Framed vertex operator algebras, codes and the moonshine module

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Abstract. For a simple vertex operator algebra whose Virasoro element is a sum of commutative Virasoro elements of central charge $\frac{1}{2}$, two codes are introduced and studied. It is proved that such vertex operator algebras are rational. For lattice vertex operator algebras and related ones, decompositions into direct sums of irreducible modules for the product of the Virasoro algebras of central charge $\frac{1}{2}$ are explicitly described. As an application, the decomposition of the moonshine vertex operator algebra is obtained for a distinguished system of 48 Virasoro algebras.

1 Introduction

Vertex operator algebras (VOAs) have been studied by mathematicians for more than a decade, but still very little is known about the general structure of VOAs. Most of the examples so far come from an auxiliary mathematical structure like affine Kac-Moody algebras, Virasoro algebras, integral lattices or are modifications of these (like orbifolds and simple current extensions). We use the definition of VOA as in [FLM], Section 8.10.

In this paper we develop a general structure theory for a class of VOAs containing a subVOA of the same rank and relatively simple form, namely a tensor product of simple Virasoro VOAs of central charge $\frac{1}{2}$. We call this the class of *framed VOAs*, abbreviated FVOAs. It contains important examples of VOAs. We show how VOAs constructed from certain integral lattices can be described as framed VOAs. In

1991 Mathematics Subject Classification. Primary 17B69.

The first author is supported by NSF grants DMS-9303374, DMS-9700923 and a research grant from the Committee on Research, UC Santa Cruz.

The second author is supported by NSF grant DMS-9623038 and the University of Michigan Faculty Recognition Grant (1993–96).

The third author is supported by a research fellowship of the DFG, grant Ho 1842/1-1.

the case that the lattice itself comes from a binary code, this can be done even more explicitly. As an application of the general structure theory we describe VOAs of small central charge as FVOAs, especially the moonshine VOA $V^\natural$ of central charge 24.

The modules of a VOA together with the intertwining operators can be put together into a larger structure which is a called an intertwining algebra [Hu1], [Hu2]. In the case where the fusion algebra of the VOA is the group algebra of an abelian group G, like for lattice VOAs, this specializes to an *abelian* intertwining algebra [DL1]; also see [Mo]. The description of VOAs containing a fixed VOA with abelian intertwining algebra is relatively simple: They correspond to the subgroups $H \leq G$ such that all the conformal weights of the VOA-modules indexed by H are integral [H3]. The Virasoro VOA of rank $\frac{1}{2}$ gives one of the easiest examples of *non abelian* intertwining algebras. Section 2 can be considered as a study of the extension problem for the tensor products of this Virasoro VOAs.

It is our hope that the ideas used in this work can be extended to structure theories for VOAs based on other classes of rational subVOAs with nonabelian intertwining algebras, like the VOAs belonging to the discrete series representations of the Virasoro algebra [W].

We continue with a more detailed description of the results in this paper.

The Virasoro algebra of central charge $\frac{1}{2}$ has just three irreducible highest weight unitary representations, with highest weights $h = 0$, $\frac{1}{2}$, $\frac{1}{16}$, and the one with $h = 0$ carries the structure of a *simple* VOA whose irreducible modules are exactly these irreducible highest weight unitary representations. The relevant fusion rules here (Theorem 2.3) are relatively simple-looking. A tensor product of r such VOAs, denoted T_r, has irreducible representations in bijection with r-tuples $(h_1, \ldots, h_r)$ such that each $h_i \in \{0, \frac{1}{2}, \frac{1}{16}\}$.

We are interested in the case of a VOA V containing a subVOA isomorphic to T_r. Such a subVOA arises from a *Virasoro frame*, a set of elements $\omega_1, \ldots, \omega_r$ such that for each i, the vertex operator components of ω_i along with the vacuum element span a copy of the simple Virasoro VOA of central charge $\frac{1}{2}$ and such that these subVOAs are mutually commutative and $\omega_1 + \cdots + \omega_r$ is the Virasoro element of V. We abbreviate VF for Virasoro frame. Such elements may be characterized internally up to a factor 2 as the unique indecomposable idempotents in the weight 2 subalgebra of T_r with respect to the algebra product $u_1 v$ induced from the VOA structure on T_r.

It was shown in [DMZ] that the moonshine VOA $V^\natural$ is a FVOA with $r = 48$.

Partial results on decompositions of $V^\natural$ into a direct sum of irreducible T_{48}-modules were obtained in [DMZ] and [H1]. These results were fundamental in proving that $V^\natural$ is holomorphic [D3]. In fact, the desire to understand $V^\natural$ was one of the original motivations for us to study FVOAs.

In Section 2, we describe how the set of r-tuples which occur lead to two linear codes $\mathcal{C}$, $\mathcal{D} \leq \mathbb{F}_2^r$ where $\mathcal{D}$ is contained in the annihilator code $\mathcal{C}^\perp$. For self-dual FVOAs we give a proof that they are equal: $\mathcal{C} = \mathcal{D}^\perp$. Associated to these codes are normal 2-subgroups $G_\mathcal{D} \leq G_\mathcal{C}$ of the subgroup G of the automorphism group $\mathrm{Aut}(V)$ of V which stabilizes the VF (as a set). The group G is finite. We get an accounting of all subVOAs of V which contain V^0, the subVOA of $G_\mathcal{D}$-invariants. We obtain a general result (Theorem 2.12) that FVOAs are rational, establishing the existence of a new broad class of rational VOAs. The rationality of FVOAs is a very important aspect of their representation theory. In particular, a FVOA has only finitely many irreducible modules.

In Section 3, we describe the Virasoro decompositions of the lattice VOAs $V_{D_1^d}$, and closely related VOAs, with respect to a natural subVOA T_{2d}.

In Section 4, we study the familiar situation of the twisted or untwisted lattice associated to a binary doubly-even code of length $d \in 8\mathbb{Z}$ and the twisted and untwisted VOA associated to a lattice. A *marking* of the code is a partition of its coordinates into 2-sets. A marking determines a D_1^d sublattice in the associated lattices and a VF in the associated VOAs. We give an explicit description of the coset decomposition of the lattices under the D_1^d sublattice, a $\mathbb{Z}_4$-code, and the decomposition of the VOA as a module for the subVOA generated by the VF. As a corollary, we give information about various multiplicities of the decompositions under this subVOA using the symmetrized marked weight enumerator of the marked code or the symmetrized weight enumerator of the $\mathbb{Z}_4$-code.

Finally, Section 5 is devoted to applications. Two examples are discussed in detail. Example I is about the Hamming code of length 8, the root lattice E_8 and the VOA V_{E_8}. Here, $r = 16$, and we find at least 5 different VFs. Example II is about the Golay code, the Leech lattice and the moonshine module, $V^\natural$, where $r = 48$. For every VF inside $V^\natural$, the code $\mathcal{C}$ has dimension at most 41. There is a special marking for which this bound of 41 is achieved, and for this marking the complete decomposition polynomial is explicitly given. The D_1-frames inside the Leech lattice which arise from a marking of the Golay code are characterized by properties of the corresponding $\mathbb{Z}_4$-codes.

Appendix A contains a few special results about orbits on markings of the Hamming code, Appendix B the stabilizer in M_{24} of the above special marking for the Golay code, and Appendix C the structure of the automorphism group of the above code of dimension 41. Appendix D classifies automorphisms of a lattice VOA which correspond to -1 on the lattice.

In [M1]–[M3], there is a new treatment of the moonshine VOA and there is some overlap with results of this article. In particular, the vertex operator subalgebra similar to our V^0 (see section 2) and its representation theory have been independently investigated in [M3].

The authors are grateful to E. Betten for discussions on markings of the Golay code. They also thank A. Feingold and G. Mason for discussions.

Notation and terminology

$\mathbf{1}$	The vacuum element of a VOA.
$\mathrm{Aut}(V)$	The automorphism group of the VOA V.
B_V	The conformal block on the torus of the VOA V.
$\mathcal{B}_2^n$	The FVOA $(M(0,0) \oplus M(\frac{1}{2}, \frac{1}{2}))^{\otimes n}$ with binary code $\mathcal{C}(\mathcal{B}_2^n) = \{(0,0),(1,1)\}^n$ of length $2n$.
$(\mathcal{B}_2^n)_0$	The subVOA of $\mathcal{B}_2^n$ belonging to the subcode of $\mathcal{C}(\mathcal{B}_2^n)$ consisting of codewords of weights divisible by 4.
c	An element of $\mathbb{F}_2^n$.
C	A linear binary code, often self-annihilating and doubly-even.
$C^\perp$	The annihilator code of C.
$\mathcal{C} = \mathcal{C}(V)$	The binary code determined by the T_r-module structure of V^0.
$\mathbb{C}[L]$	The complex group algebra of the group L.
$\mathbb{C}\{L\}$	The twisted complex group algebra of the lattice L; it is the group algebra $\mathbb{C}[\hat{L}]$ modulo the ideal generated by $\kappa + 1$.
Co_0	The Conway group which is $\mathrm{Aut}(\Lambda)$, a finite group of order $2^{22}3^9 5^4 7^2 11.13.23$; its quotient by the center $\{\pm 1\}$ is a finite simple group.
d	The length of a binary code C, usually divisible by 8.
d_4^n	The marked binary code $\{(0,0,0,0),(1,1,1,1)\}^n$ of length $4n$.
$(d_4^n)_0$	The subcode of d_4^n consisting of codewords of weights divisible by 8.
D_n	The index 2 sublattice of $\mathbb{Z}^n$ consisting of vectors whose coordinate sum is even (the "checkerboard lattice").
$\mathcal{D} = \mathcal{D}(V)$	The binary code of the $I \subseteq \{1,\ldots,r\}$ with $V^I \neq 0$.
δ_2^n	The marked Kleinian or $\mathbb{F}_4$-code $\{(0,0),(1,1)\}^n$ of length $2n$.

$(\delta_2^n)_0$	The subcode of δ_2^n consisting of codewords of weights divisible by 4.
$\delta(c)$	The number of k with $c(k) = (c_{2k-1}, c_{2k}) \in \{(0,1),(1,0)\}$.
$\Delta(L)$	The $\mathbb{Z}_4$-code associated to a lattice L with fixed D_1-frame.
E_8	The root lattice of the Lie group $E_8(\mathbb{C})$.
ϵ	A vector with components $+$ or $-$.
FVOA	Abbreviation for framed vertex operator algebra.
$\mathcal{F}$	The set $\{M(0), M(\frac{1}{2}), M(\frac{1}{16})\}$.
G	The subgroup of $\mathrm{Aut}(V)$ fixing a VF of V.
$G_{\mathcal{C}}$	The normal subgroup of G acting trivially on T_r.
$G_{\mathcal{D}}$	The normal subgroup of G acting trivially on V^0.
$\mathcal{G} = \mathcal{G}_{24}$	The Golay code of length 24.
γ	An element of $\mathbb{Z}_4^n$.
$\gamma_{\epsilon_k}^a$	A map $\mathbb{F}_2^2 \longrightarrow \mathbb{Z}_4^2$.
Γ_{ϵ}^a	A map $\mathbb{F}_2^{2n} \longrightarrow \mathbb{Z}_4^{2n}$.
h, h_i	weights of elements or modules of a VOA, usually $h_i \in \{0, \frac{1}{2}, \frac{1}{16}\}$.
H_8	The Hamming code of length 8.
$\mathcal{H} = \mathcal{H}_6$	The hexacode of length 6, a code over $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$ or over the Kleinian fourgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{0, a, b, c\}$.
I	A subset of $\{1, \ldots, r\}$.
$I + J$	The symmetric difference, for subsets of $\{1, \ldots, r\}$.
κ	A central element of order 2 in the group $\hat{L}$.
L	An integral lattice, often self-dual and even.
$\hat{L}$	A central extension of L by a central subgroup κ.
L^*	The dual lattice of L.
L_C	The even lattice constructed from a doubly-even code C.
$\widetilde{L}_C$	The “twisted” even lattice constructed from a doubly-even code C.
$L(n)$, $L^i(n)$	The generator of a Virasoro algebra given by the expansion $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$, resp. $Y(\omega_i, z) = \sum_{n\in\mathbb{Z}} L^i(n) z^{-n-2}$.
Λ	The Leech lattice.
$\mathbb{M}$	The Monster simple group.
M_{24}	The simple Mathieu group of order $2^{10}.3^3.5.7.11.23 = 244,823,040$.
$M(0)$, $M(\frac{1}{2})$, $M(\frac{1}{16})$	The irreducible modules for the Virasoro algebra with central charge $\frac{1}{2}$.
$M(1)$	The canonical irreducible module for Heisenberg algebras.
$M(h_1, \ldots, h_r)$	The irreducible T_r-module of highest weight $(h_1, \ldots, h_r)$.
$m_h(V) = m_{h_1,\ldots,h_r}$	The multiplicity of the T_r-module $M(h_1, \ldots, h_r)$ in the FVOA V; we think of this as a function of $(h_1, \ldots, h_r) \in \{0, \frac{1}{2}, \frac{1}{16}\}^r$.

$\mathcal{M}$	A marking of a binary code.
n	A natural number.
$N^{ab}_{\mu_k \epsilon_k}$	A map $\mathbb{F}_2^2 \longrightarrow \mathbb{C}[\mathcal{F}^4]$.
$\mathbf{N}^{ab}_{\mu,\epsilon}$	A map $\mathbb{F}_2^{2n} \longrightarrow \mathbb{C}[\mathcal{F}^{4n}]$.
μ	A vector with components $+$ or $-$.
$P_V(a,b,c)$	The decomposition polynomial of a FVOA V.
r	The number of elements in a VF.
$R^a_{\mu_k}$	A map $\mathbb{Z}_4 \longrightarrow \mathbb{C}[\mathcal{F}^2]$.
$\mathbf{R}^a_{\mu}$	A map $\mathbb{Z}_4^n \longrightarrow \mathbb{C}[\mathcal{F}^{2n}]$.
$\mathrm{smwe}_C(x,y,z)$	The symmetrized marked weight enumerator of a binary code C with marking $\mathcal{M}$.
$\mathrm{swe}_\Delta(A,B,C)$	The symmetrized weight enumerator of a $\mathbb{Z}_4$-code Δ.
Sym_r, Sym_Ω	The symmetric group on a set of r objects, usually the index set $\{1,\ldots,r\}$, resp. the symmetric group on the set Ω.
Σ_2^n	The $\mathbb{Z}_4$-code $\{(0,0),(2,2)\}^n$ of length $2n$.
$(\Sigma_2^n)_0$	The subcode of Σ_2^n consisting of codewords of weights divisible by 4.
T	A faithful module of dimension 2^m for an extraspecial group of order 2^{1+2m}, for some m, or for a finite quotient of some $\hat{L}$.
$T_r = M(0)^{\otimes r}$	The tensor product of r simple Virasoro VOAs of rank $\frac{1}{2}$.
V	An arbitrary VOA, often holomorphic = self-dual.
$V(c)$	The submodule of the FVOA V isomorphic to $M(\frac{c}{2})$.
V_L	The VOA constructed from an even lattice L.
V_L^T	The $\mathbb{Z}_2$-twisted module of the lattice VOA V_L.
$\widetilde{V}_L$	The "twisted" VOA constructed from an even lattice L.
VF	Abbreviation for Virasoro frame.
VOA	Abbreviation for vertex operator algebra.
V^I	The sum of irreducible T_r-submodules of V isomorphic to $M(h_1,\ldots,h_r)$ with $h_i = \frac{1}{16}$ if and only if $i \in I$.
$V^0 = V^\emptyset$	This is V^I, for $I = 0$.
$V^\natural$	The moonshine VOA, or moonshine module.
$W(R)$	The Weyl group of type R, a root system.
$Y(\,.\,,z)$	A vertex operator.
Ξ_1, Ξ_3	Two D_8^*/D_8-codes of length 1 and 3.
ω, ω_i	Virasoro elements of rank r, $\frac{1}{2}$, respectively.
Ω	The "all ones vector" $(1,1,\ldots,1)$ in $\mathbb{F}_2^n$.

2 Framed vertex operator algebras

Recall that the Virasoro algebra of central charge $\frac{1}{2}$ has three irreducible unitary representations $M(h)$ of highest weights $h = 0, \frac{1}{2}, \frac{1}{16}$ (cf. [FQS], [GKO], [KR]). Moreover, $M(0)$ can be made into a simple vertex operator algebra with central charge $\frac{1}{2}$ (cf. [FZ]).

In [DMZ], a class of simple vertex operator algebras $(V, Y, \mathbf{1}, \omega)$ containing an even number of commuting Virasoro algebras of rank $\frac{1}{2}$ were defined.

Definition 2.1 Let r be any natural number. A simple vertex operator algebra V is called a *framed vertex operator algebra* (FVOA) if the following conditions are satisfied: There exist $\omega_i \in V$ for $i = 1, \ldots, r$ such that (a) each ω_i generates a copy of the simple Virasoro vertex operator algebra of central charge $\frac{1}{2}$ and the component operators $L^i(n)$ of $Y(\omega_i, z) = \sum_{n\in\mathbb{Z}} L^i(n)z^{-n-2}$ satisfy $[L^i(m), L^i(n)] = (m-n)L^i(m+n) + \frac{m^3-m}{24}\delta_{m,-n}$, (b) the r Virasoro algebras are mutually commutative, and (c) $\omega = \omega_1 + \cdots + \omega_r$. The set $\{\omega_1, \ldots, \omega_r\}$ is called a *Virasoro frame* (VF).

In this paper we assume that V is a FVOA. It follows that V is a unitary representation for each of the r Virasoro algebras of central charge $\frac{1}{2}$.

In [DMZ] it is also assumed that V_0 is one-dimensional. This assumption is now a consequence of the simplicity of V:

Remark 2.2 *A FVOA is truncated below from zero: $V = \bigoplus_{n\geq 0} V_n$ and V_0 is one dimensional: $V_0 = \mathbb{C}\,\mathbf{1}$.*

Proof. Let $Y(\omega_i, z) = \sum_{n\in\mathbb{Z}} L^i(n)z^{-n-2}$. Since V is a unitary representation for the Virasoro algebra generated by the components for $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ as $L(n) = \sum_{i=1}^r L^i(n)$ all weights of V are nonnegative that is, $V = \bigoplus_{n\geq 0} V_n$.

Then each nonzero vector $v \in V_0$ is a highest weight vector for the r Virasoro algebras with highest weight $(0, \ldots, 0)$. The highest weight module for the i-th Virasoro algebra generated by v is necessarily isomorphic to $M(0)$. From the construction of $M(0)$ we see immediately that $L^i(0)v = 0$. So $L(-1)v = \sum_i L^i(-1)v = 0$, i.e. v is a vacuum-like vector (see [L1]). It is proved in [L1] that a simple vertex operator algebra has at most one vacuum-like vector up to a scalar. Since $\mathbf{1}$ is a vacuum like vector, we conclude that $V_0 = \mathbb{C}\,\mathbf{1}$. $\square$

The following theorem can be found in [DMZ]:

Theorem 2.3 *(1) The VOA $M(0)$ has exactly three irreducible $M(0)$-modules, $M(h)$, with $h = 0, \frac{1}{2}, \frac{1}{16}$, and any module is completely reducible.*

(2) The nontrivial fusion rules among these modules are given by: $M(\frac{1}{2}) \times M(\frac{1}{2}) = M(0)$, $M(\frac{1}{2}) \times M(\frac{1}{16}) = M(\frac{1}{16})$ *and* $M(\frac{1}{16}) \times M(\frac{1}{16}) = M(0) + M(\frac{1}{2})$.

(3) Any module for the tensor product vertex operator algebra $T_r = M(0)^{\otimes r}$, *where* r *a positive integer, is a direct sum of irreducible modules* $M(h_1, \dots, h_r) := M(h_1) \otimes \cdots \otimes M(h_r)$ *with* $h_i \in \{0, \frac{1}{2}, \frac{1}{16}\}$.

(4) As T_r*-modules,*

$$V = \bigoplus_{h_i \in \{0, \frac{1}{2}, \frac{1}{16}\}} m_{h_1,\dots,h_r} M(h_1, \dots, h_r)$$

where the nonnegative integer $m_{h_1,\dots,h_r}$ *is the multiplicity of* $M(h_1, \dots, h_r)$ *in* V*. In particular, all the multiplicities are finite and* $m_{h_1,\dots,h_r}$ *is at most* 1 *if all* h_i *are different from* $\frac{1}{16}$.

Let I be a subset of $\{1, \dots, r\}$. Define V^I as the sum of all irreducible submodules isomorphic to $M(h_1, \dots, h_r)$ such that $h_i = \frac{1}{16}$ if and only if $i \in I$. Then

$$V = \bigoplus_{I \subseteq \{1,\dots,r\}} V^I.$$

Here and elsewhere we identify a subset of $\{1, 2, \dots, r\}$ with its characteristic function, an integer vector of zeros and ones. We further identify such vectors with their image under the reduction modulo 2, i.e. we consider them as binary codewords in $\mathbb{F}_2^r$. Interpretation should be clear from context, e.g. we think of the codeword c as an r-tuple of integers in the expression $\frac{1}{2}c$.

For each $c \in \mathbb{F}_2^r$ let $V(c)$ be the sum of the irreducible submodules isomorphic to $M(\frac{1}{2}c_1, \dots, \frac{1}{2}c_r)$. Then $V^0 = \bigoplus_{c \in \mathbb{F}_2^r} V(c)$. Recall the important fact mentioned in Theorem 2.3 (4) that for $c \in \mathcal{C}$ the T_r-module $M(\frac{1}{2}c_1, \dots, \frac{1}{2}c_r)$ has multiplicity 1 in V. So, $V(c) = 0$ or is isomorphic to $M(\frac{1}{2}c_1, \dots, \frac{1}{2}c_r)$.

We can now define two important binary codes $\mathcal{C} = \mathcal{C}(V)$ and $\mathcal{D} = \mathcal{D}(V)$.

Definition 2.4 For every FVOA V, let:

$$\mathcal{C} = \mathcal{C}(V) = \{c \in \mathbb{F}_2^r \mid V(c) \neq 0\}, \quad \text{and } \mathcal{D} = \mathcal{D}(V) = \{I \in \mathbb{F}_2^r \mid V^I \neq 0\}. \tag{2.1}$$

The vector of all multiplicities $m_{h_1,\dots,h_r}$ will be denoted by $m_h(V)$. Note that the codes $\mathcal{C}$ and $\mathcal{D}$ are completely determined by $m_h(V)$.

