FRAMES, BASES AND GROUP REPRESENTATIONS

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Abstract: We develop an operator-theoretic approach to discrete frame theory on a separable Hilbert space. We then apply this to an investigation of the structural properties of systems of unitary operators on Hilbert space which are related to orthonormal wavelet theory. We also obtain applications of frame theory to group representations, and of the theory of abstract unitary systems to frames generated by Gabor type systems.

Contents

Introduction

The purpose of this manuscript is to explore and develop a number of the basic aspects of a geometric, or operator-theoretic, approach to discrete frame theory on Hilbert space which arises from the simple observation that a frame sequence is simply an inner direct summand of a Riesz basis. In other words, frames have a natural geometric interpretation as sequences of vectors which dilate(geometrically) to bases. This approach leads to simplified proofs of some of the known results in frame theory, and also leads to some new results and applications for frames.

A frame is a sequence $\{x_n\}$ of vectors in a Hilbert space H with the property that there are constants $A, B \geq 0$ such that

$$A||x||^2 \leq \sum_j | < x, x_j > |^2 \leq B||x||^2$$

for all x in the Hilbert space. A Riesz basis is a bounded unconditional basis; In Hilbert space this is equivalent to being the image of an orthonormal basis under a bounded invertible operator; another equivalence is that it is a Schauder basis which is also a frame. An inner direct summand of a Riesz basis is a sequence $\{x_n\}$ in a Hilbert space H for which there is a second sequence $\{y_n\}$ in a second Hilbert space M such that the orthogonal direct sum $\{x_n \oplus y_n\}$ is a Riesz basis for the direct sum Hilbert space $H \oplus M$.

Frame sequences have been used for a number of years by engineers and applied mathematicians for purposes of signal processing and data compression. There are papers presently in the literature which concern interpretations of discrete frame transforms from a functional analysis point of view. However, our approach seems to be different in an essential way in that the types of questions we address are somewhat different, and there is a fundamental difference in perspective arising from a dilation vantage point. This article is concerned mainly with the pure mathematics underlying frame theory. Most of the new results we present are in fact built up from basic principles.

This paper began with an idea the second author had on the plane back to College Station after participating in the stimulating conference on Operator Theory and Wavelet Theory which took place in July, 1996 in Charlotte, NC. This was the observation that a frame could be gotten by compressing a basis in a larger Hilbert space, together with the observation that this method was reversible. This quickly led the authors to an understanding of frame theory from a functional analysis point of view in a form that was very suitable for exploring its relationship to the earlier study of wandering vectors for unitary systems and orthogonal wavelets [DL] by the second author together with Xingde Dai.

The basic elements of our approach to frames are contained in Chapters 1 and 2, which contain a number of new results as well as new proofs of some well-known results. Our original plan was that these two chapters would contain only those basic results on frames that we needed to carry out our program in the remaining chapters. However, through seven preliminary versions of this manuscript, these chapters grew to accomodate some apparently new results for frames that seemed to belong to the basics of the theory. In many cases we made little or no attempt to give the most general forms possible of basic results because we felt that doing so might get in the way of the flow of the rest of the manuscript. So in these two chapters we tried to give what was needed together with the essentials of some additional results that we felt might have independent interest. Then, beginning with Chapter 3, our main thrust concerns applications to unitary systems, group representations and frame wavelets.

In [DL] the set of all complete wandering vectors for a unitary system was parameterized by the set of unitary operators in the local commutant (i. e. commutant at *a point*) of the system at a particular fixed complete wandering vector. This was the starting point for the structural theory in [DL]. In Chapter 3 we generalize this to frames. In the case where a unitary system has a complete wandering vector, we parameterize the set of all the normalized frame vectors (a vector which together with the unitary system induces a normalized tight frame for a closed subspace of the underlying Hilbert space) by the set of all the partial isometry operators in the local commutant of the unitary system at the fixed complete wandering vector. In particular the set of all complete normalized tight frame vectors for such a unitary system can be parameterized by the set of all co-isometries in the local commutant of the unitary system at this point.

Unlike the wandering vector case, we show that the set of all the normalized tight frame vectors for a unitary group can not be parameterized by the set of all the unitary operators in the commutant of the unitary group. This means that the complete normalized tight frame vectors for a representation of a countable group are not necessarily unitarily equivalent. However, instead, this set can be parameterized by the set of all the unitary operators in the von Neumann algebra generated by the representation. Several dilation results are given for unitary group systems and some other general unitary systems including the Gabor unitary systems. A simple application is that every unitary group representation which admits a complete frame vector is unitarily equivalent to a subrepresentation of the left regular representation of the group. Regarding classification of the frame vectors for a group representation, we investigate frame multiplicity for group representations in Chapter 6.

Frame wavelets have been in the literature for many years. These can be viewed as the frame vectors for the usual dilation and translation unitary system on $L^2(\mathbb{R})$. In Chapter 5 we focus our attention on strong disjointness of frame wavelets and on the frame wavelets whose Fourier transforms are normalized characteristic functions $\frac{1}{\sqrt{2\pi}}\chi_E$ for some measurable sets E. These sets are called *frame sets*. A characterization of these frame sets is given. (We note that an equation-characterization of complete normalized frame wavelets is in the literature, cf. [HW]). The study of strong disjointness in Chapter 2 leads to the concept of super-wavelets in Chapter 5. The prefix "super-" is used because they are orthonormal basis generators for a "super-space" of $L^2(\mathbb{R})$, namely the direct sum of finitely many copies of $L^2(\mathbb{R})$. Super-wavelets can be viewed as vector valued wavelets of a special type. Using the frame sets, we prove the existence of super-wavelets of any length. No superwavelet can be associated with a single multiresolution analysis (MRA) in the super-space. However, each component of a super-wavelet can be an MRA frame in the usual sense. They might have applications to signal processing, data compression and image analysis (see Remark 2.27). We also discuss the interpolation method introduced by Dai and Larson in [DL] for frame wavelets.

We wish to thank a number of our friends and colleagues for useful comments and suggestions on preliminary versions of this work. Since we were working between the two different areas of operator algebra and frame theory on this, we especially appreciated comments that helped us assess the originality and potential usefulness of our work and those that helped us improve the coherency and readability of our write-up. We first wish to thank three of Larson's research students Qing Gu, Vishnu Kamat and Shijin Lu for useful conversations and commentary with both of us throughout the year in which this manuscript was written. The list of those whose motivational and critical comments inspired us and helped us far more than they probably realize includes Pete Cazazza, Xingde Dai, Gustavo Garrigos, Bill Johnson, Michael Frank, John McCarthy, Manos Papadakis, Carl Pearcy, Lizhong

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Chapter 1

Basic Theory for Frames

We begin by giving an elementary self-contained exposition of frames suitable for our work in subsequent sections. Some of what we describe in this chapter is known and is standard in the literature. However, our dilation results and exposition characterizing tight frames and general frames as precisely the direct summands of Riesz bases seems to be new and serves to clarify some aspects of frame theory from a functional analysis point of view. For an operator theorist and also for a Banach space theorist, dilation may be the most natural point of view to take in regard to frames.

1.1 A Dilation Viewpoint on Frames

Let H be a separable complex Hilbert space. Let $B(H)$ denote the algebra of all bounded linear operators on H. Let $\mathbb{N}$ denote the natural numbers, and $\mathbb{Z}$ the integers. We will use $\mathbb{J}$ to denote a generic countable (or finite) index set such as $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}^{(2)}$, $\mathbb{N} \cup \mathbb{N}$ etc. The following are standard definitions:

A sequence $\{x_j : \ j \in \mathbb{N}\}$ of vectors in H is called a *frame* if there are constants $A, B > 0$ such that

$$A||x||^2 \leq \sum_j | < x, x_j > |^2 \leq B||x||^2 \qquad (1)$$

for all $x \in H$. The optimal constants (maximal for A and minimal for B) are called the *frame bounds*. The frame $\{x_j\}$ is called a *tight frame* if $A = B$, and is called *normalized* if $A = B = 1$. A sequence $\{x_j\}$ is called a *Riesz basis* if it is a frame and is also a basis for H in the sense that for each $x \in H$ there is a *unique* sequence $\{\alpha_j\}$ in $\mathbb{C}$ such that $x = \sum \alpha_j x_j$ with the convergence being in norm. We note that a Riesz basis is sometimes defined to be a basis which is obtained from an orthonormal basis by applying a bounded linear invertible operator (cf. [Yo]). This is equivalent to our definition (cf. Proposition 1.5). Also, it should be noted that in Hilbert spaces it is well known that Riesz bases are precisely the bounded unconditional bases.

It is clear from the absolute summation in (1) that the concept of frame (and Riesz basis) makes sense for any countable subset of H and does not depend on a sequential order. Thus there will be no confusion in discussing a frame , or Riesz basis, indexed by a countable set $\mathbb{J}$.

From the definition, a set $\{x_j : \ j \in \mathbb{J}\}$ is a normalized tight frame if and only if

$$||x||^2 = \sum_{j=1}^{\infty} | < x, \ x_j > |^2 \qquad (2)$$

for all $x \in H$. An orthonormal basis is obviously a normalized tight frame. Moreover, if $\{x_j\}$ is a normalized tight frame, then (2) implies that $||x_j|| \leq 1$ for all j. Also, if some x_k happens to be a unit vector then (2) implies that it must be orthogonal to all other vectors x_j in the frame. Thus a normalized tight frame of unit vectors is an orthonormal basis. On the other hand, some of the vectors in a tight frame may be the zero vector. If H is the zero Hilbert space, then any countable indexed set of zero vectors satisfies the definition of a normalized tight frame (provided we use the convention that $A = B = 1$ in this case). Let $\{x_n : j \in \mathbb{J}\}$ be a normalized tight frame for H. Suppose that $\{x_i : \ i \in \Lambda\}$ is a subset of $\{x_j : \ j \in \mathbb{J}\}$ which is also a normalized tight frame for H. We may assume $\Lambda \subseteq \mathbb{J}$. If $j \notin \Lambda$, then

$$||x_j||^2 = \sum_{k \in \mathbb{J}} | < x_j, \ x_k > |^2 = \sum_{i \in \Lambda} | < x_j, \ x_i > |^2.$$

Thus $\sum_{k \notin \Lambda} | < x_j, \ x_k > |^2 = 0$. So $< x_j, \ x_j >= 0$ which implies that $x_j = 0$. So the only way to enlarge a normalized tight frame in such a way that it remains a normalized tight frame is to add zero vectors.

If $\{x_n\}$ is a frame which is not a Riesz basis, and not a sequence of zeros on the zero Hilbert space, then we will call $\{x_n\}$ a *proper* frame.

We will say that frames $\{x_j : \ j \in \mathbb{J}\}$ and $\{y_j : j \in \mathbb{J}\}$ on Hilbert spaces $H, \ K$, respectively, are *unitarily equivalent* if there is a unitary $U : H \to K$ such that $Ux_j = y_j$ for all $j \in \mathbb{J}$. We will say that they are *similar* (or *isomorphic*) if there is a bounded linear invertible operator $T : H \to K$ such that $Tx_j = y_j$ for all $j \in \mathbb{J}$.

It is important to note that the two notions of equivalence of frames (unitary equivalence and isomorphism) in the above paragraph are somewhat restrictive, and are in fact more restrictive than some theorists would prefer for a notion of equivalence of frames. In particular, isomorphism of frames is not invariant under permutations. For example if $\{e_1, \ e_2\}$ is an orthonormal basis for a two-dimensional Hilbert space H_2 then both $\{e_1, \ e_2, \ 0, \ 0\}$ and $\{0, \ 0, \ e_1, \ e_2\}$ are normalized tight frames for H_2 indexed by the index set $\mathbb{J} = \{1,\ 2,\ 3,\ 4\}$. But they are not isomorphic, because isomorphism would require the existence of an invertible operator T mapping H_2 to H_2 such that $Te_1 = 0,\ Te_2 = 0,\ T0 = e_1,\ T0 = e_2$, an impossibility. For our geometric interpretation and to achieve the strength of the theorems we prove it is important to distinguish between the equivalence classes of such frames. Moreover, for a similar reason we will make no attempt to define a notion of equivalence of two frames that are not indexed by the same index set $\mathbb{J}$.

Suppose that $\{x_n\}$ is a sequence in H such that the equation

$$x = \sum_n < x,\ x_n > x_n$$

holds for all $x \in H$ (the convergence can be either in the weakly convergent sense or in the norm convergent sense). Then $\{x_n\}$ is a normalized tight frame for H since for every $x \in H$, we have

$$\begin{aligned} ||x||^2 &= lim_{n\to\infty} < \sum_{k=1}^{n} < x,\ x_k > x_k,\ x > \\ &= lim_{n\to\infty} \sum_{k=1}^{n} < x,\ x_k >< x_k,\ x > \\ &= lim_{n\to\infty} \sum_{k=1}^{n} | < x,\ x_k > |^2 = \sum_{k=1}^{\infty} | < x,\ x_k > |^2. \end{aligned}$$

Example A. Let H and K be Hilbert spaces with $H \subset K$, and let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for K. Let P denote the orthogonal projection from K onto H, and let $x_i = Pe_i$ for all i. If $x \in H$ is arbitrary, then

$$||x||^2 = \sum_j | < x,\ e_j > |^2 \qquad (3)$$

and

$$x = \sum_j < x,\ e_j > e_j. \qquad (4)$$

Since $x = Px$ and $x_j = Pe_j$ we have $< x,\ e_j > = < x,\ x_j >$, so (3) becomes (2) and hence $\{x_j\}$ is a normalized tight frame for H. Moreover, applying P to (4) then yields

$$x = \sum_j < x, x_j > x_j \qquad (5)$$

for all $x \in H$. The formula (5) is called the *reconstruction formula* for $\{x_j\}$.

Example A$_1$. For a special case let $\{e_1,\ e_2,\ e_3\}$ be an orthonormal basis for a 3-dimensional Hilbert space K. Another orthonormal basis for K is then

$$\{\frac{1}{\sqrt{3}}(e_1+e_2+e_3),\ \frac{1}{\sqrt{6}}(e_1-2e_2+e_3),\ \frac{1}{\sqrt{2}}(e_1-e_3)\}.$$

Thus from above,

$$\{\frac{1}{\sqrt{3}}(e_1+e_2),\ \frac{1}{\sqrt{6}}(e_1-2e_2),\ \frac{1}{\sqrt{2}}e_1\}$$

is a normalized tight frame for $H = span\{e_1,\ e_2\}$.

Example A$_2$. Let $K = L^2(\mathbb{T})$ where $\mathbb{T}$ is the unit circle and measure is normalized Lebesgue measure. Then $\{e^{ins} :\ n \in \mathbb{Z}\}$ is a standard orthonormal basis for $L^2(\mathbb{T})$. If $E \subseteq \mathbb{T}$ is any measurable subset then $\{e^{ins}|_E :\ n \in \mathbb{Z}\}$ is a normalized tight frame for $L^2(E)$. This can be viewed as obtained from the single vector χ_E by applying all integral powers of the (unitary) multiplication operator $M_{e^{is}}$. It turns out that these are all (for different E) unitarily *inequivalent*, and moreover every normalized tight frame that arises in a manner analogous to this is unitarily equivalent to a member of this class (see Corollary 3.10).

Example A$_3$. Take $H = \mathbb{C}^2$, $e_1 = (0,\ 1)$, $e_2 = (\frac{\sqrt{3}}{2},\ \frac{1}{2})$, $e_3 = (\frac{\sqrt{3}}{2},\ -\frac{1}{2})$. The elementary computation on page 56 of [Dau] or page 399 of [HW] shows that $\{e_1,\ e_2,\ e_3\}$ is a tight frame with frame bound $\frac{3}{2}$. An alternate way of seeing this is by observing that the set

$$\{\sqrt{\frac{2}{3}}(0,\ 1,\ \frac{1}{\sqrt{2}}),\ \sqrt{\frac{2}{3}}(\frac{\sqrt{3}}{2},\ \frac{1}{2},\ -\frac{1}{\sqrt{2}}),\ \sqrt{\frac{2}{3}}(\frac{\sqrt{3}}{2},\ -\frac{1}{2},\ \frac{1}{\sqrt{2}})\}$$

is an orthonormal basis for $\mathbb{C}^3$.

It turns out that Example A is *generic* and serves as a *model* for arbitrary normalized tight frames. One can always dilate such a frame to an orthonormal basis. One immediate consequence of the dilation is that the reconstruction formula (5) always holds for a tight frame. This is usually proven in a different way.

Proposition 1.1. *Let J be a countable (or finite) index set. Suppose that $\{x_n :\ n \in \mathbb{J}\}$ is a normalized tight frame for H. Then there exists a Hilbert space $K \supseteq H$ and an orthonormal basis $\{e_n :\ n \in \mathbb{J}\}$ for K such that $x_n = Pe_n$, where P is the orthogonal projection from K onto H.*

Proof. Let $K = l^2(\mathbb{J})$ and let $\theta : H \to K$ be the usual *frame transform* defined by

$$\theta(x) = (< x, x_n >)_{n\in\mathbb{J}} \tag{6}$$

for all $x \in H$. Since $\{x_n\}$ is a tight frame for H, we have

$$||\theta(x)||^2 = \sum_n | < x, x_n > |^2 = ||x||^2.$$

Thus θ is well defined and is an isometry. So we can embed H into K by identifying H with $\theta(H)$. Let P be the orthogonal projection from K onto $\theta(H)$. Denote the standard orthonormal basis for K by $\{e_j : \ j \in \mathbb{J}\}$. That is, for each $j \in \mathbb{J}$, e_j is defined to be the vector in $l^2(\mathbb{J})$ which is 1 in the j-th position and 0 elsewhere. We claim that $Pe_n = \theta(x_n)$. For any $m \in \mathbb{J}$, we have

$$\begin{aligned} < \theta(x_m), Pe_n > \ &= < P\theta(x_m), \ e_n > = < \theta(x_m), e_n > \\ &= < x_m, x_n > = < \theta(x_m), \theta(x_n) > . \end{aligned} \tag{7}$$

In the third equality in (7) the fact was used that, by definition of θ, for each $y \in H$ we have $< \theta(y), \ e_n > = < y, \ x_n >$. Since the vectors $\theta(x_m)$ span $\theta(H)$ it follows that $Pe_n - \theta(x_n) \perp \theta(H)$. But $ran(P) = \theta(H)$. Hence $Pe_n - \theta(x_n) = 0$, as required. $\square$

Corollary 1.2.

(i) Let $\{e_n : n \in \mathbb{J}\}$ be an orthonormal basis for H, and let V be a partial isometry in $B(H)$. Then $\{Ve_n\}$ is a normalized tight frame for the range of V.

(ii) Suppose that $\{x_n\}$ is a normalized tight frame for a Hilbert space H and $\{e_n\}$ is an orthonormal basis for a Hilbert space K. If T is the isometry defined by $Tx = \sum_{n\in\mathbb{J}} < x, x_n > e_n$, then $T^*e_n = x_n$, and $Tx_n = Pe_n$ for all $n \in \mathbb{J}$, where P is the projection from K onto the range of T. More generally, if $\{x_n\}$ is a general frame for H, then T defined above is a bounded linear operator and $T^*e_n = x_n$ for all $n \in \mathbb{J}$.

(iii) Suppose that $\{x_n\}$ is a normalized tight frame for a Hilbert space H. Then $\sum_{n\in\mathbb{J}} ||x_n||^2$ is equal to the dimension of H.

Proof. Statement (i) follows from the definition or from Proposition 1.1. For (ii), when $\{x_n\}$ is normalized, then the equality $Tx_n = Pe_n$ follows from the proof of Proposition 1.1. If $\{x_n\}$ is a general frame, boundness of T is clear. Now let $x \in H$ be arbitrary. Then we have

$$\begin{aligned} < T^*e_n, \ x > &=< e_n, \ Tx > \\ &=< e_n, \ \sum_{k\in\mathbb{J}} < x, \ x_k > e_k > \\ &= \sum_{k\in\mathbb{J}} \overline{< x, \ x_k >} < e_n, \ e_k > \\ &= \overline{< x, \ x_n >} \\ &=< x_n, \ x > . \end{aligned}$$

Hence $T^*e_n = x_n$.

To prove (iii), by Proposition 1.1, there is a Hilbert space K and an orthonormal basis $\{e_n\}$ for K such that $K \supseteq H$ and $Pe_n = x_n$, where P is the orthogonal projection from K onto H. Thus

$$\begin{aligned}\sum_{n\in\mathbb{J}} ||x_n||^2 = \sum_{n\in\mathbb{J}} &< Pe_n,\ Pe_n > \\ = \sum_{n\in\mathbb{J}} &< Pe_n,\ e_n > \\ = &\ tr(P) = dimH,\end{aligned}$$

where $tr(P)$ denotes the trace of P if H has finite dimension and is taken to be $+\infty$ if H has infinite dimension. $\square$

Some further elementary consequences of Proposition 1.2, whose proofs are given in Chapter 2 because they are used there although they could just as well have been given at this point, are contained in Proposition 2.6, Corollary 2.7, Corollary 2.8, Lemma 2.17. In particular, two frames are similar if and only if their frame transforms have the same range. Based on this, there is a one-to-one correspondence between the closed linear subspaces of $l^2(\mathbb{J})$ and the similarity classes of frames with index set $\mathbb{J}$.

Corollary 1.3. *Let $\mathbb{J}$ be a countable (or finite) index set. A set $\{x_n : \ n \in \mathbb{J}\}$ is a normalized tight frame for a Hilbert space H if and only if there exists a Hilbert space M and a normalized tight frame $\{y_n : \ n \in \mathbb{J}\}$ for M such that*

$$\{x_n \oplus y_n : \quad n \in \mathbb{N}\} \tag{8}$$

is an orthonormal basis for $H \oplus M$. (If $\{x_n\}$ is an orthonormal basis, the understanding is that M will be the zero space and each y_n will be zero vector.)

Proof. By Proposition 1.1 there is a Hilbert space $K \supseteq H$ and an orthonormal basis $\{e_n\}$ of K such that $x_n = Pe_n$, where P is the projection from K onto H. Let $M = (I - P)K$ and $y_n = (I - P)e_n,\ n \in \mathbb{J}$. $\square$

Proposition 1.4. *The extension of a tight frame to an orthonormal basis described in the statement of Corollary 1.3 is unique up to unitary equivalence. That is if N is another Hilbert space and $\{z_n\}$ is a tight frame for N such that $\{x_n \oplus z_n : \ n \in \mathbb{J}\}$ is an orthonormal basis for $H \oplus N$, then there is a unitary transformation U mapping M onto N such that $Uy_n = z_n$ for all n. In particular, $dimM = dimN$.*

Proof. Let $e_n = x_n \oplus y_n$ and $f_n = x_n \oplus z_n$. Let $\tilde{U} : H \oplus M \to H \oplus N$ be the unitary such that $\tilde{U}e_n = f_n$. Fix $x \in H$. Write $0_M, 0_N$ for the zero vector in M, N, respectively. We have

$$< x,\ x_n > = < x \oplus 0_M,\ e_n > = < x \oplus 0_N,\ f_n > .$$

So $x \oplus 0_M = \sum_n < x \oplus 0_M,\ e_n > = \sum_n < x, x_n > e_n$. Similarly $x \oplus 0_N = \sum_n < x, x_n > f_n$. Thus

$$\tilde{U}(x \oplus 0_M) = \sum_n < x,\ x_n > \tilde{U} e_n = x \oplus 0_N .$$

Identifying $H \oplus 0_M$ and $H \oplus 0_N$, it follows that $\tilde{U} = I \oplus U$, where U is a unitary in $B(M, N)$. $\square$

If $\{x_j\}$ is a normalized tight frame we will call any normalized tight frame $\{z_j\}$ such that $\{x_j \oplus z_j\}$ is an orthonormal basis for the direct sum space, as in Proposition 1.4, a *strong complementary* frame (or *strong complement*) to $\{x_j\}$. The above result says that every normalized tight frame has a strong complement which is unique up to unitary equivalence. More generally, if $\{y_j\}$ is a general frame we will call any frame $\{w_j\}$ such that $\{y_j \oplus w_j\}$ is a Riesz basis for the direct sum space a *complementary* frame (or *complement*) to $\{x_j\}$. It turns out that every frame has a normalized tight frame which is a complement (see Proposition 1.6), but there is no corresponding uniqueness result for complements as there is for strong complements (see Example B). These and related matters will be discussed at length in Chapter 2.

Now consider the technique in the proof of Proposition 1.1 carried out on a general frame $\{x_n\}$ for H with bounds A, B as in (1). Let $K = l^2(\mathbb{J})$ and let $\theta : H \to K$, $\theta(x) = (< x, x_n >)$ be the *frame transform* for $\{x_n\}$. The equation (1) implies that θ is bounded below and has closed range. Denote the range of θ by $\tilde{H}$. As earlier, let $\{e_n : \ n \in \mathbb{J}\}$ be the standard orthonormal basis for $l^2(\mathbb{J})$, and let P be the orthogonal projection of K onto $\tilde{H}$. Then for all $n, l \in \mathbb{J}$,

$$\begin{aligned} < \theta^* P e_n,\ x_l > &= < P e_n,\ \theta(x_l) > = < e_n,\ P\theta(x_l) > \\ &= < e_n,\ \theta(x_l) > = < x_n,\ x_l >, \end{aligned}$$

where $\theta^* : K \to H$ denotes the Hilbert space adjoint of θ. Since $\{x_l : \ l \in \mathbb{N}\}$ is dense in H, it follows that

$$\theta^* P e_n = x_n \tag{9}$$

for all $n \in \mathbb{J}$. Note that $\theta^*|_{\tilde{H}}$ is an invertible operator from $\tilde{H}$ to H. It agrees with the adjoint of θ when θ is regarded as an invertible operator from H to $\tilde{H}$.

Since the set $\{Pe_n : n \in \mathbb{J}\}$ is a normalized tight frame, this shows that an arbitrary frame is similar to a normalized tight frame. In the special case when $\{x_j\}$ is a Riesz basis, an elementary argument shows that $\theta(H) = K$, so $P = I$, recovering the well known fact that a Riesz basis is similar to an orthonormal basis. These arguments are each reversible.

If T is a bounded invertible operator and $\{f_n\}$ is a normalized tight frame in H, then if we set $x_n = Tf_n$ and let $u \in H$ be arbitrary, we compute

$$\sum_j |< u,\, x_j >|^2 = \sum_j |< u,\, Tf_j >|^2 = \sum_j |< T^* u, f_j >|^2 = ||T^* u||^2. \tag{9a}$$

Since

$$||T^{-1}||^{-1}||u|| \le ||T^* u|| \le ||T|| \cdot ||u||,$$

$\{x_n\}$ is a frame with frame bounds $A \ge ||T^{-1}||^{-2}$ and $B \le ||T||^2$. In fact we have $B = ||T||^2$ and $A = ||T^{-1}||^2$. To see that $B = ||T||^2$, simply let $\{u_k\}_1^\infty$ be a sequence of unit vectors with $||T^* u_k|| \to ||T^*||$, and apply equation $(9a)$. To see that $A = ||T^{-1}||^{-2}$, let $\{v_k\}_1^\infty$ be a sequence of unit vectors such that $||(T^*)^{-1} v_k|| \to ||(T^*)^{-1}||$, normalize these to get unit vectors $z_k = (T^*)^{-1} v_k / ||(T^*)^{-1} v_k||$, note that

$$||T^* v_k|| = ||(T^*)^{-1} v_k||^{-1} \to ||T^{*-1}||^{-1},$$

and apply $(9a)$. The sequence $\{x_n\}$ is a Riesz basis if and only if $\{f_n\}$ is an orthonormal basis. We capture these results formally:

Proposition 1.5. *A Riesz basis is precisely the image of an orthonormal basis under a bounded invertible operator. Likewise, a frame is precisely the image of a normalized tight frame under a bounded invertible operator. If the bounded invertible operator is denoted by T then the upper and lower frame bounds are precisely $||T||^2$ and $||T^{-1}||^{-2}$, respectively.*

There is a corresponding dilation result.

Proposition 1.6. *Let $\mathbb{J}$ be a countable (or finite) index set. If $\{x_j : j \in \mathbb{J}\}$ is a frame for a Hilbert space H, there exists a Hilbert space M and a normalized tight frame $\{y_j : j \in \mathbb{J}\}$ for M such that*

$$\{x_j \oplus y_j : j \in \mathbb{J}\}$$

is a Riesz basis for $H \oplus M$ *with the property that the bounds* A *and* B *for* $\{x_j \oplus y_j\}$ *are the same as those for* $\{x_j\}$.

Proof. By Proposition 1.5 there exists a normalized tight frame $\{f_j\}$ for H and an invertible operator T in $B(H)$ such that $x_j = Tf_j,\ \forall j$. By Corollary 1.3 there is a Hilbert space M and a normalized tight frame $\{y_j\}$ for M such that $\{f_j \oplus y_j\}$ is an orthonormal basis for $H \oplus M$. Then $T \oplus I$ is an invertible operator in $B(H \oplus K)$, and $(T \oplus I)(f_j \oplus y_j) = x_j \oplus y_j$. So $\{x_j \oplus y_j\}$ is a Riesz basis for $H \oplus M$ by Proposition 1.5. The statement regarding bounds is obvious. $\square$

One might expect that the strong complementary frame $\{y_j\}$ to the frame $\{x_j\}$ in Proposition 1.6 can be always chosen to be in H, that is M can be chosen as a subspace of H. In general this is not true. For example, let $H = \mathbb{C}$ and let $\{x_j\} = \{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\}$. Then $\{x_j\}$ is a normalized tight frame for H. Suppose that $\{y_j\}$ is a frame in H such that $\{x_j \oplus y_j\}$ is a Riesz basis for $H \oplus M$, where M is the subspace of H generated by $\{y_j\}$. Note that $\{y_j\}$ is non-trivial. Thus $M = H$. Therefore we get a Riesz basis

$$\{x_1 \oplus y_1,\ x_2 \oplus y_2,\ x_3 \oplus y_3\}$$

for the two dimensional Hilbert space $H \oplus H$, which is a contradiction. However if H is infinite dimensional, we can imbed M into H by an isometry $U : M \to H$. Thus $\{Uy_j\}$ is an indexed set in H such that $\{x_j \oplus Uy_j\}$ is an Riesz basis for $H \oplus UM$ since $\{x_j \oplus Uy_j\} = \{(I \oplus U)(x_j \oplus y_j)\}$.

