilde{\alpha}(u))$.

Theorem 3.7. *Let (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ as above. If (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ have at least $(k + 1)$-th order contact at $t = t_0, u = u_0$ then*

$$\mathcal{F}(t_0) = \widetilde{\mathcal{F}}(u_0), \quad \frac{d}{dt}\mathcal{F}(t_0) = \frac{d}{du}\widetilde{\mathcal{F}}(u_0), \quad \dots, \quad \frac{d^{k-1}}{dt^{k-1}}\mathcal{F}(t_0) = \frac{d^{k-1}}{du^{k-1}}\widetilde{\mathcal{F}}(u_0). \tag{7}$$

Conversely, if conditions (7) hold, then (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ have at least $(k + 1)$-th order contact at $t = t_0, u = u_0$, up to congruence as framed curves.

Proof. Suppose that (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu_1}, \widetilde{\nu_2})$ have at least second order contact at $t = t_0, u = u_0$. Since $\nu_1(t_0) = \widetilde{\nu_1}(u_0)$ and $\nu_2(t_0) = \widetilde{\nu_2}(u_0)$, we have $\boldsymbol{\mu}(t_0) = \bar{\boldsymbol{\mu}}(u_0)$. By the Frenet–Serret type formula,

$$\frac{d}{dt}(\gamma, \nu_1, \nu_2) = (\alpha(t)\boldsymbol{\mu}(t), \ell(t)\nu_2(t) + m(t)\boldsymbol{\mu}(t), -\ell(t)\nu_1(t) + n(t)\boldsymbol{\mu}(t)),$$

$$\frac{d}{du}(\widetilde{\gamma}, \widetilde{\nu_1}, \widetilde{\nu_2}) = (\widetilde{\alpha}(u)\bar{\boldsymbol{\mu}}(u), \widetilde{\ell}(u)\widetilde{\nu_2}(u) + \widetilde{m}(u)\bar{\boldsymbol{\mu}}(u), -\widetilde{\ell}(u)\widetilde{\nu_1}(u) + \widetilde{n}(u)\bar{\boldsymbol{\mu}}(u)).$$

It follows that $\mathcal{F}(t_0) = \widetilde{\mathcal{F}}(u_0)$. Hence, the first assertion of Theorem 3.7 holds in the case of $k = 1$.

Suppose that the assumption is true up to the k-th order of contact. Let (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu_1}, \widetilde{\nu_2})$ have at least $(k + 1)$-th order contact at $t = t_0, u = u_0$. Then they have at least k-th order of contact, so

$$\mathcal{F}(t_0) = \widetilde{\mathcal{F}}(u_0), \quad \frac{d}{dt}\mathcal{F}(t_0) = \frac{d}{du}\widetilde{\mathcal{F}}(u_0), \quad \dots, \quad \frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t_0) = \frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u_0).$$

By the Frenet–Serret type formula, we have

$$\frac{d^k}{dt^k}\gamma(t) = \Big(\frac{d^{k-1}}{dt^{k-1}}\alpha(t)\Big)\boldsymbol{\mu}(t) + f_1\Big(\mathcal{F}(t),\dots,\frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t)\Big)\nu_1(t)$$
$$+ f_2\Big(\mathcal{F}(t),\dots,\frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t)\Big)\nu_2(t) + f_3\Big(\mathcal{F}(t),\dots,\frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t)\Big)\boldsymbol{\mu}(t),$$

$$\frac{d^k}{dt^k}\nu_1(t) = \Big(\frac{d^{k-1}}{dt^{k-1}}\ell(t)\Big)\nu_2(t) + \Big(\frac{d^{k-1}}{dt^{k-1}}m(t)\Big)\boldsymbol{\mu}(t) + g_1\Big(\mathcal{F}(t),\dots,\frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t)\Big)\nu_1(t)$$
$$+ g_2\Big(\mathcal{F}(t),\dots,\frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t)\Big)\nu_2(t) + g_3\Big(\mathcal{F}(t),\dots,\frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t)\Big)\boldsymbol{\mu}(t),$$