The following Proposition generalizes Proposition 5.1 of [DMZ] and Theorem 4.2.1 of [H1]. In particular it shows $\mathcal{C}$ and $\mathcal{D}$ are linear binary codes.

Proposition 2.5 *(1) $V^0 = V^\emptyset$ is a simple vertex operator algebra and the V^I are irreducible V^0-modules. Moreover V^I and V^J are inequivalent if $I \neq J$.*

(2) For any I and J and $0 \neq v \in V^J$, $\mathrm{span}\{u_n v \mid u \in V^I\} = V^{I+J}$ where $I + J$ is the symmetric difference of I and J. Moreover, $\mathcal{D}$ is an abelian group under symmetric difference.

(3) There is one to one correspondence between the subgroups $\mathcal{D}_0$ of $\mathcal{D}$ and the vertex operator subalgebras which contain V^0 via $\mathcal{D}_0 \mapsto V^{\mathcal{D}_0}$, where we define $V^S := \oplus_{I \in S} V^I$ for any subset S of $\mathcal{D}_0$. Moreover $V^{I+\mathcal{D}_0}$ is an irreducible $V^{\mathcal{D}_0}$-module for $I \in \mathcal{D}$ and $V^{I+\mathcal{D}_0}$ and $V^{J+\mathcal{D}_0}$ are nonisomorphic if the two cosets are different.

(4) Let $I \subseteq \{1, \ldots, r\}$ be given and suppose that $(h_1, \ldots, h_r)$ and $(h'_1, \ldots, h'_r)$ are r-tuples with h_i, $h'_i \in \{0, \frac{1}{2}, \frac{1}{16}\}$ such that $h_i = \frac{1}{16}$ (resp. $h'_i = \frac{1}{16}$) if and only if $i \in I$. If both $m_{h_1,\ldots,h_r}$ and $m_{h'_1,\ldots,h'_r}$ are nonzero then $m_{h_1,\ldots,h_r} = m_{h'_1,\ldots,h'_r}$. That is, all irreducible modules inside V^I for T_r have the same multiplicities.

(5) The binary code $\mathcal{C}$ is linear and $\mathrm{span}\{u_n v \mid u \in V(c)\} = V(c + d)$ for any c, $d \in \mathcal{C}$ and $0 \neq v \in V(d)$.

(6) Moreover, there is a one to one correspondence between vertex operator subalgebras of V^0 which contain T_r and the subgroups of $\mathcal{C}$, and V is completely reducible for such vertex operator subalgebras whose irreducible modules in V^0 are indexed by the corresponding cosets in $\mathcal{C}$.

Proof. Let $v \in V^J$ be nonzero. It follows from Proposition 2.4 of [DM] or Lemma 6.1.1 of [L2] and the simplicity of V that $V = \mathrm{span}\{u_n v \mid u \in V, n \in \mathbb{Z}\}$.

From the fusion rules given in Theorem 2.3 (2) and Proposition 2.10 of [DMZ] we see that $u_n v \in V^{I+J}$ exactly for $u \in V^I$. In particular, $\mathrm{span}\{u_n v \mid u \in V^0, n \in \mathbb{Z}\} = V^J$. So, V^J can be generated by any nonzero vector and V^J is a irreducible V^0-module. Since V^I and V^J are inequivalent T_r-modules if $I \neq J$ they are certainly inequivalent V^0-modules. By Proposition 11.9 of [DL1], we know that $Y(u, z)v \neq 0$ if u and v are not 0. Thus $V^{I+J} \neq 0$ if neither V^I or V^J are 0. This shows that $\mathcal{D}$ is a group. So, we finish the proof of (1) and (2).

For (3), we first observe that for a subgroup $\mathcal{D}_0$ of $\mathcal{D}$, (2) implies that $V^{\mathcal{D}_0}$ is a subVOA which contains V^0. On the other hand, since $V = V^{\mathcal{D}}$, V is a completely reducible V^0-module. Also V^I and V^J are inequivalent V^0-modules if I and J are different. Let U be any vertex operator subalgebra of V which contains V^0. Then U is a direct sum of certain V^I. Let $\mathcal{D}_0$ be the set of $I \in \mathcal{D}$ such that $V^I \leq U$. Then $0 \in \mathcal{D}_0$. Also from (2) if I, $J \in \mathcal{D}_0$ then $I + J \in \mathcal{D}_0$. Thus $\mathcal{D}_0$ is a subgroup of $\mathcal{D}$. In order to see the simplicity of U, we take a vector $v \in V^I$ for some $I \in \mathcal{D}_0$. Then $\mathrm{span}\{u_n v \mid u \in V^J, n \in \mathbb{Z}\} = V^{I+J}$ for any $J \in \mathcal{D}_0$. It is obvious that

$\{I+J \mid J\in\mathcal{D}_0\}=\mathcal{D}_0$. Thus U is simple. The proof of the irreducibility of $V^{I+\mathcal{D}_0}$ is similar to that of simplicity of $V^{\mathcal{D}_0}$. Inequivalence of $V^{I+\mathcal{D}_0}$ and $V^{J+\mathcal{D}_0}$ is clear as they are inequivalent T_r-modules.

The proofs of (5) and (6) are similar to that of (2) and (3).

For (4) we set $p=m_{h_1,\dots,h_r}$ and $q=m_{h'_1,\dots,h'_r}$. Let W_1, …, W_p be submodules of V isomorphic to $M(h_1,\dots,h_r)$ such that $\sum_{i=1}^p W_i$ is a direct sum. Let $d=(d_1,\dots,d_r)\in\mathcal{C}$ such that $V(d)\times M(h_1,\dots,h_r)=M(h'_1,\dots,h'_r)$. Set $W'_i=\mathrm{span}\{u_nW_i \mid u\in V(d),\, n\in\mathbb{Z}\}$ for $i=1$, …, p. Then W'_i is isomorphic to $M(h'_1,\dots,h'_r)$ for all i. Note that

$$\begin{aligned}&\mathrm{span}\{u_nW'_i \mid u\in V(d), n\in\mathbb{Z}\}\\ &=\mathrm{span}\{u_nv_mW_i \mid u,v\in V(d),\, m,n\in\mathbb{Z}\}\\ &=\mathrm{span}\{u_nW_i \mid u\in T_r, n\in\mathbb{Z}\}=W_i\end{aligned}$$

(cf. Proposition 4.1 of [DM]). Thus $\sum_{i=1}^p W'_i$ must be a direct sum in V. This shows that $p\le q$. Similarly, $p\ge q$. □

Remark 2.6 We can also define framed vertex operator superalgebras. The analogue of Proposition 2.5 still holds. In particular we have the binary codes $\mathcal{C}$ and $\mathcal{D}$.

Definition 2.7 Let G be the subgroup of $\mathrm{Aut}(V)$ consisting of automorphisms which stabilize the Virasoro frame $\{\omega_i\}$. Namely,

$$G=\{g\in\mathrm{Aut}(V) \mid g\{\omega_1,\dots,\omega_r\}=\{\omega_1,\dots,\omega_r\}\}\,. \tag{2.2}$$

The two subgroups $G_{\mathcal{C}}$ and $G_{\mathcal{D}}$ are defined by:

$$\begin{aligned}G_{\mathcal{C}} &= \{g\in G \mid g|_{T_r}=1\},\\ G_{\mathcal{D}} &= \{g\in G \mid g|_{V^0}=1\}.\end{aligned}$$

Finally, we define the automorphism group $\mathrm{Aut}(m_h(V))$ as the subgroup of the group Sym_r of permutations of $\{1,\dots,r\}$ which fixes the multiplicity function $m_h(V)$, i.e. which consists of the permutations $\sigma\in Sym_r$ such that $m_{h_1,\dots,h_r}=m_{h_{\sigma(1)},\dots,h_{\sigma(r)}}$.

It is easy to see that both $G_{\mathcal{D}}$ and and $G_{\mathcal{C}}$ are normal subgroups of G and $G_{\mathcal{D}}$ is a subgroup of $G_{\mathcal{C}}$.

Following Miyamoto [M1], we define for $i=1$, …, r an involution τ_i on V which acts on V^I as -1 if $i\in I$ and as 1 otherwise. The group generated by all τ_i is a subgroup of the group of all automorphisms of V and is isomorphic to the dual group $\hat{\mathcal{D}}$ of $\mathcal{D}$.

We define another group $F_{\mathcal{C}}$ which is a subgroup of $\mathrm{Aut}(V^0)$ and is generated by σ_i which acts on $M(h_1,\ldots,h_r)$ by -1 if $h_i=\frac{1}{2}$ and 1 otherwise. The group $F_{\mathcal{C}}$ is isomorphic to the dual group $\hat{\mathcal{C}}$ of $\mathcal{C}$.

Theorem 2.8 *(1) The subgroup $G_{\mathcal{D}}$ is isomorphic to the dual group $\hat{\mathcal{D}}$ of $\mathcal{D}$.*

(2) $G_{\mathcal{C}}/G_{\mathcal{D}}$ is isomorphic to a subgroup of the dual group $\hat{\mathcal{C}}$ of $\mathcal{C}$.

(3) $G/G_{\mathcal{C}}$ is isomorphic to a subgroup of $\mathrm{Aut}(m_h(V))\le Sym_r$. In particular, G is a finite group.

(4) For any $g\in G$ and a T_r-submodule W of V isomorphic to $M(h_1,\ldots,h_r)$ then gW is isomorphic to $M(h_{\mu_g^{-1}(1)},\ldots,h_{\mu_g^{-1}(r)})$ where $\mu_g\in Sym_r$ such that $g\omega_i=\omega_{\mu_g(i)}$ for all i.

(5) If the eigenvalues of $g\in G_{\mathcal{C}}$ on V^I are i and $-i$, then i and $-i$ have the same multiplicity.

Proof. (1) Let $g\in G$ such that $g\mid_{V^0}=1$. Recall from Proposition 2.5 that $V=\bigoplus_{I\in\mathcal{D}}V^I$. Since each V^I is an irreducible V^0-module we have $V^I=\mathrm{span}\{v_nu\mid v\in V^0,n\in\mathbb{Z}\}$ for any nonzero vector $u\in V^I$. Note that g preserves each homogeneous subspace V^I_n, which is finite-dimensional. Take $u\in V^I$ to be an eigenvector of g with eigenvalue x_I and let $v\in V^0$. Then $g(v_nu)=v_ngu=x_Iv_nu$. Thus g acts on V^I as the constant x_I. For any $0\ne u\in V^I$ and $0\ne v\in V^J$ we have $0\ne Y(u,z)v\in V^{I+J}[[z,z^{-1}]]$. Since $x_{I+J}Y(u,z)v=gY(u,z)v=x_Ix_JY(u,z)v$, we see that $x_Ix_J=x_{I+J}$. In particular $x\in\hat{\mathcal{D}}$ and x takes values in $\{\pm1\}$. Clearly, each $g\in\hat{\mathcal{D}}$ acts on V^0 trivially since $\hat{\mathcal{D}}$ is generated by the τ_i. This proves (1).

For (2) we take $g\in G_{\mathcal{C}}$. A similar argument as in the first paragraph shows that $g|_{V(c)}$ is a constant $y_c=\pm1$ and $y_{c+d}=y_cy_d$. In other words we have defined an element y of $\hat{\mathcal{C}}$ which maps $c\in\mathcal{C}$ to y_c. One can easily see that this gives a group homomorphism from $G_{\mathcal{C}}$ to $\hat{\mathcal{C}}$ with kernel $G_{\mathcal{D}}$.

For (3) let $g\in G$. Then there exists a unique $\mu_g\in Sym_r$ such that $g\omega_i=\omega_{\mu_g(i)}$. Clearly we have $\mu_{g_1g_2}=\mu_{g_1}\mu_{g_2}$ for $g_1,\ g_2\in G$. It is obvious that the kernel of the map $g\mapsto\mu_g$ is $G_{\mathcal{C}}$.

In order to prove (4), we take a highest weight vector v of W. Then $L^i(0)v=h_iv$ for $i=1,\ \ldots,\ r$. So $L^i(0)gv=gL^{\mu_g^{-1}(i)}(0)v=h_{\mu_g^{-1}}v$ and gv is a highest weight vector with highest weight $(h_{\mu_g^{-1}(1)},\ldots,h_{\mu_g^{-1}(r)})$. That is, gW is isomorphic to $M(h_{\mu_g^{-1}(1)},\ldots,h_{\mu_g^{-1}(r)})$.

Finally, we turn to (5). We first mention how a general $g\in G_{\mathcal{C}}$ acts on V^I for $I\in\mathcal{D}$. Note that $g^2=1$ on V^0 by the proof of (2), that is, $g^2\in G_{\mathcal{D}}$. So $g^2=\pm1$

on each V^I. This implies that g is diagonalizable on V^I whose eigenvalues are ± 1 if $g^2 = 1$ on V^I and are $\pm i$ if $g^2 = -1$ on V^I.

In the second case, let $V^I = W_1 \oplus \cdots \oplus W_p \oplus M_1 \oplus \cdots \oplus M_q$ where all W_j, M_k are irreducible T_r-modules and $g = i$ on each W_j and $g = -i$ on each M_k. On V^0, g is not 1, since otherwise g is in $G_{\mathcal{D}}$ and g would have only ± 1 for eigenvalues, by (1). Take an irreducible T_r-submodule U of V^0 so that $g|_U = -1$. Set $W_j' = \{ u_n W_j \mid u \in U, n \in \mathbb{Z}\}$. Then $g = -i$ on each W_j'.

Claim: $\sum_{j=1}^{p} W_j'$ is a direct sum.

Using associativity, we see that

$$\begin{aligned} \operatorname{span}\{u_n W_j' \mid u \in U, n \in \mathbb{Z}\} &= \operatorname{span}\{u_m v_n W_j \mid u, v \in U,\ m, n \in \mathbb{Z}\} \\ &= \operatorname{span}\{v_n W_j \mid v \in T_r,\ n \in \mathbb{Z}\} = W_j. \end{aligned}$$

This proves the claim. Thus $p \le q$. Similarly, $q \le p$. So they must be equal. This finishes the proof. □

The results in Proposition 2.5 (2) and (3) resp. (5) and (6) can be interpreted by the "quantum Galois theory" developed in [DM] and [DLM2]. For example, Proposition 2.5 (2) and (3) is now a special case of Theorems 1 and 3 of [DM] applied for the group $G_{\mathcal{D}}$:

Remark 2.9 Note that V^0 is the space of $G_{\mathcal{D}}$-invariants. There is a one to one correspondence between the subgroups of $G_{\mathcal{D}}$ and vertex operator subalgebras of V containing V^0 via $H \mapsto V^H$. In fact, $V^H = \bigoplus_{I \in H'} V^I$ where $H' = \{I \in \mathcal{D} \mid H|_{V^I} = 1\}$. Under the identification of $G_{\mathcal{D}}$ with $\hat{\mathcal{D}}$, the subcode H' of $\mathcal{D}$ corresponds to the common kernel of the functionals in H.

Next we prove that a FVOA is always rational. Recall the definition of rationality and regularity as defined in [DLM1]. A vertex operator algebra is called *rational* if any admissible module is a direct sum of irreducible admissible modules and a rational vertex operator algebra is *regular* if any weak module is a direct sum of ordinary irreducible modules. (The reader is refereed to [DLM3] for the definitions of weak module, admissible module, and ordinary module.)

It is proved in [DLM3] that if V is a rational vertex operator algebra then V has only finitely many irreducible admissible modules and each is an ordinary irreducible module.

We need two lemmas.

Lemma 2.10 *Let V be a FVOA such that $\mathcal{D}(V) = 0$. Then any nonzero weak V-module W contains an ordinary irreducible module.*

Proof. Since $\mathcal{D}(V) = 0$ we have the decomposition $V = \bigoplus_{c \in \mathcal{C}} V(c)$. Since T_r is regular (see Proposition 3.3 of [DLM1]), W is a direct sum of ordinary irreducible T_r-modules. Let M be an irreducible T_r-submodule of W. Then

$$N := \text{span}\{u_n M \mid u \in V, \, n \in \mathbb{Z}\}$$

is an ordinary V-module as each $\text{span}\{u_n M \mid u \in V(c), \, n \in \mathbb{Z}\}$ is an ordinary irreducible T_r-module and $\mathcal{C}$ is a finite set. For an ordinary V-module X we define $m(X)$ to be the sum of the multiplicities $m_{h_1,\ldots,h_r}$ of all modules $M(h_1, \ldots, h_r)$ in X, i.e., the T_r-composition length. Let K be a V-submodule of N such that $m(K)$ is the smallest among all nonzero V-submodules of N. Then K is an irreducible ordinary V-submodule of N and of W. □

Lemma 2.11 *Any FVOA V with $\mathcal{D}(V) = 0$ is rational.*

Proof. We must show that any admissible V-module is a direct sum of irreducible ones. Let W be an admissible V-module and M the sum of all irreducible V-submodules. We prove that $W = M$. Otherwise by Lemma 2.10 the quotient module W/M has an irreducible submodule W'/M where W' is a submodule of W which contains M. Let U be an irreducible T_r submodule of W' such that $U \cap M = 0$ and set $X := \text{span}\{v_n U \mid v \in V, \, n \in \mathbb{Z}\}$. Then X is a submodule of W' and $W' = M + X$. Note that $U[c] := \text{span}\{v_n U \mid v \in V(c), \, n \in \mathbb{Z}\}$ for each $c \in \mathcal{C}$ is an irreducible T_r-module. Then either $U[c] \cap M = 0$ or $U[c] \cap M = U[c]$. If the latter happens, then $Y(v,z)(U + M/M) = 0$ in the quotient module W/M, which is impossible by Proposition 11.9 of [DL]. Thus $U[c] \cap M = 0$ for all $c \in \mathcal{C}$ and $W' = M \oplus X$. By Lemma 2.10, X has an irreducible V-submodule Y and certainly $M \oplus Y$ strictly contains M. This is a contradiction. □

Theorem 2.12 *Any FVOA V is rational.*

Proof. Let W be an admissible V-module. Then W is a direct sum of irreducible V^0-modules by Lemma 2.11. Let M be an irreducible V^0-module. It is enough to show that M is contained in an irreducible V-submodule of W. First note that there exists a subset I of $\{1, \ldots, r\}$ such that for every irreducible T_r-module $M(h_1, \ldots, h_r)$ inside M we have $h_i = \frac{1}{16}$ if and only if $i \in I$. Let X be the V-submodule generated by M. Then $X = \sum_{J \in \mathcal{D}} X[J] \leq W$ where $X[J] = \text{span}\{u_n M \mid u \in V^J, \, n \in \mathbb{Z}\}$ is a V^0-module. We will show that X is an irreducible V-module.

By the fusion rules, we know that for every irreducible T_r-submodule of V which is isomorphic to $M(h_1, \ldots, h_r)$ has $h_k = \frac{1}{16}$ if and only if $k \in I + J$. The $X[J]$ for

$J \in \mathcal{D}$ are nonisomorphic V^0-modules as they are nonisomorphic T_r-modules. Thus $X = \bigoplus_{J\in\mathcal{D}} X[J]$.

Let Y be a nonzero V-submodule of X. Then $Y = \bigoplus_{J\in\mathcal{D}} Y[J]$ where $Y[J] = Y \cap X[J]$ is a V^0-module. If $Y[J] \neq 0$ then $\mathrm{span}\{v_n Y_J \mid v \in V^J,\, n \in \mathbb{Z}\} \neq 0$. Otherwise use the associativity of vertex operators to obtain

$$0=\mathrm{span}\{u_m v_n Y[J] \mid u,v \in V^J,\, m,n \in \mathbb{Z}\}=\mathrm{span}\{v_n Y[J] \mid v \in V^0,\, n \in \mathbb{Z}\}=Y[J].$$

By associativity again we see that $\mathrm{span}\{v_n Y[J] \mid v \in V^J,\, n \in \mathbb{Z}\}$ is a nonzero V^0-submodule of M. Since M is irreducible it follows immediately that $\mathrm{span}\{v_n Y[J] \mid v \in V^J,\, n \in \mathbb{Z}\} = M$. So M is a subspace of Y. Since X is generated by M as a V-module we immediately have $X = Y$. This shows that X is indeed an irreducible V-module.

It should be pointed out that each $X[J]$ in fact is an irreducible V^0-module. Let $0 \neq u \in X[J]$. Since $X = \mathrm{span}\{v_n u \mid v \in V,\, n \in \mathbb{Z}\}$ we see that $\mathrm{span}\{v_n u \mid v \in V^0,\, n \in \mathbb{Z}\} = X[J]$. □

Corollary 2.13 *Let V be a FVOA. Then*

(1) V has only finitely many irreducible admissible modules and every irreducible admissible V-module is an ordinary irreducible V-module.

(2) V is regular, that is, any weak V-module is a direct sum of ordinary irreducible V-modules.

Proof. We have already mentioned that (1) is true for all rational vertex operator algebra (see [DLM3]). So, (1) is an immediate consequence of Theorem 2.12. In [DLM1] we proved that (2) is true for any rational vertex operator algebra which has a regular vertex operator subalgebra with the same Virasoro element. Note that T_r is such a vertex operator subalgebra of V. □

Theorem 2.12 is very useful. We will see in the later sections that the FVOAs V_Λ^+ and $V^\natural$ are rational vertex operator algebras. Theorem 2.12 simplifies the original proofs of the rationality of V_Λ^+ in [D3] and $V^\natural$ in [DLM1]. Most important, we do *not* use the self-dual property of $V^\natural$ (i.e., $V^\natural$ is the only irreducible module for itself) as proved in [D3].

It is a interesting problem to find suitable invariants for a FVOA V. Two invariants of V are the binary codes codes $\mathcal{C}$ and $\mathcal{D}$ of length r as defined before. They cannot be arbitrary but must satisfy the following conditions:

Proposition 2.14 *(1) The code $\mathcal{C}$ is even, i.e. the weight $\mathrm{wt}(c) = \sum_{i=1}^r c_i \in \mathbb{Z}_+$ of every codeword $c \in \mathcal{C}$ is divisible by 2.*

(2) The weights of all codewords $d \in \mathcal{D}$ are divisible by 8.