Given sequences of vectors $\{x_n\}$ and $\{y_n\}$ in spaces H and K, respectively, there are *two* possible notions of a direct sum for them. One is the so-called *outer* direct sum, which is the union of the sets $\{x_n \oplus 0\}_{n \in \mathbb{J}}$ and $\{0 \oplus y_n\}_{n \in \mathbb{J}}$. In this case the outer direct sum of two frames will clearly be a frame, and of two bases will be a basis. Obviously the outer direct sum of two frames will be a basis if and only if each is a basis to start with. The other is the so-called *inner* direct sum, which is $\{x_n \oplus y_n\}_{n \in \mathbb{J}}$. In this article by the *direct sum* of two sequences indexed by the same index set we will always mean the *inner* direct sum. Likewise, the term *direct summand* will mean *inner* direct summand.

A perhaps more convenient way of summing up the above dilation results is:

Theorem 1.7. *Frames are precisely the inner direct summands of Riesz bases. Normalized tight frames are precisely the inner direct summands of orthonormal bases.*

Proof. In view of Corollary 1.3 and Proposition 1.6, the only thing remaining to prove is that if $\{x_n\}$ is a Riesz basis for H, and if $P \in B(H)$ is an idempotent (a non-selfadjoint projection) then $\{Px_n : \; n \in \mathbb{J}\}$ is a frame for PH. Write $P = TQT^{-1}$ for some invertible operator $T \in B(H)$ and selfadjoint projection Q. Let $y_n = T^{-1}x_n$, another Riesz basis for H. For $x \in QH$,

$$\sum_n |< x, Qy_n >|^2 = \sum_n |< Qx, \; y_n >|^2 = \sum_n |< x, \; y_n >|^2 >,$$

so by (1), $\{Qy_n\}$ is a frame for QH. Thus since $T|_{QH}$ is a bounded invertible operator from QH onto PH, it follows that $\{TQy_n\}$ is a frame for PH, as required. □

Theorem 1.7 suggests a natural matrix-completion point of view on frames, which adds some perspective to the theory. See section 7.3.

Remark 1.8. In view of Theorem 1.7, it would seem most natural to generalize the concept of frame by defining an abstract frame in a Banach space to be simply a direct summand of a basis. In this case the index set should be $\mathbb{N}$. A particular type of basis would then suggest that particular type of frame. For instance, *Schauder frame* would be by definition a direct summand of a Schauder basis. And a *bounded unconditional frame* would mean a direct summand of a bounded unconditional basis. This latter might give the best generalization, because on Hilbert spaces the bounded unconditional bases are precisely the Riesz bases. We will comment further on this in section 7.2 of the concluding remarks chapter.

Proposition 1.9. *(i) If $\{x_n\}$ is a frame and if T is a co-isometry (that is T^* is an isometry), then $\{Tx_n\}$ is a frame. Moreover, $\{Tx_n\}$ is an normalized tight frame if $\{x_n\}$ is.*

(ii) Suppose that $\{x_n\}$ and $\{y_n\}$ are normalized tight frames, and suppose that T is a bounded linear operator which satisfies $Tx_n = y_n$ for all n. Then T is a co-isometry. If T is invertible, then it is unitary.

(iii) If $\{x_n\}$ and $\{y_n\}$ are normalized tight frames such that $\{x_n \oplus y_n\}$ is a normalized tight frame, and if $\{z_n\}$ is a normalized tight frame which is unitarily equivalent to $\{y_n\}$, then $\{x_n \oplus z_n\}$ is also a normalized tight frame.

(iv) If $\{x_n\}$ is a frame which is a Schauder basis, then it is a Riesz basis.

(v) If $\{x_n\}$ is both a Riesz basis and a normalized tight frame, then it must be an orthonormal basis.

Proof. Let A and B be the frame bounds for $\{x_n\}$. Then, since T^* is an isometry, if $x \in H$ we have

$$A||x||^2 \;=\; A||T^*x||^2 \;\leq\; \sum_n |< T^*x,\; x_n >|^2$$
$$\leq B||T^*x||^2 \;=\; B||x||^2.$$

Since $< T^*x,\; x_n >$ equals $< x,\; Tx_n >$, the result (i) follows.

For (ii), let $\{x_n\}$ and $\{y_n\}$ be normalized tight frames for Hilbert spaces H and K, respectively, and let T be a bounded operator such that $Tx_n = y_n$ for all n. Then for any $y \in K$,

$$||T^*y||^2 \;=\; \sum_n |< T^*y, x_n >|^2 \;=\; \sum_n |< y, Tx_n >|^2 \;=\; ||y||^2.$$

Thus T^* is an isometry, and so it is a unitary when T is invertible.

For (iii), let H, K_1, K_2 be the spaces of $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, respectively. Let $U \in B(K_1, K_2)$ be a unitary such that $Uy_n = z_n$, $\forall n$. Then $V := (I \oplus U)$ is a unitary such that $V(x_n \oplus y_n) = x_n \oplus z_n$. Hence $\{x_n \oplus z_n\}$ is a normalized tight frame.

To prove (iv), by Proposition 1.6, there is a frame $\{y_n\}$ for a Hilbert space M such that $\{x_n \oplus y_n\}$ is a Riesz basis for $H \oplus M$. We need to show that $M = 0$. Let P be the projection from $H \oplus M$ onto H and let $z \in M$. Write

$$z = \sum_n c_n(x_n \oplus y_n)$$

for $c_n \in \mathbb{C}$. Then

$$0 = Pz = \sum_n c_n x_n.$$

Since $\{x_n\}$ is a Schauder basis, $c_n = 0$ for all n. Thus $z = 0$, as required.

For (v), by Proposition 1.5, there is an invertible operator A such that $\{Ax_n\}$ is an orthonormal basis. Since $\{x_n\}$ is also a normalized tight frame, it follows from (ii) that A is a unitary operator. Thus $\{x_n\}$ is an orthonormal basis. $\square$

Example B. A direct sum of two normalized tight frames can be a Riesz basis which is not orthonormal. This shows that the uniqueness part of Proposition 1.4 does not generalize to the setting of Proposition 1.6. Consider Example A_1. Let

$$f_1 = \frac{1}{\sqrt{3}}(e_1 + e_2),\;\; f_2 = \frac{1}{\sqrt{6}}(e_1 - 2e_2),\;\; f_3 = \frac{1}{\sqrt{2}}e_1$$

and let $g_1 = -\frac{1}{\sqrt{2}}e_3,\ g_2 = \frac{1}{\sqrt{6}}e_3,\ g_3 = \frac{1}{\sqrt{3}}e_3$. Then $\{f_1,\ f_2,\ f_3\}$ is a normalized tight frame for $span\{e_1, e_2\}$ and $\{g_1,\ g_2,\ g_3\}$ is a normalized tight frame for $span\{e_3\}$. Let

$$\begin{aligned} h_1 &= f_1 \oplus g_1 = \frac{1}{\sqrt{3}}(e_1 + e_2) - \frac{1}{\sqrt{2}}e_3, \\ h_2 &= f_2 \oplus g_2 = \frac{1}{\sqrt{6}}(e_1 - 2e_2) + \frac{1}{\sqrt{6}}e_3, \\ h_3 &= f_3 \oplus g_3 = \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{3}}e_3. \end{aligned}$$

Then $< h_1,\ h_2 > \neq 0$, and a computation shows that $h_1,\ h_2,\ h_3$ are linearly independent. Thus $\{h_1,\ h_2,\ h_3\}$ is a Riesz basis which is not orthonormal. However if we choose

$$\tilde{g}_1 = \frac{1}{\sqrt{3}}e_3,\ \tilde{g}_2 = \frac{1}{\sqrt{6}}e_3,\ \tilde{g}_3 = -\frac{1}{\sqrt{2}}e_3,$$

then $\{f_1 \oplus \tilde{g}_1,\ f_2 \oplus \tilde{g}_2,\ f_3 \oplus \tilde{g}_3\}$ is an orthonormal basis. Thus, by Proposition 1.9 (iii), $\{g_1,\ g_2,\ g_3\}$ and $\{\tilde{g}_1,\ \tilde{g}_2,\ \tilde{g}_3\}$ are not unitarily equivalent (as is also evident by inspection in this case).

1.2 The Canonical Dual Frame

For a tight frame $\{x_j\}$ with frame bound A, the reconstruction formula is

$$x = \frac{1}{A}\sum_{j \in \mathbb{J}} < x,\ x_j > x_j.$$

If a frame is not tight, there is a similar reconstruction formula in terms of dual frames. The next proposition and the remark following it concerns the usual dual of a frame which is standard in the wavelet literature. We will call it the *canonical* dual because we will have need of alternate duals in this article. Our particular way of presenting it is different from the usual way (cf. [HW]), and is given to highlight the uniqueness aspect of the canonical dual that distinguishes it from the other alternate duals.

Proposition 1.10. *Let $\{x_n :\ n \in \mathbb{J}\}$ be a frame on a Hilbert space H. Then there exists a unique operator $S \in B(H)$ such that*

$$x = \sum_{n \in \mathbb{N}} < x, Sx_n > x_n \tag{10}$$

for all $x \in H$. An explicit formula for S is given by

$$S = A^* A$$

where A is any invertible operator in $B(H, K)$ for some Hilbert space $K$with the property that $\{Ax_n : n \in \mathbb{J}\}$ is a normalized tight frame. In particular, S is an invertible positive operator. (We will denote $x_n^* := Sx_n$ in the sequel).

Proof. Let $A \in B(H, K)$ be any invertible operator in $B(H, K)$ for some Hilbert space $K$with the property that $\{Ax_n : n \in \mathbb{J}\}$ is a normalized tight frame. The existence of such an operator follows from Proposition 1.5. Let $f_n = Ax_n$, and let $S = A^*A \in B(H)$. Then

$$\begin{aligned}\sum_n < x,\ A^*Ax_n > x_n \ &= \sum_n < Ax,\ f_n > x_n \ = \sum_n < Ax,\ f_n > A^{-1}f_n \\ &= \ A^{-1}\sum_n < Ax,\ f_n > f_n = A^{-1}Ax = x.\end{aligned}$$

So $S = A^*A$ satisfies (10).

For uniqueness, suppose that $T \in B(H)$ satisfies $x = \sum_n < x, Tx_n > x_n,\ \forall x \in H$. Then

$$\begin{aligned}x \ &= \sum_n < x,\ Tx_n > x_n = \sum_n < x,\ TA^{-1}f_n > A^{-1}f_n \\ &= A^{-1}\sum_n < (A^*)^{-1}T^*x,\ f_n > f_n \\ &= A^{-1}(A^{*-1}T^*x),\end{aligned}$$

which implies that $A^{-1}A^{*-1}T^* = I$, hence $T = A^*A$, as required. $\square$

Corollary 1.11. *Suppose that $\{x_n\}$ is a frame for Hilbert space H and $T : H \to K$ is a an invertible operator. Then $(Tx_n)^* = (T^{-1})^*x_n^*$. In particular, if T is unitary then $(Tx_n)^* = Tx_n^*$.*

Proof. Let $S \in B(H)$ such that $Sx_n = x_n^*$. Then for any $x \in H$, we have

$$\begin{aligned}Tx &= \sum_{n\in\mathbb{J}} < x,\ x_n^* > Tx_n \\ &= \sum_{n\in\mathbb{J}} < Tx,\ ((T^{-1})^*ST^{-1})Tx_n > Tx_n.\end{aligned}$$

Thus, by Proposition 1.10, $(Tx_n)^* = ((T^{-1})^*ST^{-1})Tx_n = (T^{-1}x_n^*$. $\square$

Remark 1.12 The frame defined by $x_n^* := Sx_n,\ n \in \mathbb{J}$ in Proposition 1.10 is called the *dual frame* of $\{x_n\}$ in the frame literature. Usually it is defined in terms of the *frame operator* (see below), but the way it is done in Proposition 1.10 is equivalent and points out the particular type of uniqueness property it satisfies. The way to construct it directly is to note that if we set $A = (\theta^*|_{\tilde{H}})^{-1}$, where θ is the frame transform for $\{x_n\}$, then A satisfies the condition of Proposition 1.10. That is, $\{Ax_n\}$ is a normalized tight frame. Then, viewing θ as an invertible operator from $H \to \tilde{H}$ and θ^* as an invertible operator from $\tilde{H} \to H$, the operator S in (10) becomes

$$S = A^*A = \theta^{-1}(\theta^*)^{-1} = (\theta^*\theta)^{-1}.$$

So $x_n^* = (\theta^*\theta)^{-1}x_n$ as one usually defines the dual. The operator $(\theta^*\theta)^{-1}$ is a positive operator in $B(H)$ which is frequently called the *frame operator*. (Sometimes (cf. [HW]) θ itself is called the frame operator. We have elected to distinguish between these by referring to θ as the *frame transform* and $(\theta^*\theta)^{-1}$ as the *frame operator*.) Since $\{Ax_n\}$ is a normalized tight frame, it follows that $\{S^{\frac{1}{2}}x_n\}$ is also a normalized tight frame because the polar decomposition yields $A = US^{\frac{1}{2}}$, where U is a unitary operator. As we mentioned above, in this article we will call $\{x_n^*\}$ the *canonical dual* of $\{x_n\}$, as opposed to the *alternate duals* which we will consider in the next section, and in the balance of this article. Obviously if $\{x_n\}$ is a normalized tight frame then $S = I$ and $x_n^* = x_n$.

1.3 Alternate Dual Frames

If $\{x_n\}$ is a frame which is not a basis then there are in general *many* frames $\{y_n\}$ for which the formula

$$x = \sum_n < x, y_n > x_n, \quad \forall x \in H \qquad (11)$$

holds. For the simplest case of this, first note that an arbitrary pair of nonzero complex numbers $\{x_1,\ x_2\}$ satisfying the relation $|x_1|^2 + |x_2|^2 = 1$ is a tight frame for the 1-dimensional Hilbert space $H = \mathbb{C}$, and then note that if $\{y_1,\ y_2\}$ is any pair of numbers in $\mathbb{C}$ which satisfies $x_1\overline{y_1} + x_2\overline{y_2} = 1$, then (11) is satisfied for all $x \in H$. So the *canonical dual* frame given by Proposition 1.10 is special. As mentioned above, usually it is defined in the terms of the *frame operator*.

If $\{y_n\}$ is a frame satisfying (11) for some frame $\{x_n\}$, it can still happen that $\{y_n\}$ is not the canonical dual frame for $\{x_n\}$ even in the special case where $\{x_n\}$ is a normalized tight frame. For instance, let $H = \mathbb{C}$. Then $\{\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\}$ is a normalized tight frame for H. However if we let $\{y_1,\ y_2\}$ be either $\{\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\}$ or $\{\sqrt{2},\ 0\}$, then (11) is always satisfied.

We will call a frame $\{y_n\}$ satisfying (11) an *alternate dual* frame for $\{x_n\}$. For convenience we define the class of alternate dual frames for $\{x_n\}$ to include the canonical dual of $\{x_n\}$.

It is obvious that if $\{x_n\}$ is a frame, then

$$x = \sum_n < x, x_n^* > x_n = \sum_n < x,\ x_n > x_n^*, \quad \forall x \in H.$$

Using our dilation result (Theorem 1.7), we have the following more general situation which tells us that if $\{y_n\}$ is an alternate dual of $\{x_n\}$, then $\{x_n\}$ is is an alternate dual of $\{y_n\}$.

Proposition 1.13. *Let $\{x_n\}$ and $\{y_n\}$ be frames on a Hilbert space H such that*

$$x = \sum_n < x,\ y_n > x_n$$

for all $x \in H$. Then $x = \sum_n < x,\ x_n > y_n$ for all $x \in H$.

Proof. By Theorem 1.7, there exists a Riesz basis $\{f_n\}$ on a Hilbert space K ($K \supset H$) and a projection P such that $y_n = Pf_n$. Since $\sum_n | < x, x_n > |^2 < \infty$, we can define $T \in B(H, K)$ by $Tx = \sum_n < x, x_n > f_n$, $\forall x \in H$. Then $PT \in B(H)$ and $PTx = \sum_n < x, x_n > y_n$. Write $S = PT$. Then

$$\begin{aligned} < Sx, x > &= < \sum_n < x, x_n > y_n, x > \\ &= \sum_n < x,\ x_n > \overline{< x,\ y_n >} \end{aligned}$$

and

$$\begin{aligned} < x,\ x > &= < \sum_n < x,\ y_n > x_n,\ x > \\ &= \sum_n < x,\ y_n > \overline{< x, x_n >} \\ &= \sum_n < x,\ x_n > \overline{< x,\ y_n >} \end{aligned}$$

for all $x \in H$. So $< Sx, x >= ||x||^2$, which implies that S is positive and $S^{\frac{1}{2}}$ is an isometry. Thus $S = (S^{\frac{1}{2}})^2 = (S^{\frac{1}{2}})^* S^{\frac{1}{2}} = I$, as required. $\square$

Suppose that $\{x_n :\ n \in \mathbb{J}\}$ is a frame and $\{y_n :\ n \in \mathbb{J}\}$ is simply a set indexed by $\mathbb{J}$ with the property that it satisfies (11). Then $\{y_n\}$ is not necessarily a frame. For instance let $H = \mathbb{C}$. Choose $\{x_n\}$ and $\{y_n\}$ in $\mathbb{C}$ such that

$$\sum_n |x_n|^2 = 1, \quad \sum_n x_n \overline{y_n} = 1, \quad \sum_n |y_n|^2 = \infty.$$

Then $\{x_n\}$ is a normalized tight frame for $\mathbb{C}$ and (11) is satisfied. But $\{y_n\}$ is not a frame because

$$\sum_n |< x,\ y_n >|^2 = |x| \sum_n |y_n|^2 = \infty$$

when $0 \neq x \in \mathbb{C}$.

The following elementary observation tells us that distinct alternate duals for a given frame are never *similar*.

Proposition 1.14. *Suppose that $\{x_n\}$ is a frame and $\{y_n\}$ is an alternate dual for $\{x_n\}$. If $T \in B(H)$ is an invertible operator such that $\{Ty_n\}$ is also an alternate dual for $\{x_n\}$, then $T = I$.*

Proof. This follows from:

$$T^*x = \sum_n < T^*x,\ y_n > x_n = \sum_n < x,\ Ty_n > x_n = x, \quad x \in H.$$

□

We will show in Chapter 2, Corollary 2.26, that *a frame has a unique alternate dual if and only if it is a Riesz basis.*

If $\{x_n\}$ is a frame for H and P is an orthogonal projection, then $\{Px_n\}$ is a frame for PH. It is natural to ask whether $(Px_n)^* = Px_n^*$ for all projections P? It turns out that this is not true in general unless $\{x_n\}$ is a tight frame (see Corollary 1.16). However $\{Px_n^*\}$ is always an alternate dual for $\{Px_n\}$. To see this, let $x \in PH$ be arbitrary and just note that

$$x = Px = \sum_n < Px, x_n^* > Px_n = \sum_n < x,\ Px_n^* > Px_n.$$

Proposition 1.15. *Let $\{x_n\}$ be a frame for H and let S be the (unique) positive operator in $B(H)$ such that $Sx_n = x_n^*$. If P is an orthogonal projection in $B(H)$, then $Px_n^* = (Px_n)^*$ for all n if and only if $PS = SP$.*

Proof. Assume that $Px_n^* = (Px_n)^*$ for all $n \in \mathbb{J}$. Let $T \in B(PH)$ be the (unique) positive operator such that $TPx_n = (Px_n)^*$. Then

$$TPx_n = (Px_n)^* = Px_n^* = PSx_n$$

for all $n \in \mathbb{J}$. Considering TP as an operator in $B(H)$, we have $TP = PS$ and so $TP = PSP$, which implies that $PSP = PS$. By taking adjoints on both sides, we get $PSP = SP$, and hence $PS = SP$.

Now suppose that $SP = PS$. Since $x = \sum_n < x, x_n^* > x_n$ for all $x \in H$, we have that for every $x \in H$,

$$\begin{aligned} x &= \sum_n < x, Px_n^* > Px_n \\ &= \sum_n < x, PSx_n > Px_n \\ &= \sum_n < x, SP(Px_n) > Px_n. \end{aligned}$$

Thus, by Proposition 1.10, $(Px_n)^* = SPx_n = PSx_n = Px_n^*$. $\square$

Corollary 1.16. *Let $\{x_n\}$ be a frame for H. Then $\{x_n\}$ is a tight frame if and only if $(Px_n)^* = Px_n^*$ for all orthogonal projections $P \in B(H)$.*

The following proposition characterizes the alternate duals in terms of their frame transforms.

Proposition 1.17. *Let $\{x_n\}$ and $\{y_n\}$ be frames for a Hilbert space H, and let θ_1 and θ_2 be the frame transforms for $\{x_n\}$ and $\{y_n\}$, respectively. Then $\{y_n\}$ is an alternate dual for $\{x_n\}$ if and only if $\theta_2^*\theta_1 = I$.*

Proof. For any $x, y \in H$, we have

$$\begin{aligned} < x,\ \theta_2^*\theta_1 y > &=< \theta_2(x),\ \theta_1(y) > \\ &=< \sum_{n\in\mathbb{J}} < x,\ y_n > e_n,\ \sum_{n\in\mathbb{J}} < y,\ x_n > e_n > \\ &= \sum_{n\in\mathbb{J}} < x,\ y_n >< x_n,\ y > . \end{aligned}$$

It follows that $x = \sum_n < x, y_n > x_n$ for $x \in H$ if and only if $\theta_2^*\theta_1(y) = y$ for all $y \in H$. That is, $\{y_n\}$ is an alternate dual for $\{x_n\}$ if and only if $\theta_2^*\theta_1 = I$. $\square$

Remark 1.18. One other thing that is worthwhile to note, regarding Proposition 1.10 and Remark 1.12, is that, while the notion of frame transform makes perfect sense in a more general Banach space setting (see section 2 of the concluding chapter), the *frame operator* itself (and likely the canonical dual as well) is necessarily a Hilbert space concept because the form of Proposition 1.10 forces a similarity (that is, an isomorphism) between a frame and its canonical dual, forcing in turn an isomorphism between the underlying space and its dual space, and most Banach spaces are not isomorphic to their dual spaces. However, in Banach spaces as in Hilbert spaces there are always plenty of alternate duals, which points out the essential naturality of the concept of alternate dual frame.

Chapter 2

Complementary Frames, and Disjointness

In this chapter we develop the topics of disjointness of frames and related properties as we require them in subsequent chapters. We also include some related results which we feel have significant independent interest, even though they are not necessarily used in later chapters.

2.1 Strong Disjointness, Disjointness and Weak Disjointness

Let $\{x_n\}$ be a normalized tight frame in a Hilbert space H. By Corollary 1.3 there is a Hilbert space M and a normalized tight frame $\{y_n\}$ in M such that $\{x_n \oplus y_n\}$ is an orthonormal basis for $H \oplus M$. As in Chapter 1, we will call $\{y_n\}$ a *strong complementary frame* (or *strong complement*) to $\{x_n\}$, and we will call $(\{x_n\},\ \{y_n\})$ a *strong complementary pair*. Proposition 1.4 says that the strong complement of a normalized tight frame is unique up to unitary equivalence. For instance, in Example A_1,

$$\{\frac{1}{\sqrt{3}}e_3,\ \frac{1}{\sqrt{6}}e_3,\ -\frac{1}{\sqrt{2}}e_3\}$$

is a strong complement to

$$\{\frac{1}{\sqrt{3}}(e_1+e_2),\ \frac{1}{\sqrt{6}}(e_1-2e_2),\ \frac{1}{\sqrt{2}}e_1\},$$

and in Example A_2, $\{e^{ins}|_{\mathbb{T}\setminus E} :\ n \in \mathbb{Z}\}$ is a strong complement to $\{e^{ins}|_E :\ n \in \mathbb{Z}\}$.

As in Chapter 1, if $\{x_n\}$ is a general frame we will define a *complementary frame* (or *complement*) to be any frame $\{y_n\}$ for which $\{x_n \oplus y_n\}$ is a Riesz basis for the direct sum of the underlying Hilbert spaces. It is clear that any frame similar to a complementary frame for $\{x_n\}$ is also a complementary frame for $\{x_n\}$. Proposition 1.6 shows that every frame has a complement, and in fact that the complement can be taken to be a normalized tight frame. Example B shows that the complement is not unique in any good sense (i.e. up to similarity) without additional hypotheses. We will give a remedy for this. If $\{x_{1n} :\ n \in \mathbb{N}\}$, $\{x_{2n} :\ n \in \mathbb{N}\}$, $\{y_{1n} :\ n \in$

$\mathbb{N}\}$, $\{y_{2n} : n \in \mathbb{N}\}$ are frames, we say that the *frame pairs* $(\{x_{1n}\}, \{x_{2n}\})$ and $(\{y_{1n}\}, \{y_{2n}\})$ are *similar* if there are bounded invertible operators T_1 and T_2 such that $T_1 x_{1n} = y_{1n}$ and $T_2 x_{2n} = y_{2n}$ for all n. *Note that we do not require T_1 to be equal to T_2.* We will extend similarity to ordered k-tuples in the obvious way that two k-tuples $(\{x_{1n}\}, \ldots, \{x_{kn}\})$ and $(\{y_{1n}\}, \ldots, \{y_{kn}\})$ of frames are said to be similar if there exist invertible operators T_i such that $y_{in} = T_i x_{in}$ for $i = 1, \ldots, k$ and all $n \in \mathbb{J}$. If $\{x_n\}$ is a general frame, we will define a *strong complement* to $\{x_n\}$ to be any frame $\{z_n\}$ such that the pair $(\{x_n\}, \{z_n\})$ is similar to a strong complementary pair of normalized tight frames. For example, by this definition, the complement constructed in the proof of Proposition 1.6 is a strong complement, but the one constructed in Example B is not. By definition, any frame similar to a strong complementary frame is also a strong complementary frame. We show that strong complementary frames to a given frame are similar.

Proposition 2.1. *Let $\{x_n\}$ be a frame in H, and let $\{y_n\}$ and $\{z_n\}$ be strong complementary frames to $\{x_n\}$ in Hilbert spaces M and N, respectively. Then there exists an invertible operator $A \in B(M, N)$ such that $z_n = Ay_n$ for all n.*

Proof. Let T_1, T_2, S_1, S_2 be invertible operators such that

$$\{(T_1 \oplus T_2)(x_n \oplus y_n)\}$$

and

$$\{(S_1 \oplus S_2)(x_n \oplus z_n)\}$$

are orthonormal basis for $H \oplus M$ and $H \oplus N$, respectively. Write $T_1 x_n = f_n$, $T_2 y_n = g_n$ and $S_1 x_n = h_n$, $S_2 z_n = k_n$. Then $\{f_n\}$, $\{g_n\}$, $\{h_n\}$, $\{k_n\}$ are normalized tight frames. Since

$$f_n = T_1 x_n = T_1 S_1^{-1}(S_1 x_n) = T_1 S_1^{-1} h_n,$$

we have, from Proposition 1.9 (ii), that $U := T_1 S_1^{-1}$ is unitary. Thus $\{f_n \oplus k_n\}$ $(= \{(U \oplus I)(h_n \oplus k_n\}$ is an orthonormal basis. By Proposition 1.4 there exists a unitary operator $V \in B(M, N)$ such that $k_n = V g_n$ for all n. Thus $z_n = S_2^{-1} V T_2 y_n$, as required. $\square$

Definition 2.2. *If $\{x_n\}$ is a frame in H, we define a subframe of $\{x_n\}$ to be a frame of the form $\{P x_n : n \in \mathbb{N}\}$ in the Hilbert space PH, for some projection P in $B(H)$.*

The reader will note that the above definition is analogous to the notion of a subrepresentation of a group representation. This is not accidental.

Suppose that $\{x_n\}$ and $\{y_n\}$ is a pair of frames on Hilbert spaces H and K, respectively. We will say that $\{x_n\}$ and $\{y_n\}$ are *disjoint* if $\{x_n \oplus y_n\}$ is a frame for $H \oplus K$, and *weakly disjoint* if $span\{x_n \oplus y_n\}$ is dense in $H \oplus K$. Clearly disjointness implies weak disjointness. Complementary frames are of course disjoint. We formally give these definition for k-tuples.

Definition 2.3. *A k-tuple $\{\{x_{in}\}_{n\in\mathbb{J}} : i = 1, ..., k\}$ of frames on Hilbert spaces H_i ($1 \leq i \leq k$, respectively) is called disjoint if*

$$\{x_{1n} \oplus x_{2n} \oplus . . . \oplus x_{kn}\}_{n\in\mathbb{J}}$$

is a frame for the Hilbert space $H_1 \oplus H_2 \oplus ... \oplus H_k$, and weakly disjoint if

$$span\{x_{1n} \oplus x_{2n} \oplus . . . \oplus x_{kn}\}_{n\in\mathbb{J}}$$

is dense in $H_1 \oplus H_2 \oplus ... \oplus H_k$.

There is also a notion of *strong disjointness* for a k-tuple of frames, which is a much stronger notion than disjointness, and which will be extremely important in this paper. It generalizes the notion of a strong complementary pair. A pair of *normalized tight frames* $\{x_n\}$ and $\{y_n\}$ is called *strongly disjoint* if $\{x_n \oplus y_n\}$ is a normalized tight frame for $H \oplus K$, and a pair of *general frames* $\{x_n\}$ and $\{y_n\}$ is called *strongly disjoint* if it is similar to a strongly disjoint pair of normalized tight frames. We formally give the appropriate definition for k-tuples.

Definition 2.3′. *A k-tuple $\{\{x_{in}\}_{n\in\mathbb{J}} : i = 1, ..., k\}$ of normalized tight frames on Hilbert spaces H_i ($1 \leq i \leq k$, respectively) is said to be strongly disjoint if*

$$\{x_{1n} \oplus . . . \oplus x_{kn}\}_{n\in\mathbb{J}}$$

is a normalized tight frame for $H_1 \oplus . . . \oplus H_k$. More generally, a k-tuple of general frames is said to be strongly disjoint if it is similar to a k-tuple of normalized tight frames.

It is clear that strong disjointness, disjointness and weak disjointness are invariant under similarity. For instance, if $(\{x_j\}_{j\in\mathbb{Z}}, \{y_j\}_{j\in\mathbb{Z}}, \{z_j\}_{j\in\mathbb{Z}})$ is a disjoint triple and if T_1, T_2, T_3 are bounded invertible linear operators from H_1, H_2, H_3 onto Hilbert spaces K_1, K_2, K_3, respectively, then $(\{T_1x_j\}_{j\in\mathbb{Z}}, \{T_2y_j\}_{j\in\mathbb{Z}}, \{T_3z_j\}_{j\in\mathbb{Z}})$ is disjoint because the direct sum $\{T_1x_i \oplus T_2y_i \oplus T_3z_i\}$ is the image of the frame $\{x_i \oplus y_i \oplus z_i\}$ under the bounded linear invertible operator $T_1 \oplus T_2 \oplus T_3$, which maps $H_1 \oplus H_2 \oplus H_3$ onto $K_1 \oplus K_2 \oplus K_3$, hence is itself a frame. The other cases are analogous.