$$\frac{d^k}{dt^k}\nu_2(t) = -\Big(\frac{d^{k-1}}{dt^{k-1}}\ell(t)\Big)\nu_1(t) + \Big(\frac{d^{k-1}}{dt^{k-1}}n(t)\Big)\boldsymbol{\mu}(t) + h_1\Big(\mathcal{F}(t),\dots,\frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t)\Big)\nu_1(t)$$
$$+ h_2\Big(\mathcal{F}(t),\dots,\frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t)\Big)\nu_2(t) + h_3\Big(\mathcal{F}(t),\dots,\frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t)\Big)\boldsymbol{\mu}(t)$$

for some smooth functions f_i, g_i, h_i $(i = 1, 2, 3)$. By the same calculations,

$$\frac{d^k}{du^k}\widetilde{\gamma}(u) = \Big(\frac{d^{k-1}}{du^{k-1}}\widetilde{\alpha}(u)\Big)\widetilde{\boldsymbol{\mu}}(u) + f_1\Big(\widetilde{\mathcal{F}}(u),\dots,\frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u)\Big)\widetilde{\nu_1}(u)$$
$$+ f_2\Big(\widetilde{\mathcal{F}}(u),\dots,\frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u)\Big)\widetilde{\nu_2}(u) + f_3\Big(\widetilde{\mathcal{F}}(u),\dots,\frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u)\Big)\widetilde{\boldsymbol{\mu}}(u),$$

$$\frac{d^k}{du^k}\widetilde{\nu_1}(u) = \Big(\frac{d^{k-1}}{du^{k-1}}\widetilde{\ell}(u)\Big)\widetilde{\nu_2}(u) + \Big(\frac{d^{k-1}}{du^{k-1}}\widetilde{m}(u)\Big)\widetilde{\boldsymbol{\mu}}(u) + g_1\Big(\widetilde{\mathcal{F}}(u),\dots,\frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u)\Big)\widetilde{\nu_1}(u)$$
$$+ g_2\Big(\widetilde{\mathcal{F}}(u),\dots,\frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u)\Big)\widetilde{\nu_2}(u) + g_3\Big(\widetilde{\mathcal{F}}(u),\dots,\frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u)\Big)\widetilde{\boldsymbol{\mu}}(u),$$

$$\frac{d^k}{du^k}\widetilde{\nu_2}(u) = -\Big(\frac{d^{k-1}}{du^{k-1}}\widetilde{\ell}(u)\Big)\widetilde{\nu_1}(u) + \Big(\frac{d^{k-1}}{du^{k-1}}\widetilde{n}(u)\Big)\widetilde{\boldsymbol{\mu}}(u) + h_1\Big(\widetilde{\mathcal{F}}(u),\dots,\frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u)\Big)\widetilde{\nu_1}(u)$$
$$+ h_2\Big(\widetilde{\mathcal{F}}(u),\dots,\frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u)\Big)\widetilde{\nu_2}(u) + h_3\Big(\widetilde{\mathcal{F}}(u),\dots,\frac{d^{k-2}}{du^{k-2}}\widetilde{\mathcal{F}}(u)\Big)\widetilde{\boldsymbol{\mu}}(u).$$

It follows that $(d^{k-1}/dt^{k-1})\mathcal{F}(t_0) = (d^{k-1}/du^{k-1})\widetilde{\mathcal{F}}(u_0)$. By induction, we have the first assertion.

Conversely, suppose that Condition (7) holds. By the above calculations, we have $(d^i/dt^i)(\gamma, \nu_1, \nu_2)(t_0) = (d^i/du^i)(\widetilde{\gamma}, \widetilde{\nu_1}, \widetilde{\nu_2})(u_0)$ for $i = 1, \dots, k$. Therefore, (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu_1}, \widetilde{\nu_2})$ have at least $(k + 1)$-th order contact at $t = t_0$, $u = u_0$, up to congruence as framed curves. □

3.2 Projections to planes and Legendre curves

We quickly review Legendre curves; for more detail see [6]. The Legendre curves correspond to the case of $n = 2$ for framed curves. We say that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is *a Legendre curve* if $(\gamma, \nu)^*\theta = 0$ for all $t \in I$, where θ is a canonical contact 1-form on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ (cf. [1; 2]). This condition is equivalent to $\dot{\gamma}(t) \cdot \nu(t) = 0$ for all $t \in I$. We say that $\gamma : I \to \mathbb{R}^2$ is *a frontal* if there exists a smooth mapping $\nu : I \to S^1$ such that (γ, ν) is a Legendre curve.