(3) The binary code $\mathcal{D}$ is a subcode of the annihilator code $\mathcal{C}^\perp = \{d = (d_i) \in \mathbb{F}_2^r \mid (d,c) = \sum_i d_i c_i = 0$ for all $c = (c_i) \in \mathcal{C}\}$.

Proof. Let W be a T_r-submodule isomorphic to $M(h_1,\ldots,h_r)$. Then the weight of a highest weight vector of W is $h_1+h_2+\cdots+h_r$ which is necessarily an integer as V is $\mathbb{Z}$-graded. The parts (1) and (2) now follow immediately. To see (3), note that for $c \in \mathcal{C}$ and $M \leq V^I$ isomorphic to $M(g_1,\ldots,g_r)$ one has from the fusion rules given in Theorem 2.3 (2) that

$$M' = \mathrm{span}\{u_n M(g_1,\ldots,g_r) \mid u \in V(c),\, n \in \mathbb{Z}\} \leq V^I$$

is isomorphic to $M(h_1,\ldots,h_r)$ with $h_i = g_i = \frac{1}{16}$ if $i \in I$ and $h_i = 0$ (resp. $h_i = \frac{1}{2}$) if $c_i + 2g_i = 0$ in $\mathbb{F}_2$ (resp. $c_i + 2g_i = 1$). Since the conformal weights $g_1+\cdots+g_r$ and $h_1+\cdots+h_r$ of $M(g_1,\ldots,g_r)$ and $M(h_1,\ldots,h_r)$ are both integral we see that $\#(\{i \in \{1,\ldots,r\} \mid c_i = 1\} \setminus I)$ is an even integer. Thus $\#\{i \in I \mid c_i = 1\}$ is also even as $\mathrm{wt}(c)$ is even. This implies that $(d,c) = 0$, as required, where $d \in \mathcal{D}$ is the codeword belonging to $I \subseteq \{1,\ldots,r\}$. □

Here are a few remarks on the action of $\mathrm{Aut}(V)$ on V_2, which is an action preserving the algebra product $a * b := a_3 b$ coming from the VOA structure.

Remark 2.15 (1) If V is a VOA and is generated as a VOA by V_2, then $\mathrm{Aut}(V)$ acts faithfully on V_2. This happens in the case $V = V_L^+$, where L is a lattice spanned by its vectors x such that $(x,x) = 4$.

(2) If V is a FVOA, the kernel of the action of $\mathrm{Aut}(V)$ on V_2 is contained in the intersection of the groups $G_{\mathcal{C}}$, as we vary over all frames. Hence, this kernel is a finite 2-group, of nilpotence class at most two and order dividing 2^r, where $r = \mathrm{rank}(V)$.

The framed vertex operator algebras with $\mathcal{D} = 0$ can be completely understood in an easy way.

Proposition 2.16 *For every even linear code $C \leq \mathbb{F}_2^r$ there is up to isomorphism exactly one FVOA V_C such that the associated binary codes are $\mathcal{C} = C$ and $\mathcal{D} = 0$.*

Proof. Let $V_{\mathrm{Fermi}} = M(0) \oplus M(\frac{1}{2})$ be the super vertex operator algebra as described in [KW]. The (graded) tensor product $V_{\mathrm{Fermi}}^{\otimes r}$ is a super vertex operator algebra whose code $\mathcal{C}$ is the complete code $\mathbb{F}_2^r$ (see Remark 2.6). It has the property, that the even vertex operator subalgebra is the vertex operator algebra associated to the level 1 irreducible highest weight representation for the affine Kac-Moody algebra

$D_{r/2}$ if r is even and $B_{(r-1)/2}$ if r odd (see [H1], chapter 2). The code $\mathcal{C}$ for this vertex operator algebra is the even subcode of $\mathbb{F}_2^r$. Proposition 2.5 (6) gives, for every even code $C \leq \mathbb{F}_2^r$, a FVOA V such that $\mathcal{C}(V) = C$ and $\mathcal{D}(V) = 0$. The uniqueness of the FVOA with code $\mathcal{C}(V) = C$ up to isomorphism follows from a general result on the uniqueness of simple current extensions of vertex operator algebras [H3]. □

This proposition is also proved in a different way by Miyamoto in [M2], [M3].

Recall that a holomorphic (or self-dual) VOA is a VOA V whose only irreducible module is V itself. In the case of holomorphic FVOAs, we can show that the subcode $\mathcal{D} \leq \mathcal{C}^{\perp}$ is in fact equal to $\mathcal{C}^{\perp}$.

We need some basic facts from [Z] and [DLM4] about the "conformal block on the torus" B_V [Z] of a VOA V. To apply Zhu's modular invariance theorems one has to assume that V is rational and satisfies the C_2 condition.[1] It was proved in [DLM4] that the moonshine VOA satisfies the C_2 condition. The same proof in fact works for any FVOA. We also know from Theorem 2.12 that a FVOA is rational.

Applying Zhu's result to a FVOA V yields that B_V is a finite dimensional complex vector space with a canonical base T_{M_i} indexed by the inequivalent irreducible V-modules M_i and that B_V carries a natural $SL_2(\mathbb{Z})$-module structure $\rho_V : SL_2(\mathbb{Z}) \longrightarrow GL(B_V)$.

Let V and W be two rational VOAs satisfying the C_2 condition. The following two properties of the conformal block follow directly from the definition:

(B1) $B_{V\otimes W} = B_V \otimes B_W$ as $SL_2(\mathbb{Z})$-modules and $T_{M_i\otimes M_j} = T_{M_i} \otimes T_{M_j}$.

(B2) If W is a subVOA of V with the same Virasoro element then there is a natural $SL_2(\mathbb{Z})$-module map $\iota^* : B_V \longrightarrow B_W$.

We also need the following well-known result:

(B3) For the vertex operator algebra $M(0)$, the action of $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ on $B_{M(0)}$ in the canonical basis $\{T_{M(0)}, T_{M(\frac{1}{2})}, T_{M(\frac{1}{16})}\}$ is given by the matrix

$$\begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}. \tag{2.3}$$

Here is a result about binary codes used in the proof of Theorem 2.19 below:

Lemma 2.17 *Let $\mu^{\otimes n}$ the n-fold tensor product of the matrix $\mu = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ considered as a linear endomorphism of the vector space $\mathbb{C}[\mathbb{F}_2^n] \cong \mathbb{C}[\mathbb{F}_2]^{\otimes n}$ on the canonical base $\{e_v \mid v \in \mathbb{F}_2^n\}$. For a subset $X \subseteq \mathbb{F}_2^n$ denote by $\chi_X = \sum_{v\in X} e_v$ the characteristic*

[1] The condition that V be a direct sum of highest weight representations for the Virasoro algebra was also required in [Z], but was removed in [DLM4].

function of X. Then the following relation between a linear code C and its annihilator $C^\perp$ holds:

$$\chi_{C^\perp} = \frac{1}{|C|} \cdot \mu^{\otimes n}(\chi_C).$$

Remark 2.18 $\mu^{\otimes n}$ is a Hadamard matrix of size 2^n and the corresponding linear map is called the Hadamard transform.

Proof. For every $\mathbb{Z}$-module R and function $f : \mathbb{F}_2^n \longrightarrow R$ the following relation holds (cf. Ch. 5, Lemma 2 of [MaS])

$$|C| \sum_{v\in C^\perp} f(v) = \sum_{u\in C} \sum_{v\in\mathbb{F}_2^n} (-1)^{(u,v)} f(v). \tag{2.4}$$

Now let R be the abelian group $\mathbb{C}[\mathbb{F}_2^n]$ and define f by $f(v) = e_v$ for all $v \in \mathbb{F}_2^n$. The left hand side of (2.4) is $|C| \cdot \chi_{C^\perp}$. Expansion of the right side gives:

$$\sum_{u\in C} \sum_{v_1,\ldots,v_n\in\mathbb{F}_2} \prod_{i=1}^n (-1)^{u_i v_i} e_{v_1} \otimes \cdots \otimes e_{v_n} = \sum_{u\in C} \mu^{\otimes n}(e_u) = \mu^{\otimes n}(\chi_C). \quad \square$$

Theorem 2.19 *For a holomorphic FVOA the binary codes $\mathcal{C}$ and $\mathcal{D}$ satisfy $\mathcal{D} = \mathcal{C}^\perp$.*

Proof. The vector of multiplicities $m_h(V)$ can be regarded as an element in the vector space $\mathbb{C}[\mathcal{F}^r] \cong \mathbb{C}[\mathcal{F}]^{\otimes r}$ where $\mathcal{F} = \{M(0), M(\frac{1}{2}), M(\frac{1}{16})\}$. Define two linear maps π, $\theta : \mathbb{C}[\mathcal{F}] \longrightarrow \mathbb{C}[\mathbb{F}_2] = \mathbb{C}\, e_0 \oplus \mathbb{C}\, e_1$ by

$$\pi(M(0)) = e_0, \qquad \pi(M(\frac{1}{2})) = e_0, \qquad \pi(M(\frac{1}{16})) = \sqrt{2} e_1,$$

and

$$\theta(M(0)) = e_0, \qquad \theta(M(\frac{1}{2})) = e_1, \qquad \theta(M(\frac{1}{16})) = 0.$$

Finally let $\sigma : \mathbb{C}[\mathcal{F}] \longrightarrow \mathbb{C}[\mathcal{F}]$ the linear map given by the matrix (2.3) relative to the basis $\mathcal{F}$. Now one has $\pi \circ \sigma = \mu \circ \theta$ and thus the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}[\mathcal{F}^r] & \stackrel{\sigma^{\otimes r}}{\longrightarrow} & \mathbb{C}[\mathcal{F}^r] \\ \downarrow \theta^{\otimes r} & & \downarrow \pi^{\otimes r} \\ \mathbb{C}[\mathbb{F}_2^r] & \stackrel{\mu^{\otimes r}}{\longrightarrow} & \mathbb{C}[\mathbb{F}_2^r]. \end{array} \tag{2.5}$$

By definition, the support of $\pi^{\otimes r}(m_h(V))$ is $\mathcal{D} \leq \mathbb{F}_2^r$. From Lemma 2.17 $\mu^{\otimes r} \circ \theta^{\otimes r}(m_h(V)) = |\mathcal{C}| \cdot \chi_{\mathcal{C}^\perp} \in \mathbb{C}[\mathbb{F}_2^r]$. Note that the support of $\chi_{\mathcal{C}^\perp}$ is $\mathcal{C}^\perp$. These facts together with (2.5) imply the theorem if we can show that $\sigma^{\otimes r}(m_h(V)) = m_h(V)$.

We identify $\mathbb{C}[\mathcal{F}^r]$ with the conformal block on the torus of the VOA T_r by identifying the canonical bases: $M = T_M$. Using (B1) and (B3) we observe that $\sigma^{\otimes r} = \rho_{T_r}(S)$, where ρ_{T_r} is the representation $\rho_{T_r} : SL_2(\mathbb{Z}) \longrightarrow GL(B_{T_r})$ of degree 3^r.

Define the shifted graded character $ch_V(\tau) := q^{-c/24} \sum_{n\geq 0} (\dim V_n) q^n$ where c is the central charge of V. Since V is holomorphic, the conformal block B_V is one dimensional. Then $\rho_V(S) = 1$ (the case $\rho_V(S) = -1$ is impossible since $ch_V(i) > 0$ where i is the square root of -1 in upper half plane; cf. [H1], proof of Cor. 2.1.3). Now we use (B2). The generator T_V of B_V is mapped by ι^* to $\sum m_{h_1,\ldots h_r} T_{M(h_1,\ldots,h_r)} = m_h(V)$. Since ι^* is $SL_2(\mathbb{Z})$-equivariant we get $\sigma^{\otimes r}(m_h(V)) = \rho_{T_r}(S)(m_h(V)) = \iota^*(\rho_V(S)(T_V)) = m_h(V)$. $\square$

The same kind of argument was used in the proof of Theorem 4.1.5 in [H1].

3 Vertex operator algebras $V_{D_1^d}$

Let $D_n = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid \sum_{i=i}^n x_i \text{ even}\} \leq \mathbb{R}^n$, $n \geq 1$, be the root lattice of type D_n, the "checkerboard lattice". In this section, we describe the Virasoro decomposition of modules and twisted modules for the vertex operator algebra $V_{D_1^d}$.

We work in the setting of [FLM] and [DMZ]. In particular L is an even lattice with nondegenerate symmetric $\mathbb{Z}$-bilinear form $\langle \cdot, \cdot \rangle$; $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$; $\hat{\mathfrak{h}}_{\mathbb{Z}}$ is the corresponding Heisenberg algebra; $M(1)$ is the associated irreducible induced module for $\hat{\mathfrak{h}}_{\mathbb{Z}}$ such that the canonical central element of $\hat{\mathfrak{h}}_{\mathbb{Z}}$ acts as 1; $(\hat{L}, {}^-)$ is the central extension of L by $\langle \kappa \mid \kappa^2 = 1 \rangle$, a group of order 2, with commutator map $c_0(\alpha, \beta) = \langle \alpha, \beta \rangle + 2\mathbb{Z}$; $c(\cdot, \cdot)$ is the alternating bilinear form given by $c(\alpha, \beta) = (-1)^{c_0(\alpha,\beta)}$ for $\alpha, \beta \in L$; χ is a faithful linear character of $\langle \kappa \rangle$ such that $\chi(\kappa) = -1$; $\mathbb{C}\{L\} = \operatorname{Ind}_{\langle \kappa \rangle}^{\hat{L}} \mathbb{C}_\chi$ ($\simeq \mathbb{C}[L]$, linearly), where $\mathbb{C}_\chi$ is the one-dimensional $\langle \kappa \rangle$-module defined by χ; $\iota(a) = a \otimes 1 \in \mathbb{C}\{L\}$ for $a \in \hat{L}$; $V_L = M(1) \otimes \mathbb{C}\{L\}$; $\mathbf{1} = \iota(1)$; $\omega = \frac{1}{2} \sum_{r=1}^d \beta_r(-1)^2$ where $\{\beta_1, \ldots, \beta_d\}$ is an orthonormal basis of $\mathfrak{h}$; It was proved in [B] and [FLM] that there is a linear map

$$\begin{array}{lcl} V_L & \to & (\operatorname{End} V_L)[[z, z^{-1}]], \\ v & \mapsto & Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \operatorname{End} V_L) \end{array}$$

such that $V_L = (V_L, Y, \mathbf{1}, \omega)$ is a simple vertex operator algebra. Let $L^* = \{x \in \mathfrak{h} \mid \langle x, L \rangle \leq \mathbb{Z}\}$ be the dual lattice of L. Then the irreducible modules of V_L are the $V_{L+\gamma}$

(which are defined in [D1]) indexed by the elements of the quotient group L^*/L (see [D1]). In fact, V_L is a rational vertex operator algebra (see [DLM1]).

Let θ be the automorphism of $\hat{L}$ such that $\theta(a) = a^{-1}\kappa^{\langle \bar{a},\bar{a}\rangle/2}$. Then θ is a lift of the -1 automorphism of L. We have an automorphism of V_L, denoted again by θ, such that $\theta(u \otimes \iota(a)) = \theta(u) \otimes \iota(\theta a)$ for $u \in M(1)$ and $a \in \hat{L}$. (See Appendix D for a fuller discussion.) Here the action of θ on $M(1)$ is given by $\theta(\alpha_1(n_1)\cdots\alpha_k(n_k)) = (-1)^k\alpha_1(n_1)\cdots\alpha_k(n_k)$. The θ-invariants V_L^+ of V_L form a simple vertex operator subalgebra and the -1-eigenspace V_L^- is an irreducible V_L^+-module (see Theorem 2 of [DM]). Clearly $V_L = V_L^+ \oplus V_L^-$.

Now we take for L the lattice

$$D_1^d = \bigoplus_{i=1}^{d} \mathbb{Z}\alpha_i, \;\; \langle \alpha_i, \alpha_j\rangle = 4\delta_{i,j}.$$

Then L is an even lattice and the central extension $\hat{L}$ is a direct product of D_1^d with $\langle\kappa\rangle$ and $\mathbb{C}\{L\}$ is simply the group algebra $\mathbb{C}[L]$ with basis e^α for $\alpha \in L$. It is clear that $\theta(e^\alpha) = e^{-\alpha}$ for $\alpha \in D_1^d$. We extend the action of θ from $V_{D_1^d}$ to $V_{(D_1^*)^d} = M(1)\otimes\mathbb{C}[L^*]$ such that $\theta(u \otimes e^\alpha) = (\theta u) \otimes e^{-\alpha}$ for $u \in M(1)$ and $\alpha \in L^*$. One can easily verify that θ has order 2 and $\theta Y(u,z)\theta^{-1} = Y(\theta u, z)$ for $u \in V_{D_1^d}$, where $Y(v,z)$ ($v \in V_{D_1^d}$) are the vertex operators on $V_{(D_1^*)^d}$. For any θ-invariant subspace V of V_{L^*} we use $V^\pm$ to denote the $\pm$-eigenspaces.

First we turn our attention to the case that $d = 1$. Then $L = \mathbb{Z}\alpha \cong 2\mathbb{Z} = D_1$ where $\langle\alpha,\alpha\rangle = 4$. Note that the dual lattice D_1^* is $\frac{1}{4}D_1$ and $\{0, 1, \frac{1}{2}, -\frac{1}{2}\}$ is a system of coset representatives of D_1^*/D_1.

Set

$$\begin{aligned} \omega_1 &= \frac{1}{16}\alpha(-1)^2 + \frac{1}{4}(e^\alpha + e^{-\alpha}), \\ \omega_2 &= \frac{1}{16}\alpha(-1)^2 - -\frac{1}{4}(e^\alpha + e^{-\alpha}). \end{aligned} \tag{3.1}$$

Then $\omega_i \in V_{D_1}^+$.

Lemma 3.1 *For $D_1 \cong L = \mathbb{Z}\alpha$, $\langle\alpha,\alpha\rangle = 4$, we have:*

(1) V_{D_1} is a FVOA with $r = 2$.

(2) We have the following Virasoro decompositions of $V_{D_1}^+$ and $V_{D_1}^-$:

$$V_{D_1}^+ \cong M(0,0), \quad V_{D_1}^- \cong M(\frac{1}{2},\frac{1}{2})$$

with highest weight vectors **1** *and $\alpha(-1)$, respectively.*

(3) The decompositions for $V^{\pm}_{D_1+1}$ are:

$$V^{+}_{D_1+1} \cong M(\frac{1}{2},0), \quad V^{-}_{D_1+1} \cong M(0,\frac{1}{2})$$

with highest weight vectors $(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha})$ and $(e^{\frac{1}{2}\alpha} + e^{-\frac{1}{2}\alpha})$, respectively.

(4) For $V_{D_1+\frac{1}{2}} \oplus V_{D_1-\frac{1}{2}}$ we get, in both cases,

$$(V_{D_1+\frac{1}{2}} \oplus V_{D_1-\frac{1}{2}})^{\pm} \cong M(\frac{1}{16},\frac{1}{16})$$

with highest weight vectors $e^{\frac{1}{4}\alpha} \pm e^{-\frac{1}{4}\alpha}$. In fact, both $V_{D_1+\frac{1}{2}}$ and $V_{D_1-\frac{1}{2}}$ are irreducible $V^{+}_{D_1}$-modules.

Proof. It was proved in [DMZ] (see Theorem 6.3 there) that $Y(\omega_1, z_1) = \sum_{n\in\mathbb{Z}} L^1(n)z^{-n-2}$ and $Y(\omega_2, z_2) = \sum_{n\in\mathbb{Z}} L^2(n)z^{-n-2}$ give two commuting Virasoro algebras with central charge $\frac{1}{2}$. We first show that the highest weight of $\alpha(-1)$ is $(\frac{1}{2},\frac{1}{2})$. Since $\alpha(-1) \in V^{-}_{D_1}$ has the smallest weight in $V^{-}_{D_1}$ it is immediate to see that $L^i(n)\alpha(-1) = 0$ if $n > 0$. It is a straightforward computation by using the definition of vertex operators to show that $L^1(0)\alpha(-1) = L^2(0)\alpha(-1) = \frac{1}{2}\alpha(-1)$.

Clearly, $\mathbf{1} \in (V_{D_1})^{+}$ is a highest weight vector for the Virasoro algebras with highest weight $(0,0)$. So V_{D_1} contains two highest weight modules for the two Virasoro algebras with highest weights $(0,0)$ and $(\frac{1}{2},\frac{1}{2})$. Since $M(0,0) \oplus M(\frac{1}{2},\frac{1}{2})$ and V_{D_1} have the same graded dimension we conclude that $V_{D_1} \cong M(0,0) \oplus M(\frac{1}{2},\frac{1}{2})$ and $V^{+}_{D_1} \cong M(0,0)$, $V^{-}_{D_1} \cong M(\frac{1}{2},\frac{1}{2})$. This proves (2) and shows also (1): V_{D_1} is a FVOA with $r = 2$. Additionally we see that V_{D_1} is a unitary representation of the two Virasoro algebras.

By Theorem 2.3 (3) we know that $V_{D_1+\lambda}$, for $\lambda = 0, \pm\frac{1}{2}, 1$, is a direct sum of irreducible modules $M(h_1,h_2)$ with $h_i \in \{0,\frac{1}{2},\frac{1}{16}\}$. It is easy to find all highest weight vectors in $V_{D_1+\lambda}$. Part (3) and (4) follow immediately then. $\square$

We return to the lattice $L = \bigoplus_{i=1}^{d} \mathbb{Z}\alpha_i$, $\langle\alpha_i,\alpha_j\rangle = 4\delta_{i,j}$, $L \cong D_1^d = (2\mathbb{Z})^d$. We sometimes identify L with $(2\mathbb{Z})^d$. The component $\mathbb{Z}\alpha_i$ gives two Virasoro elements ω_{2i-1} and ω_{2i}, as in (3.1), above.

Definition 3.2 The VF associated to the FVOAs derived from the D_1^d-lattice is the set $\{\omega_1,\ldots,\omega_{2d}\}$.