Note that the frame pairs $(x_n,\ y_n)$ and $(x_n^*,\ y_n^*)$ are similar. Thus, by definition, if $\{x_n\}$ and $\{y_n\}$ are strongly disjoint (resp. disjoint, weakly disjoint), then so are $\{x_n^*\}$ and $\{y_n^*\}$.

For a k-tuple of normalized tight frames $(\{z_{1n}\}_{n\in\mathbb{J}},\ \dots\ ,\ \{z_{kn}\}_{n\in\mathbb{J}})$, we call it a *complete strongly disjoint* $k-$*tuple* if $\{z_{1n} \oplus \dots \oplus z_{kn}\}$ is an orthonormal basis for $H_1 \oplus \dots \oplus H_k$. Similarly, we call a k-tuple of general frames $(\{x_{1n}\}_{n\in\mathbb{J}},\ \dots\ ,\ \{x_{kn}\}_{n\in\mathbb{J}})$ a *complete strongly disjoint k-tuple* if it is similar to a complete strongly disjoint k-tuple of normalized tight frames, or equivalently, it is a strongly disjoint k-tuple with the property that $\{x_{1n} \oplus\ \dots\ \oplus x_{kn}\}_{n\in\mathbb{J}}$ is a Riesz basis for the direct sum space. Complete disjoint k-tuples can be defined in a similar way.

We note that by Proposition 1.1 any strongly disjoint k-tuple of normalized tight frames can be extended to a *complete* strongly disjoint $(k+1)$-tuple of normalized tight frames. So any strongly disjoint k-tuple of general frames can be extended to a *complete* strongly disjoint $(k+1)$-tuple. In addition, a disjoint k-tuple of general frames can always be extended to a complete disjoint $(k+1)$-tuple by including any complementary frame to the inner direct sum of the k-tuple.

Suppose that $\{x_{in} :\ n \in \mathbb{J}\}$, $i = 1,\ \dots,\ k$, is a strongly disjoint k-tuple of frames. By applying projections to $\{x_{1n} \oplus \dots \oplus x_{kn}\}_{n\in\mathbb{J}}$, it is easy to see that each pair is also strongly disjoint (resp. disjoint, weakly disjoint). We will see in Corollary 2.12 that the converse is also true for strong disjointness. However this is no longer true for the disjointness and weakly disjointness cases, but the reason is a technical reason (see the remark following Corollary 2.12).

By definition, strong disjointness implies disjointness, and disjointness implies weak disjointness. The inverse implications are false. This will be easily seen from the characterizations of the different types of disjointness in Theorem 2.9. We will need the following simple fact:

Lemma 2.4. *Let $\{x_n\}$ and $\{y_n\}$ be normalized tight frames on Hilbert spaces H and K, respectively. If $\{x_n\}$ and $\{y_n\}$ are unitarily equivalent, then span$\{x_n \oplus y_n\}$ is not dense in $H \oplus K$.*

Proof. Let $U \in B(H, K)$ be a unitary operator such that $Ux_n = y_n$. Then $U \oplus I$ is unitary from $H \oplus K$ onto $K \oplus K$. Since $(I \oplus U)(x_n \oplus y_n) = y_n \oplus y_n$ and *span*$\{y_n \oplus y_n\}$ is not dense in $K \oplus K$, it follows that *span*$\{x_n \oplus y_n\}$ is not dense in $H \oplus K$. $\square$

For convenience we prove:

Proposition 2.5. *Suppose that* $\{x_{in} : \ n \in \mathbb{J}\}$, $i = 1, \ ..., \ k$, *are normalized tight frames for* H_i, *respectively. If*

$$\{x_{1n} \oplus . . . \oplus x_{kn}\}_{n\in\mathbb{J}}$$

is a normalized tight frame for $\overline{span}\{x_{1n} \oplus . . . \oplus x_{kn}\}$, *then in fact*

$$\overline{span}\{x_{1n} \oplus . . . \oplus x_{kn}\} = H_1 \oplus . . . \oplus H_k.$$

So $\{\{x_{in} : \ n \in \mathbb{J}\} : i = 1, ..., k\}$ *is in fact a k-tuple of strongly disjoint frames.*

Proof. We only consider pairs. The proof for the general case is similar. Fix any $l \in \mathbb{J}$. Since

$$\begin{aligned} x_l \oplus y_l &= \sum_n < x_l \oplus y_l, \ x_n \oplus y_n > x_n \oplus y_n \\ &= \sum_n (< x_l, x_n > + < y_l, y_n > x_n \oplus y_n, \end{aligned}$$

and $x_l = \sum_n < x_l, x_n > x_n$ and $y_l = \sum_n < y_l, y_n > y_l$, it follows that

$$\sum_n < x_l, x_n > y_n = \sum_n < y_l, y_n > x_n = 0.$$

Hence

$$x_l \oplus 0 = \sum_n < x_l \oplus o, \ x_n \oplus y_n > x_n \oplus y_n$$

and

$$0 \oplus y_l = \sum_n < 0 \oplus y_l, \ x_n \oplus y_n > x_n \oplus y_n$$

for all l, which implies that $span\{x_n \oplus y_n\}$ is dense in $H \oplus K$. $\square$

We note that the normalized tight condition of

$$\{x_{1n} \oplus . . . \oplus x_{kn}\}_{n\in\mathbb{J}}$$

is essential in Proposition 2.5. For example if $\{e_n\}$ is an orthonormal basis for a Hilbert space H, then $\{e_n \oplus e_n\}$ is a frame for $\overline{span}\{e_n \oplus e_n\}$, which is a *proper* subspace of $H \oplus H$.

2.2 Characterizations of Equivalence and Disjointness

To have a better understanding of the different types of disjointness, we prove the following classification result.

Proposition 2.6. *Let $\mathbb{J}$ be a countable (or finite) index set. Let H be a Hilbert space with dim $H =$ card $\mathbb{J}$, and fix an orthonormal basis $\{e_j : j \in \mathbb{J}\}$ for H. Let P and Q be projections in $B(H)$, and let $M = PH$ and $N = QH$. Suppose that $\{x_j\}$ and $\{y_j\}$ are the normalized tight frames for M and N defined by $x_j = Pe_j$ and $y_j = Qe_j$. Then $\{x_j\}$ and $\{y_j\}$ are unitarily equivalent if and only if $P = Q$.*

Proof. Suppose that $\{x_j\}$ is unitarily equivalent to $\{y_j\}$. Then there is a unitary $\tilde{V} \in B(M, N)$ such that $\tilde{V}x_j = y_j$ for all j. This determines a partial isometry $V \in B(H)$ with initial and final spaces M and N, respectively, such that $Vx_j = y_j$ for all j. Then $V^*V = P$, $VV^* = Q$ and $V = QVP = QV = VP$. Note that $VPe_j = Qe_j$. Thus, from $VP = V$, we obtain $Ve_j = Qe_j$ for all j. So since $\{e_j : j \in \mathbb{J}\}$ is an orthonormal basis , this implies that $V = Q$. Hence $P = Q$, as required. □

From Proposition 2.6 and Proposition 1.1, we have

Corollary 2.7. *Let $\mathbb{J}$ be a countable (or finite) index set. Then the set $\mathcal{F}_{\mathbb{J}}$ of the unitary equivalence classes of all normalized tight frames indexed by $\mathbb{J}$ is in one to one correspondence with the set $\mathcal{P}(H)$ of all self-adjoint projections on the Hilbert space $H = l^2(\mathbb{J})$. Likewise, the set $\mathcal{S}_{\mathbb{J}}$ of similarity equivalence classes of all frames indexed by $\mathbb{J}$ is in $1-1$ correspondence with the set of all $\mathcal{P}(H)$ of all self-adjoint projections on the Hilbert space H.*

Another way of describing this is

Corollary 2.8. *Let $\{x_j : j \in \mathbb{J}\}$ and $\{y_j : j \in \mathbb{J}\}$ be normalized tight frames for Hilbert spaces H_1 and H_2, respectively. Let θ_1 and θ_2 be the frame transforms for $\{x_j\}$ and $\{y_j\}$, respectively. Then $\{x_j\}$ is unitarily equivalent to $\{y_j\}$ if and only if θ_1 and θ_2 have the same range. Likewise, two frames are similar if and only if their frame transforms have the same range.*

The above result shows that $\mathcal{F}_{\mathbb{J}}$ can be parameterized by $\mathcal{P}(H)$ together with any choice of an orthonormal basis $\{e_j : j \in \mathbb{J}\}$ for H. So we could equip $\mathcal{F}_{\mathbb{J}}$ with the corresponding lattice structure, topological and algebraic properties of $\mathcal{P}(H)$. It can be easily shown that the lattice structure is independent of the choices of $\{e_j\}$ and H in the unitary equivalence sense. So we can define the meet, join, essential limit of frames, etc. For instance we could call a normalized tight frame $\{x_j\}_{j \in \mathbb{J}}$ an *essential limit* of a sequence of normalized tight frames $\{x_{n,j}\}_{j \in \mathbb{J}}$ if the corresponding projections P_n for $\{x_{n,j}\}_{j \in \mathbb{J}}$ converges in norm to the corresponding projection P for $\{x_j\}_{j\in\mathbb{J}}$. Theorem 2.9 tells us that the different types of disjointness of frames can be characterized by the topological and algebraic properties of $\mathcal{P}(H)$.

Theorem 2.9. *Let $\{x_j : \ j \in \mathbb{J}\}$ and $\{y_j : \ j \in \mathbb{J}\}$ be frames for Hilbert spaces H_1 and H_2, and let θ_1 and θ_2 be the frame transforms for $\{x_j\}$ and $\{y_j\}$, respectively. Let P and Q be the self-adjoint projections from $H(= l^2(\mathbb{J}))$ onto $\theta_1(H_1)$ and $\theta_2(H_2)$, respectively. Then*

(i) $\{x_j\}$ and $\{y_j\}$ are strongly disjoint if and only if $PQ = QP = 0$.

(ii) $(\{x_j\},\ \{y_j\})$ is a strong complementary pair if and only if $P = I - Q$.

(iii) $\{x_j\}$ and $\{y_j\}$ are disjoint if and only if $PH \cap QH = (0)$ and $PH + QH$ is closed.

(iv) $\{x_j\}$ and $\{y_j\}$ are weakly disjoint if and only if $PH \cap QH = \{0\}$.

(v) $(\{x_j\},\ \{y_j\})$ is a complementary pair if and only if $PH \cap QH = (0)$ and $PH + QH = H$.

Proof. Since the frame transforms of similar frames have the same range, it suffices to consider the normalized tight frame case.

Let $\{e_j : \ j \in \mathbb{J}\}$ be the standard orthonormal basis for $l^2(\mathbb{J})$. Then $\{x_j\}$ and $\{Pe_j\}$ (resp. $\{y_j\}$ and $\{Qe_j\}$) are unitarily equivalent. Since unitary equivalence preserves the different types of disjointness, we can assume that $x_j = Pe_j$ and $y_j = Qe_j$. We note that if $PH \cap QH = (0)$ and $PH + QH$ is closed, then $L : PH \oplus QH \to PH + QH$ defined by $L(u \oplus v) = u + v$ is a linear bijection, and

$$\begin{aligned} ||L(u\oplus v)||^2 &= ||u+v||^2 \le (||u|| + ||v||)^2 \\ &\le 2(||u||^2 + ||v||^2) \ = \ 2||u \oplus v||^2. \end{aligned}$$

Thus $|||u + v||| := (||u||^2 + ||v||^2)^{1/2}$ is an equivalent norm on $PH + QH$. Hence (i) - (iii) follow from the definitions and the equalities:

$$||Px \oplus Qy||^2 = ||Px||^2 + ||Qy||^2$$

and

$$\begin{aligned} \sum_j |< Px \oplus Qy,\ x_j \oplus y_j >|^2 &= \sum_j |< Px, Pe_j > + < Qy, Qe_j >|^2 \\ &= \sum_j |< Px, e_j > + < Qy, e_j >|^2 \\ &= \sum_j |< Px + Qy, e_j >|^2 \\ &= ||Px + Qy||^2. \end{aligned}$$

For (iv), suppose that $\{x_j\}$ and $\{y_j\}$ are weakly disjoint. As discussed above we can assume that $x_j = Pe_j$ and $y_j = Qe_j$ for some projections $P, Q \in B(H)$. If $PH \cap QH \neq (0)$, we choose a non-zero element $x \in PH \cap QH$. Then

$$\begin{aligned} < x \oplus (-x),\ Pe_j \oplus Qe_j > &= < x,\ Pe_j > - < x,\ Qe_j > \\ &= < Px,\ e_j > - < Qx,\ e_j > = 0 \end{aligned}$$

for all $j \in \mathbb{J}$. Thus $span\{Pe_j \oplus Qe_j\}$ is not dense in $PH \oplus QH$, which implies that $\{Px_j\}$ and $\{Qe_j\}$ are not weakly disjoint. Hence $PH \cap QH = (0)$

Conversely assume that $PH \cap QH = (0)$. To show that $span\{x_j \oplus y_j : j \in \mathbb{J}\}$ is dense in $PH \oplus QH$, let $x \in PH$ and $y \in QH$ such that $x \oplus y \perp x_j \oplus y_j$ for all $j \in \mathbb{J}$. Then

$$\begin{aligned} 0 &= < x, x_j > + < y, y_j > = < x, Pe_j > + < y, Qe_j > \\ &= < Px, e_j > + < Qy, e_j > = < x + y, e_j > \end{aligned}$$

for all j. Hence $x = -y$, which implies $x = y = 0$ since $PH \cap QH = \{0\}$. Therefore $span\{x_j \oplus y_j\}$ is dense in $PH \oplus QH$.

Prat (v) follows from (iii) and the fact that a frame is a Riesz basis if and only if the range of its frame transform is the whole space $l^2(\mathbb{J})$. □

Parts (i) and (iii) have straightforward extensions to k-tuples.

Theorem 2.9′. *Suppose that $(\{x_{1n}\}_{n\in\mathbb{J}}, \ ... \ , \ \{x_{kn}\}_{n\in\mathbb{J}})$ is a k- tuple of frames. Then it is a strongly disjoint (resp. disjoint) k-tuple if and only if the ranges of their frame transforms give an orthogonal direct sum (resp. Banach direct sum) decomposition of the closed linear span of these range spaces. In particular, it is a complete strongly disjoint (resp. complete disjoint) k-tuple if and only if the ranges of their frame transforms give an orthogonal direct sum (resp. Banach direct sum) decomposition of $l^2(\mathbb{J})$.*

Corollary 2.10. *Let $\{x_n\}$ and $\{y_n\}$ be frames for Hilbert spaces H and K, respectively. Then $\{x_n\}$ and $\{y_n\}$ are strongly disjoint if and only if one of the equations*

$$\sum_n < x,\ x_n^* > y_n = 0 \quad for\ all\ x \in H$$

or

$$\sum_n < y,\ y_n^* > x_n = 0 \quad for\ all\ y \in K$$

holds. Moreover, if one holds the other also holds.

Proof. Let θ_1 and θ_2 be the frame transforms for $\{x_n\}$ and $\{y_n\}$, respectively. By Theorem 2.9, $\{x_n\}$ and $\{y_n\}$ are strongly disjoint if and only if $\theta_1(H)$ and $\theta_2(K)$ are orthogonal. If we recall that $\theta_1(x) = (< x,\ x_n^* >)_{n\in\mathbb{J}} \in l^2(\mathbb{J})$ and $\theta_2(y) = (< y,\ y_n^* >)_{n\in\mathbb{J}} \in l^2(\mathbb{J})$, the proposition follows. $\square$

Since every frame is similar to its (canonical) dual, their frame transforms have the same range. Thus, if $\{x_n\}$ and $\{y_n\}$ are strongly disjoint frames for Hilbert spaces H and K, respectively, then, by Theorem 2.9 (i), $\{x_n^*\}$ and $\{y_n\}$ are also strongly disjoint. Hence

$$\sum_n < x,\ x_n > y_n^* = 0$$

for all $x \in H$. Similarly

$$\sum_n < y,\ y_n > x_n^* = 0$$

for all $y \in K$.

Corollary 2.11. *Let $\{x_j\}$ and $\{y_j\}$ be normalized tight frames for H_1 and H_2. Then $\{x_j\}$ and $\{y_j\}$ are weakly disjoint if and only if there is no non-zero subframe of $\{x_j\}$ which is unitarily equivalent to a subframe of $\{y_j\}$.*

Proof. We can assume that $x_j = Px_j$ and $y_j = Qe_j$. Suppose that there is no non-zero subframe of $\{x_j\}$ which is unitarily equivalent to a subframe of $\{y_j\}$. To show that $\{x_j|$ and $\{y_j\}$ are weakly disjoint, by Theorem 2.9, it suffices to show $PH \cap QH = \{0\}$. Let R be the projection from H onto $PH \cap QH$. Assume that $R \neq 0$. From $Rx_j = RPe_j = Re_j = RQe_j = Ry_j$, we have that $\{Rx_j\}$ is a non-zero subframe pf $\{x_n\}$ unitarily equivalent to a non-zero subframe $\{Ry_j\}$ of $\{y_j\}$, which contradicts our assumption. Thus $PH \cap QH = \{0\}$.

Conversely, assume that $PH \cap QH = \{0\}$. Suppose that there exist non-zero projections $P_1 \in B(PH)$ and $Q_1 \in B(QH)$ such that $\{P_1x_j\}$ and $\{Q_1y_j\}$ are unitarily equivalent. Considering P_1 and Q_1 as projections in $B(H)$, then, by Proposition 2.6, $P_1 = Q_1$ since $\{P_1e_j\}$ and $\{Q_1e_j\}$ are unitarily equivalent. So $PH \cap QH \supset P_1H \neq \{0\}$, which leads to a contradiction. $\square$

Corollary 2.12. *Suppose that $\{x_{in} :\ n \in \mathbb{J}\}(i = 1,\ ...,\ k)$ is a $k-$tuple of frames on Hilbert spaces H_i, respectively. Then*

$$(\{x_{1n}\}_{n\in\mathbb{J}},\ \dots\ ,\ \{x_{kn}\}_{n\in\mathbb{J}})$$

is a strongly disjoint $k-$tuple if and only if each pair is strongly disjoint.

Proof. By the remark following Definition 2.3′, we only need to prove the sufficiency. We can assume that each $\{x_{in}\}_{n\in\mathbb{J}}$ is a normalized tight frame.

Assume that each pair in $\{x_{in}: \ n \in \mathbb{J}\}(i = 1, \ ..., \ k)$ are strongly disjoint. Let P_i be the orthogonal projection from $l^2(\mathbb{J})$ onto the range of the frame transform θ_i for $\{x_{in}\}$. Then, by Theorem 2.9 (i), $P_i(i = 1, ..., k)$ are mutually orthogonal. Hence $\sum_{i=1}^{k} P_i e_n$ is a normalized tight frame. This implies that $\{x_{1n} \oplus ... \oplus x_{kn}\}_{n\in\mathbb{J}}$ is a normalized tight frame since

$$\theta_1 \oplus ...\theta_k : H_1 \oplus ... \oplus H_k \to l^2(\mathbb{J})$$

defined by

$$(\theta_1 \oplus ...\theta_k)(u_1 \oplus ... \oplus u_k) = \theta_1(u_1) + ... + \theta_k(u_k)$$

is an isometry and

$$(\theta_1 \oplus ...\theta_k)(x_{1n} \oplus ... \oplus x_{kn}) = P_1 e_n + ... + P_k E_n.$$

Thus

$$\{x_{1n}\oplus, ..., \ \oplus x_{kn}\}$$

is a normalized tight frame, as required. □

As we mentioned before, the above result is false for disjointness and weak disjointness. For example, let $K = H \oplus H$ and let $\{e_n\}$ be a fixed orthonormal basis for K. Let $P_1, \ P_2$ and P_3 be the orthogonal projections from K onto $M_1 = H \oplus 0, \ M_2 = 0 \oplus H$ and $M_3 = \{x \oplus x : \ x \in H\}$, respectively. Then $(\{P_1 e_n\}, \ \{P_2 e_n\}, \ \{P_3 e_n\})$ are mutually disjoint. It is easy to see that $\{P_1 e_n \oplus P_2 e_n\}_{n\in\mathbb{J}}$ is an othonormal basis for $M_1 \oplus M_2$. So the range of its frame transform is the whole space $l^2(\mathbb{J})$. This implies, by Theorem 2.9 (iii), that $\{P_1 e_n \oplus P_2 e_n\}$ and $\{P_3 e_n\}$ are not weakly disjoint (hence not disjoint). That is, $span\{P_1 e_n \oplus P_2 e_n \oplus P_2 e_n\}$ is not dense in the direct sum space $M_1 \oplus M_2 \oplus M_3$. Therefore $(\{P_1 e_n\}, \ \{P_2 e_n\}, \ \{P_3 e_n\})$ is not a disjoint (resp. weakly disjoint) triple.

Corollary 2.13. *Suppose that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are mutually strongly disjoint frames for Hilbert spaces $H, \ K$ and M, respectively. Then $\{x_n \oplus y_n\}$ and $\{z_n\}$ are strongly disjoint.*

Definition 2.14. *Let $\{x_j\}$ and $\{y_j\}$ be normalized tight frames on H_1 and H_2, respectively. We say that they commute if there exist projections $P \in B(H_1)$ and $Q \in B(H_2)$ such that $\{Px_j\}$ and $\{Qy_j\}$ are unitarily equivalent frames on PH_1 and QH_2, respectively, and $\{P^\perp x_j\}$ and $\{Q^\perp y_j\}$ are strongly disjoint frames on $P^\perp H_1$ and $Q^\perp H_2$, respectively.*

Corollary 2.15. *Let $\{x_j\}$, $\{y_j\}$, P and Q be as in Theorem 2.9. Then $\{x_j\}$ and $\{y_j\}$ commute if and only if $PQ = QP$.*

Proof. This follows immediately from Propositions 2.6 and 2.9. $\square$

See section 7.1 for more on commuting frames.

2.3 Cuntz Algebra Generators

Strong disjointness can be also characterized in terms of Cuntz algebra generators. Recall that a representation of the Cuntz algebra $\mathcal{O}_n$ on a Hilbert space H is the C*-algebra generated by an n-tuple of isometries S_i $(i = 1, \ldots, n)$ in $B(H)$ with the property that they have orthogonal ranges and $\sum_{i=1}^{n} S_i S_i^* = I$. Given two normalized tight frames $\{x_{1n} : n \in \mathbb{J}\}$ and $\{x_{2n} : n \in \mathbb{J}\}$ for a Hilbert space H, let $\{e_n\}$ be a fixed orthonormal basis for H. As before we define two isometries on H by

$$V_i x = \sum_n < x, x_{in} > e_n, \qquad n \in \mathbb{J}.$$

In general if $\{f_n\}_{j\in\mathbb{J}}$ is a normalized tight frame for H and $\{e_j\}_{j\in\mathbb{J}}$ is an orthonormal basis for K, then the formula $Tx := \sum_{j\in\mathbb{J}} < x,\ f_j > e_j$ gives the unique isometry such that $T^* e_n = f_n$ (see Corollary 1.2 (ii)).

Proposition 2.16. *Let $\{x_{in} :\ n \in \mathbb{Z}\}$ and V_i $(i = 1, 2)$ be as above. Then $\{x_{1n}\}$ and $\{x_{2n}\}$ are strongly disjoint if and only if $ran(V_1)$ and $ran(V_2)$ are orthogonal. Moreover they are strongly complementary to each other if and only if (V_1, V_2) are generators for a representation of $\mathcal{O}_2$.*

Proof. Suppose that $\{x_{1n}\}$ and $\{x_{2n}\}$ are strongly disjoint. Then, from the proof of Proposition 2.5, $\sum_n < x, x_{1n} > x_{2n} = \sum_n < x, x_{2n} > x_{1n} = 0$ for all $x \in H$. Thus for all $x, y \in H$,

$$\begin{aligned} < V_1 x, V_2 y > &= < \sum_n < x, x_{1n} > e_n, \sum_n < y, x_{2n} > e_n > \\ &= \sum_n < x, x_{1n} > \overline{< y, x_{2n} >} \\ &= < \sum_n < x, x_{1n} > x_{2n}, y > = \ 0, \end{aligned}$$

as required.

Conversely, assume that $V_1 H \perp V_2 H$. Then the above computation shows that $\sum_n < x, x_{1n} > x_{2n} = \sum_n < x, x_{2n} > x_{1n} = 0$ for all $x \in H$. So it follows that

$$x \oplus y = \sum_n < x \oplus y, x_{1n} \oplus x_{2n} > x_{1n} \oplus x_{2n}$$

for all $x, y \in H$. Thus $\{x_{1n} \oplus x_{2n}\}$ is a normalized tight frame.

If $\{x_{1n} \oplus x_n\}$ is an orthonormal basis, then

$$< x_{1k}, x_{1n} > + < x_{2k}, x_{2n} > = \ \delta_{k,n},$$

where $\delta_{k,n} = 0$ if $k \neq n$ and 1 if $k = n$. So

$$V_1 x_{1k} + V_2 x_{2k} \ = \ \sum_n (< x_{1k}, x_{1n} > + < x_{2k}, x_{2n} >) e_n \ = \ e_n.$$

Hence $V_1 V_1^* + V_2 V_2^* = I$.

Now suppose that $V_1 H \perp V_2 H$ and $V_1 V_1^* + V_2 V_2^* = I$. Define W by

$$W(x \oplus y) \ = \ V_1 x + V_2 y$$

Then

$$||W(x \oplus y)||^2 \ = \ ||V_1 x + V_2 y||^2 \ = \ ||V_1 x||^2 + ||V_2 y||^2 = ||x||^2 + ||y||^2.$$

So W is unitary since $V_1 H + V_2 H \ = \ H$. Note that $W(x_{1n} \oplus x_{2n}) = V_1 x_{1n} \ = \ V_1 V_1^* e_n + V_2 V_2^* e_n \ = \ e_n$. Thus $\{x_{1n} \oplus x_{2n} : \ n \in \mathbb{J}\} (= \{W^* e_n : \ n \in \mathbb{J}\})$ is an orthonormal basis as expected. $\square$

We note that if U_1 and U_2 are isometries which are generators for a representation of the Cuntz algebra $\mathcal{O}_2$ on H, fix an orthonormal basis $\{e_n\}$ of H, and let $x_{in} = U_i^* e_n$ $(i = 1, 2)$. Then $\{x_{1n}\}$ and $\{x_{2n}\}$ are normalized tight frames for H since the U_i^* are isometries. An elementary computation shows that

$$U_i x \ = \ \sum_n < x, \ x_{in} > e_n$$

for all $x \in H$. Hence every pair of generators of a representation of $\mathcal{O}_2$ can be obtained from a complete pair of strongly disjoint normalized tight frames. This argument and Proposition 2.16 easily extends to representations of the Cuntz algebra $\mathcal{O}_n$.

2.4 More on Alternate Duals

Now we turn to more on dual frames.

Lemma 2.17. *If $\{x_n\}$ is frame for H and if $C \in B(H)$ is an invertible operator such that $C^*Cx_n = x_n^*$ for all $n \in \mathbb{Z}$, then $\{Cx_n\}$ is a normalized tight frame.*

Proof. Since $x = \sum_n < x,\ x_n^* > x_n = \sum_n < x,\ C^*Cx_n > x_n$ for all $x \in H$, we have

$$C^{-1}x = \sum_n < C^{-1}x,\ C^*Cx_n > x_n = \sum_n < x,\ Cx_n > x_n$$

for all $x \in H$. Thus $x = \sum_n < x,\ Cx_n > Cx_n$ for all $x \in H$, which implies, by the argument preceding Example A, that $\{Cx_n\}$ is a normalized tight frame. $\square$

The following result reinforces the "correctness" of our definition of strong disjointness. It is the version of disjointness which is compatible with the definition of canonical dual.

Proposition 2.18. *Frames $\{x_n\}$ and $\{y_n\}$ are strongly disjoint if and only if $\{x_n \oplus y_n\}$ is a frame and $(x_n \oplus y_n)^* = x_n^* \oplus y_n^*$.*

Proof. ($\Rightarrow$). Suppose that $\{x_n\}$ and $\{y_n\}$ are strongly disjoint. Then let $A,\ B$ be invertible operators such that $Ax_n = f_n,\ By_n = g_n$ with the property that $\{f_n\}$, $\{g_n\}$ and $\{x_n \oplus y_n\}$ are normalized tight frames. We have $f_n \oplus g_n = (A \oplus B)(x_n \oplus y_n)$. Thus, by Proposition 1.10,

$$\begin{aligned}(x_n \oplus y_n)^* &= (A \oplus B)^*(f_n \oplus g_n)\\ &= (A^* \oplus B^*)(f_n \oplus g_n)\\ &= A^*f_n \oplus B^*g_n = x_n^* \oplus y_n^*.\end{aligned}$$

($\Leftarrow$). Assume that $x_n^* \oplus y_n^* = (x_n \oplus y_n)^*$. Again let A and B be invertible operators such that $\{Ax_n\}$ and $\{By_n\}$ are normalized tight frames. Write $f_n = Ax_n$ and $g_n = By_n$. Then, $x_n^* = A^*f_n$ and $y_n^* = B^*g_n$ by proposition 1.9. Thus

$$(A^*A \oplus B^*B)(x_n \oplus y_n) = x_n^* \oplus y_n^* = (x_n \oplus y_n)^*.$$

So, by Lemma 2.17, $(A \oplus B)(x_n \oplus y_n)$ is a normalized tight frame, which implies that $\{x_n\}$ and $\{y_n\}$ are strongly disjoint by definition. $\square$

With the similar proof as in Proposition 1.18 (or by Corollary 2.13 and Proposition 1.18), we have

Proposition 2.18′. *A k-tuple $(\{x_{1n}\}_{n\in\mathbb{J}},\ \ldots\ ,\ \{x_{kn}\}_{n\in\mathbb{J}})$ of frames are strongly disjoint if and only if it is a disjoint k-tuple and the canonical dual of the direct sum frame is equal to the direct sum of their canonical duals.*

Strongly disjoint pairs of frames on the same Hilbert space have some surprising and useful additional structural properties.