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve. Then we have the Frenet formula of the frontal γ as follows. We put $\boldsymbol{\mu}(t) = J(\nu(t))$, where J is the anti-clockwise rotation by $\pi/2$ on $\mathbb{R}^2$. We call the pair $\{\nu(t), \boldsymbol{\mu}(t)\}$ a *moving frame along the frontal $\gamma(t)$* in $\mathbb{R}^2$. The Frenet formula of the frontal (or, the Legendre curve) is given by

$$\begin{pmatrix} \dot{\nu}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix}\begin{pmatrix} \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix},$$

where $\ell(t) = \dot{\nu}(t) \cdot \boldsymbol{\mu}(t)$. Moreover, there exists a smooth function $\beta(t)$ such that

$$\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t).$$

We call the pair $(\ell(t), \beta(t))$ *the curvature of the Legendre curve* (with respect to the parameter t).

Let $(\gamma, v_1, v_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve with curvature (ℓ, m, n, α). For a fix point $t_0 \in I$, we consider three orthogonal projections from $\mathbb{R}^3$ along the direction $v_1(t_0)$, $v_2(t_0)$ and $\boldsymbol{\mu}(t_0)$.

First, we consider the projection of γ along the $v_1(t_0)$ direction given by $\gamma_{v_1} : I \to \mathbb{R}^2$ with $\gamma_{v_1}(t) = (\gamma(t) \cdot v_2(t_0), \gamma(t) \cdot \boldsymbol{\mu}(t_0))$. Then $\dot{\gamma}_{v_1}(t) = \alpha(t)(\boldsymbol{\mu}(t) \cdot v_2(t_0), \boldsymbol{\mu}(t) \cdot \boldsymbol{\mu}(t_0))$. There is a subinterval I_1 of I around t_0 such that $(\boldsymbol{\mu}(t) \cdot v_2(t_0))^2 + (\boldsymbol{\mu}(t) \cdot \boldsymbol{\mu}(t_0))^2 \neq 0$ for all $t \in I_1$. We define a smooth map $v_{v_1} : I_1 \to S^1$ by

$$v_{v_1}(t) = \frac{1}{\sqrt{(\boldsymbol{\mu}(t) \cdot v_2(t_0))^2 + (\boldsymbol{\mu}(t) \cdot \boldsymbol{\mu}(t_0))^2}}(\boldsymbol{\mu}(t) \cdot \boldsymbol{\mu}(t_0), -\boldsymbol{\mu}(t) \cdot v_2(t_0)).$$

Then $(\gamma_{v_1}, v_{v_1}) : I_1 \to \mathbb{R}^2 \times S^1$ is a Legendre curve. Since $\boldsymbol{\mu}_{v_1} : I_1 \to S^1$ is

$$\boldsymbol{\mu}_{v_1}(t) = J(v_{v_1}(t)) = \frac{1}{\sqrt{(\boldsymbol{\mu}(t) \cdot v_2(t_0))^2 + (\boldsymbol{\mu}(t) \cdot \boldsymbol{\mu}(t_0))^2}}(\boldsymbol{\mu}(t) \cdot v_2(t_0), \boldsymbol{\mu}(t) \cdot \boldsymbol{\mu}(t_0)),$$

the curvature of the Legendre curve (γ_{v_1}, v_{v_1}) is given by

$$\ell_{v_1}(t) = \frac{1}{(\boldsymbol{\mu}(t) \cdot v_2(t_0))^2 + (\boldsymbol{\mu}(t) \cdot \boldsymbol{\mu}(t_0))^2}\Big(m(t)((v_1(t) \cdot v_2(t_0))(\boldsymbol{\mu}(t) \cdot \boldsymbol{\mu}(t_0)) - (v_1(t) \cdot \boldsymbol{\mu}(t_0))(\boldsymbol{\mu}(t) \cdot v_2(t_0)))$$
$$+ n(t)((v_2(t) \cdot v_2(t_0))(\boldsymbol{\mu}(t) \cdot \boldsymbol{\mu}(t_0)) - (v_2(t) \cdot \boldsymbol{\mu}(t_0))(\boldsymbol{\mu}(t) \cdot v_2(t_0))))\Big)$$

and

$$\beta_{v_1}(t) = \alpha(t)\sqrt{(\boldsymbol{\mu}(t) \cdot v_2(t_0))^2 + (\boldsymbol{\mu}(t) \cdot \boldsymbol{\mu}(t_0))^2}.$$

Note that $\ell_{v_1}(t_0) = n(t_0)$ and $\beta_{v_1}(t_0) = \alpha(t_0)$. The projection of γ along the $v_2(t_0)$ direction is similar to the case of the $v_1(t_0)$ direction.