Corollary 3.3 *(1) The decomposition of $V^{\pm}_{D_1^d}$ into irreducible modules for T_{2d} is given by*

$$V^{\pm}_{D_1^d} \cong \bigoplus_{\substack{(h_{2i-1},h_{2i}) \in \{(0,0),(\frac{1}{2},\frac{1}{2})\} \\ (-1)^{\#\{i|h_{2i}=0\}} = \pm 1}} M(h_1,\ldots,h_{2d}).$$

In particular, $V_{D_1^d}^{\pm}$ *is a direct sum of* 2^{d-1} *irreducible modules for* T_{2d}*.*

(2) Let $\gamma = (\gamma_i) \in (D_1^*)^d$ *such that* $\gamma_i \in \{0, 1\}$*. Then we get the decomposition*

$$(V_{D_1^d+\gamma})^{\pm} \cong \bigoplus_{\substack{(h_{2i-1},h_{2i})\in\begin{cases}\{(0,0),(\frac{1}{2},\frac{1}{2})\} & \text{if } \gamma_i=0,\\ \{(0,\frac{1}{2}),(\frac{1}{2},0)\} & \text{if } \gamma_i=1\end{cases}\\ (-1)^{\#\{i|h_{2i}=0\}}=\pm 1}} M(h_1, \ldots, h_{2d}).$$

(3) Let $\gamma = (\gamma_i) \in (D_1^*)^d$*, such that* $2\gamma \notin D_1^d$*, i.e. there is at least one* i *such that* $\gamma_i = \pm\frac{1}{2}$*. Then* $(V_{D_1^d+\gamma} \oplus V_{D_1^d-\gamma})^{\pm}$*,* $V_{D_1^d\pm\gamma}$ *have the same decomposition:*

$$\bigoplus_{(h_{2i-1},h_{2i})\in\begin{cases}\{(0,0),(\frac{1}{2},\frac{1}{2})\} & \text{if } \gamma_i=0,\\ \{(\frac{1}{2},0),(0,\frac{1}{2})\} & \text{if } \gamma_i=1,\\ \{(\frac{1}{16},\frac{1}{16})\} & \text{if } \gamma_i=\pm\frac{1}{2}\end{cases}} M(h_1, \ldots, h_{2d}).$$

Proof. Note that $V_{D_1^d}$ is isomorphic to the tensor product vertex operator algebra $V_{D_1} \otimes \cdots \otimes V_{D_1}$ (d factors) and that $V_{D_1^d+\gamma}$ is isomorphic to the tensor product module $V_{\mathbb{Z}\alpha_1+\gamma_1} \otimes \cdots \otimes V_{\mathbb{Z}\alpha_d+\gamma_d}$. Thus

$$(V_{D_1^d+\gamma} \oplus V_{D_1^d-\gamma})^{\pm} = \bigoplus_{\substack{\mu\in\{+,-\}^d\\ \prod \mu_i=\pm}} V_{D_1+\gamma_1}^{\mu_1} \otimes \cdots \otimes V_{D_1+\gamma_d}^{\mu_d}.$$

The results (1) and (2) now follow from Lemma 3.1 immediately.

For (3) it is clear that the decompositions for $V_{D_1^d\pm\gamma}$ hold by Lemma 3.1. It remains to show that $V_{D_1^d\pm\gamma}$ and $(V_{D_1^d+\gamma} \oplus V_{D_1^d-\gamma})^{\pm}$ are all isomorphic T_{2d}-modules. Note from Lemma 3.1 that V_{D_1+h} and V_{D_1-h} are isomorphic T_2-modules for any $h \in \{0, 1, \pm\frac{1}{2}\}$. Thus $V_{D_1^d+\gamma}$ and $V_{D_1^d-\gamma}$ are isomorphic T_{2d}-modules. In fact, $\theta : V_{D_1^d+\gamma} \to V_{D_1^d-\gamma}$ is such an isomorphism. Thus, $(V_{D_1^d+\gamma} \oplus V_{D_1^d-\gamma})^{\pm} = \{v \pm \theta v \mid v \in V_{D_1^d+\gamma}\}$ are isomorphic to $V_{D_1^d+\gamma}$ as T_{2d}-modules. □

Next we discuss the twisted modules of V_L for an arbitrary d-dimensional positive definite even lattice L. Recall from [FLM] the definition of the twisted sectors associated to an even lattice L. Let $K = \{\theta(a)a^{-1} \mid a \in \hat{L}\}$. Then $\bar{K} = 2L$ (bar is the quotient map $\hat{L} \to L$). Also set $R := \{\alpha \in L \mid \langle \alpha, L\rangle \leq 2\mathbb{Z}\}$; then $R \geq 2L$. Then the inverse image $\hat{R}$ of R in $\hat{L}$ is the center of $\hat{L}$ and K is a subgroup of $\hat{R}$. It was proved in [FLM] (Proposition 7.4.8) there are exactly $|R/2L|$ central characters $\chi : \hat{R}/K \to \mathbb{C}^{\times}$ of $\hat{L}/K$ such that $\chi(\kappa K) = -1$. For each such χ, there is a unique (up to equivalence) irreducible $\hat{L}/K$-module T_{χ} with central character χ and every irreducible $\hat{L}/K$-module on which κK acts as -1 is equivalent to one of these. In particular, viewing T_{χ} as $\hat{L}$-module, θa and a agree as operators on T_{χ} for $a \in \hat{L}$. Let

$\hat{\mathfrak{h}}[-1]$ be the twisted Heisenberg algebra. As in Section 1.7 of [FLM] we also denote by $M(1)$ the unique irreducible $\hat{\mathfrak{h}}[-1]$-module with the canonical central element acting by 1. Define the twisted space $V_L^{T_\chi} = M(1) \otimes T_\chi$. It was shown in [FLM] and [DL2] that there is a linear map

$$\begin{array}{rcl} V_L & \to & (\mathrm{End}\, V_L^{T_\chi})[[z^{1/2}, z^{-1/2}]], \\ v & \mapsto & Y(v,z) = \sum_{n \in \frac{1}{2}\mathbb{Z}} v_n z^{-n-1} \end{array}$$

such that $V_L^{T_\chi}$ is an irreducible θ-twisted module for V_L. Moreover, every irreducible θ-twisted V_L-module is isomorphic to $V_L^{T_\chi}$ for some χ.

Define a linear operator $\hat{\theta}_d$ on $V_L^{T_\chi}$ such that

$$\hat{\theta}(\alpha_1(-n_1)\cdots\alpha_k(-n_k) \otimes t) = (-1)^k e^{d\pi i/8} \alpha_1(-n_1)\cdots\alpha_k(-n_k) \otimes t$$

for $\alpha_i \in \mathfrak{h}$, $n_i \in \frac{1}{2} + \mathbb{Z}$ and $t \in T$. Then $\hat{\theta}_d Y(u,z)(\hat{\theta}_d)^{-1} = Y(\theta u, z)$ for $u \in V_L$ (cf. [FLM]). We have the decomposition $V_L^{T_\chi} = (V_L^{T_\chi})^+ \oplus (V_L^{T_\chi})^-$ where $(V_L^{T_\chi})^+$ and $(V_L^{T_\chi})^-$ are the $\hat{\theta}_d$-eigenspaces with eigenvalues $-e^{d\pi i/8}$ and $e^{d\pi i/8}$ respectively. Then both $(V_L^{T_\chi})^+$ and $(V_L^{T_\chi})^-$ are irreducible V_L^+-modules (cf. Theorem 5.5 of [DLi]).

As before, we now take $L = \mathbb{Z}\alpha \cong D_1$ with $\langle \alpha, \alpha \rangle = 4$. Then $K = 2L$, $R = L$ and $R/K \cong \mathbb{Z}_2$. Let χ_1 be the trivial character of R/K and χ_{-1} the nontrivial character. Then both T_{χ_1} and $T_{\chi_{-1}}$ are one-dimensional L modules and α acts on $T_{\chi_{\pm 1}}$ as ± 1.

Lemma 3.4 *We have the Virasoro decompositions:*

$$(1) \qquad (V_{D_1}^{T_{\chi_1}})^+ \cong M(\frac{1}{16}, \frac{1}{2}), \qquad (V_{D_1}^{T_{\chi_1}})^- \cong M(\frac{1}{16}, 0).$$

$$(2) \qquad (V_{D_1}^{T_{\chi_{-1}}})^+ \cong M(\frac{1}{2}, \frac{1}{16}), \qquad (V_{D_1}^{T_{\chi_{-1}}})^- \cong M(0, \frac{1}{16}).$$

Proof. Recall from [DL2] that

$$V_L^{T_{\chi_1}} = \sum_{n \in \frac{1}{2}\mathbb{Z},\ n \geq 0} (V_L^{T_{\chi_1}})_{\frac{1}{16}+n}$$

(see Proposition 6.3 and formula (6.28) of [DL2]). Note that

$$(V_L^{T_{\chi_1}})^+ = \sum_{n \in \mathbb{Z},\ n \geq 0} (V_L^{T_{\chi_1}})_{\frac{1}{16}+\frac{1}{2}+n}$$

and that

$$(V_L^{T_{\chi_1}})^- = \sum_{n \in \mathbb{Z},\ n \geq 0} (V_L^{T_{\chi_1}})_{\frac{1}{16}+n}.$$

Since both $(V_L^{T_{\chi_1}})^+$ and $(V_L^{T_{\chi_1}})^-$ are irreducible V_L^+-modules we only need to calculate highest weights for nonzero highest weight vectors in these spaces. Note that T_{χ_1} is a space of highest weight vectors of $(V_L^{T_{\chi_1}})^-$. One can easily verify that $L^1(0) = \frac{1}{16}$ and $L^2(0) = 0$ on T_{χ_1}. Thus $(V_L^{T_{\chi_1}})^- \cong M(\frac{1}{16}, 0)$.

Also observe that $\alpha(-1/2) \otimes T_{\chi_1}$ is a space of highest weight vectors of $(V_L^{T_{\chi_1}})^+$. From Lemma 3.1 we know that $\alpha(-1) \in V_L^- \cong M(\frac{1}{2}, \frac{1}{2})$. Now use the fusion rule given in Theorem 2.3 to conclude that $(V_L^{T_{\chi_1}})^+ \cong M(\frac{1}{16}, \frac{1}{2})$. Part (2) is proved similarly. □

As we did in the untwisted case, we now consider the twisted modules for the lattice $L = \bigoplus_{i=1}^d \mathbb{Z}\alpha_i \cong D_1^d$, $\langle \alpha_i, \alpha_j \rangle = 4\delta_{i,j}$, where d is now a positive integer divisible by 8. Then $K = 2L$, $R = L$ and $R/2L \cong \mathbb{Z}_2^d$. Thus, there are 2^d irreducible characters for $R/2L$ which are denoted by χ_J (where J is a subset of $\{1, \ldots, d\}$) sending α_j to -1 if $j \in J$ and to 1 otherwise. Then we have $\chi_J = \prod_j \chi_{x_j}$ where χ_{x_j} is a character of $\mathbb{Z}\alpha_j/\mathbb{Z}2\alpha_j$ and $x_j = \chi_J(a_j)$. Moreover, $T_{\chi_J} \cong T_{\chi_{x_1}} \otimes \cdots \otimes T_{\chi_{x_d}}$. In particular, each T_{χ_J} is one dimensional.

Corollary 3.5 *We have the Virasoro decompositions:*

$$(V_{D_1^d}^{T_{\chi_J}})^\pm = \bigoplus_{\substack{(h_{2i-i}, h_{2i}) \in \left\{ \begin{array}{l} \{(\frac{1}{16},0),(\frac{1}{16},\frac{1}{2})\} \text{ if } i \notin J \\ \{(0,\frac{1}{16}),(\frac{1}{2},\frac{1}{16})\} \text{ if } i \in J \end{array} \right. \\ (-1)^{\#\{j | h_j = \frac{1}{2}\}} = \pm(-1)^{d/8}}} M(h_1, \ldots, h_{2d}).$$

Proof. Recall from the proof of Corollary 3.3 that $V_{D_1^d}$ is isomorphic to the tensor product vertex operator algebra $V_{D_1} \otimes \cdots \otimes V_{D_1}$. Note that $V_{D_1^d}^{T_{\chi_J}}$ is isomorphic to the tensor product $V_{D_1}^{T_{\chi_{x_1}}} \otimes \cdots \otimes V_{D_1}^{T_{\chi_{x_d}}}$ and $\hat{\theta}_d$ is also a tensor product $\hat{\theta}_1 \otimes \cdots \otimes \hat{\theta}_d$. By Lemma 3.4,

$$V_{D_1^d}^{T_{\chi_J}} = \bigoplus_{(h_{2i-i}, h_{2i}) \in \left\{ \begin{array}{l} \{(\frac{1}{16},0),(\frac{1}{16},\frac{1}{2})\} \text{ if } i \notin J \\ \{(0,\frac{1}{16}),(\frac{1}{2},\frac{1}{16})\} \text{ if } i \in J \end{array} \right.} M(h_1, \ldots, h_{2d}).$$

Since $\hat{\theta}_d = (-1)^{\#\{j | h_j = \frac{1}{2}\}}(-1)^{d/8} = (-1)^{\#\{j | h_j = 0\}}(-1)^{d/8}$ on $M(h_1, \ldots, h_{2d})$ we see that $M(h_1, \ldots, h_{2d})$ embeds in $(V_L^{T_{\chi_J}})^\pm$ if and only if $(-1)^{\#\{j | h_j = \frac{1}{2}\}}(-1)^{d/8} = \pm 1$. The proof is complete. □

Remark 3.6 Note that $V_{D_1^d}^{T_{\chi_J}}$ is $\frac{1}{2}\mathbb{Z}$ graded if d is divisible by 8 (cf. [DL2]). In fact $(V_{D_1^d}^{T_{\chi_J}})^+$ is then the subspace of $V_{D_1^d}^{T_{\chi_J}}$ consisting of vectors of integral weights while $(V_{D_1^d}^{T_{\chi_J}})^-$ is the subspace of $V_{D_1^d}^{T_{\chi_J}}$ consisting of vectors of non-integral weights.

4 Vertex operator algebras associated to binary codes

Let C be a doubly-even linear binary code of length $d \in 8\mathbb{Z}$ containing the all ones vector $\Omega = (1, \ldots, 1)$. As mentioned in Section 2, we can regard a vector of $\mathbb{F}_2^d$ as an element in $\mathbb{Z}^d$ in an obvious way. One can associate (cf. [CS1]) to such a code the two even lattices

$$L_C = \{\frac{1}{\sqrt{2}}(c+x) \mid c \in C,\ x \in (2\mathbb{Z})^d\}$$

and

$$\widetilde{L}_C = \{\frac{1}{\sqrt{2}}(c+y) \mid c \in C,\ y \in (2\mathbb{Z})^d,\ 4 \mid \textstyle\sum y_i\} \cup$$
$$\{\frac{1}{\sqrt{2}}(c+y+(\tfrac{1}{2}, \ldots, \tfrac{1}{2})) \mid c \in C,\ y \in (2\mathbb{Z})^d,\ 4 \mid (1-(-1)^{d/8} + \textstyle\sum y_i)\}$$

and for every *self-dual* even lattice there are two vertex operator algebras V_L and $\widetilde{V}_L = V_L^+ \oplus (V_L^T)^+$ (see [FLM], [DGM1]).

Definition 4.1 A *marking* for the code C is a partition $\mathcal{M} = \{(i_1, i_2), \ldots, (i_{d-1}, i_d)\}$ of the positions 1, 2, …, d into $\frac{d}{2}$ pairs.

A marking $\mathcal{M} = \{(i_1, i_2), \ldots, (i_{d-1}, i_d)\}$ determines the D_1^d sublattice $\bigoplus_{l=1}^d \mathbb{Z}\alpha_l$ inside L_C and $\widetilde{L}_C$, where $\alpha_{2k-1} = \sqrt{2}(e_{i_{2k-1}} + e_{i_{2k}})$ and $\alpha_{2k} = \sqrt{2}(e_{i_{2k-1}} - e_{i_{2k}})$ for $k = 1, \ldots, \frac{d}{2}$ using $\{e_i\}$ as the standard base of $L_C \otimes \mathbb{Q} = \mathbb{Q}^\Omega$. Let us simplify notation and arrange for the marking to be $\mathcal{M} = \{(1,2), (3,4), \ldots, (d-1, d)\}$.

From Definition 3.2, we see that every such marking defines a system of $2d$ commuting Virasoro algebras inside the vertex operator algebras V_{L_C}, $V_{\widetilde{L}_C} \cong \widetilde{V}_{L_C}$ and $\widetilde{V}_{\widetilde{L}_C}$. As the main theorem we describe explicitly the decomposition into irreducible T_{2d}-modules in terms of the marked code. The triality symmetry of $\widetilde{V}_{\widetilde{L}_C}$ given in [FLM] and [DGM1] is directly visible in this decomposition. (See also [G1].)

In order to give the Virasoro decompositions in a readable way, we need the next lemma which describes L_C and $\widetilde{L}_C$ in the coordinate system spanned by the α_i. We use the following notation. Let γ_+^0, γ_-^0, γ_+^1 and γ_-^1 be the maps $\mathbb{F}_2^2 \longrightarrow (D_1^*/D_1)^2 = \{0, \frac{1}{2}, -\frac{1}{2}, 1\}^2$ defined by the table

	$(0,0)$	$(1,1)$	$(1,0)$	$(0,1)$
γ_+^0	$(0,0)$	$(1,0)$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2})$
γ_-^0	$(1,1)$	$(0,1)$	$(-\frac{1}{2},-\frac{1}{2})$	$(-\frac{1}{2},\frac{1}{2})$
γ_+^1	$(\frac{1}{2},0)$	$(-\frac{1}{2},0)$	$(1,\frac{1}{2})$	$(1,-\frac{1}{2})$
γ_-^1	$(-\frac{1}{2},1)$	$(\frac{1}{2},1)$	$(0,-\frac{1}{2})$	$(0,\frac{1}{2})$

and write $c(k)$ ($k = 1, \ldots, \frac{d}{2}$) for the pair (c_{2k-1}, c_{2k}) of coordinates of a codeword $c \in C$. Finally let for $b = 0$ or 1 and $\epsilon = (\epsilon_1, \ldots, \epsilon_{d/2}) \in \{+,-\}^{d/2}$

$$\Gamma^a_\epsilon(b) = \bigoplus_{k=1}^{d/2} D_1^2 + \gamma^b_{\epsilon_k}(c(k)) \tag{4.1}$$

be a coset of D_1^d.

Lemma 4.2 *We have the following coset decomposition of* L_C *and* $\widetilde{L}_C$ *under the above* D_1^d *sublattice:*

$$\begin{aligned} L_C &= \bigcup_{c\in C} \bigcup_{\epsilon\in\{+,-\}^{d/2}} \Gamma^0_\epsilon(c), \\ \widetilde{L}_C &= \bigcup_{c\in C} \left(\bigcup_{\substack{\epsilon\in\{+,-\}^{d/2} \\ \prod \epsilon_i = +}} \Gamma^0_\epsilon(c) \cup \bigcup_{\substack{\epsilon\in\{+,-\}^{d/2} \\ \prod \epsilon_i = (-)^{d/8}}} \Gamma^1_\epsilon(c) \right). \end{aligned}$$

Proof. The result follows from the definition of these two lattices. □

We next interpret the decompositions in terms of codes over $\mathbb{Z}_4 = \{0,1,2,3\}$ associated to L_C and $\tilde{L}_C$. See [CS2] for the relevant definitions for $\mathbb{Z}_4$-codes.

Let L be a positive definite even lattice of rank d which contains a D_1^d as a sublattice. We call such a sublattice a *D_1-frame*. Note that $(D_1^*/D_1)^d$ is isomorphic to $\mathbb{Z}_4^d$. Then $\Delta(L) := L/D_1^d \leq (D_1^*/D_1)^d$ is a code over $\mathbb{Z}_4$ and $\Delta(L)$ is self-annihilating if and only if L is self-dual. For the lattices L_C and $\tilde{L}_C$ we give the following explicit description of the corresponding codes Δ:

Let $\widehat{\ }$ be the map from $\mathbb{F}_2^d$ to $\mathbb{Z}_4^d$ induced from $\hat{}: \mathbb{F}_2^2 \cong D_2^*/D_2 \longrightarrow (D_1^*/D_1)^2 \cong \mathbb{Z}_4^2$, $00 \mapsto 00$, $11 \mapsto 20$, $10 \mapsto 11$ and $01 \mapsto 31$. Let $(\Sigma_2^n)_0$ be the subcode of the $\mathbb{Z}_4$-code $\Sigma_2^n = \{(00),(22)\}^n$ of length $2n$ consisting of codewords of weights divisible by 4. Then we have

$$\begin{aligned} \Gamma &= \Delta(L_C) = \widehat{C} + \Sigma_2^{d/2}, \\ \widetilde{\Gamma} &= \Delta(\tilde{L}_C) = \widehat{C} + (\Sigma_2^{d/2})_0 \cup \widehat{C} + (\Sigma_2^{d/2})_0 + \begin{cases} (1,0,\ldots,1,0,1,0), & \text{if } d \equiv 0 \pmod{16}, \\ (1,0,\ldots,1,0,3,2), & \text{if } d \equiv 8 \pmod{16}. \end{cases} \end{aligned} \tag{4.2}$$

An important invariant of a $\mathbb{Z}_4$-code Δ is the symmetrized weight enumerator, as defined in [CS2].

Definition 4.3 The *symmetrized weight enumerator* of a $\mathbb{Z}_4$-code Δ of length d is given by

$$\mathrm{swe}_\Delta(A,B,C) = \sum_{0\leq r,s\leq d} U_{r,s}\, A^{d-r-s} B^r C^s,$$

where $U_{r,s}$ is the number of codewords $\gamma \in \Delta$ having at r positions the value ± 1 and at s positions the value 2.

To describe the symmetrized weight enumerator for our codes Γ and $\widetilde{\Gamma}$ in terms of the marked binary code C, we introduce the analogous invariant for marked binary codes.

Definition 4.4 The *symmetrized marked weight enumerator* of a binary code C of length d with a marking $\mathcal{M}$ is the homogeneous polynomial

$$\mathrm{smwe}_C(x,y,z) = \sum_{k=0}^{d/2} \sum_{l=0}^{[k/2]} W_{k,l}\, x^{d/2-k+l} y^{k-2l} z^l,$$

where $W_{k,l}$ is the number of codewords $c \in C$ of Hamming weight k having the value $(c_{i_{2m-1}}, c_{i_{2m}}) = (1,1)$ for exactly l of the $d/2$ pairs (i_{2m-1}, i_{2m}) of the marking $\mathcal{M}$.