Proposition 2.19. *Suppose that $\{x_n\}$ and $\{y_n\}$ are strongly disjoint frames for the same Hilbert space H, then $\{x_n + y_n\}$ is a frame for H. In particular, if $\{x_n\}$ and $\{y_n\}$ are strongly disjoint proper normalized tight frames for H, then $\{x_n+y_n\}$ is a tight frame with frame bound 2. More generally, if $\{x_n\}$ and $\{y_n\}$ are strongly disjoint frames for closed subspaces M and N of H, respectively, then $\{x_n+y_n\}$ is a frame for the closed linear span of M and N.*

Proof. Let $x \in \overline{span}(M \cup N)$, and let P and Q be the orthogonal projections onto M and N, respectively. Then $||Px||^2 + ||Qx||^2 \geq ||x||^2$. Suppose that $\{x_n\}$ has frame bounds a, b, and $\{y_n\}$ has frame bound $c,\ d$. We have

$$\begin{aligned}
||x||^2 &= \sum_n |< x,\ x_n + y_n >|^2 \\
&= \sum_n |< Px,\ x_n > + < Qx,\ y_n >|^2 \\
&= \sum_n |< Px,\ x_n >|^2 + \sum_n |< Qx,\ y_n >|^2 \\
&+ Re \sum_n < Px,\ x_n >< y_n, Qx > .
\end{aligned}$$

Note that, from either Theorem 2.9 (i) or Corollary 2.10, $\sum_n < Px,\ x_n >< y_n, Qx >= 0$. Therefore

$$\begin{aligned}
\sum_n |< x,\ x_n + y_n >|^2 &= \sum_n |< Px,\ x_n >|^2 + \sum_n |< Qx,\ y_n >|^2 \\
&\geq b||Px||^2 + d||Qx||^2 \geq min(b, d)||x||^2
\end{aligned}$$

and similarly

$$\sum_n |< x,\ x_n + y_n >|^2 \leq 2max(a, c)||x||^2.$$

Thus $\{x_n + y_n\}$ is a frame.

In the case that $\{x_n\}$ and $\{y_n\}$ are strongly disjoint proper normalized tight frames for H, the above argument implies that

$$\sum_n |< x,\ x_n + y_n >|^2 = 2||x||^2$$

for all $x \in H$. Hence $\{x_n + y_n\}$ is a tight frame with frame bound 2. □

Similarly for k-tuples we have:

Theorem 2.19′. *Suppose that $(\{x_{1n}\}_{n\in\mathbb{J}},\ \ldots\ ,\ \{x_{kn}\}_{n\in\mathbb{J}})$ is strongly disjoint k-tuple of frames on the same Hilbert space H. Then the sum of these frames is a* *frame for H. In particular when it is a strongly disjoint k-tuple of proper normalized tight frames, then the sum of these frames is a tight frame for H with frame bound k.*

For the weaker notion of disjointness we also have

Proposition 2.20. *If $\{x_n\}$ and $\{y_n\}$ are disjoint frames for H, then $\{x_n + y_n\}$ is also a frame for H.*

Proof. Let θ_1 and θ_2 be the frame transforms for $\{x_n$ and $\{y_n\}$, respectively. Then, by Corollary 1.2 (ii), $x_n = \theta_1^* e_n$ and $y_n = \theta_2^* e_n$ for all $n \in \mathbb{J}$, where $\{e_n\}$ is the standard orthonormal basis for $l^2(\mathbb{J})$. Thus for any $x \in H$, we have

$$\begin{aligned}\sum_n |< x,\ x_n + y_n > |^2 &= \sum_n < x, \theta_1^* e_n + \theta_2^* e_n > |^2 \\ &= \sum_n |< \theta_1(x) + \theta_2(x),\ e_n > |^2 \\ &= ||\theta_1(x) + \theta_2(x)||^2.\end{aligned}$$

By Theorem 2.9 (iii), there exists a positive constant a such that

$$||\theta_1(x) + \theta_2(y)y||^2 \geq a(||\theta_1(x)||^2 + ||\theta_2(y)||^2)$$

for all $x, y \in H$. Also note that if a_1, b_1 and a_2, b_2 are frame bounds for $\{x_n\}$ and $\{y_n\}$, respectively, then

$$a_1||x||^2 \leq ||\theta_1(x)||^2 \leq b_1||x||^2$$

and

$$a_2||x||^2 \leq ||\theta_2(x)||^2 \leq b_2||x||^2$$

for all $x \in H$. Thus for all $x \in H$, we get

$$a(a_1 + a_2)||x||^2 \leq \sum_n |< x,\ x_n + y_n > |^2 \leq (\sqrt{b_1} + \sqrt{b_2})^2||x||^2.$$

Therefore $\{x_n + y_n\}$ is also a frame for H. $\square$

The second statement of Proposition 2.19 generalizes considerably:

Proposition 2.21. *Suppose that $\{x_n\}$ and $\{y_n\}$ are strongly disjoint normalized tight frames for H and $A, B \in B(H)$ are operators such that $AA^* + BB^* = I$. Then $\{Ax_n + By_n\}$ is a normalized tight frame for H. In particular $\{\alpha x_n + \beta y_n\}$ is a normalized tight frame whenever α and β are scalars such that $|\alpha|^2 + |\beta|^2 = 1$.*

Proof. Let $\{e_n\}$ be the standard orthonormal basis for $l^2(\mathbb{J})$, and let θ_1 and θ_2 be the frame transforms for $\{x_n\}$ and $\{y_n\}$, respectively. Then θ_1 and θ_2 are isometries with orthogonal range such that $\theta_1^* e_n = x_n$ and $\theta_2^* e_n = y_n$ for all $n \in \mathbb{J}$. Let $T = A\theta_1^* + B\theta_2^*$. We claim that T^* is an isometry. In fact, since $\theta_1^*\theta_2 = \theta_2^*\theta_1 = 0$, we have

$$\begin{aligned} TT^* &= A\theta^*\theta_1 A^* + B\theta_2^*\theta_2 B^* \\ &= AA^* + BB^* = I \end{aligned}$$

Thus $\{Ax_n + By_n\}$ ($= \{Te_n\}$) is an normalized tight frame for H by Proposition 1.9 (i). □

More generally by an analogous argument we have the following:

Proposition 2.21′. *Suppose that* $(\{x_{1n}\}_{n\in\mathbb{J}}, \ \dots \ , \ \{x_{kn}\}_{n\in\mathbb{J}})$ *is strongly disjoint* k*-tuple of normalized tight frames on the same Hilbert space* H *and* $A_i \in B(H)$ *such that* $\sum_{i=1}^k A_i A_i^* = I$. *Then* $\{\sum_{i=1}^k A_i x_{in}\}_{n\in\mathbb{Z}}$ *is a normalized tight frame for* H.

Given a frame $\{x_n\}$ for H, we sometime want to find a tight alternate dual frame for $\{x_n\}$. In general tight alternate duals might not exist. For example if $\{x_n\}$ is a Riesz basis which is not tight, then it has a unique alternate dual which also fails to be tight. Thus there is no tight alternate dual in this case. However, the following result tells us that in many (in fact in most) cases tight alternate duals do exist,

Proposition 2.22. *Let* $\{x_n\}$ *be a frame for a Hilbert space* H, *and let* $A \in B(H)$ *be an invertible operator such that* $\{A^{-1}x_n^*\}$ *is a normalized tight frame. If* $||A|| < 1$, *then* $\{x_n\}$ *has a normalized tight alternate dual if and only if the range of the frame transform for* $\{x_n\}$ *has co-dimension greater than or equal to the dimension of* H.

Proof. Write $z_n = A^{-1}x_n^*$. Let θ_1 and θ be the frame transforms for $\{x_n\}$ and $\{z_n\}$, respectively. Then $\theta_1(H) = \theta(H)$ since $\{z_n\}$ and $\{x_n\}$ are similar. Let P be the projection from $l^2(\mathbb{J})$ onto the range of θ, and let $\{e_n\}$ be the standard orthonormal basis for $l^2(\mathbb{J})$. Since $\theta(z_n) = Pe_n$, we have that $\{z_n \oplus P^\perp e_n\}$ is an orthonormal basis for $H \oplus M$, where $M = P^\perp l^2(\mathbb{J})$.

First assume that $dim(\theta(H)^\perp) \geq dimH$. Choose a closed subspace N of $\theta(H)^\perp$ such that $dimN = dimH$. Let $W : N \to H$ be a fixed unitary, and let $w_n = WQe_n$, where Q is the orthogonal projection from $l^2(\mathbb{J})$ onto N. Then $\{w_n\}$ is a normalized tight frame for H, which is strongly disjoint with $\{z_n\}$. Let $B = \sqrt{I - AA^*}$. Then $AA^* + BB^* = I$. Thus, by Proposition 2.21, $\{z_n + Bw_n\}$ is a normalized tight frame for H. Note that B is invertible. Hence $\{Bw_n\}$ is a frame which is strongly disjoint with $\{z_n\}$. Therefore $\{Bw_n\}$ and $\{x_n\}$ are also strongly disjoint since $\{z_n\}$

and $\{x_n\}$ are similar. This implies

$$\sum_n < x,\ Bw_n > x_n = 0$$

for all $x \in H$. Note that $Az_n = x_n^*$. Then we have

$$x = \sum_n < x,\ x_n^* > x_n = \sum_n < x,\ Az_n + Bw_n > x_n$$

for all $x \in H$. That is, $\{Az_n + Bw_n\}$ is a normalized tight alternate dual for $\{x_n\}$, as required.

Conversely, assume that $\{x_n\}$ has a normalized tight alternate dual $\{y_n\}$. Define $T : H \to H \oplus M$ by

$$Tx = \sum_n < x,\ y_n > (z_n \oplus P^\perp e_n), \quad x \in H.$$

Then T is an isometry since $\{z_n \oplus P^\perp e_n\}$ is an orthonormal basis, and $T^*(z_n \oplus P^\perp e_n) = y_n$. Write $T = \begin{pmatrix} C \\ D \end{pmatrix}$ with $C \in B(H)$ and $D \in B(H, M)$. Then $C^*C + D^*D = I$ and $y_n = C^*z_n + D^*P^\perp e_n$. We prove that $C^* = A$. Let $x_n^* = Sx_n$, where $S \in B(H)$ is the frame operator for $\{x_n\}$. Since $\{x_n\}$ and $\{P^\perp e_n\}$ are strongly disjoint, we have

$$\sum_n < x,\ D^*P^\perp e_n > x_n = \sum_n < Dx,\ P^\perp e_n > x_n = 0$$

for all $x \in H$. So for any $x \in H$, we have

$$\begin{aligned} x &= \sum_n < x,\ y_n > x_n \\ &= \sum_n < x,\ C^*z_n + D^*P^\perp e_n > x_n \\ &= \sum_n < x,\ C^*A^{-1}Sx_n > x_n. \end{aligned}$$

By Proposition 1.10, we have that $C^*A^{-1}S = S$, which implies that $C^* = A$. Hence $D^*D = I - AA^*$ is invertible. Therefore $dimM \geq dimH$. $\square$

Let $\{x_n\}$ be a frame for a Hilbert space H, and let $A \in B(H)$ be an invertible operator such that $\{A^{-1}x_n^*\}$ is a normalized tight frame. Suppose that the range of the frame transform for $\{x_n\}$ has codimension $\geq dimH$. From the proof of Proposition 2.22, if we let $B = \sqrt{1 - ||A||^{-2}AA^*}$, then there is a normalized tight frame $\{w_n\}$ such that it is strongly disjoint with $\{x_n\}$, and $\{x_n + Bw_n\}$ is a tight frame with frame bound $||A||^2$. Although $\{Bw_n\}$ is not necessarily a frame for H, it is a sequence which is strongly disjoint with $\{x_n\}$ in the sense that

$$\sum_n < x, x_n > Bw_n = \sum_n < x, Bw_n > x_n = 0$$

for all $x \in H$. If we require $\{By_n\}$ to be a frame for H, we have

Corollary 2.23. *Let $\{x_n\}$ be a frame for a Hilbert space H. Suppose that the range of its frame transform has co-dimension $\geq dimH$. Then there is a frame $\{y_n\}$ for H which is strongly disjoint with $\{x_n\}$ so that $\{x_n + y_n\}$ is a tight frame for H with frame bound arbitrarily close to the upper frame bound of $\{x_n\}$. Moreover, $\{x_n\}$ has a tight alternate dual with frame bound equal to the upper frame bound of $\{x_n^*\}$.*

Proof. The second statement follows immediately from the preceding argument by replacing $\{x_n\}$ by $\{x_n^*\}$. For the first statement, let $A \in B(H)$ such that $\{A^{-1}x_n\}$ is a normalized tight frame. Scale $\{x_n\}$ by $c > 0$ such that $c||A|| < 1$. Then, by the proof of the above result, there is a frame $\{u_n\}$ for H which is strongly disjoint with $\{x_n\}$ such that $\{cx_n + u_n\}$ is a normalized tight frame for H. Hence $\{x_n + c^{-1}u_n\}$ is a tight frame for H. Note that $||A||^2$ is the upper frame bound for $\{x_n\}$, and thus $\{x_n + \frac{1}{c}u_n\}$ has frame bound c^{-2} which can be arbitrarily close to $||A||^2$. □

The following is an immediate corollary of Proposition 2.19′ and Corollary 2.10.

Corollary 2.24. *Suppose that $(\{x_{1n}\}, \ . \ . \ . \ , \ \{x_{kn}\})$ is a strongly disjoint k-tuple of frames acting on the same Hilbert space H. Then these k frames have a common alternate dual.*

Proof. By similarity, we also have that $(\{x_{1n}^*\}, \ . \ . \ . \ , \ \{x_{kn}\}^*)$ is a strongly disjoint k-tuple. Hence, by Proposition 2.19′, $\{\sum_{i=}^{k} x_{in}^*\}_{n\in\mathbb{J}}$ is a frame for H. Note that, by Corollary 2.10, $\sum_n < x, \ x_{ln}^* > x_{in} = 0$ when $i \neq l$. Thus $\{\sum_{i=}^{k} x_{in}^*\}_{n\in\mathbb{J}}$ is a common alternate dual for all $\{x_{in}\}$ $(i = 1, \ ... \ , , k)$. □

We note that if $\{x_n\}$ is a frame for H and P is an orthogonal projection from H onto a subspace M, then $\{Px_n\}$ and $\{P^\perp x_n\}$ are always disjoint. To see this, let $U : H \to M \oplus M^\perp$ be the unitary operator defined by

$$Ux = Px \oplus P^\perp x.$$

Then $Ux_n = Px_n \oplus P^\perp x_n$. Thus $\{Px_n \oplus P^\perp x_n\}$ is a frame for $M \oplus M^\perp$, which implies that $\{Px_n\}$ and $\{P^\perp x_n\}$ are disjoint. However they are not always strongly disjoint. In fact, from Corollary 1.16 and Proposition 2.18, we have the following characterization.

Corollary 2.25. *Let $\{x_n\}$ be a frame for H and suppose that P is an orthogonal projection in $B(H)$. Let $S \in B(H)$ be the operator such that $Sx_n = x_n^*$. Then $\{Px_n\}$ and $\{P^\perp x_n\}$ are strongly disjoint if and only if $PS = SP$.*

The following is a consequence of Proposition 2.19 which we mentioned in Section 1.3.

Corollary 2.26. *Suppose that $\{x_j\}$ is a frame on a Hilbert space H. Then $\{x_j\}$ has a unique alternate dual if and only if it is a Riesz basis.*

Proof. It suffices to prove the necessity. First assume that $\{x_j\}$ is a normalized tight frame which is not an orthonormal basis. By Proposition 1.1, there is a normalized tight frame $\{y_j\}$ for a Hilbert space M such that $\{x_j \oplus y_j\}$ is an orthonormal basis for $H \oplus M$. Choose y_k such that $y_k \neq 0$ and let P be the projection from M onto the one dimensional subspace generated by y_k. Thus $\{x_j\}$ and $\{Py_j\}$ are strongly disjoint since

$$\{x_j \oplus Py_j\} = \{(I \oplus P)(x_j \oplus y_j)\},$$

which is a normalized tight frame for $H \oplus PM$. Embed M into H by an isometry U. Then $\{x_j\}$ and $\{UPy_j\}$ are strongly disjoint. Thus, By Corollary 2.10,

$$\sum_j < x,\ UPy_j > x_j = 0$$

for all $x \in UM$. However if $x \in (UM)^{\perp}$, we have $< x,\ Uy_j >= 0$. Thus

$$\sum_j < x,\ UPy_j > x_j = 0$$

for all $x \in H$. It follows that

$$x = \sum_j < x,\ x_j > x_j = \sum_j < x,\ x_j + UPy_j > x_j, \quad x \in H.$$

By Proposition 2.19, $\{x_j + UPy_j\}$ is a frame for H. Therefore it is an alternate dual for $\{x_j\}$, which is different from the classical dual since $x_k \neq x_k + UPy_k$.

Now let $\{x_j\}$ be an arbitrary frame which is not a Riesz basis. By Proposition 1.10, there is an invertible operator $A \in B(H)$ such that $\{Ax_j\}$ is a normalized tight frame for H and

$$x = \sum_j < x,\ Sx_j > x_j$$

for all $x \in H$, where $S = A^*A$ and $\{Sx_j\}$ is the classical dual frame of $\{x_j\}$. Note that $\{Ax_j\}$ is not an orthonormal basis. Thus from what we just proved there is an alternate dual $\{y_j\}$ of $\{Ax_j\}$ which is different from $\{Ax_j\}$. Let $z_j = A^*y_j$. Then $\{z_j\}$ is a frame different from the classical dual frame $\{A^*Ax_j\}$ and

$$x = A^{-1}Ax = A^{-1}\sum_j < Ax,\ y_j > Ax_j = \sum_j < x,\ A^*y_j > x_j.$$

Thus $\{A^*y_j\}$ is an alternate dual for $\{x_j\}$. Hence $\{x_j\}$ has different frame duals $\{Sx_j\}$ and $\{A^*y_j\}$, as required. $\square$

In the case that $\{x_n\}$ is a normalized tight frame, then it can the shown that there is a unique normalized tight alternate dual frame, namely, $\{x_n\}$ itself. However if $\{x_n\}$ in addition satisfies the codimension condition in Corollary 2.23, then, except for the canonical dual $\{x_n\}$, it has another tight alternate dual which has the form $\{x_n + y_n\}$, where $\{y_n\}$ is a tight frame for H which is strongly disjoint with $\{x_n\}$.

Remark 2.27. We address some potential application aspects of strongly disjoint k-tuples. For simplicity we only consider normalized tight frames. However, all the following discussions carry through if one replaces the frames in all relevant inner products by its canonical duals. Suppose that $\{x_n :\ n \in \mathbb{N}\}$ and $\{y_n\}$ are strongly disjoint normalized tight frames for Hilbert spaces H and K, respectively. Then given any pair of vectors $x \in H,\ y \in K$, we have that

$$x = \sum_n < x, x_n > x_n, \quad y = \sum_n < y,\ y_n > y_n.$$

If we let $a_n =< x,\ x_n >$ and $b_n =< y,\ y_n >$, and then let $c_n = a_n + b_n$, we have

$$\sum_n a_n y_n = 0, \quad \sum_n b_n x_n = 0,$$

by the strong disjointness, and therefore we have

$$x = \sum_n c_n x_n, \quad y = \sum_n c_n y_n.$$

This says that, by using one set of data $\{c_n\}$, we can recover two vectors x and y (they may even lie in different Hilbert spaces) by applying the respective inverse transforms corresponding to the two frames $\{x_n\}$ and $\{y_n\}$. The above argument obviously extends to the k-tuple case: If $\{f_{in} :\ n \in \mathbb{J}\}$, $i = 1,\ ...,\ k$, is a strongly disjoint k-tuple of normalized tight frames for Hilbert spaces $H_1,\ ...,\ H_k$, and if $(x_1,\ ...,\ x_k)$ is an arbitrary k-tuple of vectors with $x_i \in H_i,\ 1 \leq i \leq k$, then we have

$$x_i = \sum_{n \in \mathbb{J}} < x_i,\ f_{in} > f_{in}$$

for each $1 \leq i \leq k$. So if we define a single "master" sequence of complex numbers $\{c_n :\ n \in \mathbb{J}\}$ by

$$c_n = \sum_{i=1}^{k} < x_i,\ f_{in} >,$$

then the strong disjointness implies that for each *individual* i we have

$$x_i = \sum_{n \in \mathbb{J}} c_n f_{in}.$$

This simple observation might be useful in applications to data compression. We will discuss this phenomenon for unitary groups and wavelet systems in sections 4 and 5. It leads to our notion of superwavelets.

The concept of Remark 2.27 can be condensed into the following proposition, which says that the frame transforms corresponding to strongly disjoint normalized tight frames indexed by the same set $\mathbb{J}$ act orthogonally as operators in the sense that the support of each inverse transform is orthogonal to the ranges of the others.

Proposition 2.28. *Let $\{f_{in} : n \in \mathbb{J}\}$, $i = 1, ..., k$, be a strongly disjoint k-tuple of normalized tight frames for Hilbert spaces $H_1, \ . \ . \ . \ , H_k$. Let $\theta_i : H_i \to l^2(\mathbb{J})$ be the corresponding frame transforms, and for each i let $\Gamma_i := \theta_i^*$ denote the inverse transform mapping $l^2(\mathbb{J}) \to H_i$. Then $\Gamma_i \theta_j = 0$ if $i \neq j$, and $\Gamma_i \theta_i = I_{H_i}$, $i, j = 1, ..., k$.*

Chapter 3

Frame Vectors for Unitary Systems

Following Dai and Larson [DL], a *unitary system* $\mathcal{U}$ is a subset of the unitary operators acting on a separable Hilbert space H which contains the identity operator I. So a *unitary group* ia a special case of unitary system. An element $\psi \in H$ is called a *wandering vector* for $\mathcal{U}$ if $\mathcal{U}\psi := \{U\psi : \quad U \in \mathcal{U}\}$ is an orthonormal set; that is $< U\psi, V\psi > = 0$ if $U, V \in \mathcal{U}$ and $U \neq V$. If $\mathcal{U}\psi$ is an orthonormal basis for H, then ψ is called a *complete wandering vector* for $\mathcal{U}$. The set of all complete wandering vectors for $\mathcal{U}$ is denoted by $\mathcal{W}(\mathcal{U})$.

Analogously, a vector $x \in H$ is called a *normalized tight frame vector* (resp. *frame vector* with bounds a and b) for a unitary system $\mathcal{U}$ if $\mathcal{U}x$ forms a tight frame (resp. frame with bounds a and b) for $\overline{span}(\mathcal{U}x)$. It is called a *complete normalized tight frame vector* (resp. *complete frame vector* with bounds a and b) when $\mathcal{U}x$ is a normalized tight frame (resp. frame with bounds a and b) for H.

If $\mathcal{U}$ is a unitary system and $\psi \in \mathcal{W}(\mathcal{U})$, the *local commutant* $C_\psi(\mathcal{U})$ at ψ is defined by $\{T \in B(H) : (TU - UT)\psi = 0, U \in \mathcal{U}\}$. Clearly $C_\psi(\mathcal{U})$ contains the commutant $\mathcal{U}'$ of $\mathcal{U}$. When $\mathcal{U}$ is a unitary group, it is actually the commutant of $\mathcal{U}$. A useful result is the one to one correspondence between the complete wandering vectors and the unitary operators in $C_\psi(\mathcal{U})$. In particular, if $\psi \in \mathcal{W}(\mathcal{U})$, then $\mathcal{W}(\mathcal{U}) = \mathbb{U}(C_\psi(\mathcal{U}))\psi = \{T\psi : T \in \mathbb{U}(C_\psi(\mathcal{U}))\}$ (see[DL], Proposition 1.3), where $\mathbb{U}(\mathcal{S})$ denotes the set of all unitary operators in $\mathcal{S}$ for any subset $\mathcal{S} \subseteq B(H)$. It is also known that ψ separates $C_\psi(\mathcal{U})$ in the sense that the mapping $A \to A\psi$ from $C_\psi(\mathcal{U})$ to H is injective. In [La] it was pointed out that an analogous result holds for *complete Riesz vectors* (those vectors ψ for which $\mathcal{U}\psi$ is a Riesz basis for H). In the same way, the set of all complete Riesz vectors for $\mathcal{U}$ is in one to one correspondence with the set of all *invertible* operators in $C_\psi(\mathcal{U})$. The following result characterizes all the normalized tight frame vectors for $\mathcal{U}$ in terms of the partial isometries in the local commutant at a fixed complete wandering vector.

3.1 The Local Commutant and Frame Vectors

In this section we will characterize frame vectors in terms of operators in the local commutant at a fixed complete wandering vector. We first prove:

Proposition 3.1. *Suppose that ψ is a complete wandering vector for a unitary system $\mathcal{U}$. Then*

(i) a vector η is a normalized tight frame vector for $\mathcal{U}$ if and only if there is a (unique) partial isometry $A \in C_{\psi}(\mathcal{U})$ such that $A\psi = \eta$.

(ii) a vector η is a complete normalized tight frame vector for $\mathcal{U}$ if and only if there is a (unique) co-isometry $A \in C_{\psi}(\mathcal{U})$ such that $A\psi = \eta$.

Proof. The uniqueness follows from the fact that ψ separates $C_{\psi}(\mathcal{U})$. The statement (ii) follows from (i) since if A is a partial isometry in $C_{\psi}(\mathcal{U})$, then $\{U\eta : U \in \mathcal{U}\} = \{AU\psi : U \in \mathcal{U}\}$ generates H if and only if A^* is an isometry. So we only need to prove (i).

Suppose that η is a normalized tight frame vector for $\mathcal{U}$. Define a linear operator B by

$$Bx = \sum_{U \in \mathcal{U}} < x, U\eta > U\psi$$

for $x \in \overline{span}(\mathcal{U}\eta)$, and $Bx = 0$ when $x \perp \overline{span}(\mathcal{U}\eta)$. Since η is a normalized tight frame vector, we have that B is isometric on $\overline{span}(\mathcal{U}\eta)$. Thus B is a partial isometry with closed range BH. Let P be the orthogonal projection onto BH, and let $A = B^* = B^*P$. We will show that A is a partial isometry with the required property.

We first claim that $BU\eta = PU\psi$, for all $U \in \mathcal{U}$.

In fact, let $V \in \mathcal{U}$. We have

$$\begin{aligned} < BV\eta, PU\psi > &= < PBV\eta, U\psi > = < BV\eta,\ U\psi > \\ &= < \sum_{S \in \mathcal{U}} < V\eta, S\eta > S\psi,\ U\psi > \\ &= < V\eta,\ U\eta > . \end{aligned}$$

Since B is isometric on $\overline{span}(\mathcal{U}\eta)$, it follows that

$$< BV\eta, PU\psi > = < V\eta, U\eta > = < BV\eta, BU\eta > .$$

Thus $BU\eta = PU\psi$.

Next we claim that $A = B^*P \in C_{\psi}(\mathcal{U})$ and $A\psi = \eta$.

From the above paragraph, we have $B^*BU\eta = B^*PU\psi$ for all $U \in \mathcal{U}$. Note that $U\eta$ is contained in the initial subspace of the partial isometry B. Thus $B^*BU\eta = U\eta$. So $U\eta = AU\psi$ for each $U \in \mathcal{U}$. Thus $\eta = A\psi$ and $A \in C_\psi(\mathcal{U})$.

Note that $PH(= BH)$ is the final subspace of B and B^* is isometric on PH and takes the value zero on $P^\perp H$. Thus $B^*P = B^*$, which implies that A is a partial isometry.

Conversely let A be a partial isometry in $C_\psi(\mathcal{U})$ and let $\eta = A\psi$. Note that A^* is isometric on AH. Then for any $x \in AH$, we have

$$\begin{aligned} ||x||^2 = ||A^*x||^2 &= \sum_{U\in\mathcal{U}} | < A^*x, U\psi > |^2 \\ &= \sum_{U\in\mathcal{U}} | < x, AU\psi > |^2 \\ &= \sum_{U\in\mathcal{U}} | < x, UA\psi > |^2 \\ &= \sum_{U\in\mathcal{U}} | < x, U\eta > |^2. \end{aligned}$$

Thus η is a normalized tight frame vector for $\mathcal{U}$ on $\overline{span}(\mathcal{U}\eta)(= AH)$. $\square$

As in the wandering vector case (cf [DL] Lemma 1.6), we have

Proposition 3.2. *Let $\mathcal{S}$ be a unital semigroup of unitaries in $B(H)$. If $\mathcal{S}$ has a complete normalized tight frame vector, then $\mathcal{S}$ is a group.*

Proof. Let $U \in \mathcal{S}$. We want to show that $U\mathcal{S} = \mathcal{S}$. Let η be a complete normalized tight frame vector for $\mathcal{S}$. Then for any $x \in H$, we have

$$||U^{-1}x||^2 = \sum_{S\in\mathcal{S}} | < U^{-1}x,\ S\eta > |^2 = \sum_{S\in\mathcal{S}} | < x,\ US\eta > |^2$$

and

$$||U^{-1}x||^2 = ||x||^2 = \sum_{S\in\mathcal{S}} | < x,\ S\eta > |^2.$$

If $U^{-1} \notin \mathcal{S}$, then $I \notin U\mathcal{S}$. Thus $< x,\ \eta >= 0$ since $U\mathcal{S} \subset \mathcal{S}$. Let $x = \eta$. We get a contradiction. So $\mathcal{S}$ is a group. $\square$

Suppose that η and ξ are complete normalized tight frame vectors for a unitary system $\mathcal{U}$. If $\{U\eta\}_{U\in\mathcal{U}}$ and $\{U\xi\}_{U\in\mathcal{U}}$ are unitarily equivalent, then there is a unitary operator W satisfying $WU\eta = U\xi$ for every $U \in \mathcal{U}$. In particular, $W\eta = \xi$. Hence $WU\eta = UW\eta$ for all $U \in \mathcal{U}$. So when $\mathcal{U}$ is a unitary group, we have that $\mathcal{U}\eta$ and

$\mathcal{U}\xi$ are unitarily equivalent frames if and only if there is a unitary operator $W \in \mathcal{U}'$ such that $W\eta = \xi$. Thus a frame unitary equivalence determines an equivalence relation for complete normalized tight frame vectors. For a set $\mathcal{S} \subseteq B(H)$, we use $w^*(\mathcal{S})$ to denote the von Neumann algebra generated by $\mathcal{S}$, and use $\mathcal{S}'$ to denote the commutant of $\mathcal{S}$, that is

$$\mathcal{S}' = \{T \in B(H) : \ ST - TS = 0, \quad \forall S \in \mathcal{S}\}.$$

An element $x \in H$ is call a *trace vector* for a von Neumann algebra $\mathcal{R}$ acting on H if $< ABx, \ x > = < BAx, \ x >$ for all $A, B \in \mathcal{R}$. A trace vector x for $\mathcal{R}$ is said to be *faithful* if the mapping $A \to Ax$ $(A \in \mathcal{R})$ is injective. The following lemma can be found in [La], and will be frequently used in this paper.