Next, we consider the projection of γ along $\boldsymbol{\mu}(t_0)$, given by $\gamma_\mu : I \to \mathbb{R}^2$ with $\gamma_\mu(t) = (\gamma(t) \cdot v_1(t_0), \gamma(t) \cdot v_2(t_0))$. Then $\dot{\gamma}_\mu(t) = \alpha(t)(\boldsymbol{\mu}(t) \cdot v_1(t_0), \boldsymbol{\mu}(t) \cdot v_2(t_0))$. In this case, γ_μ is not always a frontal, that is, there does not exists a smooth mapping $v_\mu : I \to S^1$ such that $(\gamma_\mu, v_\mu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre curve, see Example 4.2. However, if γ_μ is not infinitely flat around t_0, namely, if either $\gamma(t) \cdot v_1(t_0)$ or $\gamma(t) \cdot v_2(t_0)$ does not belong to $\mathfrak{m}_1^\infty$, then γ_μ is a frontal (cf. [6]).

In general, let $t_0 \in I$ and fix a positive orthonormal basis $\{v_1, v_2, v_3\}$ on $\mathbb{R}^3$, with $(v_1, v_2) \in \Delta_2$ and $v_3 = v_1 \times v_2$, such that $v_3 \neq \pm\boldsymbol{\mu}(t_0)$. Then we consider the orthogonal projection along v_3 to the (v_1, v_2)-plane. We denote $\gamma_v : I \to \mathbb{R}^2$ given by $\gamma_v(t) = (\gamma(t) \cdot v_1, \gamma(t) \cdot v_2)$. Then $\dot{\gamma}_v(t) = \alpha(t)(\boldsymbol{\mu}(t) \cdot v_1, \boldsymbol{\mu}(t) \cdot v_2)$. By the assumption, there is a subinterval $\tilde{I}$ of I around t_0 such that $(\boldsymbol{\mu}(t) \cdot v_1, \boldsymbol{\mu}(t) \cdot v_2) \neq (0, 0)$ for all $t \in \tilde{I}$. We define a smooth map $v_v : \tilde{I} \to S^1$ by

$$v_v(t) = \frac{1}{\sqrt{(\boldsymbol{\mu}(t) \cdot v_1)^2 + (\boldsymbol{\mu}(t) \cdot v_2)^2}}(\boldsymbol{\mu}(t) \cdot v_2, -\boldsymbol{\mu}(t) \cdot v_1).$$

Then $(\gamma_v, v_v) : \tilde{I} \to \mathbb{R}^2 \times S^1$ is a Legendre curve. Since $\boldsymbol{\mu}_v : \tilde{I} \to S^1$ is

$$\boldsymbol{\mu}_v(t) = J(v_v(t)) = \frac{1}{\sqrt{(\boldsymbol{\mu}(t) \cdot v_1)^2 + (\boldsymbol{\mu}(t) \cdot v_2)^2}}(\boldsymbol{\mu}(t) \cdot v_1, \boldsymbol{\mu}(t) \cdot v_2),$$

the curvature of the Legendre curve (γ_v, v_v) is given by

$$\ell_v(t) = \frac{1}{(\boldsymbol{\mu}(t) \cdot v_1)^2 + (\boldsymbol{\mu}(t) \cdot v_2)^2}\Big(m(t)((v_1(t) \cdot v_1)(\boldsymbol{\mu}(t) \cdot v_2) - (v_1(t) \cdot v_2)(\boldsymbol{\mu}(t) \cdot v_1))$$
$$+ n(t)((v_2(t) \cdot v_1)(\boldsymbol{\mu}(t) \cdot v_2) - (v_2(t) \cdot v_2)(\boldsymbol{\mu}(t) \cdot v_1)))\Big)$$

and

$$\beta_v(t) = \alpha(t)\sqrt{(\boldsymbol{\mu}(t) \cdot v_1)^2 + (\boldsymbol{\mu}(t) \cdot v_2)^2}.$$

Remark 3.8. If we take a positive orthonormal basis $\{v_1, v_2, v_3\}$ on $\mathbb{R}^3$ such that $v_3 \in S^2 \setminus \{\pm\boldsymbol{\mu}(I)\}$, then we may consider $\tilde{I} = I$. In this case, the Legendre curve (γ_v, v_v) can be defined globally.