Remark 4.5 The concept of marked binary codes can be considered as the third step in the sequence D_8^*/D_8-codes, Kleinian codes, marked binary codes, $\mathbb{Z}_4$-codes and VFOAs (cf. Section 5 and [H2], last section). It is very useful and one obtains easily the usual code-theoretic type of results. For example, the following one:

(1) *Mass formula.*

$$\sum_{[(C,\mathcal{M})]} \frac{2^{d/2}(d/2)!}{|Aut_{\mathcal{M}}(C)|} = 2\cdot 3\cdot 5\cdot \ldots \cdot (2^{(d/2)-2}+1),$$

where the sum runs over equivalence classes of pairs of doubly even self annihilating codes C with marking $\mathcal{M}$ and $Aut_{\mathcal{M}}(C)$ denotes the group of automorphisms of C that fix $\mathcal{M}$. For an application, see Appendix A.

(2) *Ring of invariants.* The symmetrized marked weight enumerator belongs to a ring $\mathbb{C}[u_4, v_4, u_8] \oplus \mathbb{C}[u_4, v_4, u_8]\cdot u_{12}$ generated by $u_4 = x^4 + 6\,x^2z^2 + z^4 + 8\,y^4$, $v_4 = x^4 + z^4 + 12\,xzy^2 + 2\,y^4$ and two polynomials u_8 and u_{12} of degree 8 resp. 12, subject to one relation for u_8^2.

From Lemma 4.2 we get:

Corollary 4.6 *The symmetrized weight enumerators of the $\mathbb{Z}_4$-codes Γ and $\widetilde{\Gamma}$ are given by*

$$\mathrm{swe}_\Gamma(A,B,C) = \mathrm{smwe}_C(A^2+C^2, 2B^2, 2AC), \tag{4.3}$$

$$\mathrm{swe}_{\widetilde{\Gamma}}(A,B,C) = \frac{1}{2}\cdot \mathrm{smwe}_C(A^2+C^2, 2B^2, 2AC) + \frac{1}{2}\left((A^2-C^2)^{d/2}\right) \tag{4.4}$$

$$+\frac{1}{2}\cdot 2^{d/2}\left((A+C)^{d/2} + (-1)^{d/8}(A-C)^{d/2}\right)B^{d/2}. \qquad \square$$

Motivated by Lemmas 3.1 and 3.4, define for $a \in \{0,1\}$, $\alpha \in \{+,-\}$ and $x \in \{0, \pm\frac{1}{2}, 1\}$ the 16 formal linear combinations of T_2-modules $R^a_\alpha(x)$ by the following table:

	0	1	$\frac{1}{2}, -\frac{1}{2}$
R^0_+	$M(0,0)$	$M(\frac{1}{2},0)$	$\frac{1}{2}M(\frac{1}{16},\frac{1}{16})$
R^0_-	$M(\frac{1}{2},\frac{1}{2})$	$M(0,\frac{1}{2})$	$\frac{1}{2}M(\frac{1}{16},\frac{1}{16})$
R^1_+	$\frac{1}{\sqrt{2}}M(\frac{1}{16},\frac{1}{2})$		$\frac{1}{\sqrt{2}}M(\frac{1}{2},\frac{1}{16})$
R^1_-	$\frac{1}{\sqrt{2}}M(\frac{1}{16},0)$		$\frac{1}{\sqrt{2}}M(0,\frac{1}{16})$

For $\mu \in \{+,-\}^n$ and an element $\gamma = (\gamma_k) \in \mathbb{Z}_4^n$ which is identified with $\{0,1,\pm\frac{1}{2}\}$ we write shortly

$$\mathbf{R}^a_\mu(\gamma) = \bigotimes_{k=1}^n R^a_{\mu_k}(\gamma_k). \tag{4.5}$$

We see from Lemmas 3.1 and 3.4 that $R^0_\pm = V^\pm_{D_1+x}$ and $R^1_\pm = \frac{1}{\sqrt{2}}(V^{T_{\chi_{(-1)^{2x}}}}_{D_1})^\pm$. The introduction of the extra factor $\frac{1}{\sqrt{2}}$ in the twisted case enables us to write the Virasoro decompositions for the twisted sector V_L^T in a neat way. The index 0 in $R^0_\pm$ refers to the untwisted case while the index 1 in $R^1_\pm$ refers to the twisted case.

Let L be a self-dual even lattice of rank d containing a D_1-frame. So, L is defined by the self-annihilating $\mathbb{Z}_4$-code $\Delta = L/D_1^d \leq (D_1^*/D_1)^d$ of length d which is now even in the sense that $\mathrm{swe}_\Delta(1, x^{\frac{1}{2}}, x)$ is a polynomial in x^2 (cf. [BS]).

Theorem 4.7 *The vertex operator algebras V_L and $\widetilde{V}_L$ have the following decompositions as modules for T_{2d}:*

$$\begin{aligned} V_L &= \bigoplus_{\gamma\in\Delta} \bigoplus_{\mu\in\{+,-\}^d} \mathbf{R}^0_\mu(\gamma), \\ \widetilde{V}_L &= \bigoplus_{\gamma\in\Delta} \bigoplus_{\substack{\mu\in\{+,-\}^d \\ \prod \mu_k = +}} \mathbf{R}^0_\mu(\gamma) \oplus \bigoplus_{\gamma\in\Delta} \bigoplus_{\substack{\mu\in\{+,-\}^d \\ \prod \mu_k = (-)^{d/8}}} \mathbf{R}^1_\mu(\gamma). \end{aligned}$$

In order to determine the decomposition for $\widetilde{V}_L$, we first study the decomposition of V_L^T.

Since L is self-dual, V_L has a unique irreducible θ-twisted module V_L^T [D2]. In this case T can be constructed in the following way. Let Q be a subgroup of L containing the D_1^d which is maximal such that $\langle\alpha,\beta\rangle \in 2\mathbb{Z}$ for α, $\beta \in Q$ (it exists since L has ascending chain conditions on subgroups). Let $\hat{Q}$ be the inverse image of Q in $\hat{L}$. Note that $|L/Q| = 2^{d/2}$. Then $\hat{Q}$ is a maximal abelian subgroup of $\hat{L}$ which contains $\hat{D}_1^d \cong D_1^d \times \langle\kappa\rangle$ and which contains K. Let $\psi : \hat{Q} \to \langle\pm 1\rangle$ be a character of $\hat{Q}$ such that $\psi(\kappa K) = -1$. Then T can be realized as the induced $\hat{L}$-module

$T = \mathbb{C}[\hat{L}] \otimes_{\mathbb{C}[\hat{Q}]} \mathbb{C}_\psi$ where $\mathbb{C}_\psi$ is one dimensional $\hat{Q}$-module defined by the character ψ. For $a \in \hat{L}$, set $t(a) = a \otimes 1 \in T$. It is easy to see that we can choose $a_i \in \hat{D}_1^d$ such that $\bar{a}_i = \alpha_i$ for all i and $\psi(a_i) = 1$. Then for $a \in \hat{L}$

$$a_i t(a) = a_i a \otimes 1 = (-1)^{\langle \alpha_i, \bar{a} \rangle} a a_i \otimes 1 = (-1)^{\langle \alpha_i, \bar{a} \rangle} a \otimes 1. \tag{4.6}$$

Thus, $\mathbb{C}\ t(a)$ is a one-dimensional representation for D_1^d, with character χ given by $\chi(a_i) = (-1)^{\langle \alpha_i, \bar{a} \rangle}$. In fact $\mathbb{C}\ t(a)$ is isomorphic to T_χ as D_1^d-modules. Let $\beta_l \in L$ for $l = 1, \ldots, 2^{d/2}$ represent the distinct cosets of Q in L. Choose $b_l \in \hat{L}$ with $\bar{b}_l = \beta_l$ for all l. Then $\{t(b_l) \mid l = 1, \ldots, 2^{d/2}\}$ forms a basis of T and each $t(b_l)$ spans a one-dimensional module for D_1^d. Denote the character of D_1^d on $\mathbb{C}\ t(b_l)$ by χ_l. Then $M(1) \otimes t(a)$ is isomorphic to $V_{D_1^d}^{T_{\chi_l}}$ as θ-twisted $V_{D_1^d}$-modules and as T_{2d}-modules. Thus, $V_L^T \cong \bigoplus_{l=1}^{2^{d/2}} V_{D_1^d}^{T_{\chi_l}}$.

Proposition 4.8 *Let $\Theta \subseteq \Delta$ be a complete coset system for the induced $\mathbb{Z}_4$-subcode Q/D_1^d of Δ. We have the Virasoro decomposition*

$$V_L^T = \bigoplus_{\gamma \in \Theta} V_{D_1^d}^{T_{\varphi_\gamma}} = \bigoplus_{\mu \in \{+,-\}^d} \bigoplus_{\gamma \in \Delta} \mathbf{R}_\mu^1(\gamma),$$

where the character φ_γ is determined by $\varphi_\gamma(a_i) = (-1)^{2\gamma_i}$ if we identify $D_1^*/D_1 \cong \mathbb{Z}_4$ with $\{0, 1, \pm\frac{1}{2}\}$.

Proof. The first equality has been proven in the previous discussion. In order to see the second equality note that by Lemma 3.4 we have

$$\mathbf{R}_\mu^1(\gamma) = 2^{-d/2} \bigotimes_{k=1}^{d} (V_{D_1}^{T_{\chi_{x_k}}})^{\mu_k} \tag{4.7}$$

where $x_k = (-1)^{2\gamma_k}$. Observe that $\langle \alpha, \beta \rangle \in 2\mathbb{Z}$ for any α, $\beta \in Q$. Let a, $b \in \hat{L}$ such that $\bar{a} + Q = \bar{b} + Q$. Then from (4.6), $M(1) \otimes t(a)$ and $M(1) \otimes t(b)$ are isomorphic θ-twisted $V_{D_1^d}$-modules and are isomorphic T_{2d}-modules. Thus $\mathbf{R}_\mu^1(\gamma) = \mathbf{R}_\mu^1(\gamma')$ if γ and γ' are in the same coset of Q/D_1^d in Δ. Since the coset of Q/D_1^d in Δ has exactly $2^{d/2}$ elements, we immediately see from (4.7) that for $\gamma \in \Delta$

$$V_{D_1^d}^{T_{\varphi_\gamma}} = \bigoplus_{\mu \in \{+,-\}^d} \bigoplus_{\sigma \in Q/D_1^d} \mathbf{R}_\mu^1(\gamma + \sigma).$$

This proves the second equality. □

Proof of Theorem 4.7: For a subset N of L we denote by $\hat{N}$ the inverse image of N in $\hat{L}$ and set $V[N] = M(1) \otimes \mathbb{C}\{\hat{N}\}$ Then $V_L = \bigoplus_{\gamma \in \Delta} V[D_1^d + \gamma]$ and $V[D_1^d + \gamma]$

is isomorphic to $V_{D_1^d+\gamma}$ (defined in Section 3) as $V_{D_1^d}$-modules. The decomposition for V_L follows immediately from Corollary 3.3 and Lemma 4.2.

Now we study the decomposition of V_L^+. If $\gamma_j = \pm\frac{1}{2}$ for some j then

$$(V[D_1^d+\gamma] \oplus V[D_1^d-\gamma])^+$$

is isomorphic to $V[D_1^d+\gamma]$ as T_{2d}-modules and has the desired decomposition. So we can assume that all $\gamma_j = 0,\, 1$. In Lemma 4.9 below we will prove that the θ defined on $V_{D_1^d+\gamma}$ in Section 3 coincides with the θ on $V[D_1^d+\gamma]$. We again use Corollary 3.3 to see that $V[D_1^d+\gamma]^+$ has the desired decomposition.

For the twisted part, we use Proposition 4.8 and Corollary 3.5 to obtain

$$(V_L^T)^+ = \bigoplus_{\gamma\in\Delta} \left(\mathbf{R}_\mu^1(\gamma)\right)^+ = \bigoplus_{\gamma\in\Delta} \bigoplus_{\substack{\mu\in\{+,-\}^d \\ \prod\mu_k=(-)^{d/8}}} \mathbf{R}_\mu^1(\gamma). \qquad \square$$

Lemma 4.9 *With the same notations as in the the proof of Theorem 4.7, the θ defined on $V_{D_1^d+\gamma}$ in Section 3 coincides with the θ on $V[D_1^d+\gamma]$ if all $\gamma_j = 0,\, 1$.*

Proof. Let X be a sublattice of L containing D_1^d such that $\langle x,y\rangle \in 2\mathbb{Z}$ for $x,y\in X$. Then the inverse image $\hat{X}$ of X in $\hat{L}$ is an abelian group. We can choose a section $e : X \to \hat{X}$ such that $e_xe_y = \kappa^{\langle x,y\rangle/2}e_{x+y}$. Then $e_x^{-1} = \kappa^{\langle x,x\rangle/2}e_{-x}$ for $x\in X$. Thus $\theta\iota(e_x) = \iota(e_{-x})$.

Take X to be the sublattice generated by D_1^d and γ. Then $V[D_1^d+\gamma]$ is generated by $\iota(e_\gamma)$ as $V_{D_1^d}$-module and $V[D_1^d+\gamma]^+$ is generated by $\iota(e_\gamma)+\iota(e_{-\gamma})$ as $V_{D_1^d}^+$-module. In fact $V[D_1^d+\gamma]^+$ is an irreducible $V_{D_1^d}^+$-module. Let $\mu : V_{D_1^d+\gamma} \mapsto V[D_1^d+\gamma]$ be an $V_{D_1^d}$-module isomorphism such that $\mu e^\gamma = \iota(e_\gamma)$. We must prove that μ maps $V_{D_1^d+\gamma}^+$ to $(V[D_1^d+\gamma])^+$. As both $V_{D_1^d+\gamma}^+$ and $(V[D_1^d+\gamma])^+$ are irreducible $V_{D_1^d}^+$-modules, it is enough to show that $\mu(e^\gamma+e^{-\gamma}) = \iota(e_\gamma)+\iota(e_{-\gamma})$ or equivalently $\mu e^{-\gamma} = \iota(e_{-\gamma})$.

Let J be the subset of $\{1,\ldots,d\}$ consisting of j such that $\gamma_j = 1$. Note that $e^{-\gamma}$ is the coefficient of $z^{-2|J|}$ in $Y(e^{-2\gamma},z)e^\gamma$ and $\iota(e_{-\gamma})$ is the coefficient of $z^{-2|J|}$ in $Y(\iota(e_{-2\gamma}),z)\iota(e_\gamma)$ as $(-1)^{\langle 2\gamma,\gamma\rangle/2}$ is even. Also note that $2\gamma\in D_1^d$. Thus $\mu e^{-\gamma} = \iota(e_{-\gamma})$. $\square$

Now we return our lattices L_C and $\widetilde{L}_C$ associated to the code C. We assume that C is a *self-annihilating* (i.e., $C = C^\perp$) doubly-even binary code. Then the $\mathbb{Z}_4$-codes Γ and $\widetilde{\Gamma}$ are self-annihilating and even, or equivalently as mentioned above, the lattices L_C and $\widetilde{L}_C$ are self-dual and even.

Combining Lemma 4.2 and Theorem 4.7 we will see how to read off the Virasoro decomposition directly from the marked code C. For a, $b \in \{0,1\}$, α, $\beta \in \{+,-\}$ and x, $y \in \{0,1\}$, define the formal linear combinations of T_4-modules $N^{ab}_{\alpha\beta}((x,y))$ by

$$N^{ab}_{\alpha\beta}((x,y)) = \bigoplus_{\substack{\alpha',\alpha''\in\{+,-\}\\ \alpha'\alpha''=\alpha}} \mathbf{R}^a_{(\alpha',\alpha'')}(\gamma^b_\beta((x,y)))$$

where $\mathbf{R}^a_{(\alpha',\alpha'')}$ was defined in (4.5). Explicitly, we get the 64 formal linear combinations as shown in the following table:

	$(0,0)$	$(1,1)$	$(0,1),(1,0)$
N^{00}_{++}	$M(0,0,0,0)\oplus M(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$M(\frac{1}{2},0,0,0)\oplus M(0,\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$\frac{1}{2}M(\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16})$
N^{00}_{-+}	$M(0,0,\frac{1}{2},\frac{1}{2})\oplus M(\frac{1}{2},\frac{1}{2},0,0)$	$M(\frac{1}{2},0,\frac{1}{2},\frac{1}{2})\oplus M(0,\frac{1}{2},0,0)$	$\frac{1}{2}M(\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16})$
N^{00}_{+-}	$M(\frac{1}{2},0,\frac{1}{2},0)\oplus M(0,\frac{1}{2},0,\frac{1}{2})$	$M(0,0,\frac{1}{2},0)\oplus M(\frac{1}{2},\frac{1}{2},0,\frac{1}{2})$	$\frac{1}{2}M(\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16})$
N^{00}_{--}	$M(\frac{1}{2},0,0,\frac{1}{2})\oplus M(0,\frac{1}{2},\frac{1}{2},0)$	$M(0,0,0,\frac{1}{2})\oplus M(\frac{1}{2},\frac{1}{2},\frac{1}{2},0)$	$\frac{1}{2}M(\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16})$
N^{01}_{++}, N^{01}_{-+}	$\frac{1}{2}M(\frac{1}{16},\frac{1}{16},0,0)\oplus\frac{1}{2}M(\frac{1}{16},\frac{1}{16},\frac{1}{2},\frac{1}{2})$		$\frac{1}{2}M(0,0,\frac{1}{16},\frac{1}{16})\oplus\frac{1}{2}M(\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{1}{16})$
N^{01}_{+-}, N^{01}_{--}	$\frac{1}{2}M(\frac{1}{16},\frac{1}{16},\frac{1}{2},0)\oplus\frac{1}{2}M(\frac{1}{16},\frac{1}{16},0,\frac{1}{2})$		$\frac{1}{2}M(\frac{1}{2},0,\frac{1}{16},\frac{1}{16})\oplus\frac{1}{2}M(0,\frac{1}{2},\frac{1}{16},\frac{1}{16})$
N^{10}_{++}, N^{10}_{+-}	$\frac{1}{2}M(\frac{1}{16},0,\frac{1}{16},0)\oplus\frac{1}{2}M(\frac{1}{16},\frac{1}{2},\frac{1}{16},\frac{1}{2})$		$\frac{1}{2}M(0,\frac{1}{16},0,\frac{1}{16})\oplus\frac{1}{2}M(\frac{1}{2},\frac{1}{16},\frac{1}{2},\frac{1}{16})$
N^{10}_{-+}, N^{10}_{--}	$\frac{1}{2}M(\frac{1}{16},\frac{1}{2},\frac{1}{16},0)\oplus\frac{1}{2}M(\frac{1}{16},0,\frac{1}{16},\frac{1}{2})$		$\frac{1}{2}M(\frac{1}{2},\frac{1}{16},0,\frac{1}{16})\oplus\frac{1}{2}M(0,\frac{1}{16},\frac{1}{2},\frac{1}{16})$
N^{11}_{++}, N^{11}_{+-}	$\frac{1}{2}M(0,\frac{1}{16},\frac{1}{16},0)\oplus\frac{1}{2}M(\frac{1}{2},\frac{1}{16},\frac{1}{16},\frac{1}{2})$		$\frac{1}{2}M(\frac{1}{16},0,0,\frac{1}{16})\oplus\frac{1}{2}M(\frac{1}{16},\frac{1}{2},\frac{1}{2},\frac{1}{16})$
N^{11}_{-+}, N^{11}_{--}	$\frac{1}{2}M(\frac{1}{2},\frac{1}{16},\frac{1}{16},0)\oplus\frac{1}{2}M(0,\frac{1}{16},\frac{1}{16},\frac{1}{2})$		$\frac{1}{2}M(\frac{1}{16},\frac{1}{2},0,\frac{1}{16})\oplus\frac{1}{2}M(\frac{1}{16},0,\frac{1}{2},\frac{1}{16})$

For μ, $\epsilon \in \{+,-\}^{n/2}$ and an element $c \in \mathbb{F}_2^n$ we write

$$\mathbf{N}^{ab}_{\mu,\epsilon}(c) = \bigotimes_{k=1}^{n/2} N^{ab}_{\mu_k\epsilon_k}(c(k)),$$

where $c(k) = (c_{2k-1}, c_{2k})$ as before. Let $\delta(c)$ be the number of k with $c(k) \in \{(0,1),(1,0)\}$. Recall that the lattice D_1^d determines $2d$ commuting Virasoro algebras inside the four vertex operator algebras V_{L_C}, $V_{\widetilde{L}_C}$, $\widetilde{V}_{L_C}$ and $\widetilde{V}_{\widetilde{L}_C}$. The following is the main theorem of this paper.