Lemma 3.3. *Let $\mathcal{U}$ be a unitary group such that $\mathcal{W}(\mathcal{U})$ is nonempty. Then both $w^*(\mathcal{U})$ and $\mathcal{U}'$ are finite von Neumann algebras. Moreover each element in $\mathcal{W}(\mathcal{U})$ is a faithful trace vector for both $w^*(\mathcal{U})$ and $\mathcal{U}'$.*

Corollary 3.4. *Suppose that $\mathcal{U}$ is a unitary group such that $\mathcal{W}(\mathcal{U})$ is non-empty. Then every complete normalized tight frame vector must be a complete wandering vector.*

Proof. Let $\psi \in \mathcal{W}(\mathcal{U})$ and let η be a complete normalized tight frame vector. Then, Proposition 3.1 (2), there exists a co-isometry $A \in C_{\psi}(\mathcal{U}) = \mathcal{U}'$ such that $\eta = A\psi$. Since $\mathcal{U}'$ is a finite von Neumann algebra, it follows that A is a unitary operator. Hence η is a complete wandering vector for $\mathcal{U}$ by Proposition 1.3 in [DL]. $\square$

By Corollary 3.4, if a unitary group system $\mathcal{U}$ has a complete tight frame which is not a wandering vector, then $\mathcal{W}(\mathcal{U})$ is empty.

With a minor modification of the proof for Proposition 3.1, one can easily get the following more general result. We leave the details to the interested reader.

Proposition 3.5. *Let $\mathcal{U}$ and ψ be as in Proposition 3.1. Then a vector η is a frame vector with frame bounds a and b if and only if there exists an (unique) operator $A \in C_{\psi}(\mathcal{U})$ such that $\eta = A\psi$ and $aP \leq AA^* \leq bP$ for some orthogonal projection P.*

In fact P is the orthogonal projection onto $\overline{span}(\mathcal{U}\eta)$. So we have

Corollary 3.6. *The vector η is a complete frame vector with frame bounds a and b if and only if there exists an (unique) operator $A \in C_{\psi}(\mathcal{U})$ such that $\eta = A\psi$ and $aI \leq AA^* \leq bI$.*

By a similar argument used in the proof of Proposition 2.1 in [DL], it is easy to prove that $\{U\eta : U \in \mathcal{U}\}$ is a Riesz basis if and only if $\eta = S\psi$ for some invertible operator $S \in C_\psi(\mathcal{U})$ (see Proposition 2.1 in [La]).

Corollary 3.7. *Let $\mathcal{U}$ and ψ be as in Proposition 3.1. Suppose that $\mathcal{M}$ is a finite von Neumann algebra contained in $C_\psi(\mathcal{U})$ and $\eta = A\psi$ for some operator $A \in \mathcal{M}$. Then*

(i) η is a complete frame vector if and only if $\mathcal{U}\eta$ is a Riesz basis for H.

(ii) η is a complete normalized tight frame vector if and only if η is a complete wandering vector.

Proof. For (i), suppose that η is a complete frame vector for $\mathcal{U}$. Then AA^* is invertible by Corollary 3.6. Let $A^* = U(AA^*)^{\frac{1}{2}}$ be the polar decomposition of A. Then U is a partial isometry with initial space $H(= (AA^*)^{\frac{1}{2}}H)$ and $U \in \mathcal{M}$. Thus U is unitary since $\mathcal{M}$ is a finite von Neumann algebra. Therefore A is invertible, which implies that $\mathcal{U}\eta$ is a Riesz basis. The other direction is shown in an analogous way. For (ii), assume that η is a complete normalized tight frame vector, then $AA^* = I$ by Proposition 3.1 (ii). Thus A is a unitary operator since $\mathcal{M}$ is finite, and both A and A^* are in $\mathcal{M}$. The other direction is trivial. □

3.2 Dilation Theorems for Frame Vectors

The general dilation result (Proposition 1.1) tells us that if η is a complete normalized tight frame vector for a unitary system $\mathcal{U}$, then $\{U\eta : \ U \in \mathcal{U}\}$ can be dilated to an orthonormal basis. What we expect here is to dilate η to a complete wandering vector. In this section we show that this can be done for unitary groups and some other special cases.

Theorem 3.8. *Suppose that $\mathcal{U}$ is a unitary group on H and η is a complete normalized tight frame vector for $\mathcal{U}$. Then there exist a Hilbert space $K \supseteq H$ and a unitary group $\mathcal{G}$ on K such that $\mathcal{G}$ has complete wandering vectors, H is an invariant subspace of $\mathcal{G}$ such that $\mathcal{G}|_H = \mathcal{U}$, and the map $g \to g|_H$ is a group isomorphism from $\mathcal{G}$ onto $\mathcal{U}$.*

Proof. Let $K = l^2(\mathcal{U})$, and for each $U \in \mathcal{U}$, let λ_U be the left regular representation defined by $\lambda_U \chi_V = \chi_{UV}, \quad V \in \mathcal{U}$, where χ_V is the characteristic function at the single point set $\{V\}$. Consider the unitary group $\mathcal{G} = \{\lambda_U : \ U \in \mathcal{U}\}$. Then $\mathcal{U}$ and $\mathcal{G}$ are group isomorphic , and $e_V \in \mathcal{W}(\mathcal{U})$ for all $V \in \mathcal{U}$. Now we define $\theta : H \to K$ by

$$W(x) = \sum_{U \in \mathcal{U}} < x, U\eta > \chi_U.$$

Let P be the orthogonal projection onto $W(H)$. Then W is an isometry and $P\chi_U = W(U\eta)$ as discussed in the proof of Proposition 1.1.

We first prove that $\lambda_U W = WU$ on H for each $U \in \mathcal{U}$. Let $V \in \mathcal{U}$ be arbitrary. Then

$$\begin{aligned}
\lambda_U W(V\eta) &= \lambda_U(\sum_{S\in\mathcal{U}} < V\eta, S\eta > \chi_S) \\
&= \sum_{S\in\mathcal{U}} < V\eta, S\eta > \chi_{US} \\
&= \sum_{S\in\mathcal{U}} < V\eta, U^*S\eta > \chi_S \\
&= \sum_{S\in\mathcal{U}} < UV\eta, S\eta > \chi_S \\
&= WU(V\eta).
\end{aligned}$$

Thus $W^*\lambda_U W = U$ on WH since $\{V\eta : \ V \in \mathcal{U}\}$ generates H.

Secondly, we verify that $P \in \mathcal{G}'$. In fact, for any $U \in \mathcal{U}$, we have

$$P\lambda_U\chi_I = P\chi_U = W(U\eta) = WUW^*W\eta = WUW^*P\chi_I.$$

Since we just verified that $WUW^* = \lambda_U$ on PK, we obtain $P\lambda_U\chi_I = \lambda_U P\chi_I$. Hence $P \in C_{\chi_I}(\mathcal{G})(= \mathcal{G}')$. By identifying H with WH, we complete the proof. $\square$

Suppose that there exists another unitary group $\mathcal{G}_1$ acting on a Hilbert space K_1 satisfying all the requirements of Theorem 3.8. Then $\mathcal{G}_1$ is unitarily equivalent to its left regular representation on $l^2(\mathcal{G}_1)$. So we can assume that $K_1 = l^2(\mathcal{G}_1) = l^2(\mathcal{U})$. Therefore $\mathcal{G}$ and K in Theorem 3.8 are unique in the unitary equivalence sense. For convenience, Theorem 3.8 can be restated as follows:

Theorem 3.8$'$. *Let $\mathcal{G}$ be a group and π be a representation of $\mathcal{G}$ on a Hilbert space H such that $\pi(\mathcal{G})$ has a complete normalized tight frame vector η. Then there exists a representation π' on a Hilbert space K and a complete normalized tight frame vector ξ for $\pi'(\mathcal{G})$ such that $\eta \oplus \xi$ is a complete wandering vector for $(\pi \oplus \pi')(\mathcal{G})$ on $H \oplus K$. Moreover π', K and ξ are unique up to unitary equivalence.*

So if P is the orthogonal projection from $H \oplus K$ onto $H \oplus 0$, then P is in the commutant of $(\pi \oplus \pi')(\mathcal{G})$ and $\eta = P(\eta \oplus \xi)$, $\xi = P(\eta \oplus \xi)$.

Corollary 3.9. *Suppose that $\mathcal{U}$ is a unitary group which has a complete normalized tight frame vector. Then the von Neumann algebra $w^*(\mathcal{U})$ generated by $\mathcal{U}$ is finite.*

Proof. Let $\mathcal{G}$, K and P be as in Theorem 3.8. Then, by Lemma 3.3, $w^*(\mathcal{G})$ is a finite von Neumann algebra. Thus $w^*(\mathcal{U})(= w^*(\mathcal{G}|_{PH})$ is also finite. $\square$

Corollary 3.10. *Let $T \in B(H)$ be a unitary operator and let $\eta \in H$ be a vector such that $\{T^n\eta : \ n \in \mathbb{Z}\}$ is a normalized tight frame for H. Then there is a unique (modulo a null set) measurable set $E \subset \mathbb{T}$ such that $\{T^n\eta : n \in \mathbb{Z}\}$ and $\{e^{ins}|_E : n \in \mathbb{Z}\}$ are unitarily equivalent frames.*

Proof. First note that the powers T^n, $n \in \mathbb{Z}$, are distinct, so H must be infinite dimensional and $\{T^n : \ n \in \mathbb{Z}\}$ is group-isomorphic to $\mathbb{Z}$. Indeed, if $T^k = I$ for some $k \neq 0$, then in the equation

$$||\eta||^2 = \sum_{n\in\mathbb{Z}} | < \eta, T^n\eta > |^2,$$

infinitely many terms in the right hand side reduce to $| < \eta, \ \eta > |^2 \ = \ ||\eta||^4$, contradicting the fact that $\eta \neq 0$.

Let λ be the left regular representation of $\mathbb{Z}$ on $l^2(\mathbb{Z})$. Write $\psi = e_0$. Then, by Theorem 3. 8, there is a projection $P \in l^\infty(\mathbb{Z})$ such that $\{T^n\eta : n \in \mathbb{Z}\}$ and $\{\lambda(n)P\psi : n \in \mathbb{Z}\}$ are unitarily equivalent. Note that $\{\lambda(n) : n \in \mathbb{Z}\}$ and $\{M_{e^{ins}} : n \in \mathbb{Z}\}$ are unitarily equivalent, where M_f denotes the multiplication operator on $L^2(\mathbb{T})$ with symbol f. Let $W : l^2(\mathbb{Z}) \to L^2(\mathbb{T})$ be the unitary operator inducing the equivalence and satisfying $W\psi = 1$. Then $Q = WPW^*$ is a projection in $L^\infty(\mathbb{T})$. Write $Q = \chi_E$ for some $E \subset \mathbb{T}$. Then $\{\lambda(n)P\psi : n \in \mathbb{Z}\}$ and $\{M_{e^{ins}}\chi_E : n \in \mathbb{Z}\}$ are unitarily equivalent, as required. The uniqueness of E follows from Proposition 2.6. $\square$

For a unitary system $\mathcal{U}$ on a Hilbert space H, we recall that a closed subspace M of H is called a *complete wandering subspace* for $\mathcal{U}$ if *span*$\{UM : \ U \in \mathcal{U}\}$ is dense in H, and $UM \perp VM$ when $U \neq V$. Let $\{e_i : \ i \in \mathcal{I}\}$ be an orthonormal basis for M. Then M is a complete wandering subspace for $\mathcal{U}$ if and only if $\{Ue_i : \ U \in \mathcal{U}, i \in \mathcal{I}\}$ is an orthonormal basis for H. We call $\{e_i\}$ a *complete multi-wandering vector*. Analogously, an n-tuple $(\eta_1, \ ..., \ \eta_n)$ of non-zero vectors (here n can be ∞) is called a *complete normalized tight multi-frame vector* for $\mathcal{U}$ if $\{U\eta_i : \ U \in \mathcal{U}, i = 1, ..., n\}$ forms a complete normalized tight frame for H. Equivalently, if

$$||x||^2 = \sum_{i=1}^{n} \sum_{U\in\mathcal{U}} | < x, U\eta_i > |^2$$

for every $x \in H$.

Let $\mathcal{G}$ be a group and let λ be the left regular representation of $\mathcal{G}$. Then $\{\lambda_g \otimes I_n : g \in \mathcal{G}\}$ has a complete multi-wandering vector $(f_1, \ ..., \ f_n)$, where $f_1 = (\chi_e, 0, ..., 0), \ ..., \ f_n = (0, 0, \ ..., \ \chi_e)$. Let P be any projection in the commutant of $(\lambda \otimes I_n)(\mathcal{G})$. Then $(Pf_1, \ ..., \ Pf_n)$ is a complete normalized multi-tight farme vector for the subrepresentation $(\lambda \otimes I_n)|_P$. As in Theorem 3.8, it turns out that every representation with a complete normalized multi-tight frame vector arises in this way.

Theorem 3.11. *Let $\mathcal{G}$ be a countable group and let π be a representation of $\mathcal{G}$ on a Hilbert space H such that $\pi(\mathcal{G})$ has a complete normalized tight multi-frame vector $\{\eta_1, \ ..., \ \eta_n\}$. Then π is unitarily equivalent to a subrepresentation of $\lambda \otimes I_n$*

Proof. (This is similar to the proof of Theorem 3.8. We give it for completeness.) Write $\sigma = \lambda \otimes I_n$. Define $W : H \to l^2(\mathcal{G}) \oplus ... \oplus l^2(\mathcal{G})$ by

$$W(x) = \sum_{i=1}^{n} \sum_{g \in \}} < x, \pi(g)\eta_i > \sigma(g) f_i$$

for all $x \in H$. Then W is an isometry from H to $W(H)$. Let P be the projection from $l^2(\mathcal{G}) \oplus ... \oplus l^2(\mathcal{G})$ onto $W(H)$. It follows from the proof of Proposition 1.1 that $P\sigma(g)f_i = W\pi(g)\eta_i$.

Next we claim that $\sigma(g)W = W\pi(g)$ on H for every $g \in \mathcal{G}$. This follows from the computation

$$\begin{aligned}
\sigma(g)W(x) &= \sigma(g) \sum_{i=1}^{n} \sum_{h \in \mathcal{G}} < x, \ \pi(h)\eta_i > \sigma(h) f_i \\
&= \sum_{i=1}^{n} \sum_{h \in \mathcal{G}} < x, \ \pi(h)\eta_i > \sigma(gh) f_i \\
&= \sum_{i=1}^{n} \sum_{h \in \mathcal{G}} < x, \ \pi(g^{-1}h)\eta_i > \sigma(h) f_i \\
&= \sum_{i=1}^{n} \sum_{h \in \mathcal{G}} < \pi(g)x, \ \pi(h)\eta_i > \sigma(h) f_i \\
&= W\pi(g)x
\end{aligned}$$

for all $x \in H$.

Finally, for any $g, h \in \mathcal{G}$ and any i, the above results imply

$$\begin{aligned}
P\sigma(g)\sigma(h)f_i &= P\sigma(gh)f_i = W\pi(gh)\eta_i \\
&= W\pi(g)\pi(h)\eta_i = \sigma(g)W\pi(h)\eta_i \\
&= \sigma(g)P\sigma_h f_i.
\end{aligned}$$

Hence $P \in \sigma(\mathcal{G})'$ $\square$

We will call two unitary systems $\mathcal{U}$ and $\tilde{\mathcal{U}}$ *isomorphic* if there is a bijection $\sigma : \mathcal{U} \rightarrow \tilde{\mathcal{U}}$ such that $\sigma(UV) = \sigma(U)\sigma(V)$ whenever $U, V, UV \in \mathcal{U}$. Given a unitary system $\mathcal{U}$ on a Hilbert space H. We say that $\mathcal{U}$ has the *dilation property* if for every complete normalized tight frame vector η for $\mathcal{U}$, there exists a Hilbert space K and a unitary system $\tilde{\mathcal{U}}$ on K such that there is a complete wandering vector ψ for $\tilde{\mathcal{U}}$, and there is isomorphism $\sigma : \mathcal{U} \rightarrow \tilde{\mathcal{U}}$ and a projection $P \in C_{\psi}(\tilde{\mathcal{U}})$ with the property that $\{U\eta : U \in \mathcal{U}\}$ and $\{\sigma(U)P\psi : U \in \mathcal{U}\}$ are unitarily equivalent frames. Roughly speaking, $\{U\eta : U \in \mathcal{U}\}$ can be dilated to an orthonormal basis induced by an isomorphic unitary system and a complete wandering vector. We have shown that unitary groups always have this dilation property, and in Chapter 4 we will point out that Gabor type unitary systems (see the definition in Chapter 4) also have this property. In general, we ask

Problem A: Does every unitary system have the dilation property?

Let $\mathcal{U}$ be a unitary system of the form

$$\mathcal{U} = \mathcal{U}_1\mathcal{U}_0 := \{V_1V_0 : \ V_1 \in \mathcal{U}_1, \ V_0 \in \mathcal{U}_0\},$$

where $\mathcal{U}_1$ and $\mathcal{U}_2$ are unitary groups such that $\mathcal{U}_1 \cap \mathcal{U}_0 = \{I\}$. Suppose that $\mathcal{W}(\mathcal{U})$ is not empty. Then for some special complete normalized tight frame vector η, we have the following dilation result, which will be use in section 4. For a vector $x \in H$, we use $[\mathcal{U}_0 x]$ to denote the closure of $span(\mathcal{U}_0 x)$.

Proposition 3.12. *Let $\mathcal{U}$ be as above. Suppose that η is a complete normalized tight frame vector such that $[\mathcal{U}_0\eta]$ is a wandering subspace for $\mathcal{U}_1$. Then there is a vector $\xi \in H$ such that*

$$\{U\eta \oplus U\xi : \ U \in \mathcal{U}\}$$

is an orthonormal basis for $H \oplus [\mathcal{U}\xi]$.

proof. Fix $\psi \in \mathcal{W}(\mathcal{U})$. As before we define $\theta : H \rightarrow H$ by

$$\theta(x) \ = \ \sum_{U \in \mathcal{U}} < x, \ U\eta > U\psi$$

for all $U \in \mathcal{U}$. Then θ is a unitary operator from H onto $\theta(H)$ and $PU\psi = \theta(U\eta)$ for every $U \in \mathcal{U}$, where P is the projection from H onto $\theta(H)$. Let $\tilde{U} = \theta U \theta^*$. We claim that $\tilde{U}\theta(\eta) = U\theta(\eta)$.

In fact, let $U = U_1U_0$ with $U_1 \in \mathcal{U}_1$ and $U_0 \in \mathcal{U}_0$. Note that, by hypothesis, $< U_0\eta,\ V_1U_0\eta >= 0$ when $V_1 \in \mathcal{U}_1$ and $V_1 \neq I$. Thus

$$\begin{aligned}
\tilde{U}_0\theta(\eta) &= \sum_{V_1\in\mathcal{U}_1, V_0\in\mathcal{U}_0} < U_0\eta,\ V_1V_0\eta > V_1V_0\psi \\
&= \sum_{V_0\in\mathcal{U}_0} < U_0\eta,\ V_0\eta > V_0\psi \\
&= U_0 \sum_{V_0\in\mathcal{U}_0} < \eta,\ U_0^{-1}V_0\eta > U_0^{-1}V_0\psi \\
&= U\theta(\eta).
\end{aligned}$$

In the last equality we use the fact that $U_0^{-1}\mathcal{U}_0 = \mathcal{U}_0$. Therefore we have

$$\begin{aligned}
\tilde{U}\theta(\eta) &= \sum_{V_1\in\mathcal{U}_1, V_0\in\mathcal{U}_0} < U_1U_0\eta,\ V_1V_0\eta > V_1V_0\psi \\
&= \sum_{V_0\in\mathcal{U}_0} < U_1U_0\eta,\ U_1V_0\eta > U_1V_0\psi \\
&= U_1 \sum_{V_0\in\mathcal{U}_0} < \eta,\ V_0\eta > V_0\psi \\
&= U_1U_0\theta(\eta) = U\theta(\eta).
\end{aligned}$$

Since $P\psi = \theta(\eta)$, we get $PU\psi = \tilde{U}\theta(\eta) = U\theta(\eta) = UP\psi$, which implies that $P \in C_\psi(\mathcal{U})$. So $P^\perp \in C_\psi(\mathcal{U})$. Let $\xi = P^\perp\eta$. Then

$$\begin{aligned}
\{\tilde{U}\theta(\eta) \oplus U\xi :\ U \in \mathcal{U}\} &= \{UP\psi \oplus UP^\perp\psi :\ U \in \mathcal{U}\} \\
&= \{PU\psi \oplus P^\perp U\psi :\ U \in \mathcal{U}\}.
\end{aligned}$$

So $\{\tilde{U}\theta(\eta) \oplus U\xi :\ U \in \mathcal{U}\}$ is a an orthonormal set. Since

$$\{U\eta \oplus U\xi :\ U \in \mathcal{U}\} = (\theta^* \oplus I)\{\tilde{U}\theta(\eta) \oplus U\xi :\ U \in \mathcal{U}\}(\theta \oplus I),$$

it follows that $\{U\eta \oplus U\xi :\ U \in \mathcal{U}\}$ is an orthonormal set, and thus it is an orthonormal basis for $H \oplus [\mathcal{U}\xi]$ by Proposition 2.5. $\square$

We note that if $\mathcal{U}$ is a unitary group such that $\mathcal{W}(\mathcal{U})$ is not empty, then, by Corollary 3.4, every complete normalized tight frame vector must be a wandering vector. Thus Theorem 3.8 can not be considered as a special case of Proposition 3.12.

3.3 Equivalent Classes of Frame Vectors

For an arbitrary unitary system $\mathcal{U}$ and a complete wandering vector $\psi \in \mathcal{W}(\mathcal{U})$, as we mentioned before that for every complete wandering vector $\phi \in \mathcal{W}(\mathcal{U})$ there is an invertible (in fact unitary) operator $V \in C_\psi(\mathcal{U})$ such that $\phi = V\psi$. The reader would expect that an analogous result when replacing ψ, ϕ by frame vectors should be true. Unfortunately this is no longer true even for the unitary group case. We have the following characterization:

Proposition 3.13. *Let $\mathcal{G}$, π, π', η, ξ, P, H and K be as in Theorem 3.8′. Let $\mathcal{M}$ be the von Neumann algebra generated by $\{\pi(g) \oplus \pi'(g) : g \in \mathcal{G}\}$. Then the following are equivalent*

(i) P is in the center of $\mathcal{M}$, i. e. $P \in \mathcal{M} \cap \mathcal{M}'$.

(ii) For every complete normalized tight frame vector x for $\pi(\mathcal{G})$, there is an (unique) unitary operator $V \in \pi(\mathcal{G})'$ such that $x = V\eta$.

(iii) For every complete frame vector x for $\pi(\mathcal{G})$, there is an (unique) invertible operator $V \in \pi(\mathcal{G})'$ such that $x = V\eta$.

Proof. Let $\psi = \eta \oplus \xi$. For $(i) \Rightarrow (iii)$, by Proposition 3.5 there is an operator $A \in \mathcal{M}'$ such that $A\psi = x$ and $aP \leq AA^* \leq bP$, where a and b are frame bounds for $\{\pi(\mathcal{G})x\}$. Let $V = PAP$. Then $V \in \pi(\mathcal{G})'$ is invertible and $V\eta = PAP\eta = PAP(P\psi) = PA\psi = Px = x$, as required.

Suppose (iii) holds. Then, by Proposition 1.9 (ii), V is unitary if x is a a normalized tight frame vector, and hence (ii) follows.

Now prove $(ii) \Rightarrow (i)$. To show that P is in the center of $\mathcal{M}$, it suffices to prove that $P \in \mathcal{M}$. Let $A \in \mathcal{M}'$ be an arbitrary unitary operator. Then $PA\psi$ is a complete normalized tight frame vector for $\pi(\mathcal{G})$ on H. Thus there is a unitary operator $V \in \pi(\mathcal{U})'$ such that $V\eta = PA\psi$. That is

$$VP\psi = PA\psi.$$

Note that $\pi(\mathcal{G})' = P\mathcal{M}'P$ (see Proposition 1 on page 17 of [Di]). We write $V = PBP$ for some operator $B \in \mathcal{M}'$. Then PBP, $PA \in \mathcal{M}'$ and $PBP\psi = PA\psi$. Hence $PBP = PA$ since ψ separates $\mathcal{M}'$. This implies that $A^*P = PB^*P$ and so that P is an invariant projection for A^*. Since all the unitary operators in $\mathcal{M}'$ generates $\mathcal{M}'$, we have that $P \in \mathcal{M}$, as expected.

The uniqueness of the operator in (ii) and (iii) follows from the fact that η separates $\pi(\mathcal{G})'$. $\square$

Corollary 3.14. *Let $\mathcal{G}$ be an abelian group and let π be a representation of $\mathcal{G}$ on a Hilbert space H. Suppose that η is a complete normalized tight frame vector* *for $\pi(\mathcal{G})$. Then for every complete frame vector ξ , there is an (unique) invertible operator $V \in \pi(\mathcal{G})'$ such that $\xi = V\eta$.*

From Proposition 3.13, one can easily construct a unitary group which does not satisfy (*ii*) or equivalently (*iii*). For instance, let $\mathcal{G}$ be any non-abelian group and let π be the regular left representation on $l^2(\mathcal{G})$. Then $\psi := e_I$ (where e_I is the characteristic function at the singleton point set $\{I\}$) is a complete wandering vector for $\pi(\mathcal{G})$. Let P be a projection in $\pi(\mathcal{G})'$ such that P is not in the von Neumann algebra generated by $\pi(\mathcal{G})$ (the existence of such a P follows from the fact that $\pi(\mathcal{G})$ is non-abelian). Let $\mathcal{U} := \pi(\mathcal{G})|_{ranP}$. Then $\eta := P\psi$ is a complete normalized tight frame vector for $\mathcal{U}$ and $\mathcal{U}$ does not have property (*ii*) and (*iii*) in Proposition 1.9.

Chapter 4

Gabor Type Unitary Systems

The study of Gabor frames was initiated by D. Gabor in 1946 ([Gab]) with a proposed use for communication purpose, and in recent years is has been one of the major subjects in the study of frame theory and wavelet theory (cf. [BW], [Dau1], [DGM]). We recall that if $a, b > 0$ and $g \in L^2(\mathbb{R})$, then we call $\{g_{m,n} : \ m, n \in \mathbb{Z}\}$ a Gabor system associated with g and a, b, where

$$g_{m,n}(\xi) = e^{2\pi imb\xi} g(\xi - na), \quad n, m \in \mathbb{Z}.$$

When $\{g_{m,n} : m, n \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$, then we call $\{g_{m,n} : m, n \in \mathbb{Z}\}$ a *Gabor frame* associated with g and $a, \ b$. If we define operators $U, V \in B(L^2(\mathbb{R}))$ by

$$(Uf)(\xi) = e^{2\pi ib\xi} f(\xi)$$

and

$$(Vf)(\xi) = f(\xi - a)$$

for all $f \in \text{Ł}^2(\mathbb{R})$, then U and V are unitary operators. An elementary computation shows that

$$UV = e^{-2\pi iab} VU.$$

In particular $UV = VU$ if and only if ab is an integer. It is clear that $g_{m,n} = \{U^m V^n g : \ m, n \in \mathbb{Z}\}$. Therefore $\{g_{m,n}\}$ is a Gabor frame if and only if g is a complete frame vector for the unitary system $\{U^m V^n : \ m, n \in \mathbb{Z}\}$. In general, suppose that U and V are unitary operators on a Hilbert space H and satisfy the relation

$$UV = \lambda VU$$

for some unimodular scalar λ, then we call the unitary system

$$\mathcal{U} = \{U^m V^n : \ n, m \in \mathbb{Z}\}$$

a *Gabor type unitary system.* Note that $U^m V^n = \lambda^{mn} V^n U^m$ for all $n, m \in \mathbb{Z}$. In the case $\lambda = e^{2\pi i\theta}$ for some irrational number θ, then $\mathcal{U}$ is called an *irrational rotation unitary system* which is closely related to the important C* - algebra class-irrational rotation C*-algebras, and has been studied in [Han]. For a vector $\eta \in H$, we call $\{U^m V^n \eta : \ m, n \in \mathbb{Z}\}$ a *Gabor type frame* if $\{U^m V^n \eta : \ m, n \in \mathbb{Z}\}$ is a frame for H. Note that if $\lambda \neq 1$, then $\mathcal{U}$ can not be a group. The main purpose of this chapter is to show that Gabor type unitary systems share most of the important properties with unitary group systems. Some of these results have been studied in [Han] for irrational rotation unitary systems. We first prove the following uniqueness result.

Proposition 4.1. *Suppose that* $\{U_1^m V_1^n : \ m, n \in \mathbb{Z}\}$ *and* $\{U_2^m V_2^n : \ m, n \in \mathbb{Z}\}$ *are two Gabor type unitary systems with respect to the same scalar* λ*, and suppose that both of them have complete wandering vectors. Then there is a unitary operator* W *such that* $WU_1W^* = U_2$ *and* $WV_1W^* = V_2$*.*

Proof. Assume that $\{U_1^m V_1^n : \ m, n \in \mathbb{Z}\}$ and $\{U_2^m V_2^n : \ m, n \in \mathbb{Z}\}$ are acting on Hilbert spaces H_1 and H_2 with complete wandering vectors ψ_1 and ψ_2, respectively. Write $\psi_{m,n}^{(i)} = U_i^m V_i^n \psi_i$ for $i = 1, 2$ and $m, n \in \mathbb{Z}$. Then $\{\psi_{m,n}^{(i)} : m, n \in \mathbb{Z}\}$ is an orthonormal basis for H_i. Define $W : H_1 \longrightarrow H_2$ by $W\psi_{m,n}^{(1)} = \psi_{m,n}^{(2)}$ for all n and m. Then W is a unitary operator and we have

$$\begin{aligned} WU_1\psi_{m,n}^{(1)} &= WU_1U_1^m V_1^n \psi_1 \\ &= U_2 U_2^m V_2^n \psi_2 \\ &= U_2 W \psi_{m,n}^{(1)} \end{aligned}$$

and

$$\begin{aligned} WV_1\psi_{m,n}^{(1)} &= WV_1U_1^m V_1^n \psi_1 \\ &= \lambda^{-m} W U_1^m V_1^{n+1} \psi_1 \\ &= \lambda^{-m} U_2^m V_2^{n+1} \psi_2. \\ &= V_2 U_2^m V_2^n \psi_2 \\ &= V_2 W \psi_{m,n}^{(1)} \end{aligned}$$

Thus $WU_1W^* = U_2$ and $WV_1W^* = V_2$ since these relations hold on an orthonormal basis for H_1. □

Remark 4.2. For any unimodular scalar λ, there exists a Gabor type unitary system $\mathcal{U}$ with respect to λ such that $\mathcal{W}(\mathcal{U})$ is non empty. For instance, let H be the Hilbert space $l^2(\mathbb{Z} \times \mathbb{Z})$, and let $e_{n,m}$ be the standard orthonormal basis for H.