3.3 Framed immersions

Let I and $\tilde{I}$ be intervals. A smooth function $s : \tilde{I} \to I$ is a (*positive*) *change of parameter* when s is surjective and has a positive derivative at every point. It follows that s is a diffeomorphism.

Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta_2$ and $(\tilde{\gamma}, \tilde{\nu}_1, \tilde{\nu}_2) : \tilde{I} \to \mathbb{R}^3 \times \Delta_2$ be framed curves whose curvatures are (ℓ, m, n, α) and $(\tilde{\ell}, \tilde{m}, \tilde{n}, \tilde{\alpha})$ respectively. Suppose (γ, ν_1, ν_2) and $(\tilde{\gamma}, \tilde{\nu}_1, \tilde{\nu}_2)$ are parametrically equivalent via the change of parameter $s : \tilde{I} \to I$. Thus $(\tilde{\gamma}(t), \tilde{\nu}_1(t), \tilde{\nu}_2(t)) = (\gamma(s(t)), \nu_1(s(t)), \nu_2(s(t)))$ for all $t \in \tilde{I}$. By differentiation, we have

$$\tilde{\ell}(t) = \ell(s(t))\dot{s}(t), \quad \tilde{m}(t) = m(s(t))\dot{s}(t), \quad \tilde{n}(t) = n(s(t))\dot{s}(t), \quad \tilde{\alpha}(t) = \alpha(s(t))\dot{s}(t).$$

Hence the curvature is dependent on the parametrization. Note that (γ, ν_1, ν_2) is a framed immersion if and only if $(\ell(t), m(t), n(t), \alpha(t)) \neq (0, 0, 0, 0)$ for all $t \in I$. Moreover, for an adapted frame, we may assume that $\ell(t) = 0$ for all $t \in I$.

In general, we cannot consider the arc-length parameter of the framed base curve γ, since γ may have singularities. However, if (γ, ν_1, ν_2) is an immersion, we introduce the arc-length parameter for the framed immersion (γ, ν_1, ν_2). The *speed* $s(t)$ of the framed immersion at the parameter t is defined to be the length of the tangent vector at t, namely,

$$s(t) = |(\dot{\gamma}(t), \dot{\nu}_1(t), \dot{\nu}_2(t))| = \sqrt{\dot{\gamma}(t) \cdot \dot{\gamma}(t) + \dot{\nu}_1(t) \cdot \dot{\nu}_1(t) + \dot{\nu}_2(t) \cdot \dot{\nu}_2(t)}.$$

Given scalars $a, b \in I$, we define the arc-length from $t = a$ to $t = b$ to be the integral of the speed,

$$L(\gamma, \nu) = \int_a^b s(t)\, dt.$$

By the same method for the are-length parameter of a regular curve, one can prove the following (cf. [5; 7; 8]).

Proposition 3.9. *Let* (γ, ν_1, ν_2) *be a framed immersion, and let* $t_0 \in I$. *Then* (γ, ν_1, ν_2) *is parametrically equivalent to a unit speed curve* $(\tilde{\gamma}, \tilde{\nu}_1, \tilde{\nu}_2) : \tilde{I} \to \mathbb{R}^3 \times \Delta_2$ *under a change of parameter* $t : \tilde{I} \to I$ *with* $t(0) = t_0$ *and with* $t'(s) > 0$.

We call the parameter s in Proposition 3.9 *the arc-length parameter for the framed immersion*. Let s be the are-length parameter for (γ, ν_1, ν_2). By definition, we have $\gamma'(s) \cdot \gamma'(s) + \nu'_1(s) \cdot \nu'_1(s) + \nu'_2(s) \cdot \nu'_2(s) = 1$, where $'$ is the derivation with respect to s. It follows that $2\ell(s)^2 + m(s)^2 + n(s)^2 + \alpha(s)^2 = 1$.