Theorem 4.10 *For the Virasoro frame coming from the marked code C, we have the following decompositions*

$$V_{L_C} = \bigoplus_{c\in C} \bigoplus_{\mu,\epsilon\in\{+,-\}^{d/2}} \mathbf{N}^{00}_{\mu,\epsilon}(c),$$

$$V_{\widetilde{L}_C} = \bigoplus_{\substack{c\in C\\ \delta(c)=0}} \bigoplus_{\substack{\mu,\epsilon\in\{+,-\}^{d/2}\\ \prod\epsilon_k=+}} \mathbf{N}^{00}_{\mu,\epsilon}(c) \oplus \frac{1}{2}\cdot \bigoplus_{\substack{c\in C,\\ \delta(c)>0}} \bigoplus_{\mu,\epsilon\in\{+,-\}^{d/2}} \mathbf{N}^{00}_{\mu,\epsilon}(c)$$

$$\oplus \bigoplus_{c\in C} \bigoplus_{\substack{\mu,\epsilon\in\{+,-\}^{d/2}\\ \prod\epsilon_k=(-)^{d/8}}} \mathbf{N}^{01}_{\mu,\epsilon}(c),$$

$$\widetilde{V}_{L_C} = \bigoplus_{\substack{c\in C\\ \delta(c)=0}} \bigoplus_{\substack{\mu,\epsilon\in\{+,-\}^{d/2}\\ \prod\mu_k=+}} \mathbf{N}^{00}_{\mu,\epsilon}(c) \oplus \frac{1}{2}\cdot \bigoplus_{\substack{c\in C,\\ \delta(c)>0}} \bigoplus_{\mu,\epsilon\in\{+,-\}^{d/2}} \mathbf{N}^{00}_{\mu,\epsilon}(c)$$

$$\oplus \bigoplus_{c\in C} \bigoplus_{\substack{\mu,\epsilon\in\{+,-\}^{d/2}\\ \prod\mu_k=(-)^{d/8}}} \mathbf{N}^{10}_{\mu,\epsilon}(c),$$

$$\widetilde{V}_{\widetilde{L}_C} = \bigoplus_{\substack{c\in C,\\ \delta(c)=0}} \bigoplus_{\substack{\mu,\epsilon\in\{+,-\}^{d/2}\\ \prod\mu_k=\prod\epsilon_k=+}} \mathbf{N}^{00}_{\mu,\epsilon}(c) \oplus \frac{1}{4}\cdot \bigoplus_{\substack{c\in C,\\ \delta(c)>0}} \bigoplus_{\mu,\epsilon\in\{+,-\}^{d/2}} \mathbf{N}^{00}_{\mu,\epsilon}(c)$$

$$\oplus\frac{1}{2}\cdot\bigoplus_{c\in C}[\bigoplus_{\substack{\mu,\epsilon\in\{+,-\}^{d/2}\\ \prod\epsilon_k=(-)^{d/8}}} \mathbf{N}^{01}_{\mu,\epsilon}(c) \oplus \bigoplus_{\substack{\mu,\epsilon\in\{+,-\}^{d/2}\\ \prod\mu_k=(-)^{d/8}}} \mathbf{N}^{10}_{\mu,\epsilon}(c) \oplus \bigoplus_{\substack{\mu,\epsilon\in\{+,-\}^{d/2}\\ \prod\epsilon_k=\prod\mu_k=(-)^{d/8}}} \mathbf{N}^{11}_{\mu,\epsilon}(c)].$$

Proof. Recall (4.1). For a codeword $c \in C$ and $\epsilon \in \{+,-\}^{d/2}$ let $\gamma = \Gamma^b_\epsilon(c)$ and fix $s \in \{+,-\}$. Using the definition of $\mathbf{N}^{ab}_{\mu,\epsilon}(c)$ we get

$$\begin{aligned}
\bigoplus_{\substack{\mu\in\{+,-\}^d\\ \prod\mu_k=s}} \mathbf{R}^a_\mu(\gamma) &= \bigoplus_{\substack{\mu',\mu''\in\{+,-\}^{d/2}\\ \prod\mu'_k\mu''_k=s}} \bigotimes_{i=1}^{d/2} \mathbf{R}^a_{(\mu'_i,\mu''_i)}((\gamma_{2i-1},\gamma_{2i}))\\
&= \bigoplus_{\substack{\mu\in\{+,-\}^{d/2}\\ \prod\mu_k=s}} \bigotimes_{i=1}^{d/2} \bigoplus_{\substack{\mu'_i\,\mu''_i\in\{+,-\}\\ \mu'_i\mu''_i=\mu_i}} \mathbf{R}^a_{(\mu'_i,\mu''_i)}((\gamma_{2i-1},\gamma_{2i}))\\
&= \bigoplus_{\substack{\mu\in\{+,-\}^{d/2}\\ \prod\mu_k=s}} \bigotimes_{i=1}^{d/2} N^{ab}_{\mu_i\epsilon_i}(c(i))\\
&= \bigoplus_{\substack{\mu\in\{+,-\}^{d/2}\\ \prod\mu_k=s}} \mathbf{N}^{ab}_{\mu,\epsilon}(c).
\end{aligned}$$

By using the identification of $\Gamma^a_\epsilon(c)$ with codewords in Γ and $\tilde{\Gamma}$ the decomposition follows from Lemma 4.2 and Theorem 4.7 if we use the following two observations:

First note that $N^{00}_{\mu_k\epsilon_k}(c(k)) = N^{00}_{\pm\mu_k\,\pm\epsilon_k}(c(k))$ for $c(k) \in \{(0,1),(1,0)\}$. So for $\delta(c) > 0$ we can for suppress the distinction between $\pm\epsilon_k$ (resp. $\pm\mu_k$) in the decomposition and compensate it with one factor $\frac{1}{2}$.

Second, the value of $\mathbf{N}^{01}_{\mu,\epsilon}(c)$ (resp. $\mathbf{N}^{10}_{\mu,\epsilon}(c)$ and $\mathbf{N}^{11}_{\mu,\epsilon}(c)$) depends for fixed c only on ϵ (resp. μ). $\square$

Now we discuss an action of Sym_3 (the permutation group on three letters) defined in [FLM] and [DGM2] on V_{L_C} and $\widetilde{V}_{\widetilde{L}_C}$ in terms of our decompositions. The resulting group of automorphisms is sometimes called the *triality group*.

Recall from Chapters 11 and 12 of [FLM] and Sections 7 and 8 of [DGM2] that the triality group is generated by distinct involutions σ and τ. Also recall from Section 4 that $\alpha_1, \ldots, \alpha_d$ form a D_1-frame in L. A straightforward computation shows that $\sigma\omega_{4i-3} = \omega_{4i-3}$, $\sigma\omega_{4i} = \omega_{4i}$ and σ interchanges $\omega_{4i-2} = \omega_{4i-1}$ for all $i = 1, \ldots, \frac{d}{2}$. Similarly, τ interchanges $\omega_{4i-3} = \omega_{4i-2}$ and fixes ω_{4i-1} and ω_{4i}. Thus the triality group is a subgroup of G defined in (2.2) for both V_{L_C} and $\widetilde{V}_{\widetilde{L}_C}$. Its image in $G/G_{\mathcal{C}} \leq Sym_{2d}$ is the above described permutation of the elements of the VF $\{\omega_\nu\}$.

Additionally, the involution σ defines an isomorphism between $V_{\widetilde{L}_C}$ and $\widetilde{V}_{L_C}$.

Definition 4.11 Following [DMZ], the *decomposition polynomial* of a FVOA $V = \bigoplus m_{h_1,\ldots,h_r} M(h_1, \ldots, h_r)$ is defined as

$$P_V(a, b, c) = \sum_{i,j,k} A_{i,j,k}\, a^i b^j c^k$$

where $A_{i,j,k}$ is the number of T_r-modules $M(h_1, \ldots, h_r)$ in a T_r composition series of V for which the number of coordinates in $(h_1, \ldots, h_r)$ equal to 0, $\frac{1}{2}$, $\frac{1}{16}$ is i, j, k, respectively.

The polynomial is homogeneous of degree r and, in general, depends on the chosen Virasoro frame $\{\omega_1, \ldots, \omega_r\}$ inside of V.

The following corollary is an immediate consequence of Theorem 4.10.

Corollary 4.12 *Using the symmetrized marked weight enumerator* $\mathrm{smwe}_C(x, y, z)$ *one has*

$$\begin{aligned}
P_{V_{L_C}}(a,b,c) &= \mathrm{smwe}_C(a^4 + 6a^2b^2 + b^4, 2c^4, 4a^3b + 4ab^3),\\
P_{V_{\widetilde{L}_C}}(a,b,c) &= \frac{1}{2}\left(a^4 - 2a^2b^2 + b^4\right)^{\frac{d}{2}} + \frac{1}{2}\,\mathrm{smwe}_C(a^4 + 6a^2b^2 + b^4, 2c^4, 4a^3b + 4ab^3)\\
&\quad + \frac{1}{2}\cdot 2^{d/2}\left((a+b)^d + (-1)^{d/8}(a-b)^d\right)c^d,\\
P_{\widetilde{V}_{L_C}}(a,b,c) &= P_{V_{\widetilde{L}_C}}(a,b,c),\\
P_{\widetilde{V}_{\widetilde{L}_C}}(a,b,c) &= \frac{1}{4}\cdot 3\left(a^4 - 2a^2b^2 + b^4\right)^{\frac{d}{2}}\\
&\quad + \frac{1}{4}\,\mathrm{smwe}_C(a^4 + 6a^2b^2 + b^4, 2c^4, 4a^3b + 4ab^3)\\
&\quad + 3\cdot\frac{1}{4}\cdot 2^{d/2}\left((a+b)^d + (-1)^{d/8}(a-b)^d\right)c^d. \qquad \square
\end{aligned}$$

Remark 4.13 From Theorem 4.10 we can deduce that $\widetilde{V}_{\widetilde{L}_C}$ is a self-dual rational vertex operator algebra. The proof for the special case of $V^\natural$ given in [D3] works in general since the Virasoro decompositions were the only information needed.

5 Applications

In this section we discuss some important applications for Theorem 4.10. The simplest example is for the Hamming code H_8 of length 8. When C is the Golay code $\mathcal{G}_{24}$ of length 24 there is a special marking and we obtain a particular interesting decomposition of the moonshine module $V^\natural = \widetilde{V}_{\widetilde{L}_{\mathcal{G}_{24}}}$ under 48 Virasoro algebras.

Example I: The Hamming code H_8, the root lattice E_8, and the lattice vertex operator algebra V_{E_8}

The Hamming code H_8 is the unique self-annihilating doubly-even binary code of length 8. Its automorphism group is isomorphic to $AGL(\mathbb{F}_2^3)$. The root lattice E_8 of the exceptional Lie group $E_8(\mathbb{C})$ is the unique even unimodular lattice of rank 8. It has the Weyl group $W(E_8)$ as its automorphism group. The lattice vertex operator algebra V_{E_8}, whose underlying vector space is the irreducible level 1 highest weight representation of the affine Kac-Moody algebra $E_8^{(1)}$, is a self-dual vertex operator algebra of rank 8 whose automorphism group is the Lie group $E_8(\mathbb{C})$. One can show, under some additional conditions on the vertex operator algebra , that V_{E_8} is the unique self-dual VOA of rank 8 (cf. [H1], Ch. 2).

The uniqueness of the lattice E_8 implies $E_8 \cong L_{H_8} \cong \widetilde{L}_{H_8}$ and $V_{E_8} \cong V_{L_{H_8}} \cong V_{\widetilde{L}_{H_8}} \cong \widetilde{V}_{L_{H_8}} \cong \widetilde{V}_{\widetilde{L}_{H_8}}$ for the vertex operator algebras, since one has $V_{\widetilde{L}_C} \cong \widetilde{V}_{L_C}$ in general (see [DGM1], [DGM2] and the remark after Theorem 4.10).

We will determine up to automorphism all markings for the Hamming code, all D_1-frames of the E_8 lattice, and five Virasoro frames inside V_{E_8} and describe the corresponding decompositions. They are all coming from markings of the Hamming code.

To fix notation we choose $\{(00001111), (00110011), (11000011), (01010101)\}$ as a set of base vectors for the Hamming code.

Theorem 5.1 *There are* 3 *orbits of markings for the Hamming code* H_8 *under* $\mathrm{Aut}(H_8)$. *Their main properties can be found in the next table. The last column shows the symmetrized marked weight enumerator.*

orbit	orbit representatives	stabilizer	orbit size	$\mathrm{smwe}_{H_8}(x,y,z)$
α	$\{(1,2),(3,4),(5,6),(7,8)\}$	$2^3{:}Sym_4$	7	$x^4+6\,x^2z^2+z^4+8\,y^4$
β	$\{(1,2),(3,4),(5,7),(6,8)\}$	$2^2.Dih_8$	42	$x^4+2\,x^2z^2+z^4+8\,xzy^2+4\,y^4$
γ	$\{(1,2),(3,5),(4,7),(6,8)\}$	Sym_4	56	$x^4+z^4+12\,xzy^2+2\,y^4$

The proof is an easy counting exercise (see Appendix A).

We remark that every pair (i,j) of the eight positions is the component of exactly one of the seven markings of type α: Every marking contains 4 pairs, so we cover

$7 \cdot 4 = 28$ pairs. There are $\binom{8}{2} = 28$ different such pairs on which $\mathrm{Aut}(H_8)$ transitively acts.

As explained in the last section before Lemma 4.2 in general, every marking of H_8 determines a D_1^8 sublattice inside $L_{H_8} \cong E_8$ and $\widetilde{L}_{H_8} \cong E_8$.

The following theorem shows that all possible D_1-frames in E_8 are obtained in this way.

Theorem 5.2 (Conway-Sloane [CS2]) *There are 4 orbits of D_1^8 sublattices inside E_8 under the action of $W(E_8)$. Their main properties are shown in the next table. The column "origin" lists the corresponding (untwisted, resp. twisted lattice) Hamming code marking and $\mathrm{swe}_\Delta(A, B, C)$ is the symmetrized weight enumerator of the decomposition code $\Delta = E_8/D_1^8 \leq (D_1^*/D_1)^8 \cong \mathbb{Z}_4^8$.*

orbit	origin	*stabilizer*	orbit size	$\mathrm{swe}_\Delta(A, B, C)$
$\mathcal{K}_8$	α	$\mathbb{Z}_2^7. \cdot Sym_8$	135	$A^8 + 28A^2C^6 + 70A^4C^4 + 28A^6C^2 + C^8 + 128B^8$
$\mathcal{K}_8'$	β, $\widetilde{\alpha}$	$2^7(4!)^2$	9450	$A^8 + C^8 + 12A^2C^2(A^4 + C^4) + 38A^4C^4 +$ $64AC(A^2 + C^2)B^4 + 64B^8$
$\mathcal{L}_8$	γ, $\widetilde{\beta}$	$2^8 \cdot 4!$	113400	$A^8 + C^8 + 4A^2C^2(A^4 + C^4) + 22A^4C^4 +$ $96AC(A^2 + C^2)B^4 + 32B^8$
$\mathcal{O}_8$	$\widetilde{\gamma}$	$2.AGL(3, 2)$	259200	$A^8 + C^8 + 14A^4C^4 + 112AC(A^2 + C^2)B^4 + 16B^8$

Proof. It was also explained in the last section that every D_1^8 sublattice inside E_8 defines a $\mathbb{Z}_4$-code $\Delta \leq (D_1^*/D_1)^8 \cong \mathbb{Z}_4^8$. Since E_8 is self-dual and even, Δ is self-annihilating and even as a code over $\mathbb{Z}_4$. All self-annihilating $\mathbb{Z}_4$-codes of length 8 are classified in [CS2], Theorem 2. Only $\mathcal{K}_8$, $\mathcal{K}_8'$, $\mathcal{L}_8$ and $\mathcal{O}_8$ are even (see also [BS]). The order of $\mathrm{Aut}(\Delta)$ and $\mathrm{swe}_\Delta(A, B, C)$ are also described in [CS2]. To show that these codes arise from the markings of the Hamming code as in the table we apply Corollary 4.6. □

The remark after Theorem 5.1 about the Hamming code has an analogue here: every vector of squared length 4 inside E_8 is contained in exactly one D_1-frame belonging to the orbit of type $\mathcal{K}_8$ since $135 \cdot 16 = 2160$, the number of vectors of squared length 4, and $W(E_8)$ acts transitively on such vectors. These 135 D_1^8-sublattices are in bijection with cosets of $2E_8$ in E_8 which have coset representatives of norm 4.

Every D_1-frame inside E_8 determines 16 commuting Virasoro vertex operator algebras of rank $\frac{1}{2}$ inside V_{E_8} and $\widetilde{V}_{E_8} \cong V_{E_8}$. Altogether, one gets at least five different systems of commuting Virasoro subVOAs:

Theorem 5.3 *Let $\{\omega_1, \ldots, \omega_{16}\}$ be a Virasoro frame inside V_{E_8}. The possible decomposition polynomials are displayed in the next table. They correspond by the untwisted* *or twisted lattice construction to the D_1-frames inside E_8 as indicated in the column origin. Furthermore, the first four cases belong to four distinct orbits of Virasoro frames under the action of the Lie group $E_8(\mathbb{C})$. In the fifth case, Ω, at least the T_{16}-module structure is unique.*

case	origin	$P_{V_{E_8}}(a,b,c)$
Γ	$\mathcal{K}_8$	$\frac{1}{2}\left[(a+b)^{16}+(a-b)^{16}\right]+128\,c^{16}$
Σ	$\mathcal{K}_8'$, $\widetilde{\mathcal{K}}_8$	$a^{16}+b^{16}+56\,(a^{14}\,b^2+\,a^2\,b^{14})+924\,(a^{12}\,b^4+\,a^4\,b^{12})+$ $3976\,(a^{10}\,b^6+\,a^6\,b^{10})+6470\,a^8\,b^8+$ $(128\,(a^7\,b+\,a\,b^7)+896\,(a^5\,b^3\,+a^3\,b^b))c^8+64\,c^{16}$
Ψ	$\mathcal{L}_8$, $\widetilde{\mathcal{K}}_8'$	$a^{16}+b^{16}+24\,(a^{14}\,b^2+a^2\,b^{14})+476\,(a^{12}\,b^4+a^4\,b^{12})+$ $1960\,(a^{10}\,b^6+a^6\,b^{10})+3270\,a^8\,b^8+$ $(192\,(a^7\,b+a\,b^7)+1344\,(a^5\,b^3+a^3\,b^5))c^8+32\,c^{16}$
Θ	$\mathcal{O}_8$, $\widetilde{\mathcal{L}}_8$	$a^{16}+b^{16}+8\,(a^{14}\,b^2+a^2\,b^{14})+252\,(a^{12}\,b^4+a^4\,b^{12})+$ $952\,(a^{10}\,b^6+a^6\,b^{10})1670\,a^8\,b^8+$ $(224\,(a^7\,b+a\,b^7)+1568\,(a^5\,b^3+a^3\,b^5))c^8+16\,c^{16}$
Ω	$\widetilde{\mathcal{O}}_8$	$a^{16}+b^{16}+140\,(a^{12}\,b^4+a^4\,b^{12})+448\,(a^{10}\,b^6+a^6\,b^{10})+870\,a^8\,b^8+$ $(240\,(a^7\,b\,+a\,b^7)+1680\,(a^5\,b^3+a^3\,b^5))c^8+8\,c^{16}$

Proof. Using Corollary 4.12 to compute $P_{V_{E_8}}(a,b,c)$ for the different Virasoro subVOAs T_{16} coming from V_{E_8}, and $\widetilde{V}_{E_8}$ and a given D_1^8 sublattice in E_8 one checks that the polynomials for Γ, Σ, Ψ, Θ and Ω correspond to the D_1-frames of E_8 as indicated.

We show that there are no other possibilities for the decomposition polynomial $P_{V_{E_8}}(a,b,c)$ and we will see directly that there is a unique $\mathrm{Aut}(V_{E_8})$-orbit of T_{16} subVOAs corresponding to each of the cases Γ, Σ, Ψ and Θ.

Assume a vertex operator subalgebra T_{16} in V_{E_8} is given. First we determine the possible decomposition polynomials.

As described in the proof of Theorem 4.1.5 in [H1], $SL_2(\mathbb{Z})=\langle S,T\rangle$ with $S=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ and $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ acts on $\mathbb{C}[a,b,c]$ by

$$\rho(S)=\begin{pmatrix}1/2&1/2&1/\sqrt{2}\\1/2&1/2&-1/\sqrt{2}\\1/\sqrt{2}&-1/\sqrt{2}&0\end{pmatrix},\qquad \rho(T)=e^{-2\pi i/48}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&e^{2\pi i/16}\end{pmatrix}$$

via substitution. Since V_{E_8} is a self-dual VOA of rank 8, the decomposition polynomial must be invariant under the action of $\rho(S)$ and $\rho(T^3)$ (cf. proof of Theorem 2.19 or Th. 2.1.2 and Th. 4.1.5 in [H1]). They generate a matrix group $G=\langle\rho(S),\rho(T)^3\rangle$ of order 384 as can easily be seen with the help of the program Gap [Sgap]. The dimension of the space of invariant polynomials of degree n is the multiplicity of the trivial representation in the n^{th} symmetric power of ρ. This multiplicity is given by the coefficient of t^n in the expression

$$\rho_G(t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(\mathrm{id} - gt)} \, .$$

For degree 16 we obtain that the space of invariant polynomials is two dimensional; a possible base is given by $P^{\Gamma}_{V_{E_8}}(a,b,c)$ and $P^{\Omega}_{V_{E_8}}(a,b,c)$. The only polynomials $P(a,b,c)$ inside this space having positive coefficients and satisfying the necessary conditions $P(1,0,0) = 1$ and $P(1,1,0) = |\mathcal{C}| = 2^l$, $0 \leq l \leq 15$ with integral l, are the five polynomials given in the theorem.

Next we claim that the code $\mathcal{C}$ is uniquely determined from its weight enumerator $P(a,b,0)$: The weight enumerator of its annihilator code $\mathcal{C}^{\perp}$ is $a^{16} + (2^k - 2)a^8 b^8 + b^{16}$, with $k = 16 - l = 1, \ldots, 5$. For $k = 5$ the uniqueness of $\mathcal{C}^{\perp}$ and so of $\mathcal{C}$ is the uniqueness of the simplex code (see Appendix C, Prop. **??**). For smaller k, it follows also from the proof of Prop. **??**.

The code $\mathcal{C}$ contains for $k = 1$, 2, 3 and 4 the subcode $\mathcal{C}_0 = \{(00),(11)\}^8$. By Corollary 3.3 and the uniqueness statement of Proposition 2.16 the corresponding subVOA must be $V_{D_1^8}$. Recall that $V_{E_8} = M(1) \otimes \mathbb{C}\{E_8\}$. The weight one subspace $(V_{E_8})_1$ is a Lie algebra under $[u,v] = u_0 v$ which is isomorphic to the Lie algebra of type E_8 and $(V_{D_1^8})_1$ is a Cartan subalgebra of $(V_{E_8})_1$. From the construction of $V_{E_8} = M(1) \otimes \mathbb{C}\{E_8\}$ we have a canonical Cartan subalgebra $M(1)_1$ of $(V_{E_8})_1$ which is identified with $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} E_8$. Since all Cartan subalgebras are conjugate under the adjoint action of the Lie group $E_8(\mathbb{C})$ we can assume that $(V_{D_1^8})_1 = M(1)_1 \leq (V_{E_8})_1$.