Define unitary operators U, V on H by $Ue_{m,n} = e_{m+1,n}$ and $Ve_{m,n} = \lambda^{-m}e_{m,n+1}$. Then $UV = \lambda VU$ follows from

$$\begin{aligned} UVe_{m,n} &= U(\lambda^{-m}e_{m,n+1}) = \lambda^{-m}e_{m+1,n+1} \\ &= \lambda\lambda^{-(m+1)}e_{m+1,n+1} = \lambda VUe_{m,n}. \end{aligned}$$

Thus $\mathcal{U}_{U,V}$ is a Gabor type unitary system. Since

$$U^mV^ne_{0,0} = e_{m,n}$$

for any $n, m \in \mathbb{Z}$, $e_{0,0}$ is a complete wandering vector for $\mathcal{U}$.

Theorem 4.3. *Suppose that $\mathcal{U}$ is a Gabor type unitary system with respect to λ with a complete wandering vector ψ. Then*

(i) $C_\psi(\mathcal{U}) = \mathcal{U}'$,

(ii) the vector ψ is a faithful trace vector for both $w^*(\mathcal{U})$ and $\mathcal{U}'$,

(iii) both $w^*(\mathcal{U})$ and $\mathcal{U}'$ are finite von Neumann algebras,

(iv) $\mathcal{W}(\mathcal{U}) = \mathbb{U}(w^*(\mathcal{U}))\psi$,

(v) $\mathcal{W}(\mathcal{U})$ is a connected, closed subset in H and span$\mathcal{W}(\mathcal{U})$ is dense in H.

Proof. The proof is similar to that of Theorem 1 in [Han]. For completeness we include the proof of (ii) and (iii).

Let $\psi \in \mathcal{W}(\mathcal{U})$ be arbitrary. First we show that $< AB\psi, \psi >=< BA\psi, \psi >$ for all $A, B \in w^*(\mathcal{U})$. It is enough to verify that this holds for $A = U^nV^m, B = U^kV^l$ with $n, m, k, l \in \mathbb{Z}$ since the linear span of $\mathcal{U}$ is an algebra. In fact, this follows from

$$< U^nV^mU^kV^l\psi, \psi > = \lambda^{-mk} < U^{n+k}V^{m+l}\psi, \psi >$$

$$= \begin{cases} 0 & (n+k, m+l) \neq (0,0) \\ \lambda^{-mk} & (n+k, m+l) = (0,0) \end{cases}$$

and

$$\begin{aligned} < U^kV^lU^nV^m\psi, \psi > &= \lambda^{-ln} < U^{k+n}V^{l+m}\psi, \psi > \\ &= \begin{cases} 0 & (n+k, m+l) \neq (0,0) \\ \lambda^{-ln} & (n+k, m+l) = (0,0) \end{cases}. \end{aligned}$$

Thus ψ is a trace vector of $w^*(\mathcal{U})$. Note that ψ is also a cyclic vector for $w^*(\mathcal{U})$ since $\mathcal{U}\psi$ is an orthonormal basis for H. Thus, by Lemma 7.2.14 in [KR], ψ is a joint cyclic trace vector for $w^*(\mathcal{U})$ and $\mathcal{U}'$. By Theorem 7.2.15 in [KR], this implies that both $w^*(\mathcal{U})$ and $\mathcal{U}'$ are finite von Neumann algebras. $\square$

Corollary 4.4. *Suppose that $\mathcal{U}$ is a Gabor type unitary system such that $\mathcal{W}(\mathcal{U})$ is nonempty. Then*

(i) Every complete normalized tight frame vector for $\mathcal{U}$ must be a complete wandering vector.

(ii) Every wandering vector must be complete.

Proof. The statement (i) follows from (i) and (iii) in Theorem 4.3 and the proof of Corollary 3.4. For (ii), let $\psi \in \mathcal{W}(\mathcal{U})$ and let η be a wandering vector. Then $WU^mV^n\psi = U^mV^n\eta$ defines an isometry in $\mathcal{U}'(= C_\psi(\mathcal{U})$. Thus W is a unitary since $\mathcal{U}'$ is finite. Therefore $\eta(= W\psi)$ is a complete wandering vector for $\mathcal{U}$. $\square$

Corollary 4.5. *Suppose that $\mathcal{U}$ is a Gabor type unitary system such that $\mathcal{W}(\mathcal{U})$ is non empty. If $\{U^mV^n\eta : n, m \in \mathbb{Z}\}$ is a frame for H, then it is a Riesz basis for H.*

Proof. Take $\psi \in \mathcal{W}(\mathcal{U})$. By Corollary 3.6, there is an operator $A \in \mathcal{U}'$ such that $\eta = A\psi$. Thus the result follows from Theorem 4.3 (iii) and Corollary 3.7. $\square$

Let $g \in L^2(\mathbb{R})$ and let

$$g_{m,n} = e^{2\pi imb\xi}g(\xi - na)$$

for all $n, m \in \mathbb{Z}$. If $ab = 1$ and $g = \frac{1}{\sqrt{a}}\chi_{[0,a]}$, then $\{g_{m,n}\}$ is an orthonormal basis. Thus from Corollary 4.5, we obtain

Corollary 4.6. *Suppose that $ab = 1$. Then following are equivalent:*

(i) The set $\{g_{m,n}\}$ is a frame for $L^2(\mathbb{R})$.

(ii) The set $\{g_{m,n}\}$ is a Riesz basis for $L^2(\mathbb{R})$.

Corollary 4.7. *(For case $a = b = 1$, see Proposition 2.1, [HW], Page 403) Suppose that $\{g_{m,n}\}$ is a frame for $L^2(\mathbb{R})$. Then the dual frame of $\{g_{m,n}\}$ is also a Gabor frame.*

Proof. By Corollary 4.6, $\{g_{m,n}\}$ is a Riesz basis for $L^2(\mathbb{R})$. Thus, by the Remark following Corollary 3.6, there is an invertible operator A in $\mathcal{U}'$ such that $g = A(\frac{1}{\sqrt{a}}\chi_{[0,a]})$. Hence $\{(A^{-1})^*U^mV^n(\frac{1}{\sqrt{a}}\chi_{[0,a]}\})$ is the dual frame of $\{g_{m,n}\}$. Note that $(A^{-1})^*$ is also in $\mathcal{U}'$. Let $h = (A^{-1})^*(\frac{1}{\sqrt{a}}\chi_{[0,a]})$. Then the dual frame of $\{g_{m,n}\}$ is $\{h_{m,n}\}$, as required. $\square$

Now we prove that the any Gabor type unitary system has the dilation property.

Theorem 4.8. *Let $\mathcal{U}_1(= \{U_1^mV_1^n : m, n \in \mathbb{Z}\})$ be a Gabor type unitary system on a Hilbert space H_1 associated with λ. Then for any complete normalized tight frame* *vector η for $\mathcal{U}_1$, there is a Gabor type unitary system $\mathcal{U}_2(=\{U_2^m V_2^n :\ m,n \in \mathbb{Z}\})$ on a Hilbert space H_2 associated with λ and a normalized tight frame vector ξ for $\mathcal{U}_2$ such that*

$$\{U_1^m V_1^n \eta \oplus U_2^m V_2^n \xi :\ m,n \in \mathbb{Z}\}$$

is an orthonormal basis for $H_1 \oplus H_2$.

Proof. Let $H, \mathcal{U}$ and ψ be as in the remark following Proposition 4.1. Define an operator $W : H_1 \to H$ by

$$Wx = \sum_{m,n\in\mathbb{Z}} < x,\ U_1^m V_1^n \eta > U^m V^n \psi$$

for all $x \in H$. Then W is an isometry from H_1 onto WH_1. Let P be the orthogonal projection from H onto WH_1. We need to show that $P \in \mathcal{U}'$. Since $C_\psi(\mathcal{U}) = \mathcal{U}'$, similar to the proof of Theorem 3.8, it suffices to show that $W^*(U^m V^n)W = U_1^m V_1^n$ on the H_1 for all $m,n \in \mathbb{Z}$. However this follows from the following calculation:

$$\begin{aligned}
U^m V^n (Wx) &= U^m V^n \sum_{k,l\in\mathbb{Z}} < x,\ U_1^k V_1^l \eta > U^k V^l \psi \\
&= \sum_{k,l\in\mathbb{Z}} < x,\ U_1^k V_1^l \eta > U^m V^n U^k V^l \psi \\
&= \sum_{k,l\in\mathbb{Z}} \lambda^{-nk} < x,\ U_1^k V_1^l \eta > U^{k+m} V^{l+n} \psi \\
&= \sum_{k,l\in\mathbb{Z}} \lambda^{-nk} < U_1^m V_1^n x,\ U_1^m V_1^n U_1^k V_1^l \eta > U^{k+m} V^{l+n} \psi \\
&= \sum_{k,l\in\mathbb{Z}} < U_1^m V_1^n x,\ U_1^{k+m} V_1^{l+n} \eta > U^{k+m} V^{l+n} \psi \\
&= W(U_1^m V_1^n x)
\end{aligned}$$

for all $x \in H_1$ and all $m,n \in \mathbb{Z}$. Let $U_2 = P^\perp U P^\perp$, $V_2 = P^\perp V P^\perp$, $H_2 = P^\perp H$ and let $\xi = P^\perp \psi$. Then $\mathcal{U}_2$ and ξ satisfy our requirement. $\square$

The following result tells us that to study a Gabor type unitary system (resp. unitary group system) which has a complete frame vector is equivalent to studying the unitary system which has a complete normalized tight frame vector.

Proposition 4.9. *If a Gabor type unitary system (resp. unitary group system) has a complete frame vector, then it has a complete normalized tight frame vector. In particular, if a Gabor unitary system $\mathcal{U}$ has a vector η such that $\{U^m V^n \eta :\ n,m \in \mathbb{Z}\}$ is a Riesz basis, then $\mathcal{U}$ has a complete wandering vector.*

Proof. Let $\mathcal{U}$ be a Gabor unitary system on a Hilbert space space H. Suppose that it has a complete frame vector η. Then, by Proposition 1.10, there is a positive invertible operator $S \in B(H)$ such that

$$x = \sum_{m,n\in\mathbb{Z}} < x,\ SU^mV^n\eta > U^mV^n\eta$$

for all $x \in H$. Replacing x by $S^{-1}x$, we obtain

$$\begin{aligned} S^{-1}x &= \sum_{m,n\in\mathbb{Z}} < S^{-1}x,\ SU^mV^n\eta > U^mV^n\eta \\ &= \sum_{m,n\in\mathbb{Z}} < x,\ U^mV^n\eta > U^mV^n\eta \end{aligned}$$

since S is positive. The same argument as in the proof of Theorem 4.8 shows that $S^{-1}U^kV^lx = U^kV^lS^{-1}x$ for all $x \in H$. Thus $S^{-1} \in \mathcal{U}'$, and hence $S \in \mathcal{U}'$ since $\mathcal{U}'$ is a von Neumann algebra. Let $A = S^{\frac{1}{2}}$. Then A is also $\mathcal{U}'$. From

$$x = \sum_{m,n\in\mathbb{Z}} < Ax, AU^mV^n\eta > U^mV^n\eta$$

for all $x \in H$, we have (replace x by $A^{-1}x$)

$$A^{-1}x = \sum_{m,n\in\mathbb{Z}} < x, AU^mV^n\eta > U^mV^n\eta.$$

Hence

$$x = \sum_{m,n\in\mathbb{Z}} < x, U^mV^nA\eta > U^mV^nA\eta$$

for all $x \in H$. By Lemma 2.17, $\{U^mV^nA\eta :\ m, n \in \mathbb{Z}\}$ is a normalized tight frame for H, which implies that $A\eta$ is a complete normalized tight frame vector for $\mathcal{U}$, as required. The same argument applies to the unitary group system case.

If in addition $\{U^mV^n\eta :\ m, n \in \mathbb{Z}\}$ is a Riesz basis for H. Let A be as above such that $\{AU^mV^n\eta :\ m, n \in \mathbb{Z}\}$ is a complete normalized frame. Note that it is also a Riesz basis. Thus, by Proposition 1.9 (v), it is an orthonormal basis for H. Therefore $A\eta$ is a complete wandering vector for $\mathcal{U}$. $\square$

From Theorem 4.8 and Proposition 4.9, we immediately obtain

Corollary 4.10. *If a Gabor type unitary system (resp. unitary group system) has a complete frame vector, then both $w^*(\mathcal{U})$ and $\mathcal{U}'$ are finite von Neumann algebras.*

Remarks 4.11. (i) For a Gabor unitary system $\mathcal{U}$ with a complete normalized tight frame vector η, if $\lambda = 1$, then $\mathcal{U}$ is an abelian group. Thus, by Corollary 3.14, for any complete frame vector ξ for $\mathcal{U}$, there is an (unique) invertible operator A in the von Neumann algebra $w^*(\mathcal{U})(= \mathcal{U}')$ such that $\xi = A\eta$. When $\lambda \neq 1$, this is not true in general. For instance, let $\mathcal{U}$ and ψ be as in the Remark following Proposition 4.1. Choose a projection $P \in \mathcal{U}'$ but $P \notin w^*(\mathcal{U})$ (this can be done since $w^*(\mathcal{U})$ is not an abelian von Neumann algebra in this case). Let $\mathcal{U}_1 = P\mathcal{U}P$ on the Hilbert space PH. Then $P\psi$ is a complete normalized tight frame vector for $\mathcal{U}_1$. But the same argument as in the proof of Proposition 3.13 shows that there exists complete normalized tight frames which can not be expressed as the form of $A\eta$ for some $A \in \mathcal{U}'$. However, as in the group case (Theorem 6.17 in Chapter 6, we can show that the set of all the complete normalized tight frame vectors for $\mathcal{U}$ is equal to the set $\{A\eta : A \in \mathbb{U}(w^*(\mathcal{U}))$, where $\mathbb{U}(w^*(\mathcal{U}))$ denotes the set of all the unitary operators in $w^*(\mathcal{U})$.

(ii) For a Gabor system $\{g_{m,n}\}$ associated with a, b, if $ab > 1$, it was proven by I. Daubechies and M. Rieffel (cf [Dau1], [Dau2], [Dau3], [Ri]) that the linear span of $\{g_{m,n}\}$ can not be dense in $L^2(\mathbb{R})$ for all $g \in L^2(\mathbb{R})$. Thus $\{g_{m,n}\}$ can not be a frame, whatever the choice of g. In the case $ab \leq 1$, there exists g such that $\{g_{m,n}\}$ is a complete normalized frame. For example $9 = \sqrt{b}\chi_{[0,a]}$ generates a complete normalized tight frame. It is also easy to see that $\{g_{m,n}\}$ can be an orthonormal basis only when $ab = 1$.

(iii) We use U_b and V_a to denote the multiplication unitary operator by $e^{2\pi bs}$ and the translation operator by a, respectively. If $ab = c$, then the unitary systems $\{U_b^m V_a^n : \ m, n \in \mathbb{Z}\}$ and $\{U_1^m V_c^n : \ m, n \in \mathbb{Z}\}$ are unitarily equivalent by the unitary operator $W \in B(L^2(\mathbb{R}))$ defined by

$$(Wf)(s) = \frac{1}{\sqrt{a}} f(\frac{1}{a}s), \quad f \in L^2(\mathbb{R}).$$

Thus ab determines a unique class of concrete Gabor unitary systems.

(iv) If $ab < 1$, it is possible to construct two frame vectors f and g such that $\{f_{m,n}\}$ and $\{g_{m,n}\}$ are strongly disjoint frames. For example when $ab \leq \frac{1}{2}$. Let $f = \sqrt{b}\chi_{[0,a]}$ and $g = \sqrt{b}\chi_{[a,2a]}$. Then $\{f_{m,n}\}$ and $\{g_{m,n}\}$ are complete normalized tight frames for $L^2(\mathbb{R})$. To see that they are strongly disjoint, suppose $ka \leq \frac{1}{b} < (k+1)a$. For simplicity, assume that $k = 2$ and let $h = \sqrt{b}\chi_{[2a,\frac{1}{b}]}$. Then

$$\{f_{m,n} \oplus g_{m,n} \oplus h_{m,n} : \ m, n \in \mathbb{Z}\}$$

is an orthonormal set. Note that $\{h_{m,n}\}$ is also a normalized tight frame for $M := \overline{span}\{h_{m,n} : m, n \in \mathbb{Z}\}$. Thus, by the proof of Proposition 2.5, we have that

$$\{f_{m,n} \oplus g_{m,n} \oplus h_{m,n} : \ m, n \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})\oplus L^2(\mathbb{R})\oplus M$. Hence $\{f_{m,n}\oplus g_{m,n}\}$ is a normalized tight frame for $L^2(\mathbb{R})\oplus L^2(\mathbb{R})$, as required. When $ab=\frac{1}{2}$, $\{f_{m,n}\}$ and $\{g_{m,n}\}$ are complementary normalized tight frames. Even in the $ab\leq\frac{1}{2}$ case, we don't know if it is always possible that for an arbitrary complete normalized tight frame $\{f_{m,n}\}$, there exists another complete normalized tight frame $\{g_{m,n}\}$ for $L^2(\mathbb{R})$ such that $\{f_{m,n}\oplus g_{m,n}\}$ is an orthonormal basis for $L^2(\mathbb{R})\oplus L^2(\mathbb{R})$. The same question remains for the $\frac{1}{2}<ab<1$ case.

Chapter 5

Frame Wavelets, Super-wavelets and Frame Sets

In this chapter we study the frame wavelets. For simplicity, we only consider the one-dimensional dyadic wavelet system case. Practically, however, the reader will note that by use of the appropriate abstract techniques of Chapter 3 many aspects of the material we present in this Chapter carry over to $\mathbb{R}^n$ theory.

We recall that (cf [HW]) that a function $\psi \in L^2(\mathbb{R})$ is an *orthonormal wavelet* (resp. *frame wavelet*) if $\{\psi_{j,k} : \ j,k \in \mathbb{Z}\}$ is an orthonormal basis (resp. frame) for $L^2(\mathbb{R})$, where

$$\psi_{j,k}(s) \ = \ 2^{\frac{j}{2}}\psi(2^j s - k)$$

for all $j,k \in \mathbb{Z}$. Let T and D be the translation and dilation unitary operators, respectively, on $L^2(\mathbb{R})$ defined by $(Tf)(t) = f(t-1)$ and $(Df)(t) = \sqrt{2}f(2t)$. Then ψ is a wavelet if and only if $\{D^jT^k\psi : j,k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. The Fourier transform, $\hat{f}$, of $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is defined by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(s)e^{-is\xi}ds.$$

This transformation can be uniquely extended to a unitary operator $\mathcal{F}$ on $L^2(\mathbb{R})$. We write $\hat{D} =: \mathcal{F}D\mathcal{F}^{-1}$ and $\hat{T} =: \mathcal{F}T\mathcal{F}^{-1}$. For convenience, if A and B are unitary operators on a Hilbert space H, we will use $\mathcal{U}_{A,B}$ to denote the set $\{A^nB^m;\ n,m \in \mathbb{Z}\}$. A function is called a *tight frame wavelet* (resp. *normalized tight frame wavelet*) if it is a complete tight frame vector (resp. complete normalized tight frame vector) for $\mathcal{U}_{D,T}$ in $L^2(\mathbb{R})$. The following is Proposition 2.1 for the frame wavelet case.

Proposition 5.1. *Let ψ be a fixed wavelet. Then $f \in L^2(\mathbb{R})$ is a normalized tight frame wavelet if and only if there is a co-isometry $A \in C_\psi(\mathcal{U}_{D,T})$ such that $f = A\psi$.*

5.1 Frame Sets

In [HWW1], [HWW2], [FW], the Minimally-Supported-Frequencies wavelets are extensively studied. These are the wavelets whose Fourier transforms have minimal support of measure 2π. X. Dai and D. Larson in [DL] independently studied the same class of wavelets and introduced the concept of wavelet set. Recall from

[DL] that a measurable set $E \subset \mathbb{R}$ is called a *wavelet set* if $\frac{1}{\sqrt{2\pi}}\chi_E$, where χ_E is the characteristic function of E, is the Fourier transform of a wavelet. The corresponding wavelet is called an *s-elementary wavelet.* It is quite easy to check that E is a wavelet set if and only if it is the support set of the Fourier transform of an MSF wavelet.

Two measurable sets E and F are *translation congruent modulo* 2π if there exists a measurable bijection $\phi : E \to F$ such that $\phi(s) - s$ is an integral multiple of 2π for each $s \in E$. Analogously, two measurable sets G and H are *dilation congruent modulo* 2 if there exists a measurable bijection $\tau : G \to H$ such that for any $s \in G$ there is $n \in \mathbb{Z}$ satisfying $\tau(s) = 2^n S$. Lemma 5.3 in [DL] tells us that a measurable set E is a wavelet set if and only if E is both a 2-dilation generator of a partitionn (modulo null sets) of $\mathbb{R}$ and a 2π-translation generator of a partitionn (modulo null sets) of $\mathbb{R}$ in the sense that both $\{E + 2k\pi : \ k \in \mathbb{Z}\}$ and $\{2^n E : \ n \in \mathbb{Z}\}$ form partitionns of $\mathbb{R}$ modulo null sets. Equivalently, E is a wavelet set if and only if E is both translation congruent to $[0, 2\pi)$ modulo 2π and dilation congruent to $[-2\pi, -\pi) \cup [\pi, 2\pi)$ modulo 2. We note in passing that $\frac{1}{\sqrt{2\pi}}\chi_{[-2\pi,-\pi)\cup[\pi,2\pi)}$ is the Fourier transform of the Shannon wavelet, and $[-2\pi, -\pi) \cup [\pi, 2\pi)$ is the simplest wavelet set.

Definition 5.2. *A measurable subset E of $\mathbb{R}$ is called a frame set if $\frac{1}{\sqrt{2\pi}}\chi_E$ is a complete normalized tight frame vector for $\mathcal{U}_{\hat{D},\hat{T}}$, where χ_E is the characteristic function of E.*

We will give a complete characterization for frame sets.

Lemma 5.3. *Let $f \in L^2(\mathbb{R})$ and $E = supp(f)$. Then the following are equivalent:*

(1) $\{\hat{T}^n f : n \in \mathbb{Z}\}$ is a normalized tight frame for $L^2(E)$,

(2) $|f| = \frac{1}{\sqrt{2\pi}}\chi_E$, and E is 2π-translation congruent to a subset F of $[0, 2\pi]$.

Proof. For (2) $\Rightarrow$ (1), write $f(t) = \theta(t)|f(t)|$ such that θ is a unimodular measurable function. Let $K = E \cup ([0, 2\pi]\backslash F)$. Then K is 2π-translation congruent to $[0, 2\pi]$. Thus

$$\{\hat{T}^n \theta(t)\frac{1}{\sqrt{2\pi}}\chi_K(t) : n \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(K)$.

Let P be the orthogonal projection from $L^2(K)$ to $L^2(E)$. Then

$$P(e^{int}\theta(t)\frac{1}{\sqrt{2\pi}}\chi_K(t)) = e^{int}\theta(t)\frac{1}{\sqrt{2\pi}}\chi_E(t).$$

Therefore $\{e^{int}\theta(t)\frac{1}{\sqrt{2\pi}}\chi_E(t) :\in \mathbb{Z}\}$ is a normalized tight frame for $L^2(E)$.

For (1) $\Rightarrow$ (2), we first show that E is 2π-translation congruent to a subset F of $[0, 2\pi]$.

Suppose that E is not 2π-translation congruent to any subset of $[0, 2\pi]$. Then there exist a subset F of E and some integer $k \in \mathbb{Z}$ with the property that $F \cap (F + 2k\pi)$ is empty, $F + 2k\pi \subset E$ and f is bounded on $F \cup (F + 2k\pi)$. Write $G = F \cup (F + 2k\pi)$. Then $\chi_G f \in L^2(E)$. Thus

$$\chi_G f^2 = \sum_{n \in \mathbb{Z}} < \chi_G f^2, e^{int} f > e^{int} f$$

and

$$||\chi_G f^2||^2 = \sum_{n \in \mathbb{Z}} | < \chi_G f^2, e^{int} f > |^2.$$

Let $g = \sum_{n \in \mathbb{Z}} < \chi_G f^2, e^{int} f > e^{int}$ defined on $[0, 2\pi]$ and extend it to $\mathbb{R}$ by 2π-periodical property. Then

$$\chi_G(t) f^2(t) = g(t) f(t), \qquad a.e. \;\; t \in \mathbb{R}.$$

This implies that $\chi_G f = g$ a.e. on E. Therefore $f(t + 2\pi) = f(s)$ on F. Let $h = \chi_F - \chi_{F+2k\pi}$. Then $h \in L^2(E)$ and $h \neq 0$ and

$$< h, e^{int} f >=< \chi_F(t), f^{int} f(t) > - < \chi_{E+2k\pi}(t), e^{int} f >= 0,$$

which contradicts the fact that $\{e^{ist} f \; : \; n \in \mathbb{Z}\}$ is a normalized tight frame for $L^2(E)$. So we conclude that E is 2π-translation congruent to some subset F of $[0, 2\pi]$.

Now we show that $|f| = \frac{1}{\sqrt{2\pi}} \chi_E$. Let $\Omega = E \cup ([0, 2\pi] \backslash F)$ and $\psi = \frac{1}{\sqrt{2\pi}} \chi_\Omega$. Then Ω is 2π-translation congruent to $[0, 2\pi]$. Thus

$$\{\hat{T}^n \psi : n \in \mathbb{Z}\}$$

is an orthormal basis for $L^2(\Omega)$. Therefore, by Proposition 3.1, there is a partial isometry $V \in C_\psi(\{\hat{T}^n|_{L^2(\Omega)} : \; n \in \mathbb{Z}\})$ such that $f = V\psi$ and $VL^2(\Omega) = L^2(E)$. Let $\mathcal{A}$ be the von Neumann algebra generated by $\{\hat{T}^n|_{L^2(\Omega)} : \; n \in \mathbb{Z}\}$. Then $\mathcal{A}$ is abelian with a cyclic vector ψ. Thus

$$C_\psi(\{\hat{T}^n|_{L^2(\Omega)} : \; n \in \mathbb{Z}\}) = \mathcal{A}' = \mathcal{A} = L^\infty(\Omega).$$

So we can write $V = M_h$ for some function $h \in L^\infty(\Omega)$, where $M_h g = hg$ for all $g \in L^2(\Omega)$. Since $M_h M_h^* = M_{|h|^2}$ is a projection and $M_h M_h^* L^2(\Omega) = L^2(E)$, we have that $|h|^2 = \chi_E$. Write $h(t) = \theta(t)\chi_E(t)$ for some unimodular function θ. Then we have

$$f(t) = V\psi(t) = h(t)\psi(t) = \frac{1}{\sqrt{2\pi}} \theta(t) \chi_E(t).$$

Hence $|f| = \frac{1}{\sqrt{2\pi}} \chi_E$ as required. $\square$

Theorem 5.4. *Let E be a measurable subset of $\mathbb{R}$. Then E is a frame set if and only if E is both 2π-translation congruent to a subset F of $[0, 2\pi]$ and 2-dilation congruent to $[-2\pi, \pi] \cup [\pi, 2\pi]$.*

Proof. Suppose that E is 2π-translation congruent to a subset F of $[0, 2\pi]$ and 2-dilation congruent to $[-2\pi, \pi] \cup [\pi, 2\pi]$. Then, by Lemma 5.3, $\frac{1}{\sqrt{2\pi}}\chi_E$ is a complete normalized tight frame vector for $\{\hat{T}^n|_{L^2(E)} : \ n \in \mathbb{Z}\}$. The condition that E is 2-dilation congruent to $[-2\pi, \pi] \cup [\pi, 2\pi]$ implies that $L^2(E)$ is a complete wandering subspace for $\{\hat{D}^n : \ n \in \mathbb{Z}\}$. Thus $\frac{1}{\sqrt{2\pi}}\chi_E$ is a complete normalized tight frame vector for $\mathcal{U}_{\hat{D},\hat{T}}$. So E is a frame set.

Conversely, suppose that E is a frame set and let $\psi = \frac{1}{\sqrt{2\pi}}\chi_E$. Then

$$\cup_{n\in\mathbb{Z}} 2^n E = \mathbb{R}.$$

Then there exists a measurable subset K of E such that K is 2-dilation congruent to $F := [-2\pi, -\pi] \cup [\pi, 2\pi]$. In fact, let $E_n = E \cap 2^n F, \ n \in \mathbb{Z}$. For any set $A \subset R$, write $\tilde{A} = \cup_{n\in\mathbb{Z}} 2^n A$. Let

$$K_0 = E_0, \qquad K_1 = E_1 \backslash \tilde{K_0}, \qquad K_{-1} = E_{-1} \backslash (\tilde{K_0} \cup \tilde{K_1}),$$
$$K_2 = E_2 \backslash (\tilde{K_0} \cup \tilde{K_1} \cup \tilde{K_{-1}}), \quad \ldots \ etc.$$

Then $K = \cup_{n\in\mathbb{Z}} K_n$ will satisfy our requirement. We claim that $K = E$ (modulo a null set). Let $K^+ = \{x \in K : x > 0\}$ and $K^- = \{x \in K : x < 0\}$. Then

$$\int_{K^+} \frac{1}{t} dt = \int_{[\pi,2\pi]} \frac{1}{t} dt = ln2 \qquad (a)$$

and

$$\int_{K^-} \frac{1}{t} dt = \int_{[-2\pi,-\pi]} \frac{1}{t} dt = ln2 \qquad (b)$$

However, from Theorem 5.3.1 in [Dau2], we know that

$$\int_0^\infty \frac{|\psi(t)|^2}{t} dt = \frac{ln2}{2\pi} \qquad (c)$$

and

$$\int_{-\infty}^o \frac{|\psi(t)|^2}{|t|} dt = \frac{ln2}{2\pi} \qquad (d)$$

Combining (a), (b) with (c), (d), we have

$$\mu(E^+ \backslash K^+) = 0, \quad \mu(E^- \backslash K^-) = 0,$$

where μ is Lebesgue measure. Hence $K = K^+ \cup K^- = E^+ \cup E^- = E$. So E is 2-dilation congruent to a subset of $[-2\pi, -\pi] \cup [\pi, 2\pi]$. Therefore $\{\hat{T}^n \psi : n \in \mathbb{Z}\}$ generates $L^2(E)$. This implies that ψ is a complete frame vector for $\{\hat{T}^n|_{L^2(E)} : n \in \mathbb{Z}\}$. So, by Lemma 5.3, E is 2π-translation congruent to a subset of $[0, 2\pi]$. $\square$

5.2 Super-wavelets

In this section we discuss super-wavelets and the disjointness of frame wavelets.