If we consider the framed immersion with an adapted frame, then $\ell(s) = 0$ for all $s \in I$. It follows that we have $m(s)^2 + n(s)^2 + \alpha(s)^2 = 1$.

4 Examples

Example 4.1. Let n_1, n_2, n_3, k_1 and k_2 be natural numbers with $n_2 = n_1 + k_1$ and $n_3 = n_2 + k_2$. Let $(\gamma, \nu_1, \nu_2) : \mathbb{R} \to \mathbb{R}^3 \times \Delta_2$ be defined by

$$\gamma(t) = \left(\frac{1}{n_1} t^{n_1}, \frac{1}{n_2} t^{n_2}, \frac{1}{n_3} t^{n_3} \right),$$

$$\nu_1(t) = \frac{1}{\sqrt{1 + t^{2k_1}}} (-t^{k_1}, 1, 0),$$

$$\nu_2(t) = \frac{1}{\sqrt{(1 + t^{2k_1})(1 + t^{2k_1} + t^{2k_1 + 2k_2})}} (-t^{k_1 + k_2}, -t^{2k_1 + k_2}, 1 + t^{2k_1}).$$

We see that (γ, ν_1, ν_2) is a framed curve, and a framed immersion when $n_1 = 1$ or $k_1 = 1$. We say that γ is of type (n_1, n_2, n_3). By definition, $\boldsymbol{\mu} : \mathbb{R} \to S^2$ is given by

$$\boldsymbol{\mu}(t) = \frac{1}{\sqrt{1 + t^{2k_1} + t^{2k_1 + 2k_2}}} (1, t^{k_1}, t^{k_1 + k_2})$$

and the components of the curvature are

$$\ell(t) = \frac{k_1 t^{2k_1+k_2-1}}{(1+t^{2k_1})\sqrt{1+t^{2k_1}+t^{2k_1+2k_2}}},$$

$$m(t) = \frac{-k_1 t^{k_1-1}}{\sqrt{(1+t^{2k_1})(1+t^{2k_1}+t^{2k_1+2k_2})}},$$

$$n(t) = \frac{-t^{k_1+k_2-1}(k_1+k_2+k_2 t^{2k_1})}{(1+t^{2k_1}+t^{2k_1+2k_2})\sqrt{1+t^{2k_1}}},$$

$$\alpha(t) = t^{n_1-1}\sqrt{1+t^{2k_1}+t^{2k_1+2k_2}}.$$

Example 4.2. Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be the smooth mapping $\gamma(t) = \begin{cases} (t, 0, e^{-1/t^2}) & \text{if } t > 0, \\ (0, 0, 0) & \text{if } t = 0, \\ (t, e^{-1/t^2}, 0) & \text{if } t < 0. \end{cases}$

The curve γ is regular but does not satisfy the linear independent condition at $t = 0$. However, γ is a framed base curve. We have the smooth mapping $(v_1, v_2) : \mathbb{R} \to \Delta_2$ with

$$v_1(t) = \begin{cases} \left(1/\sqrt{2+\dot{f}(t)^2}\right)(\dot{f}(t), -1, -1) & \text{if } t \neq 0, \\ (1/\sqrt{2})(0, -1, -1) & \text{if } t = 0, \end{cases}$$

$$v_2(t) = \begin{cases} \left(1/\sqrt{(1+\dot{f}(t)^2)(2+\dot{f}(t)^2)}\right)(\dot{f}(t), 1+\dot{f}(t)^2, -1) & \text{if } t > 0, \\ (1/\sqrt{2})(0, 1, -1) & \text{if } t = 0, \\ \left(1/\sqrt{(1+\dot{f}(t)^2)(2+\dot{f}(t)^2)}\right)(-\dot{f}(t), 1, -1-\dot{f}(t)^2) & \text{if } t < 0, \end{cases}$$

where $f(t) = e^{-1/t^2}$ for $t \neq 0$. It is easy to see that (γ, v_1, v_2) is a framed curve. Since $\mu : \mathbb{R} \to S^2$ is given by