It is well-known that $\mathbb{C}\{E_8\} = 1 \otimes \mathbb{C}\{E_8\} = \{u \in V_{E_8} \mid h_n u = 0,\, h \in \mathfrak{h},\, n > 0\}$ which is the vacuum space for the Heisenberg algebra $\hat{\mathfrak{h}}_{\mathbb{Z}}$ (see Section 3). Similarly, $\mathbb{C}\{D_1^8\} = \{u \in V_{D_1^8} \mid h_n u = 0,\, h \in \mathfrak{h},\, n > 0\}$. Thus, $\mathbb{C}\{D_1^8\}$ is a subspace of $\mathbb{C}\{E_8\}$. Note that $\mathbb{C}\{E_8\}$ and $\mathbb{C}\{D_1^8\}$ are direct sums of weight spaces for the Cartan algebra $\mathfrak{h}$ and the corresponding weight lattices are exactly E_8 and D_1^8. This determines a D_1^8 sublattice of E_8, unique up to the action of $W(E_8)$. That is, for $k = 1$, 2, 3, 4 the Virasoro frames $\{\omega_1, \ldots, \omega_{16}\}$ come from one of the four D_1-frames inside E_8 by the untwisted lattice-VOA construction.

It remains to show that for $k = 5$, i.e. in the case Ω, the obtained Virasoro decomposition is unique. As stated before the code $\mathcal{C}^{\perp}$, which is equal to $\mathcal{D}$ by Theorem 2.19, is the simplex code and so $\mathcal{C}$ is the extended Hamming code of length 16. The uniqueness of the code $\mathcal{C}$ implies by Theorem 2.3 (4) that $V^0_{E_8}$ is unique as a T_{16}-module. Let $I \in \mathcal{D}$ such that $|I| = 8$. Take an irreducible T_{16}-module W in V^I.

Using the action of V^0 on W we see that all such $M(h_1, \ldots, h_{16})$ occur in V^I where $h_i = \frac{1}{16}$ if $i \in I$ and $h_i \in \{0, \frac{1}{2}\}$ if $i \notin I$ and the number of i with $h_i = \frac{1}{2}$ is odd. So, there are 2^7 nonisomorphic T_{16}-modules inside V^I. Since $\mathcal{D}$ has 30 codewords of weight 8, we get at least $30 \cdot 2^7$ such nonisomorphic T_{16}-modules. But $30 \cdot 2^7$ is exactly the coefficient of c^8 in $P(1, 1, c)$. This shows that all these modules have multiplicity one. Finally the multiplicity of $M(\frac{1}{16}, \ldots, \frac{1}{16})$ is 8. Therefore the decomposition in the last case is unique. □

Remark 5.4 (1) We expect that also in the fifth case the vertex operator algebra structure is unique, i.e. Ω corresponds to a unique $E_8(\mathbb{C})$-orbit of Virasoro frames.

(2) A different proof would follow if the list of the 71 unitary self-dual VOA *candidates* of rank 24 given by Schellekens [Sch] is complete:
The fusion algebras for $M(0)$ and the Kac-Moody VOA $V_{B_{1,1}}$ are isomorphic and one can identify the corresponding intertwiner spaces (cf. [MoS], Appendix D). From this, one can define for every VOA V of rank c containing $M(0)^{\otimes 2c}$ a VOA W of rank $3c$ containing $V_{B_{1,1}}^{\otimes 2c}$. There are five candidates W on Schellekens list containing $V_{B_{1,1}}^{\otimes 16}$, namely $W_{D_{24,1}}$, $W_{D_{12,1}^2}$, $W_{D_{6,1}^4}$, $W_{A_{3,1}^8}$ and $W_{A_{1,2}^{16}}$. They correspond to Γ, Σ, Ψ, Θ and Ω in this order. Again the uniqueness of $W_{A_{1,2}^{16}}$ is unknown. The decomposition of $W_{A_{1,2}^{16}}$ as a $V_{B_{1,1}}^{\otimes 16}$-module obtained in [Sch] by a computer calculation follows from our analysis of the case Ω.

The next table summarizes the relation between the markings for H_8, the D_1-frames inside E_8 and the Virasoro frames $\{\omega_1, \ldots, \omega_{16}\}$ inside V_{E_8} as obtained in the last three theorems. The arrow $\swarrow$ (resp. $\searrow$) denotes the untwisted (resp. twisted) construction. For a detailed explanation of the second row of the table see [H2]. Self-dual *Kleinian codes* are a generalization of the so called *type* IV codes over $\mathbb{F}_4$. Especially, the notation of a *marking* of a Kleinian code is defined in [H2]. Finally, Ξ_1 is the D_8^*/D_8-code $\{0, s\}$ of length 1 where the D_8-coset $s \in D_8^*/D_8$ has minimal squared length 2.

type	object	marking/frame								
D_8^*/D_8-code:	Ξ_1					A				
					$\swarrow$		$\searrow$			
Kleinian codes:	ϵ_2				a		b			
				$\swarrow$		$\searrow$ $\swarrow$		$\searrow$		
binary codes:	H_8			α		β		γ		
			$\swarrow$		$\searrow$ $\swarrow$		$\searrow$ $\swarrow$		$\searrow$	
lattices:	E_8		$\mathcal{K}_8$		$\mathcal{K}'_8$		$\mathcal{L}_8$		$\mathcal{O}_8$	
		$\swarrow$		$\searrow$ $\swarrow$		$\searrow$ $\swarrow$		$\searrow$ $\swarrow$		$\searrow$
VOAs:	V_{E_8}	Γ		Σ		Ψ		Θ		Ω

Example II: The Golay code $\mathcal{G}_{24}$, the Leech lattice Λ, and the moonshine module $V^\natural$

The moonshine module $V^\natural$ is the $\mathbb{Z}_2$-orbifold vertex operator algebra of V_Λ associated to the Leech lattice Λ which is itself the twisted lattice $\widetilde{L}_{\mathcal{G}_{24}}$ coming from the Golay code $\mathcal{G}_{24}$ (cf. [B], [FLM]). That is, $V^\natural = \widetilde{V}_{\widetilde{L}_{\mathcal{G}_{24}}}$.

To describe Virasoro decompositions of the moonshine module coming from markings of the Golay code, we must study these markings first. For the decomposition polynomial $P_{V^\natural}(a,b,c)$ only, it is enough to compute the coefficients $W_{k,l}$ of the symmetrized marked weight enumerator. The possible values for $W_{8,l}$ (and so for $W_{16,l}$) for the Golay code were computed by Conder and McKay in [CM]. They found 90 possibilities. It is not clear if the numbers $W_{12,l}$, which are also needed, can be determined from the $W_{8,l}$ alone.

The markings for the Golay code are classified by the double cosets $\mathbb{Z}_2^{12}.Sym_{12}\backslash Sym_{24}/M_{24}$. (The first subgroup is the stabilizer of a partition of the 24-set into 2-sets; the second is M_{24}, the automorphism group of $\mathcal{G}_{24}$.) In fact there are 1858 different classes of markings [Be].

The binary linear code $\mathcal{C} \leq \mathbb{F}_2^{48}$ as defined in Section 2 depends also on the chosen marking. Since for the moonshine module we have $\dim V_1^\natural = 0$ the minimal weight of $\mathcal{C}$ is at least four. The following easy result gives an restriction on the dimension of $\mathcal{C}$.

Lemma 5.5 *For every frame of* 48 *Virasoro vertex operator algebras of rank* $\frac{1}{2}$ *inside the moonshine module the dimension of* $\mathcal{C}$ *is smaller than or equal to* 41.

Proof. Deleting one coordinate of the codewords of a k-dimensional code $\mathcal{C}$ of minimal weight 4 leads to a code of length 47, dimension k and minimal weight at least 3. Minimal weight 3 implies that the spheres of radius one around the codewords of this code are all disjoint, i.e. we have the sphere packing condition $2^k\cdot(1+47) \leq 2^{47}$ or $k \leq 41$. $\square$

There is indeed a special marking $\mathcal{M}^*$ where $\mathcal{C}$ meets this bound. A good way to define it, is to describe the Golay code itself by a "double twist" construction. Starting from the glue code Ξ_3 of the Niemeier lattice with root sublattice D_8^3 one gets first the hexacode $\mathcal{H}_6$, a code over the Kleinian fourgroup, and from the hexacode one obtains the Golay code $\mathcal{G}_{24}$:

As a code over $D_8^*/D_8 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{0,1,s,\bar{s}\}$ (where 1, s, $\bar{s}$ are the D_8-cosets represented by $(0^7,1)$, $((\frac{1}{2})^7, \pm\frac{1}{2})$, respectively) one has (cf. [V])

$$\Xi_3 = \{(000),(s11),(1s1),(11s),(0\bar{s}\bar{s}),(\bar{s}0\bar{s}),(\bar{s}\bar{s}0),(sss)\}\,.$$

The hexacode as a code over $D_4^*/D_4 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{0, a, b, c\}$ (where $a = [(0,0,0,1)]$, $b = [(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})]$ and $c = [(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})]$) can be defined by

$$\mathcal{H}_6 = \widetilde{\Xi}_3 := \left(\widehat{\Xi}_3 + (\delta_2^3)_0\right) \cup \left(\widehat{\Xi}_3 + (\delta_2^3)_0 + (b0b0ca)\right).$$

Here $\widehat{\ }$ is the map induced from $\hat{\ } : D_8^*/D_8 \longrightarrow (D_4^*/D_4)^2$, $0 \mapsto 00$, $1 \mapsto a0$, $s \mapsto bb$ and $\bar{s} \mapsto cb$, and $(\delta_2^n)_0$ is the subcode of the Kleinian code $\delta_2^n := \{(00),(aa)\}^n$ of length $2n$ consisting of codewords of weights divisible by 4.

In a similar way one gets

$$\mathcal{G}_{24} = \widetilde{\mathcal{H}}_6 := \left(\widehat{\mathcal{H}}_6 + (d_4^6)_0\right) \cup \left(\widehat{\mathcal{H}}_6 + (d_4^6)_0 + (1000\ 1000\ \ldots\ 1000\ 0111)\right),$$

where $\widehat{\ }$ is the map induced from $\hat{\ } : D_4^*/D_4 \longrightarrow (D_2^*/D_2)^2 \cong \mathbb{F}_2^4$, $0 \mapsto 0000$, $a \mapsto 1100$, $b \mapsto 1010$, $c \mapsto 0110$, and $(d_4^n)_0$ is the subcode of the binary code $d_4^n := \{(0000),(1111)\}^n$ of length $4n$ consisting of codewords of weights divisible by 8. This is the usual MOG or hexacode construction of the Golay code and is a special case of the twisted construction of binary codes from Kleinian codes (cf. [H2], last section).

In this description of the Golay code we let $\mathcal{M}^* = \{(1,2),\ldots,(47,48)\}$ the special marking mentioned above. The marking used in [DMZ] and [H1] arose from the way the Golay code was written there as a cyclic code.

The symmetrized marked weight enumerator for the marking $\mathcal{M}^*$ of the Golay code is easily computed (using for example the above description) and one gets

$$\begin{aligned} \mathrm{smwe}_{\mathcal{G}_{24}}(x,y,z) &= x^{12} + z^{12} + 39\,(x^4 z^8 + x^8 z^4) + 48\,x^6 z^6 \\ &\quad + \left(96\,(x^6 z^2 + x^2 z^6) + 192\,x^4 z^4\right) y^4 \\ &\quad + \left(576\,(x^5 z + x z^5) + 1920\,x^3 z^3\right) y^6 \\ &\quad + \left(48\,(x^4 + z^4) + 288\,x^2 z^2\right) y^8 + 128\,y^{12}. \end{aligned} \tag{5.8}$$

Another property of the marking $\mathcal{M}^*$ is, that it has the largest stabilizer inside M_{24} among all the different markings, namely $2^6{:}\,[Sym_4 \times Sym_3]$ of order $2^{10}3^2 = 9216$ (see Appendix B), as was noted in [CM].

Remark 5.6 Assume that a marking is represented by the standard partition $\{(1,2),(3,4),\ldots,(23,24)\}$. The markings of the Golay code that arise from markings of the hexacode in the sense of Kleinian codes (cf. end of last subsection) are exactly the ones for which the code $(d_4^6)_0$ is a subcode of $\mathcal{G}'_{24}$, a code equivalent to $\mathcal{G}_{24}$.

From Lemma 4.2, we get the decomposition of the Leech lattice $\Lambda \cong \widetilde{L}_{\mathcal{G}_{24}}$ under the D_1-frame belonging to the marking $\mathcal{M}^*$. For the symmetrized weight enumerator of the corresponding code $\widetilde{\Gamma} \leq \mathbb{Z}_4^{24}$ (see (4.2)). Corollary 4.2 gives:

$$
\begin{aligned}
\mathrm{swe}_{\widetilde{\Gamma}}(A,B,C) \;=\;& A^{24}+\mathrm{C}^{24}+23439\,(A^{16}\,\mathrm{C}^{8}+A^{8}\,\mathrm{C}^{16})+4032\,(A^{6}\,\mathrm{C}^{18}+A^{18}\,\mathrm{C}^{6})\\
&+378\,(A^{4}\,\mathrm{C}^{20}+\,A^{20}\,\mathrm{C}^{4})+60480\,(A^{10}\,\mathrm{C}^{14}+A^{14}\,\mathrm{C}^{10})+85484\,A^{12}\,\mathrm{C}^{12}\\
&+\Big(3072\,(A^{2}\,\mathrm{C}^{14}+A^{14}\,\mathrm{C}^{2})+43008\,(A^{12}\,\mathrm{C}^{4}+A^{4}\,\mathrm{C}^{12})\\
&+193536\,(A^{10}\,\mathrm{C}^{6}+A^{6}\,\mathrm{C}^{10})+307200\,A^{8}\,\mathrm{C}^{8}\Big)\,B^{8}\\
&+\Big(86016\,(A^{11}\,\mathrm{C}+A\,\mathrm{C}^{11})+1576960\,(A^{9}\,\mathrm{C}^{3}+A^{3}\,\mathrm{C}^{9})\\
&+5677056\,(A^{7}\,\mathrm{C}^{5}+A^{5}\,\mathrm{C}^{7})\Big)\,B^{12}\\
&+\Big(6144\,(A^{8}\;+\mathrm{C}^{8})+172032\,(A^{6}\,\mathrm{C}^{2}+\,A^{2}\,\mathrm{C}^{6})+430080\,A^{4}\,\mathrm{C}^{4}\Big)\;B^{16}\\
&+262144\,B^{24}.
\end{aligned}
$$

As stated before, the markings for the Golay code are classified by the double cosets $\mathbb{Z}_2^{12}.Sym_{12}\backslash Sym_{24}/M_{24}$.

The classification of all D_1-frames in the Leech lattice would seem to be more complicated. From equation (4.2), we see that in the case where the D_1-frame comes from a marking of the Golay code the corresponding $\mathbb{Z}_4$-code $\widetilde{\Gamma}$ contains the subcode $(\Sigma_2^{12})_0$. The following result gives the converse. Recall that the *Euclidean weight* of a codeword is the minimal Euclidean squared norm of a coset representative in $(D_1^*)^{24}$.

Lemma 5.7 *Every self-annihilating even $\mathbb{Z}_4$-code Δ of length 24 and minimal Euclidean weight 4 containing the subcode $(\Sigma_2^{12})_0$ can be obtained from a marking of the Golay code as in equation (4.2).*

Proof. Let $K=\bigoplus_{i=1}^{24}\mathbb{Z}\,a_i$ a lattice of type A_1^{24} in $\mathbb{R}^{24}$, i.e. the a_i are pairwise orthogonal vectors of squared length 2. Set $L=\bigoplus_{i=1}^{24}\mathbb{Z}\,b_i$, with $b_{2i-1}=a_{2i-1}+a_{2i}$ and $b_{2i}=a_{2i-1}-a_{2i}$ for $i=1,\ \ldots,\ 12$, i.e. L is a lattice of type D_1^{24}. Finally let $M=2K$. On K, the group $\mathbb{Z}_2^{24}{:}\,Sym_{24}$ acts by monomial matrices with entries ±1 with respect to the basis $\{a_i\mid i=1,\ldots,24\}$. The lattice L is fixed at least by the group of sign changes. Clearly $K^*/K\cong\mathbb{Z}_2^{24}$, $L^*/L\cong\mathbb{Z}_4^{24}$, $M^*/M\cong\mathbb{Z}_8^{24}$, with the induced action of $\mathbb{Z}_2^{24}{:}\,Sym_{24}$ on $\mathbb{Z}_2^{24}$ and $\mathbb{Z}_8^{24}$ and of $\mathbb{Z}_2^{24}$ on K.

The code $\Delta\leq L^*/L$ determines a self-dual even lattice Λ of rank 24 and minimal length 4. (This must be the Leech lattice since it is the unique self-dual even rank 24 lattice of minimal length 4.)

To prove the lemma we have to find a doubly-even self-annihilating binary code $\mathcal{G}_{24}'\leq K^*/K$ equivalent to the Golay code $\mathcal{G}_{24}$ such that $\mathcal{G}_{24}'$ determines $\Delta=\widetilde{\Gamma}$ as in (4.2). (Instead of changing the marking $\mathcal{M}=\{(1,2),\ldots,(23,24)\}$, the choice which is determined by the relation between K and L, we are permuting the code $\mathcal{G}_{24}$; these procedures are equivalent.)

The lattice Λ defines a self-annihilating even $\mathbb{Z}_8$-code $\Omega = \Lambda/M \leq M^*/M$ of minimal Euclidean weight 4. If we start with the with our standard copy of the Golay code $\mathcal{G}_{24}$ we get a lattice $\widetilde{\Lambda}$, a $\mathbb{Z}_4$-code $\widetilde{\Delta} \subset L^*/L$, and a $\mathbb{Z}_8$-code $\widetilde{\Omega} \subset M^*/M$.

Since $(\Sigma_2^{12})_0$ is contained in Δ, we see easily that the code Ω contains all $\binom{24}{2}$ vectors of type $(4^2 0^{22})$. As a main step in the uniqueness proof of Λ in [Co], it was shown that such a code is unique up to the action of $\mathbb{Z}_2^{24}{:}\,Sym_{24}$, i.e. we have a π in this group such that $\pi(\widetilde{\Lambda})/M = \Sigma$. The copy $\mathcal{G}'_{24} = \pi(\mathcal{G}_{24})$ of the Golay code gives the code Δ in L^*/L. □

Finally we come to the Virasoro decomposition of the moonshine module $V^\natural = \widetilde{V}_{\widetilde{L}_{\mathcal{G}_{24}}}$.

The following theorem gives an precise description of the codes $\mathcal{C}$ and $\mathcal{D}$ as defined in section 2.

Theorem 5.8 *The code $\mathcal{C}$ associated to the special marking $\mathcal{M}^*$ of the Golay code has length 48 and dimension 41. Its annihilator code $\mathcal{C}^\perp = \{d \in \mathbb{F}_2^{48} \mid (d,c) = 0$ for all $c \in \mathcal{C}\}$ is of dimension 7 and equals the code $\mathcal{D}$ which has generator matrix*

$$\begin{pmatrix}
1111111111111111 & 0000000000000000 & 0000000000000000\\
0000000000000000 & 1111111111111111 & 0000000000000000\\
0000000000000000 & 0000000000000000 & 1111111111111111\\
0000000011111111 & 0000000011111111 & 0000000011111111\\
0000111100001111 & 0000111100001111 & 0000111100001111\\
0011001100110011 & 0011001100110011 & 0011001100110011\\
0101010101010101 & 0101010101010101 & 0101010101010101
\end{pmatrix}.$$

Proof. Recall the description of the Golay code given above. The codes $\mathcal{H}_6$ and $\mathcal{G}_{24}$ are unions of two parts. The first part we call the untwisted part and the second is called the twisted part.

First we show that the above matrix is a parity check matrix for $\mathcal{C}$. From Theorem 4.10 we see that a codeword $c \in \mathcal{G}_{24}$ gives us an irreducible T_{48}-module $M(h_1, \ldots, h_{48})$ with all h_i different from $\frac{1}{16}$ if and only if $c(k) \in \{(0,0),(1,1)\}$ for all k. The codewords with this property are exactly the ones that are coming from the codeword $(000) \in \Xi_3$. This gives the first three rows of the parity check matrix. The next two rows correspond to the selection of the subcodes $(\delta_2^3)_0 \subset \delta_2^3$ and $(d_4^6)_0 \subset d_4^6$. Let $\mathcal{B}_2^n$ the FVOA $(M(0,0) \oplus M(\frac{1}{2},\frac{1}{2}))^{\otimes n}$ with binary code $\mathcal{C}(\mathcal{B}_2^n) = \{(0,0),(1,1)\}^n$ of length $2n$. The subVOA $(\mathcal{B}_2^n)_0$ is the FVOA belonging to the subcode of $\mathcal{C}(\mathcal{B}_2^n)$ consisting of codewords of weights divisible by 4 (cf. Proposition 2.16). Then the last two rows of the parity check matrix correspond to the selection of the subcodes $(\Sigma_2^{12})_0 \subset \Sigma_2^{12}$ and $\mathcal{C}((\mathcal{B}_2^{24})_0) \subset \mathcal{C}(\mathcal{B}_2^{24})$: this are the conditions $\prod \epsilon_k = +$ and $\prod \mu_k = +$. There are no further conditions.

To determine $\mathcal{D}$ note first that the inclusion $\mathcal{D} \leq \mathcal{C}^\perp$ is Proposition 2.14 (3). To see $\mathcal{C}^\perp \leq \mathcal{D}$ observe that the codewords $\{(s11), (1s1), (11s)\} \subset \Xi_3$ correspond to the first three lines of the generator matrix, the twisted parts of $\mathcal{H}_6$ and $\mathcal{G}_{24}$ to the next two, and two of the last three summands of $V^\natural = \widetilde{V}_{\widetilde{L}_{\mathcal{G}_{24}}}$ in Theorem 4.10 correspond to the last two lines of the generator matrix.

Alternatively, one can compute $\mathcal{D}$ by using the self-duality of the moonshine VOA [D3] and apply Theorem 2.19. □

The code $\mathcal{C}$ is also the lexicographic code of length 48 and minimal weight 4 (see [CS3], Th. 6). As mentioned there, it is a "shortened extended Hamming code" of length 64 in the following sense: If we extend the generator matrix of $\mathcal{D}$ by the block

$$\begin{pmatrix} 1111111111111111 \\ 1111111111111111 \\ 1111111111111111 \\ 0000000011111111 \\ 0000111100001111 \\ 0011001100110011 \\ 0101010101010101 \end{pmatrix},$$

we obtain a parity check matrix for the extended Hamming code H_{64} of length 64. The vectors $c \in \mathbb{F}_2^{64}$ with 0's in the last 16 coordinates belong to H_{64} if and only if the vector of the first 48 coordinates belongs to $\mathcal{C}$.