Definition 5.5. *Suppose that $\eta_1,\ ..., \eta_n$ are normalized tight frame wavelets. We will call the n-tuple $(\eta_1,\ ..., \eta_n)$ a super-wavelet of length n if $\{D^k T^l \eta_1 \oplus ... \oplus D^k T^l \eta_n : k, l \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R}) \oplus ... \oplus L^2(\mathbb{R})$. If E and F are frame sets, then E and F are called strongly disjoint if $\{\hat{D}^k \hat{T}^l \frac{1}{\sqrt{2\pi}} \chi_E : k, l \in \mathbb{Z}\}$ and $\{\hat{D}^k \hat{T}^l \frac{1}{\sqrt{2\pi}} \chi_F : k, l \in \mathbb{Z}\}$ are strongly disjoint. We call (E, F) a strong complementary pair if $(\mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}} \chi_E),\ \mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}} \chi_F))$ is a super-wavelet.*

As we mentioned in the last remark of Chapter 2, the concept of super-wavelet might have applications in signal processing, data compression and image analysis. The prefix "super-" is used because they are orthonormal basis generators for a "super-space" of $L^2(\mathbb{R})$, namely the direct sum of finitely many copies of $L^2(\mathbb{R})$. We first prove the existence of super-wavelets of any length. These can be viewed as vector valued wavelets of a special type. We need the following Lemma which is a special case of Theorem 1 in [DLS].

Lemma 5.6. *Let E and F be bounded measurable sets in $\mathbb{R}$ such that E contains a neighborhood of 0, and F has nonempty interior and is bounded away from θ. Then there is a measurable set $G \subset \mathbb{R}$, which is 2-dilation congruent to F and 2π-translation congruent to E.*

For $n \geq 2$, let

$$E_1 = [-\pi, -\frac{1}{2}\pi) \cup [\frac{1}{2}\pi, \pi), \quad E_2 = [-\frac{1}{2}\pi, -\frac{1}{4}\pi) \cup [\frac{1}{4}\pi, \frac{1}{2}\pi),$$

$$\cdots\cdots$$

$$E_{n-1} = [-\frac{1}{2^{n-2}}\pi,\ -\frac{1}{2^{n-1}}\pi) \cup [\frac{1}{2^{n-1}}\pi,\ \frac{1}{2^{n-2}}\pi).$$

Then, by Theorem 5.4, E_i is a frame set for $i = 1, 2, ..., n-1$. Let

$$E = [-\frac{1}{2^{n-1}}\pi,\ \frac{1}{2^{n-1}}\pi)$$

and $F = [-\pi, -\frac{1}{2}\pi) \cup [\frac{1}{2}\pi, \pi)$. Then, from Lemma 5.6, there exists a measurable set G such that G is 2-dilation congruent to F and 2π-translation congruent to E, and hence G is a frame set. We claim that $(\eta_1,\ ...\ ,\ \eta_n)$ is a super-wavelet for $\mathcal{U}_{\hat{D},\hat{T}}$, where $\eta_i = \frac{1}{\sqrt{2\pi}}\chi_{E_i}$ for $i = 1,\ ...,\ n-1$ and $\eta_n = \frac{1}{\sqrt{2\pi}}\chi_G$. Since $< \hat{D}^k\hat{T}^l\psi_i,\ \hat{T}^j\psi_i >= 0$ for all $j, l \in \mathbb{Z}$ and all $n \neq 0$ $(i = 1,\ ...,\ n)$. We only need to check the orthonormality of $\{\hat{T}^l\eta_1 \oplus ... \oplus \hat{T}^l\eta_n : k, l \in \mathbb{Z}\}$. In fact, the orthonormality follows from the following equality immediately,

$$
\begin{aligned}
< \hat{T}^l\psi_1 \oplus ... \oplus \hat{T}^l\psi_n,\ \psi_1 \oplus ... \oplus \psi_n >&= \sum_{k=1}^{n} < e^{ils}\psi_k,\ \psi_k > \\
&= \frac{1}{2\pi}\sum_{k=1}^{n-1}\int_{E_k} e^{ils}\chi_{E_k}(s)ds + \frac{1}{2\pi}\int_G e^{ils}\chi_G(s)ds \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ils}ds.
\end{aligned}
$$

For $n = \infty$, we let $E_k = E_{k-1} = [-\frac{1}{2^{k-2}}\pi,\ -\frac{1}{2^{k-1}}\pi) \cup [\frac{1}{2^{k-1}}\pi,\ \frac{1}{2^{k-2}}\pi)$ for all k, and let $\psi_k = \frac{1}{\sqrt{2\pi}}\chi_{E_k}$. Then, the similar argument shows that $(\psi_1,\ \psi_2,\ ...\ )$ is a super-wavelet. Thus we have

Proposition 5.7. *For any n (n can be ∞), there is a super-wavelet of length n.*

Example C. Let $E = [-\pi, -\frac{1}{2}\pi) \cup [\frac{1}{2}\pi, \pi)$. The argument before Proposition 5.7 gives us the existence of the *strong complement frame set* of E. Now we construct a concrete one. Consider a set of type $[a, \frac{\pi}{2}) \cup [2\pi,\ a + 2\pi)$. This set is a 2-dilation generator of a partitionn $[0, \infty)$ if $\frac{1}{4}(a + 2\pi) = 2a$. So we get $a = \frac{2\pi}{7}$. Thus $[\frac{2\pi}{7},\ \frac{\pi}{2}) \cup [2\pi,\ \frac{16\pi}{7})$ is a 2-dilation generator of a partitionn of $[0,\ \infty)$. Symmetrically, $[-\frac{16\pi}{7},\ -2\pi) \cup [-\frac{\pi}{2},\ -\frac{2\pi}{7})$ is a 2-dilation generator of a partitionn of $(-\infty,\ 0]$. Write

$$
\begin{aligned}
A &= [-\frac{16\pi}{7},\ -2\pi), \quad B = [-\frac{\pi}{2},\ -\frac{2\pi}{7}), \\
C &= [\frac{2\pi}{7},\ \frac{\pi}{2}), \quad C = [2\pi,\ \frac{16\pi}{7}),
\end{aligned}
$$

and let $L = A \cup B \cup C \cup D$. Then

$$
(A + 2\pi) \cup B \cup C \cup (D - 2\pi) = [-\frac{\pi}{2},\ \frac{\pi}{2})
$$

and

$$
\frac{1}{4}A \cup B \cup C \cup \frac{1}{4}D = [-\frac{4\pi}{7},\ -\frac{2\pi}{7}) \cup [\frac{2\pi}{7},\ \frac{4\pi}{7}).
$$

Thus L is a frame set which is a strong complement of E. Similarily one can verify that for $a = \frac{2n\pi}{8n-1}$,

$$
[-(2n\pi + a),\ -2n\pi) \cup [-\frac{\pi}{2},\ -a) \cup [a,\ \frac{\pi}{2}) \cup [2n\pi,\ 2n\pi + a)
$$

is also a strong complementary frame set of E.

Let E be a frame set. Then E is 2π-translation congruent to a subset, denoted by $\tau(E)$, of $[0,\ 2\pi)$.

Proposition 5.8. *Let E and F be frame sets. Then*

(i) E and F are strongly disjoint if and only if $\tau(E) \cap \tau(F)$ has measure zero.

(ii) (E, F) is a strong complementary pair if and only if both $\tau(E) \cup \tau(F) = [0,\ 2\pi)$ and $\tau(E) \cap \tau(F)$ has measure zero.

(iii) $\{\hat{D}^n \hat{T}^m (\frac{1}{\sqrt{2\pi}} \chi_E) : n, m \in \mathbb{Z}\}$ and $\{\hat{D}^n \hat{T}^m (\frac{1}{\sqrt{2\pi}} \chi_F) : n, m \in \mathbb{Z}\}$ are unitarily equivalent if and only if $\tau(E) = \tau(F)$.

Proof. We write $\psi = \frac{1}{\sqrt{2\pi}} \chi_E$, $\eta = \frac{1}{\sqrt{2\pi}} \chi_F$ and $G = \tau(E) \cap \tau(F)$.

(i) Assume that E and F are strongly disjoint. Then for any function $f \in L^2(\mathbb{R})$, by Corollary 2.10,

$$\sum_{k,l} < f, \hat{D}^k \hat{T}^l \psi > \hat{D}^k \hat{T}^l \eta = 0.$$

Let $E_1 \subset E$ and $F_1 \subset F$ such that $E_1 \sim_\tau G$ and $F_1 \sim_\tau G$. Then

$$\sum_{k,l} < \chi_{E_1}, \hat{D}^k \hat{T}^l \psi > \hat{D}^k \hat{T}^l \eta = 0.$$

Hence

$$\sum_{k,l} < \chi_{E_1}, \hat{D}^k \hat{T}^l \psi >< \hat{D}^k \hat{T}^l \eta,\ \chi_{F_1} >= 0.$$

Note that, by Theorem 5.4, $< \chi_{E_1}, \hat{D}^k \hat{T}^l \psi >= 0$ and $< \hat{D}^k \hat{T}^l \eta,\ \chi_{F_1} >= 0$ when $k \neq 0$. Thus

$$\begin{aligned}
0 &= \sum_{k,l} < \chi_{E_1}, \hat{D}^k \hat{T}^l \psi >< \hat{D}^k \hat{T}^l \eta,\ \chi_{F_1} > \\
&= \sum_{l} < \chi_{E_1},\ \hat{T}^l \psi >< \hat{T}^l \eta,\ \chi_{F_1} > \\
&= \frac{1}{2\pi} \sum_{l} \int_{E_1} e^{ils} ds \cdot \int_{F_1} e^{-ils} ds \\
&= \sum_{L} \int_{G} e^{ils} ds \cdot \int_{G} e^{-ils} ds \\
&= \sum_{l} | \int_{G} e^{ils} ds |^2.
\end{aligned}$$

Hence G has measure zero.

Conversely assume that G has measure zero. Then $E \cup F$ is 2π-translation congruent to $\tau(E) \cup \tau(F)$, which is a subset of $[0,\ 2\pi)$. This also implies that $E \cap F$ has measure zero. Let $f \in L^2(E)$ and $g \in L^2(F)$. Then, by Theorem 5.4, we have

$$< f \oplus g,\ \hat{D}^k \hat{T} \psi \oplus \hat{D}^k \hat{T} \eta >= 0$$

if $k \neq 0$. Let $h \in L^2(E \cup F)$ be defined by

$$h(s) = \begin{cases} f(s),\ s \in E \\ g(s),\ s \in F \end{cases}$$

Then, by Lemma 5.3,

$$||h||^2 = \sum_{l \in \mathbb{Z}} |< h,\ \hat{T}^l(\psi + \eta) > |^2.$$

However

$$\begin{aligned} |< h,\ \hat{T}^l(\psi + \eta) > |^2 &= \frac{1}{2\pi} | \int_{E \cup F} h(s) e^{-ils} ds|^2 \\ &= \frac{1}{2\pi} | \int_E f(s) e^{-ils} ds + \int_F g(s) e^{-ils} ds|^2 \\ &= |< f \oplus g,\ \hat{T}^l \psi \oplus \hat{T}^l \eta > |^2. \end{aligned}$$

Thus we have

$$\begin{aligned} ||f \oplus g||^2 &= \sum_{l \in \mathbb{Z}} |< f \oplus g,\ \hat{T}^l \psi \oplus \hat{T}^l \eta > |^2 \\ &= \sum_{k,l \in \mathbb{Z}} |< f \oplus g,\ \hat{D}^k \hat{T}^l \psi \oplus \hat{D}^k \hat{T}^l \eta > |^2. \end{aligned}$$

Now let $g, f \in L^2(\mathbb{R})$ be arbitrary. Since E and F are frame sets, we have decomposition $f = \oplus_{n \in \mathbb{Z}} f_n$ and $g = \oplus_{n \in \mathbb{Z}} g_n$ with $f_n \in \hat{D}^n L^2(E)$ and $g_n \in \hat{D}^n L^2(F)$ for all n. Thus

$$\begin{aligned} ||f \oplus g||^2 &= ||f||^2 + ||g||^2 \\ &= \sum_{n \in \mathbb{Z}} ||f_n||^2 + \sum_{n \in \mathbb{Z}} ||g_n||^2 \\ &= \sum_{n \in \mathbb{Z}} (||f_n||^2 + ||g_n||^2) \\ &= \sum_{n \in \mathbb{Z}} (\sum_{k,l \in \mathbb{Z}} |< f_n \oplus g_n,\ \hat{D}^k \hat{T}^l \psi \oplus \hat{D}^k \hat{T}^l \eta > |^2) \\ &= \sum_{k,l \in \mathbb{Z}} |< f_k \oplus g_k,\ \hat{D}^k \hat{T}^l \psi \oplus \hat{D}^k \hat{T}^l \eta > |^2. \end{aligned}$$

On the other hand, we have

$$\sum_{k,l\in\mathbb{Z}} |< f \oplus g,\ \hat{D}^k\hat{T}^l\psi \oplus \hat{D}^k\hat{T}^l\eta >|^2$$
$$= \sum_{k,l\in\mathbb{Z}} (|\sum_{n\in\mathbb{Z}} < f_n,\ \hat{D}^k\hat{T}^l\psi > + \sum_{n\in\mathbb{Z}} < g_n,\ \hat{D}^k\hat{T}^l\eta >|^2)$$
$$= \sum_{k,l\in\mathbb{Z}} |< f_k,\ \hat{D}^k\hat{T}^l\psi > + < g_n,\ \hat{D}^k\hat{T}^l\eta >|^2$$

Thus
$$||f \oplus g||^2 = \sum_{k,l\in\mathbb{Z}} |< f \oplus g,\ \hat{D}^k\hat{T}^l\psi \oplus \hat{D}^k\hat{T}^l\eta >|^2$$
for all $f, g \in L^2(\mathbb{R})$. Therefore E and F are strongly disjoint.

(ii) Suppose that (E, F) is a strong complementary pair. Then, by (i), $\tau(E) \cap \tau(F)$ has measure zero. Since $\{\hat{D}^k\hat{T}^l\psi \oplus \hat{D}^k\hat{T}^l\eta :\ k, l \in \mathbb{Z}\}$ is an orthonormal basis, $||\psi \oplus \eta|| = 1$. This implies that

$$2\pi = \int_E ds + \int_F ds = \int_{\tau(E)} ds + \int_{\tau(F)} ds = \mu(\tau(E) \cup \tau(F)).$$

Hence $\tau(E) \cup \tau(F) = [0,\ 2\pi)$, as required.

Conversely if $\{\tau(E),\ \tau(F)\}$ is a partitionn of $[0,\ 2\pi)$, then $||\psi \oplus \eta|| = 1$. Thus $||\hat{D}^k\hat{T}^l\psi \oplus \hat{D}^k\hat{T}^l\eta|| = 1$ for all $k, l \in \mathbb{Z}$. However, by (i), $\{\hat{D}^k\hat{T}^l\psi \oplus \hat{D}^k\hat{T}^l\eta :\ k, l \in \mathbb{Z}\}$ is a normalized tight frame. Thus it is an orthonormal basis. So (E, F) is a strong complementary pair.

(iii) First assume that $\{\hat{D}^n\hat{T}^m(\frac{1}{\sqrt{2\pi}}\chi_E) : n, m \in \mathbb{Z}\}$ and $\{\hat{D}^n\hat{T}^m(\frac{1}{\sqrt{2\pi}}\chi_F) : n, m \in \mathbb{Z}\}$ are unitarily equivalent. Then there is a unitary operator $W \in B(L^2(\mathbb{R}))$ such that
$$W\hat{D}^n\hat{T}^m(\frac{1}{\sqrt{2\pi}}\chi_E) \ = \ \hat{D}^n\hat{T}^m(\frac{1}{\sqrt{2\pi}}\chi_F)$$
for all $n, m \in \mathbb{Z}$. In particular, we have

$$W\hat{T}^m(\frac{1}{\sqrt{2\pi}}\chi_E) \ = \ \hat{T}^m(\frac{1}{\sqrt{2\pi}}\chi_F)$$

for all $n, m \in \mathbb{Z}$. Hence $\{\hat{T}^m(\frac{1}{\sqrt{2\pi}}\chi_E) :\ m \in \mathbb{Z}\}$ and $\{\hat{T}^m(\frac{1}{\sqrt{2\pi}}\chi_F) :\ m \in \mathbb{Z}\}$ are unitarily equivalent. Note that $\{\hat{T}^m(\frac{1}{\sqrt{2\pi}}\chi_E)\}_{m\in\mathbb{Z}}$ and $\{\hat{T}^m(\frac{1}{\sqrt{2\pi}}\chi_{\tau(E)})\}_{m\in\mathbb{Z}}$ are unitarily equivalent, and so are $\{\hat{T}^m(\frac{1}{\sqrt{2\pi}}\chi_F)\}_{m\in\mathbb{Z}}$ and $\{\hat{T}^m(\frac{1}{\sqrt{2\pi}}\chi_{\tau(F)}\}_{m\in\mathbb{Z}}$. Thus $\{\hat{T}^m(\frac{1}{\sqrt{2\pi}}\chi_{\tau(E)}) :\ m \in \mathbb{Z}\}$ and $\{\hat{T}^m(\frac{1}{\sqrt{2\pi}}\chi_{\tau(F)}) :\ m \in \mathbb{Z}\}$ are unitarily equivalent. By Corollary 3.10, we have that $\tau(E) = \tau(F)$.

Now suppose that $\tau(E) = \tau(F)$. Then

$$\{\hat{T}^m(\frac{1}{\sqrt{2\pi}}\chi_E) : \ m \in \mathbb{Z}\} \quad and \quad \{\hat{T}^m(\frac{1}{\sqrt{2\pi}}\chi_F) : \ m \in \mathbb{Z}\}$$

are unitarily equivalent. Let

$$W : L^2(E) \to L^2(F)$$

be the unitary opeartor inducing the unitary equivalence. Since $D^n L^2(E) \perp D^m L^2(E)$ and $D^n L^2(F) \perp D^m L^2(F)$ when $n \neq m$, we can extend W to a unitary operator in $B(L^2(\mathbb{R}))$, which induces a unitary equivalence between

$$\{\hat{D}^n\hat{T}^m(\frac{1}{\sqrt{2\pi}}\chi_E) : n, m \in \mathbb{Z}\} \quad and \quad \{\hat{D}^n\hat{T}^m(\frac{1}{\sqrt{2\pi}}\chi_F) : n, m \in \mathbb{Z}\}.$$

□

Proposition 5.8 can be extended to the n-tuple frame sets $(E_1, \ E_2, \ \ldots, E_n)$ case in an obvious way. It is known that (cf [HKLS]) each two-interval wavelet set has the form:

$$[2a - 4\pi, \ a - 2\pi] \cup [a, \ 2a],$$

where $0 < a < 2\pi$. These are special two-interval frame sets. The other two-interval frame sets can be characterized as following:

Proposition 5.9. *Suppose that E is a two-interval set which is not a wavelet set. Then*

(i) E is a frame set if and only if it has the form $[-2a, \ -a] \cup [b, \ 2b]$ with the property that $a, \ b > 0$ and $a + b \leq \pi$.

(ii) If F is another two-interval frame set, then E and F are unitarily equivalent if and only if $E = F$.

Proof. (i) Suppose that E is a two-interval frame set. Then, by Theorem 5.4, there exist $a, \ b > 0$ such that $E = [-2a, \ -a] \cup [b, \ 2b]$. Because E is not a wavelet set, we have $a + b < 2\pi$. We first claim that $a, \ b < \pi$. Assume, to the contrary, that $a \geq \pi$. Then $-2\pi \in [-2\pi, \ -a] \subset [-2a, \ -a]$ and $[-2\pi, \ -a] + 2\pi = [0, 2\pi - a]$. Since $b < 2\pi - a$, we get $[b, \ \pi] \cap [b, \ 2\pi - a]$ has non-zero measure, which contradicts the assumption that E is 2π-translation congruent to $[-\pi, \ \pi]$. Hence $a \leq \pi$ and similarily $b < \pi$. Now we show that $a + b \leq \pi$. Assume that $\pi > a \geq \pi/2$. Note that $[-2a, \ -\pi] + 2\pi = [2(\pi - a), \ \pi]$, $b \leq \pi$ and $[-2a, \ -a] \cup [b, \ 2b]$ is 2π-translation congruent to a subset of $[-\pi, \ \pi]$. We have $2b \leq 2(\pi - a)$. Hence $a + b < leq\pi$. Similarly $a + b \leq \pi$ when $b \geq \pi/2$.

Conversely, suppose that $a + b \leq \pi$. Then either $a \leq \pi/2$ or $b \leq \pi/2$. So we can assume that $a \leq \pi/2$. This implies that $[-2a,\ -a] \subset [-\pi, 0]$ If $b \leq \pi/2$. Then, by Theorem 5.4, $[-2a,\ -a] \cup [b,\ 2b]$ is frame set, as required. If $b > \pi/2$, then $(\pi,\ 2b] - 2\pi] = [-\pi,\ -(2\pi - b)]$ does not intersect with $(-2a,\ -a]$ since $a + b \leq \pi$. Hence E is 2π-translation congruent to a subset of $[-\pi, \pi]$. And so E is a frame set.

(ii) Suppose that $F = [-2c,\ -c] \cup [d,\ 2d]$ is a two-interval frame set such that E and F are 2π-translation congruent.

If both $a \geq \pi/2$ and $c \geq \pi/2$, then $[-\pi,\ -a] = [-\pi,\ -c]$. Hence $a = c$. Note that $a + b = c + d$. we get $b = d$. Then other cases are similar. $\square$

Example D. By Lemma 5.6, there is a frame set E which is 2π-translation congruent to $[-\pi, \frac{1}{2}\pi)$. We claim that there is no frame set F such that $(E,\ F)$ is a strong complementary pair. In fact assume, to the contrary, that there is a frame set F with the property that $(E,\ F)$ is a strong complementary pair. Then, by Proposition 5.8, $\tau(F) = [\frac{\pi}{2},\ \pi)$. Let $F_+ = F \cap [0,\ \infty)$ and $F_- = F \cap (-\infty,\ 0]$. Then F_+ is 2-dilation congruent $[\frac{\pi}{2}, \pi)$. Thus

$$\int_{F_+} \frac{1}{x} dx = \int_{\frac{\pi}{2}}^{\pi} \frac{1}{x} dx = ln2.$$

Since F_+ is 2π-translation congruent to a proper subset of $[\frac{\pi}{2}, \pi)$, we must have $F_+ \subset [\frac{\pi}{2}, \infty)$. Let $G = F_+ \cap [\frac{\pi}{2}, \pi)$ and $K = F_+ \cap [\pi, \infty)$. Then K is 2π-translation congruent to a *proper* subset, say L , of $[\frac{\pi}{2},\ \pi) \backslash G$ since, otherwise, F_+ is 2π-translation congruent to $[\frac{\pi}{2}, \pi)$, which contradicts the assumption that F_- has positive measure. Hence

$$\begin{aligned}
\int_{F_+} \frac{1}{x} dx &= \int_G \frac{1}{x} dx + \int_K \frac{1}{x} dx \\
&\leq \int_G \frac{1}{x} dx + \int_L \frac{1}{x} dx \\
&< \int_{F_+} \frac{1}{x} dx = ln2
\end{aligned}$$

Therefore F can not be a frame set. We do not know whether there is a normalized tight frame function η such that $(\frac{1}{\sqrt{2\pi}} \chi_E, \eta)$ is a super-wavelet. Thus we ask:

Problem B: Let η_1 be a complete frame vector for $\mathcal{U}_{D,T}$ which is not a wavelet. Is there a super-wavelet $(\eta_1, \eta_2, ...\eta_n)$ for some n ? for all $n \geq 2$?

However from Proposition 3.12, we have

Proposition 5.10. *Let f be a normalized tight frame wavelet such that for all $j, l, n, m \in \mathbb{Z}$, $D^n T^l f$ and $D^m T^j f$ are orthogonal when $n \neq m$. Then there exits a function $g \in L^2(\mathbb{R})$ so that*

$$\{D^n T^m f \oplus D^n T^m g : \ n, m \in \mathbb{Z}\}$$

is an orthonormal set.

If we let $\hat{f} = \frac{1}{\sqrt{2\pi}}\chi_E$ for some frame set E. Then the condition in Proposition 5.10 is always satisfied. In this case we can choose $\hat{g} = \frac{1}{\sqrt{2\pi}}\chi_F$ for some measurable set F. In fact, by Theorem 5.4, E is 2π-translation congruent to a subset G of $[-2\pi, \ -\pi) \cup [\pi, \ 2\pi)$. Let $F = ([-2\pi, \ -\pi) \cup [\pi, \ 2\pi))\backslash G$. Then F will satisfy our requirement. In view of Problem B and Proposition 5.10, we ask

Problem C: Suppose that $\{D^n T^m f : \ n, m \in \mathbb{Z}\}$ is a normalized tight frame for $[\mathcal{U}_{D,T} f]$ $(f \neq 0)$. Is there a normalized tight frame wavelet g such that $\{D^n T^m f : n, m \in \mathbb{Z}\}$ and $\{D^n T^m g : \ n, m \in \mathbb{Z}\}$ are unitarily equivalent? In other words, is there a unitary transformation W from the Hilbert space $[\mathcal{U}_{D,t} f]$ onto the Hilbert space $L^2(\mathbb{R})$ such that $W D^n T^m f = D^n T^m W f$ for all $n, m \in \mathbb{Z}$?

From Proposition 2.19, the sum of a finite number of strongly disjoint normalized tight frame wavelets is a complete tight frame vector. For example, let $E = [-\pi, -\frac{1}{4}\pi) \cup [\frac{1}{4}\pi, \ \pi)$. Then $\frac{1}{\sqrt{2\pi}}\chi_E$ is the Fourier transform of a complete tight frame vector with frame bound 2. the set E is not a frame set according to our definition because the frame bound is not 1. This shows that sets exist whose normalized characteristic functions are Fourier transforms of complete tight frame vectors with frame bounds different than 1. This raises a number of problems.

Problem D: Characterize all the measurable sets E for which $\{\hat{D}^n \hat{T}^m \frac{1}{\sqrt{2\pi}}\chi_E : n, m \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$. (Let us call these *general* frame sets. If the frame is tight call the set a *general tight frame set.* According to our Definition 5.2, when the frame is a normalized tight frame we call the set simply a *frame set.* We feel that these are the most important ones. Theorem 5.4 is a characterization of these. A characterization of the *general frame sets* along these lines seems elusive, however.)

Two subproblems are the following:

Problem $\mathbf{D}_1$: Characterize all general tight frame sets.

Problem $\mathbf{D}_2$: Characterize all numbers A, B for which there exists a general frame set with frame bounds A and B.

5.3 A Characterization of Super-wavelets

In [HW], E. Hernadez and G. Weiss characterized all the normalized tight frame wavelets in terms of two simple equations.

Theorem 5.11. *(Theorem 1.6, [HW]) A function $\psi \in L^2(\mathbb{R})$ is a normalized tight frame wavelet if and only if*

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j s)|^2 = \frac{1}{2\pi}, \quad for \quad a.e.\ s \in \mathbb{R}$$

and

$$\sum_{j=0}^{\infty} \hat{\psi}(2^j s)\overline{\hat{\psi}(2^j(s+2m\pi))} = 0 \quad for\ a.e.\ s \in \mathbb{R},\ m \in 2\mathbb{Z}+1.$$

In order to characterize all the super-wavelets, we need the following lemma.