$$\mu(t) = v_1(t) \times v_2(t) = \begin{cases} \left(1/\sqrt{1+\dot{f}(t)^2}\right)(1, 0, \dot{f}(t)) & \text{if } t > 0, \\ (1, 0, 0) & \text{if } t = 0, \\ \left(1/\sqrt{1+\dot{f}(t)^2}\right)(1, \dot{f}(t), 0) & \text{if } t < 0, \end{cases}$$

the curvature of the framed curve is given by

$$\ell(t) = \begin{cases} \dot{f}(t)\ddot{f}(t)/(2+\dot{f}(t)^2)\sqrt{1+\dot{f}(t)^2} & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -\dot{f}(t)\ddot{f}(t)/(2+\dot{f}(t)^2)\sqrt{1+\dot{f}(t)^2} & \text{if } t < 0, \end{cases}$$

$$m(t) = \begin{cases} \ddot{f}(t)/\sqrt{(1+\dot{f}(t)^2)(2+\dot{f}(t)^2)} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0, \end{cases}$$

$$n(t) = \begin{cases} \ddot{f}(t)/\left((1+\dot{f}(t)^2)\sqrt{2+\dot{f}(t)^2}\right) & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -\ddot{f}(t)/\left((1+\dot{f}(t)^2)\sqrt{2+\dot{f}(t)^2}\right) & \text{if } t < 0, \end{cases}$$

$$\alpha(t) = \begin{cases} \sqrt{1+\dot{f}(t)^2} & \text{if } t \neq 0, \\ 1 & \text{if } t = 0. \end{cases}$$

Note that consider the projection to $\mu(0) = (1, 0, 0)$ direction. Then $\gamma_\mu : \mathbb{R} \to \mathbb{R}^2$ is given by

$$\gamma_\mu(t) = \begin{cases} -(1/\sqrt{2})(e^{-1/t^2}, e^{-1/t^2}) & \text{if } t > 0, \\ (0, 0) & \text{if } t = 0, \\ (1/\sqrt{2})(-e^{-1/t^2}, e^{-1/t^2}) & \text{if } t < 0. \end{cases}$$

It follows that $\gamma_\mu : \mathbb{R} \to \mathbb{R}^2$ is not a frontal (cf. [6]).

References

[1] V. I. Arnol'd, *Singularities of caustics and wave fronts*, volume 62 of *Mathematics and its Applications* (*Soviet Series*).
     Kluwer 1990. MR1151185 (93b:58019) Zbl 0734.53001

[2] V. I. Arnol'd, S. M. Guseĭn-Zade, A. N. Varchenko, *Singularities of differentiable maps. Vol. I*. Birkhäuser 1985.
     MR777682 (86f:58018) Zbl 0554.58001

[3] R. L. Bishop, There is more than one way to frame a curve. *Amer. Math. Monthly* **82** (1975), 246–251.
     MR0370377 (51 #6604) Zbl 0298.53001

[4] M. Bôcher, Certain cases in which the vanishing of the Wronskian is a sufficient condition for linear dependence.
     Trans. Amer. Math. Soc. **2** (1901), 139–149. MR1500560 JFM 32.0313.02

[5] J. W. Bruce, P. J. Giblin, *Curves and singularities*. Cambridge Univ. Press 1992. MR1206472 (93k:58020) Zbl 0770.53002

[6] T. Fukunaga, M. Takahashi, Existence and uniqueness for Legendre curves. *J. Geom.* **104** (2013), 297–307.
     MR3089782 Zbl 1276.53002

[7] C. G. Gibson, *Elementary geometry of differentiable curves*. Cambridge Univ. Press 2001.
     MR1855907 (2002i:53005) Zbl 1031.53002

[8] A. Gray, E. Abbena, S. Salamon, *Modern differential geometry of curves and surfaces with Mathematica*.
     Chapman & Hall/CRC, Boca Raton, FL 2006. MR2253203 (2007d:53001) Zbl 1123.53001

[9] G. Peano, Sur le déterminant Wronskien. *Mathesis* **9** (1889), 75–76, 110–112. Zbl 21.0153.01

[10] K. Wolsson, A condition equivalent to linear dependence for functions with vanishing Wronskian.
     Linear Algebra Appl. **116** (1989), 1–8. MR989712 (90i:34012) Zbl 0671.15005