The automorphism group of this code is of type $2^{12}[GL(4,2) \times Sym_3]$ and has order 495452160 (see Appendix C for a proof).

For future references we give the decomposition polynomial as obtained from Corollary 4.12 in full. Remember that a, b and c count the modules of conformal weight 0, $\frac{1}{2}$, resp. $\frac{1}{16}$ (see Definition 4.11).

Corollary 5.9 *The complete decomposition polynomial for the moonshine module belonging to the special marking $\mathcal{M}^*$ is given by*

$$\begin{aligned} P_{V^\natural}^{\mathcal{M}^*}(a,b,c) = {} & a^{48} + b^{48} + 3300\,(a^{44}\,b^4 + a^4\,b^{44}) + 189504\,(a^{42}\,b^6 + a^6\,b^{42}) \\ & +5907810\,(a^{40}\,b^8 + a^8\,b^{40}) + 102156864\,(a^{38}\,b^{10} + a^{10}\,b^{38}) \\ & +1088684372\,(a^{36}\,b^{12} + a^{12}\,b^{36}) + 7535996160\,(a^{34}\,b^{14} + a^{14}\,b^{34}) \\ & +35232581487\,(a^{32}\,b^{16} + a^{16}\,b^{32}) + 114215080192\,(a^{30}\,b^{18} + a^{18}\,b^{30}) \end{aligned}$$

$$+261496913352\,(a^{28}\,b^{20}+a^{20}\,b^{28})+427898196864\,(a^{26}\,b^{22}+a^{22}\,b^{26})$$
$$+503871835740\,a^{24}\,b^{24}$$
$$+\Big(6144\,(a^{30}\,b^{2}+a^{2}\,b^{30})+430080\,(a^{28}\,b^{4}+a^{6}\,b^{28})$$
$$+10881024\,(a^{26}\,b^{6}+a^{6}\,b^{26})+126197760\,(a^{24}\,b^{8}+a^{8}\,b^{24})$$
$$+774199296\,(a^{22}\,b^{10}+a^{10}\,b^{22})+2709417984\,(a^{20}\,b^{12}+a^{12}\,b^{20})$$
$$+5657364480\,(a^{18}\,b^{14}+a^{14}\,b^{18})+7212810240\,a^{16}\,b^{16}\Big)\,c^{16}$$
$$+\Big(184320\,(a^{23}\,b+a\,b^{23})+15544320\,a^{21}\,b^{3}+a^{3}\,b^{21})$$
$$+326430720\,(a^{19}\,b^{5}+a^{5}\,b^{19})+2658078720\,(a^{17}\,b^{7}+a^{7}\,b^{17})$$
$$+10041630720\,(a^{15}\,b^{9}+a^{9}\,b^{15})+19170385920\,(a^{13}\,b^{11}+a^{11}\,b^{13})\Big)\,c^{24}$$
$$+\Big(3072\,(a^{16})+b^{16})+368640\,(a^{14}\,b^{2}+a^{2}\,b^{14})$$
$$+5591040\,(a^{12}\,b^{4}+a^{4}\,b^{12})+24600576\,(a^{10}\,b^{6}+a^{6}\,b^{10})$$
$$+39536640\,(a^{8}\,b^{8})\Big)\;c^{32}+131072\,c^{48}.$$

It was shown in Chapter 4 of [H1] that for a self-dual vertex operator algebra V the decomposition polynomial belongs to the ring $\mathbb{C}[a,b,c]^G$ of invariants for some 3×3-matrix group G of order 1152. The space of invariant homogeneous polynomials of degree 48 is 7-dimensional and it can be checked that the above polynomial indeed belongs to this space by using the explicit base given in [H1].

We expect that the analog of Remark 5.6 and Lemma 5.7 holds: Every self-dual FVOA of central charge 24 and minimal weight 2 (i.e. $\dim V_1 = 0$) containing the subVOA $(\mathcal{B}_2^{24})_0$ can be obtained from a D_1-frame of the Leech lattice as in the second equation of Theorem 4.7.

Appendix

A Orbits on markings of a Hamming Code

Notation A.1 Let H be the unique binary code with parameters $[8,4,4]$, the Hamming code (see Appendix C). We take it to be the span of (00001111), (00110011), (01010101), (11111111) (see (**??**)). Let $A := \mathrm{Aut}(H) \cong AGL(3,2)$ (**??**). A *marking* is a partition of the index set into 2-sets.

The number of markings is $\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}/4! = 2520/24 = 105$. We show that there are three orbits of A on the set of markings and determine the stabilizers. This group is triply but not quadruply transitive on the eight indices.

Notation A.2 It helps to interpret the index set as $V \cong \mathbb{F}_2^3$ with the obvious action of A. So, 2-sets correspond to affine subspaces of dimension 1. Take a linear subspace $U \leq V$ of dimension 1. Let T the translation subgroup of A and let $L := Stab_A(0) \cong GL(3,2)$. Let M be a marking, $S := Stab_A(M)$. By double transitivity, we may assume $U \in M$. Let $R \leq T$ be the group of order 2 corresponding to U.

Case α. We assume that all four parts of the marking M are cosets of U. Then $S = TStab_L(U) \cong 2^3{:}Sym_4$, a group of index 7 in A.

Case β. We assume that exactly two parts of the marking are cosets of U, say U and W. Let P and Q be the other two parts. Then $X := U \cup W$ is a dimension 2 linear subspace of V and $Y := P \cup Q$ is its complement. Both P and Q are cosets of a common linear 1-dimensional subspace $U^* \neq U$ of X.

Let R^* be the fours group in T which corresponds to X; $R^* > R$. Then, R^* stabilizes both $\{U, W\}$ and $\{P, Q\}$, whence $R^* \leq S$; in fact, $R^* = T \cap S$.

Since A acts transitively on pairs of parallel affine 1-spaces, S acts transitively on $\{X, Y\}$; let $S_0 := Stab_S(X) = Stab_S(Y)$. Then S_0 has index 2 in S and acts transitively on $\{U, W\}$; let S_1 be the common stabilizer, index 2 in S_0. Also, S_0 acts transitively on $\{P, Q\}$; let S_2 be the common stabilizer, index 2 in S_0. Then $S_1 \neq S_2$ (since R stabilizes U and W but interchanges P and Q), and $S_4 := S_1 \cap S_2 \triangleleft S$ and $S/S_4 \cong Dih_8$, a Sylow 2-group of Sym_4 (via its action on the marking).

It suffices to show that $|S_4| = 4$. Clearly, elements of S_4 have square 1. The involution which is trivial on X and interchanges the points within each of P and Q is in L. The same idea, with U, W replaced by P, Q gives an involution which is in a conjugate of L, say in L^g, where $g \in A$ interchanges X and Y. Since these involutions are different, $|S_4| \geq 3$. If $1 \neq u \in S_4$ has a fixed point, say $v \in V$, it may be interpreted as a linear transformation by taking v as the origin; since u is an involution, its fixed point subspace has dimension 2, and is a union of members of M, so is one of X or Y; this means u is one of the two involutions already defined. Therefore, $|S_4| \leq 4$, whence equality.

Case γ. We assume that all parts of the marking besides U are not cosets of U. It follows that $S \cap T = 1$, so S embeds in L. Clearly, 7 does not divide $|S|$, so S embeds as a proper subgroup of order dividing 24. Thus, the orbit here has length divisible by $8 \cdot 7 = 56$. By our above count of the number of markings, this must be the exact number. We conclude that $S \cong Sym_4$, since the only subgroups of odd index in $GL(3,2)$ are parabolic subgroups [Ca], 8.3.2.

B Automorphisms of a marked Golay code

We settle the stabilizer in M_{24} of the special marking $\mathcal{M}^*$ we obtained in our description of the Golay code and identify $\mathcal{M}^*$ with the exceptional marking of Blackburn, Conder and McKay [CM] with parameters $(48, 576, 96, 0, 39)$.

As noted in section 5 our construction of $\mathcal{G}$ is equivalent to the usual hexacode construction, as in [G2] (5.25). The marking $\mathcal{M}^*$ in this notation is gotten from the usual sextet partition of the 24-set Ω

0	•	•	•	•	•	•
1	•	•	•	•	•	•
ω	•	•	•	•	•	•
$\bar{\omega}$	•	•	•	•	•	•

by intersecting the columns with the unions $Row_0 \cup Row_1$ and $Row_\omega \cup Row_{\bar{\omega}}$:

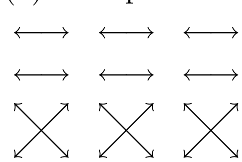
0	•	•	•	•	•	•
1	•	•	•	•	•	•
ω	•	•	•	•	•	•
$\bar{\omega}$	•	•	•	•	•	•

The set of twelve resulting 2-sets form $\mathcal{M}$.

In this appendix, we settle the stabilizer in M_{24} of the marking above.

In [G2], the action of the associated sextet group on Ω is described. The group has shape $H := 2^6 3 \cdot Sym_6$ and may be thought of as $\mathcal{H}\colon \mathrm{Aut}^*(\mathcal{H})$, the affine hexacode group ($\mathcal{H}$ denotes the hexacode) (5.25). As in [G2], Chapters 5 and 6, we use the notation K_i for the 4-set in Ω occurring as the i^{th} column above and $K_{ij\ldots}$ denotes the union $K_i \cup K_j \cup \cdots$.

The obvious subgroup of H which preserves $\mathcal{M}$ is $\mathcal{H} : P$, where $P = S \times \langle t \rangle \cong Sym_4 \times 2$, where S is generated by the groups of permutations (1) the four-group of row-respecting column permutations which interchange columns within evenly many coordinate blocks K_{12}, K_{34}, K_{56}; (2) the copy of Sym_3 obtained by permuting the three coordinate blocks (respecting the order within the blocks); (3) the permutation t is given by the diagram



[G2] (5.38), UP2.

The corresponding subgroup $Sym_4 \times 2$ of Sym_6 is maximal (since it is the stabilizer of a 2-set in a sextuply transitive action). Since the "scalar" transformation UP9 (5.38) [G2] (fixes top row elementwise, cycles rows 2, 3, 4 downward)

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

does not stabilize $\mathcal{M}$, it follows that $\mathcal{H} : \mathcal{P}$ is the stabilizer of $\mathcal{M}$ in H.

Notation B.1 $R := Stab_G(\mathcal{M})$, $G := M_{24}$.

The next step is to determine R; we know that $R \cap H = \mathcal{H}\colon P$.

We take a clue from the symmetrized marked weight enumerator $\text{smwe}_{\mathcal{G}}(x, y, z)$ of the Golay code as given in (5.8) and see that the parameters in the sense of [CM] are $(48, 576, 96, 0, 39)$. The next result is an exercise.

Lemma B.2 *(i) The octads which contribute to contribute to $c_0 = 48$ are those with even parity and which are labeled by a hexacode word of the form $(00xxxx)$, where $x = \omega$ or $\bar{\omega}$ and where the zeroes occur in any of the three coordinate blocks; these octads are unions of 2-sets which are subsets of columns labeled by x.*

(ii) The octads which contribute to contribute to $c_4 = 39$ have even parity and are one of K_{ij} (15 of these) or are octads labeled by hexacode words (001111), with the zeroes occurring in any of the three coordinate blocks; these octads are unions of parts of $\mathcal{M}$ which occur in columns labeled by 1 (24 such octads).

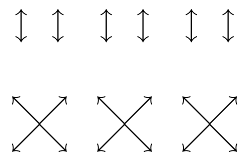
Clearly, R permutes the sets of octads (i) and (ii).

Lemma B.3 *The orbits of $\mathcal{H} : P$ on X, the set of octads in (ii), are the following:*
(a) K_{12}, K_{34}, K_{56} (length 3);
(b) K_{ij}, for all $\{ij\} \neq \{12\}, \{34\}, \{56\}$ (length 12);
(c) octads labeled by some (001111) (length 24).

Theorem B.4 *R is a subgroup of index 7 in the stabilizer of the trio (a), whence $|R : R \cap H| = 3$ and $R \cong 2^6\colon[Sym_4 \times Sym_3]$.*

Proof. We consider the action of R on X. The octads in X which have only a 0- or a 4-set as intersection with all members of X are the three in (a). So, R preserves this trio and so is in the trio group, J, of the form $2^6[GL(3,2) \times Sym_3]$. The group $\mathcal{H}$: P is a subgroup of J of index 21. We consider the possibility that 7 divides $|R|$. Let

$g \in R$ be an element of order 7. Then g fixes at least 1 of the remaining 36 members of X. An element of order 7 in G fixes exactly three octads and clearly these are just the octads of our trio (a), a contradiction. So, R has order $2^{10}3$ or $2^{10}3^2$. We eliminate the former by exhibiting a permutation in $R \backslash H$; UP13 from (5.38) [G2] does the job.



. □

Finally we can identify our marking $\mathcal{M}^*$ with the one in [CM]. This is not completely obvious since the labelings chosen in [CM] are different from the standard ones, e.g. in [Atlas] or [G2].

There are $|G|/|R| = 26565$ markings equivalent to $\mathcal{M}^*$, but this is exactly the number of markings obtained in [CM] with parameters $(48, 576, 96, 0, 39)$, i.e. there is only one orbit of markings with this parameters.

C Automorphism group of certain codes $\mathcal{C}$ and $\mathcal{D}$ of length $3 \cdot 2^d$

We are studying binary codes $\mathcal{C} \leq \mathbb{F}_2^{\Omega}$, where $|\Omega| = 3 \cdot 2^d$ and $\mathcal{D} := \mathcal{C}^{\perp}$ is spanned by the $d+3$ rows of the matrix

$$M := \begin{pmatrix} 1111\ldots\ldots1111 & 0000\ldots\ldots0000 & 0000\ldots\ldots0000 \\ 0000\ldots\ldots0000 & 1111\ldots\ldots1111 & 0000\ldots\ldots0000 \\ 0000\ldots\ldots0000 & 0000\ldots\ldots0000 & 1111\ldots\ldots1111 \\ 00\ldots0011\ldots11 & 00\ldots0011\ldots11 & 00\ldots0011\ldots11 \\ \vdots & \vdots & \vdots \\ 0011\ldots\ldots0011 & 0011\ldots\ldots0011 & 0011\ldots\ldots0011 \\ 0101\ldots\ldots0101 & 0101\ldots\ldots0101 & 0101\ldots\ldots0101 \end{pmatrix}. \tag{C.9}$$

Our problem is to find $F := \mathrm{Aut}(\mathcal{C}) = \mathrm{Aut}(\mathcal{D}) \leq Sym_{\Omega}$.

Notation C.1 *We partition Ω into three coordinate blocks $\Gamma_1 := \{1, 2, \ldots, 2^d\}$, $\Gamma_3 := \{2^d+1, 2^d+2, \ldots, 2 \cdot 2^d\}$ $\Gamma_3 := \{2 \cdot 2^d + 1, 2 \cdot 2^d + 2, \ldots, 3 \cdot 2^d\}$.*

Here is our main result; it was referred to after Theorem 5.8, for $d = 4$.

Theorem C.2 *$F \cong 2^{3d}[GL(d,2) \times Sym_3]$, where the 2^{3d} may be interpreted as a tensor product of a d and 3 dimensional module for the factors of $GL(d,2) \times Sym_3$.*

The two main parts of the proof consist of showing that F preserves the partition $\{\Gamma_i\}$ then getting the automorphism groups of the related length 2^d codes.

Theorem C.3 *(i) The code J spanned by the d vectors*

$$\begin{array}{c} 0000\ldots0000\ 1111\ldots1111 \\ 0000\ldots1111\ 0000\ldots1111 \\ \vdots \\ 0011\ldots0011\ 0011\ldots0011 \\ 0101\ldots0101\ 0101\ldots0101 \end{array}.$$

has automorphism group $GL(d,2)$.

(ii) The code spanned by J and the all ones vector has automorphism group $AGL(d,2)$; the normal translation subgroup is the group of automorphisms which are trivial modulo the span of the all ones vector.

(iii) There is a unique binary code of length 2^d and dimension d in which all nonzero weights are 2^{d-1}. It is equivalent to the code of (i).

Proof. [AK], Chapter 5. □

Notation C.4 Let $\mathcal{R}$ be the span of the first three rows of M and let $\mathcal{S}$ be the span of the last d. Note that the projections of $\mathcal{S}$ or $\mathcal{D}$ to any Γ_i block is a code as described by (C.3). We observe that every element of $\mathcal{R}$ has cardinality 0, 2^d, $2 \cdot 2^d$ or $3 \cdot 2^d$ and that every element of $\mathcal{D} \setminus \mathcal{R}$ has cardinality $3 \cdot 2^{d-1}$. To check this, just verify it for elements of $\mathcal{S}$, a triply thickened [G2] (3.19) extended Hamming code, and note that the effect of adding an element of $\mathcal{R}$ to an element $d \in \mathcal{D}$ is, for each i, to take the i^{th} projection of d to itself or its complement with respect to Γ_i. So, F fixes $\mathcal{R}$.

Lemma C.5 *$F := \mathrm{Aut}(\mathcal{D})$ permutes the partition Γ_i, $i = 1, 2, 3$, as Sym_3.*

Proof. Since F preserves $\mathcal{R}$, we deduce that F preserves the partition by examining the three minimal weight elements of $\mathcal{R}$. On the other hand, any blockwise permutation fixes the set of rows of M (permutes the first three, fixes the rest). □

Notation C.6 *Let H be the subgroup of F which fixes each Γ_i; the code $\mathcal{S}$ (C.4) is a triply thickened version of the d-dimensional length 2^d code associated to $GL(d,2)$, as in (C.3). It is clear that the natural action of a group $F_0 \cong GL(d,2) \times Sym_3$ (first factor F_1 acting diagonally and the second F_2 as block permutations) is in F and stabilizes $\mathcal{S}$. Note that the second factor acts trivially on $\mathcal{S}$.* □

Proof of Theorem C.2:

There is a group T_i acting as translations on Γ_i, identified with $\mathbb{F}_2^d$ as in (C.3), and trivially on Γ_j, for $j \neq i$; we choose these identifications to be compatible with the action of F_2. The direct product $T := T_1 \times T_2 \times T_3$ is in F.

Since H fixes $\mathcal{R}$, we consider the action of H on $\mathcal{D}/\mathcal{R}$. The kernel of this action corresponds naturally to a subgroup $Hom(\mathcal{D}/\mathcal{R}, \mathcal{R})$, order 2^{3d}, and may be interpreted as an element of T as in the above paragraph. Since $T \leq H$ and $|T| = 2^{3d}$, this kernel is T. Since $F_1 \leq H$ induces $GL(\mathcal{D}/\mathcal{R})$ on $\mathcal{D}/\mathcal{R}$, $H = TF_1$ and $F = TF_0$. □

D Lifting minus the identity

Definition D.1 Let L be an even integral lattice. A *lift of* -1 is an automorphism θ of the lattice VOA V_L such that for all $x \in L$, there is a scalar c_x so that $\theta\colon e^x \mapsto c_x e^{-x}$. (Here, e^x means $1 \otimes e^x$, where 1 is the constant polynomial.)

As usual, there is an epsilon function in the description of the lattice VOA V_L, $\varepsilon : L \times L \to \mathbf{C}^\times$, which is bimultiplicative and satisfies $\varepsilon(x,y)\varepsilon(x,y)^{-1} = (-1)^{(x,y)}$.

Lemma D.2 *Let x, $y \in L$. For some integer k and scalar c, $e^x_k e^y = ce^{x+y}$ ($a_k b$ means the value of the k^{th} binary composition on a, b). In fact, we take $k = -1-(x,y)$ and $c = \varepsilon(x,y)$, which is always nonzero.*

Proof. This is obvious from the form of the vertex operator representing e^x. □

Lemma D.3 *If the set $S = -S$ spans L, then the set of all e^x, for $x \in S$, generates the associated lattice VOA V_L.*

Proof. By Lemma D.2, we may assume that $S = L$. Let V' be the subVOA so generated. Note that for any $\alpha \in L$, $\alpha(-1) = e^\alpha_{(\alpha,\alpha)-2} e^{-\alpha}$. Thus V' contains all e^α and $\alpha(-1)$ for $\alpha \in S$. It is clear that V_L is irreducible under the component operators of $Y(e^\alpha, z)$ and $Y(\alpha(-1), z)$ for $\alpha \in S$, hence V' contains all $p \otimes e^\alpha$, where p is a polynomial expression in $\alpha(n)$, for $n < 0$. It follows immediately that $V' = V_L$. □

Notation D.4 Let M be the set of lifts of -1 and T the rank ℓ torus of automorphisms of V_L associated to L. There is an identification $T = \mathbb{C}^\ell / L^*$ so that $t = v + L^* \in T$ sends e^x to $e^{2\pi i(v,x)} e^x$.

Lemma D.5 *Let A be an abelian group, $\langle u \rangle$ be a group of order 2 which acts on A by letting u invert every element of A. Set $B := A\langle u \rangle$, the semidirect product. Every element of the coset Au is an involution, and two such involutions cu and du are conjugate in B (equivalently, by an element of A) iff cd^{-1} is the square of an element of A. This last condition follows if A is divisible, e.g. a torus.*

Theorem D.6 *M forms an orbit under conjugation by T in $\mathrm{Aut}(V_L)$.*

Proof. Let $x_1, \ldots, x_\ell$ form a basis of L. Given an element of M, we may compose it with an element $r \in T$ to assume it satisfies $e^{\pm x_i} \mapsto e^{\mp x_i}$, for all i. The conditions $e^{\pm x_i} \mapsto e^{\mp x_i}$ characterize an automorphism, since these 2ℓ elements generate the VOA, by Lemma D.3. This composition is the same as conjugation by $s \in T$ such that $s^2 = r$ or r^{-1}. So, we are done if we prove that $e^x \mapsto e^{-x}$ for all $x \in L$. But this is clear from Lemma D.2 since $\varepsilon(-x, -y) = \varepsilon(x, y)$.

Corollary D.7 *Given two lifts of -1 on V_L, their fixed point subVOAs are isomorphic. In fact, these subVOAs are in the same orbit of $\mathrm{Aut}(V_L)$.*

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