Lemma 5.12. *Let $\psi_1,\ \psi_2,\ \ldots,\ \psi_m \in L^2(\mathbb{R})$. Then*

$$\{\hat{D}^n\hat{T}^l\psi_1 \oplus \hat{D}^n\hat{T}^l\psi_2 \oplus ... \oplus \hat{D}^n\hat{T}^l\psi_m :\ n, l \in \mathbb{Z}\}$$

is an orthonormal set if and only if

$$\sum_{k\in\mathbb{Z}}\sum_{i=1}^{m} |\psi_i(s+2k\pi)|^2 = \frac{1}{2\pi}, \quad a.e.\ s \in \mathbb{R}$$

and

$$\sum_{k\in\mathbb{Z}}\sum_{i=1}^{m} \psi_i(2^j(s+2k\pi))\overline{\psi_i(s+2k\pi)} = 0, \quad a.e.\ s \in \mathbb{R}, \quad j \geq 1.$$

Proof. For simplicity we only check the $m = 2$ case. Since

$$\begin{aligned} < \hat{T}^l\psi_1 \oplus \hat{T}^l\psi_2,\ \psi_1 \oplus \psi_2 > &= \int_{\mathbb{R}} e^{ils}|\psi_1(s)|^2 ds + \int_{\mathbb{R}} e^{ils}|\psi_2(s)|^2 ds \\ &= \int_0^{2\pi} e^{ils} \sum_{k\in\mathbb{Z}} (|\psi_1(s+2k\pi)|^2 + |\psi_2(s+2k\pi)|^2), \end{aligned}$$

it follows that $\{\hat{T}^l\psi_1 \oplus \hat{T}^l\psi_2 :\ l \in \mathbb{Z}\}$ is an orthonormal set if and only if

$$\sum_{k\in\mathbb{Z}} (|\psi_1(s+2k\pi)|^2 + |\psi_2(s+2k\pi)|^2) = \frac{1}{2\pi}, \quad a.e.\ s \in \mathbb{R}.$$

If we note that $T^l D^n = D^n T^{2^n l}$ when $n \geq 0$, it is easy to see that the orthogonality between $\hat{D}^n\hat{T}^l\psi_1 \oplus \hat{D}^n\hat{T}^l\psi_1$ and $\hat{D}^j\hat{T}^k\psi_1 \oplus \hat{D}^j\hat{T}^k\psi_1$ for $j > n$ and $k, l \in \mathbb{Z}$, can be reduced to the orthogonality between $\hat{D}^j\hat{T}^k\psi_1 \oplus \hat{D}^j\hat{T}^k\psi_1$ and $\psi_1 \oplus \psi_2$ for $j > 0$ and $k \in \mathbb{Z}$. Let $j > 0$ and $k \in \mathbb{Z}$. Then

$$
\begin{aligned}
< \psi_1 \oplus \Psi_2,\, \hat{D}^j\hat{T}^k\psi_1 \oplus \hat{D}^j\hat{T}^k\psi_2 > &= < \hat{D}^{-j}\psi_1 \oplus \hat{D}^{-j}\psi_2, \hat{T}^k\psi_1 \oplus \hat{T}^k\psi_2 > \\
&= \int_{\mathbb{R}} 2^{j/2}\overline{\psi_1(s)}\psi_1(2^j s)e^{iks}ds + \int_{\mathbb{R}} 2^{j/2}\overline{\psi_2}\psi_2(2^j s)e^{iks}ds \\
&= 2^j \int_0^{2\pi} \sum_{l\in\mathbb{Z}}(\sum_{m=1}^{2} \psi_m(2^j(s+2l\pi)\overline{\psi_m(s+2l\pi)})e^{iks}ds.
\end{aligned}
$$

Thus the orthogonality of between $\hat{D}^j\hat{T}^k\psi_1 \oplus \hat{D}^j\hat{T}^k\psi_1$ and $\psi_1 \oplus \psi_2$ for $j > 0$ and $k \in \mathbb{Z}$ is equivalent to the condition

$$
\sum_{k\in\mathbb{Z}}\sum_{m=1}^{2} \psi_m(2^j(s+2k\pi))\overline{\psi_m(s+2k\pi)} = 0, \quad a.e.\ s \in \mathbb{R}, \quad j \geq 1.
$$

Thus the lemma follows. $\square$

Theorem 5.13. *Let $\psi_1, \ . \ . \ . \ , \ \psi_m \in L^2(\mathbb{R})$. Then $(\psi_1, \ . \ . \ . \ , \ \psi_m)$ is a super-wavelet if and only if the following equations hold*

(1) $\sum_{j\in\mathbb{Z}} |\hat{\psi}_i(2^j s)|^2 = \frac{1}{2\pi}, \quad for \quad a.e.\ s \in \mathbb{R},\ i = 1,\ .\ .\ .\ ,\ m,$

(2) $\sum_{j=0}^{\infty} \hat{\psi}_i(2^j s)\overline{\hat{\psi}_i(2^j(s+2k\pi))} = 0 \quad for\ a.e.\ s \in \mathbb{R},\ k \in 2\mathbb{Z}+1,\ i = 1,\ .\ .\ .\ ,\ m,$

(3) $\sum_{k\in\mathbb{Z}}\sum_{i=1}^{m} |\hat{\psi}_i(s+2k\pi)|^2 = \frac{1}{2\pi}, \quad a.e.\ s \in \mathbb{R},$

(4) $\sum_{k\in\mathbb{Z}}\sum_{i=1}^{m} \hat{\psi}_i(2^j(s+2k\pi))\overline{\hat{\psi}_i(s+2k\pi)} = 0, \quad a.e.\ s \in \mathbb{R}, \quad j \geq 1.$

Proof. The necessity follows from Theorem 5.11 and Lemma 5.12. Suppose that $(1) - - - (4)$ hold. Then, by Theorem 5.11 and Lemma 5.12, we get that

$$
\{D^n D^l\psi_1 \oplus D^n T^l\psi_2 \oplus ... \oplus D^n T^l\psi_m :\ n, l \in \mathbb{Z}\}
$$

is an orthonormal set, and for each i, $\{D^n T^l\psi_i :\ \ k, l \in \mathbb{Z}\}$ is a normalized tight frame for $L^2(\mathbb{R})$. Thus, by Proposition 2.5, we have that

$$
span\{D^n D^l\psi_1 \oplus D^n T^l\psi_2 \oplus ... \oplus D^n T^l\psi_m :\ n, l \in \mathbb{Z}\}
$$

is dense in $L^2(\mathbb{R}) \oplus ... \oplus L^2(\mathbb{R})$. Hence $(\psi_1,\ .\ .\ .\ , \psi_m)$ is a super-wavelet, as required. $\square$

5.4 Some Frazier-Jawerth Frames

M. Frazier and B. Jawerth studied (cf. [FJ]) the following frame wavelets:

Let $\psi \in L^2(\mathbb{R})$ be such that $supp(\hat{\psi})$ is contained in $\{s \in \mathbb{R} : \frac{1}{2} \leq |s| \leq 2\}$ and

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j s)|^2 = \frac{1}{2\pi} \quad for\ all\ s \neq 0.$$

Then the second equation in Theorem 5.11 is automatically satisfied. Thus ψ is a normalized tight frame wavelet. We call this type of frame wavelets the Frazier-Jawerth type frame wavelets.

We can use Theorem 5.13 to construct super-wavelets of length n $(n \geq 3)$starting from a Frazier-Jawerth frame wavelet. Fix a Frazier-Jawerth type frame wavelet ψ_1. We define $\psi_2 \in L^2(\mathbb{R})$ by

$$\hat{\psi}_2(s) = \begin{cases} \hat{\psi}_1(2s) & s \in [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1] \\ \beta(s)\hat{\psi}_1(\frac{s}{2}) & s \in [-2, -1) \cup (1, 2] \\ 0 & elsewhere \end{cases}$$

where $|\beta(s)| = 1$ and

$$\beta(s) = -\frac{\hat{\psi}_1(s)\overline{\hat{\psi}_1(2s)}}{\overline{\hat{\psi}_1(s)}\hat{\psi}_1(2s)}$$

whenever $\hat{\psi}_1(s)\overline{\hat{\psi}_1(2s)} \neq 0$.

For every $s \neq 0$. Let $k \in \mathbb{Z}$ such that $2^k s \in [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$. Then

$$\begin{aligned} \sum_{j\in\mathbb{Z}} |\hat{\psi}_2(2^j s)|^2 &= |\hat{\psi}_2(2^k s)|^2 + |\hat{\psi}_2(2^{k+1} s)|^2 \\ &= |\hat{\psi}(2^{k+1} s)|^2 + |\beta(2^{k+1} s)\hat{\psi}_1(2^k s)|^2 \\ &= |\hat{\psi}(2^{k+1} s)|^2 + |\hat{\psi}_1(2^k s)|^2 \\ &= \frac{1}{2\pi}. \end{aligned}$$

Since $supp(\hat{\psi}_2)$ is also contained in $\{s \in \mathbb{R} : \frac{1}{2} \leq |s| \leq 2\}$, we get from Theorem 5.11 that ψ_2 is a Frazier-Jawerth type frame wavelet. We claim that ψ_1 and ψ_2 are strongly disjoint frame wavelets. This can be deduced from the following more general result:

Proposition 5.14. *Let ψ be a fixed Frazier-Jawerth type frame wavelet. Then for each $m \geq 3$, there exist $\psi_2, ...\psi_m \in L^2(\mathbb{R})$ such that $(\psi_1, \ . \ . \ . \ , \ \psi_m)$ is a super-wavelet. Moreover ψ_2 can be chosen as a Frazier-Jawerth type frame wavelet.*

Proof. Let ψ_2 be defined as above. Then it is a Frazier-Jawerth type frame wavelet. Let

$$E_3 = (-\frac{1}{2}, \ -\frac{1}{4}] \cup [\frac{1}{4}, \ \frac{1}{2}), \quad . \ . \ . \ , \ E_{m-1} = (-\frac{1}{2^{m-2}}, \ -\frac{1}{2^{m-1}}] \cup [\frac{1}{2^{m-1}}, \ \frac{1}{2^{m-2}}).$$

Then, by Theorem 5.4, $E_3, \ . \ . \ . \ , \ E_{m-1}$ are frame sets. By Lemma 5.6, there is measurable set, say E_m, which is 2-dilation congruent to $[-2, \ -1) \cup [1, \ 2)$ and 2π-translation conguent to the complement of $\{s: \ \frac{1}{2^{m-1}} \leq |s| \leq 2\}$ in $[-\pi, \ \pi]$. So, again by Theorem 5.4, E_m is also a frame set. Now let $\hat{\psi}_j = \frac{1}{\sqrt{2\pi}} \chi_{E_j}$ for $3 \leq j \leq m$. We claim that $(\psi_1, \ . \ . \ . \ , \ \psi_m)$ is a super-wavelet. For simplicity, we only check the case $m = 3$. Since ψ_j is frame wavelet for each j, it sufficies to check the equations (3) and (4) in Theorem 5.13.

Let $s \neq 0$ be arbitary and let $G = \{s: \ \frac{1}{2} \leq |s| \leq 2\}$. Note that $\cup_{k\in\mathbb{Z}}(G + 2k\pi)$ and $\cup_{k\in\mathbb{Z}}(E_3 + 2k\pi)$ are disjoint sets, and their union is $\mathbb{R} \setminus \{0\}$. First suppose that $s = t + 2l\pi$ for some $l \in \mathbb{Z}$ and some $t \in G$. Then $s + 2k\pi \notin E_3$ for all $k \in \mathbb{Z}$. Hence

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \sum_{i=1}^{3} |\hat{\psi}_i(s + 2k\pi)|^2 &= \sum_{k\in\mathbb{Z}} \sum_{i=1}^{2} |\hat{\psi}_i(s + 2k\pi)|^2 \\
&= |\hat{\psi}_1(t + 2l\pi)|^2 + |\hat{\psi}_2(t + 2l\pi)|^2 \\
&= \begin{cases} |\hat{\psi}_1(t + 2l\pi)|^2 + |\hat{\psi}_1(2(t + 2l\pi))|^2, & \frac{1}{2} \leq |t + 2l\pi| \leq 1 \\ |\hat{\psi}_1(t + 2l\pi)|^2 + |\hat{\psi}_1(\frac{1}{2}(t + 2l\pi))|^2, & 1 < |t + 2l\pi| \leq 2 \end{cases} \\
&= \frac{1}{2\pi}
\end{aligned}$$

When $s + 2l\pi \in E_3$ for some $l \in \mathbb{Z}$, then $s + 2k\pi \notin G$ for all $k \in \mathbb{Z}$. Hence

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \sum_{i=1}^{3} |\hat{\psi}_i(s + 2k\pi)|^2 &= \sum_{k\in\mathbb{Z}} |\hat{\psi}_3(s + 2k\pi)|^2 \\
&= |\hat{\psi}_3(s + 2l\pi)|^2 = \frac{1}{2\pi}.
\end{aligned}$$

Thus (3) in Theorem 5.13 holds.

To check (4), since E_3 is a frame set we always have

$$\sum_{k\in\mathbb{Z}} \hat{\psi}_3(2^j(s + 2k\pi))\overline{\hat{\psi}_3(s + 2k\pi)} = 0, \ a.e. \ s \in \mathbb{R}, \ j \geq 1.$$

Thus it suffices to check

$$\sum_{k\in\mathbb{Z}} \sum_{i=1}^{2} \hat{\psi}_i(2^j(s + 2k\pi))\overline{\hat{\psi}_i(s + 2k\pi)} = 0, \ a.e. \ s \in \mathbb{R}, \ j \geq 1.$$

Let $s \in \mathbb{R}$. If either $S \in \cup_{k\in\mathbb{Z}}(E_3 + 2k\pi)$ or $s + 2l\pi \in \{s : 1 < |s| \leq 2\}$ for some $l \in \mathbb{Z}$, then

$$\hat{\psi}_i(2^j(s + 2k\pi))\overline{\hat{\psi}_i(s + 2k\pi)} = 0$$

for all $j \geq 1$ and all $k \in \mathbb{Z}$. Hence (4) holds in this case.

If $s+2l\pi \in \{s : \frac{1}{2} \le |s| \le 1\}$ for some (unique) $l \in \mathbb{Z}$, then

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\sum_{i=1}^{2}\hat{\psi}_i(2^j(s+2k\pi))\overline{\hat{\psi}_i(s+2k\pi)} \\
&\qquad = \hat{\psi}_1(2(s+2l\pi))\overline{\hat{\psi}_1(s+2l\pi)} + \hat{\psi}_2(2(s+2l\pi))\overline{\hat{\psi}_2(s+2l\pi)} \\
&\qquad = \hat{\psi}_1(2(s+2l\pi))\overline{\hat{\psi}_1(s+2l\pi)} + \beta(s+2l\pi)\hat{\psi}_1(s+2l\pi)\overline{\hat{\psi}_1(2(s+2l\pi))} \\
&\qquad = 0
\end{aligned}$$

by the definition of $\beta(s)$. The proof is complete. $\square$

Note that if $\hat{\psi}_1$ is continuous, then $\hat{\psi}_2$ constructed above is not continuous. However we can find another normalized tight frame wavelet η which has the required regularity. Define $\eta_1 \in L^2(\mathbb{R})$ such that $supp(\hat{\eta_1})$ is contained in $\{s : \frac{1}{8} \le |s| \le \frac{1}{2}\}$ and

$$\sum_{j\in\mathbb{Z}} |\hat{\eta}_1(2^j s)|^2 = 1, \quad s \neq 0.$$

Then η is a normalized tight frame wavelet. Let ψ_2 be as in Proposition 5.14 and define η_2 in a similar way. Using Lemma 5.6, we can find a frame set E which is 2π-translation congruent to $[-\pi,\ \pi] \setminus \{s : \frac{1}{8} \le |s| \le 2\}$. Let $\hat{\psi}_3 = \frac{1}{\sqrt{2\pi}}\chi_E$. Then, by a similar argument as in the proof of Proposition 5.14, $\{\psi_1,\ \psi_2,\ \eta_1,\ \eta_2,\ \psi_3\}$ is a super-wavelet of length 5. Hence ψ_1 and η_1 are strongly disjoint normalized tight frame wavelets. We can choose ψ_1 and η_2 with any required regularity. For instance, let ν be a C^k or C^∞ function such that

$$\nu(s) = \begin{cases} 0, & s \ge 1 \\ 1, & s \le 0 \end{cases}$$

Then define ψ_1 and η_1 by

$$\hat{\psi}_1(s) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{is/2} sin[\frac{\pi}{2}\nu(2|s|-1)], & 1/2 \le |s| \le 1 \\ \frac{1}{\sqrt{2\pi}} e^{is/2} cos\frac{\pi}{2}\nu(|s|-1)], & 1 \le |s| \le 2 \\ 0, & otherwise \end{cases}$$

and

$$\hat{\eta}_1(s) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{is/2} sin[\frac{\pi}{2}\nu(8|s|-1)], & 1/8 \le |s| \le 1/4 \\ \frac{1}{\sqrt{2\pi}} e^{is/2} cos[\frac{\pi}{2}\nu(4|s|-1)], & 1/4 \le |s| \le 1/2 \\ 0, & otherwise \end{cases}$$

It is not hard to check that ψ_1 and η_1 are normalized tight frame wavelets with the same regularity as ν.

For some of the Frazier-Jawerth frame wavelets, they can be extended to a length-2 super-wavelets. For instance, let $\hat{\eta}(s) = \frac{1}{\sqrt{2\pi}}\chi_{[-2,-1]\cup[1,2]}(s)$. Then Lemma 5.6 implies that η can be extended to a length-2 super-wavelet. We ask the following question which is a subproblem of Problem B.

Problem $\mathbf{B}_1$: Can we extend every Frazier-Jawerth frame wavelet to a length-2 super-wavelet?

The following proposition characterizes all the unitary equivalent classes for the Frazier-Jawerth frame wavelets. A function f on $\mathbb{R}$ is called 2-dilation periodic if $f(2s) = f(s)$ for *a.e.* $s \in \mathbb{R}$.

Proposition 5.15. *Let ψ_1 and ψ_2 be Frazier-Jawerth frame wavelets. Then ψ_1 and ψ_2 are unitarily equivalent if and only if $\hat{\psi}_2(s) = \alpha(s)\hat{\psi}_1(s)$ for some unimodular 2-dilation periodic function α.*

Proof. Note that ψ_1 and ψ_2 are unitarily equivalent if and only if

$$< \hat{D}^n\hat{T}^l\hat{\psi}_1,\ \hat{T}^j\hat{\psi}_1 > = < \hat{D}^n\hat{T}^l\hat{\psi}_2,\ \hat{T}^j\hat{\psi}_2 >$$

for all $l, j \in \mathbb{Z}$ and all $n \geq 0$.

If α is a unimodule 2-dilation periodic function, then M_α is a unitary operator in the commutant of $\{\hat{D},\ \hat{T}\}$. Thus

$$\begin{aligned} < \hat{D}^n\hat{T}^l\hat{\psi}_2,\ \hat{T}^j\hat{\psi}_2 > &= < M_\alpha\hat{D}^n\hat{T}^l\hat{\psi}_1,\ M_\alpha\hat{T}^j\hat{\psi}_1 > \\ &= < \hat{D}^n\hat{T}^l\hat{\psi}_1,\ \hat{T}^j\hat{\psi}_1 >, \end{aligned}$$

which implies that ψ_1 and ψ_2 are unitarily equivalent.

Conversely, suppose that ψ_1 and ψ_2 are unitarily equivalent. Then from

$$< \hat{T}^l\hat{\psi}_1,\ \hat{\psi}_1 > = < \hat{T}^l\hat{\psi}_2,\ \hat{\psi}_2 >$$

for all $l \in \mathbb{Z}$, we have

$$\int_{\frac{1}{2}\leq|s|\leq 2} e^{ils}(|\hat{\psi}_1(s)|^2 - |\hat{\psi}_2(s)|^2)ds = 0, \quad l \in \mathbb{Z}.$$

Hence $|\hat{\psi}_1(s)| = |\hat{\psi}_2(s)|$ for *a.e.* $s \in \mathbb{R}$. Similarly from

$$< \hat{T}^l\hat{\psi}_1,\ \hat{D}^{-l}\hat{\psi}_1 > = < \hat{T}^l\hat{\psi}_2,\ \hat{D}^{-l}\hat{\psi}_2 >$$

for all $l \in \mathbb{Z}$, we get

$$\hat{\psi}_1(s)\overline{\hat{\psi}_1(2s)} = \hat{\psi}_2(s)\overline{\hat{\psi}_2(2s)}, \quad \frac{1}{2} \leq |s| < 1.$$

Define $\alpha(s)$ on $\{s : \frac{1}{2} \leq |s| < 1\}$ by

$$\alpha(s) = \begin{cases} \overline{\hat{\psi}_2(2s)}/\overline{\hat{\psi}_1(2s)}, & \hat{\psi}_1(2s) \neq 0 \\ \overline{\hat{\psi}_2(s)}/\overline{\hat{\psi}_1(s)}, & \hat{\psi}_1(2s) = 0 \end{cases}$$

Then extend α to $\mathbb{R} \setminus \{0\}$ by 2-dilation periodic property. Clearly when $s \in (-1,\ -\frac{1}{2}] \cup [\frac{1}{2},\ 1)$, $\hat{\psi}_2(s) = \alpha(s)\hat{\psi}_1(s)$. Let $s \in (-2,\ -1] \cup [1,\ 2)$ and assume that $\hat{\psi}_1(s) \neq 0$. Then, since $|\hat{\psi}_1(s)| = |\hat{\psi}_2(s)|$, we have

$$\begin{aligned} \hat{\phi}_2(s) &= (\overline{\hat{\psi}_2(s)}/\overline{\hat{\psi}_1(s)})\hat{\psi}_1(s) \\ &= \alpha(\frac{s}{2})\hat{\psi}_1(s) \\ &= \alpha(s)\hat{\psi}_1(s). \end{aligned}$$

Hence we have $\hat{\psi}_2(s) = \alpha(s)\hat{\psi}_1(s)$ for all s. $\square$

5.5 MRA Super-wavelets

An importamt concept in wavelet theory is multiresolution analysis which is used to derive wavelets. We recal that A *multiresolution analysis (MRA)* for $L^2(\mathbb{R})$ consists of a sequence $\{V_j :\ j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying

(1) $V_j \subset V_{j+1},\ j \in \mathbb{Z}$,

(2) $\cap_{j\in\mathbb{Z}} V_j = \{0\},\ \overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$,

(3) $f \in V_j$ if and only if $Df \in V_{j+1},\ j \in \mathbb{Z}$,

(4) there exists $\phi \in V_0$ such that $\{T^k\phi :\ k \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

The function ϕ in (4) is called a *scaling function* for the multiresolution analysis. It is well known (cf. [HW]) that if ϕ is a scaling function for an MRA, then there is a 2π-periodic measurable function m such that

$$\hat{\phi}(2\xi) = m(\xi)\hat{\phi}(\xi)$$

for a.e. $\xi \in \mathbb{R}$. The function m is called the *low-pass filter* for ϕ, and is uniquely determined by ϕ. It is known that the function ψ given by

$$\hat{\psi}(\xi) = e^{\frac{i\xi}{2}}\overline{m(\frac{1}{2}\xi + \pi)}\hat{\phi}(\frac{1}{2}\xi) \qquad (*)$$

is a wavelet, and moreover, it is known that every function of the form

$$\hat{\psi}(\xi) = e^{\frac{i\xi}{2}} k(s)\overline{m(\frac{1}{2}\xi + \pi)}\hat{\phi}(\frac{1}{2}\xi) \qquad (**)$$

where k is any measurable unimodular 2π-periodic function, is a wavelet. These are all contained in the difference space $W_0 = V_1 \ominus V_0$, and moreover, every wavelet contained in W has the form $(**)$. By definition, a wavelet which has this form for some some MRA is called an MRA wavelet.

Let ψ be a wavelet (resp. normalized tight frame wavelet). Let W_j be the subspace generated by $\{D^j T^l \psi : \ l \in \mathbb{Z}\}$ and let

$$V_j = \oplus_{k<j} V_k.$$

Then $\{V_j : \ j \in \mathbb{Z}\}$ satisfies $(1) - - - (3)$. If (4) is satisfied, then ψ is an MRA wavelet. If ψ is a normalized tight frame wavelet and if there is a function ϕ in V_0 such that $\{T^l\phi : l \in \mathbb{Z}\}$ is a normalized tight frame for V_0, then we call ψ an MRA frame wavelet.

Let $\phi_1, \ \phi_2 \in L^2(\mathbb{R})$ and let V_0 be the closed subspace generated by $\{T^l\psi_1 \oplus T^l\psi_2 : l \in \mathbb{Z}\}$. The following result tells us that there is no MRA super-wavelet in the usual sense.

Proposition 5.16. *Suppose that* $V_0 \subset (D\oplus D)V_0$ *and that* $\{T^l\psi_1 \oplus T^l\psi_2 : \ l \in \mathbb{Z}\}$ *is an orthonormal basis for* V_0. *Then* $\cup_{j\in\mathbb{Z}}(D^j \oplus D^j)V_0$ *is not dense in* $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

Proof. Since $\phi_1(\frac{s}{2}) \oplus \phi_2(\frac{s}{2})$ is in V_0 and $\{T^l\psi_1 \oplus T^l\psi_2 : \ l \in \mathbb{Z}\}$ is an orthonormal basis for V_0, there is sequence $\{\alpha_k\}$ of complex numbers such that $\sum_{k\in\mathbb{Z}} |\alpha_k|^2 < \infty$ and

$$\phi_1(\frac{s}{2}) \oplus \phi_2(\frac{s}{2}) = \sum_{k\in\mathbb{Z}} \alpha_k (T^k\phi_1(s) \oplus T^k\phi_2(s)).$$

Taking Fourier transforms, we obtain

$$\hat{\phi_1}(2s) = m(s)\hat{\phi_1}(s)$$

and

$$\hat{\phi_2}(2s) = m(s)\hat{\phi_2}(s),$$

where $m(s) = \sum_{k\in\mathbb{Z}} \alpha_k e^{iks}$ is a 2π-periodic function.

The orthonormality of $\{T^l\psi_1 \oplus T^l\psi_2 : \ l \in \mathbb{Z}\}$ also implies that

$$\sum_{k\in\mathbb{Z}} (|\hat{\phi_1}(s + 2k\pi)|^2 + |\hat{\phi_2}(s + 2k\pi)|^2) = \frac{1}{2\pi}, \quad a.e. \ s \in \mathbb{R}.$$

This condition together with the relation between m and $\hat{\phi}_i$ $(i=1,2)$ implies that

$$|m(s)|^2 + |m(s+\pi)|^2 = 1, \quad a.e.\ s \in \mathbb{R}.$$

In particular we have $|m(s)| \le 1$. Hence $|\hat{\phi}_i(2^{-j}s)|$ $(i=1,2)$ is non-decreasing for almost every $s \in \mathbb{R}$ as $j \to \infty$. Let

$$g_i(s) = lim_{j\to\infty}|\hat{\phi}_i(2^{-j}s)| \ \ (i=1,2).$$

Assume, to the contrary, that $\cup_{j\in\mathbb{Z}}(D^j \oplus D^j)V_0$ is dense in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. we will prove that $g_i(s) = \frac{1}{\sqrt{2\pi}}$, $a.e.\ s \in \mathbb{R}$ $(i=1,2)$. The techiques we used here can be found in the proofs of Theorem 1.7 (page 48) and Theorem 5.2 (page 382) in [HW]. Let P_j be the orthogonal projection from $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ onto $(\hat{D}^j \oplus \hat{D}^j)V_0$. Then $P_j \to I(j \to \infty)$ in the strong operator topology. Let $f = \chi_{[-1,1]} \oplus 0$. We have $||f||^2 = 2$ and $||P_j f|| \to ||f||$. On the other hand we have

$$\begin{aligned}
||P_j f||^2 &= ||\sum_{k\in\mathbb{Z}} < P_j f,\ \hat{D}^j\hat{T}^k\hat{\phi_1} \oplus \hat{D}^j\hat{T}^k\hat{\phi_2} > \hat{D}^j\hat{T}^k\hat{\phi_1} \oplus \hat{D}^j\hat{T}^k\hat{\phi_2}||^2 \\
&= \sum_{k\in\mathbb{Z}} |< f,\ \hat{D}^j\hat{T}^k\hat{\phi_1} \oplus \hat{D}^j\hat{T}^k\hat{\phi_2} >|^2 \\
&= \sum_{k\in\mathbb{Z}} |< \chi_{[-1,1]},\ \hat{D}^j\hat{T}^k\hat{\phi_1} >|^2 \\
&= \sum_{k\in\mathbb{Z}} |\int_{\mathbb{R}} 2^{-j/2}\chi_{[-1,1]}(s)\overline{\hat{\phi_1}(2^{-j}s)}e^{-i2^{-j}ks}ds|^2 \\
&= 2\pi \cdot 2^j \sum_{k\in\mathbb{Z}} |\int_{-2^{-j}}^{2^{-j}} \overline{\hat{\phi_1}(\xi)}\frac{1}{\sqrt{2\pi}}e^{-ik\xi}d\xi|^2.
\end{aligned}$$

Note that $\{\frac{1}{\sqrt{2\pi}}e^{ik\xi} : \ k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2([-\pi,\pi])$ and that $[-2^{-j},\ 2^{-j}] \subset [-\pi,\pi]$ when j large enough. Thus

$$\begin{aligned}
||P_j f||^2 &= 2^{j+1}\pi \int_{-2^{-j}}^{2^{-j}} |\hat{\phi_1}(\xi)|^2 d\xi \\
&= 2\pi \int_{-1}^{1} |\hat{\phi_1}(2^{-j}s)|^2 ds \to 2.
\end{aligned}$$

Hence

$$\frac{1}{2}\int_{-1}^{1} |\hat{\phi_1}(2^{-j}s)|^2 ds \to \frac{1}{2\pi}$$

which implies that

$$\frac{1}{2}\int_{-1}^{1} |g_1(s)|^2 ds \to \frac{1}{2\pi}.$$

Thus $g_1(s) = \frac{1}{\sqrt{2\pi}},\ a.e.\ s \in \mathbb{R}$ since $g_1(s) \leq \frac{1}{\sqrt{2\pi}},\ a.e.\ s \in \mathbb{R}$. Similarly, $g_2(s) = \frac{1}{\sqrt{2\pi}},\ a.e.\ s \in \mathbb{R}$.

From

$$\hat{\phi_1}(s) = \hat{\phi_1}(2^{-j}s)\pi_{k=1}^{j} m(2^k s)$$

and

$$\hat{\phi_2}(s) = \hat{\phi_2}(2^{-j}s)\pi_{k=1}^{j} m(2^k s)$$

for all $j \in \mathbb{N}$, we obtain $|\hat{\phi_1}(s)| = \hat{\phi_2}(s)|,\ a.e.\ s \in \mathbb{R}$. Therefore

$$\sum_{k\in\mathbb{Z}} |\hat{\phi_1}(s+2k\pi)|^2 = \sum_{k\in\mathbb{Z}} |\hat{\phi_2}(s+2k\pi)|^2 = \frac{1}{\pi}.$$

This implies that $\{T^k\sqrt{2}\phi_1 : k \in \mathbb{Z}\}$ is an orthonormal set. Let $V_{0,1}$ be the closed subspace generated by this set and let $V_{j,1} = D^j V_{0,1}$. Then our assumption implies that $V_{j,1} \subset V_{j+1,1}$ and $\cup_j V_{0,j}$ is dense in $L^2(\mathbb{R})$. Thus the above argument also implies that

$$lim_{j\to\infty}|\sqrt{2}\hat{\phi_1}(2^{-j}s)| = \frac{1}{\sqrt{2\pi}},\quad a.e.\quad s \in \mathbb{R}$$

which contradicts the following equality

$$lim_{j\to\infty}|\hat{\phi_1}(2^{-j}s)| = \frac{1}{\sqrt{2\pi}},\quad a.e.\quad s \in \mathbb{R}.$$

Therefore $\cup_{j\in\mathbb{Z}}(D^j \oplus D^j)V_0$ is not dense in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. $\square$

In view of the above proposition we call a super-wavelet $(\eta_1,\ \ldots,\ \eta_k)$ an *MRA super-wavelet* if every $\eta_i (i = 1, ..., k)$ is an MRA frame wavelet. For example, let E and L be as in Example C and let

$$E^s = \cup_{j=1}^{\infty} 2^{-j}E, \qquad L^s = \cup_{j=1}^{\infty} 2^{-j}L.$$

Then $E^s = [-\frac{\pi}{2},\ 0) \cup (0,\ \frac{\pi}{2})$ and

$$L^s = [-\frac{8\pi}{7},\ -\pi) \cup [-\frac{4\pi}{7},\ -\frac{\pi}{2}) \cup [-\frac{2\pi}{7},\ 0) \cup [o, \frac{2\pi}{7}) \cup [\frac{\pi}{2},\ \frac{4\pi}{7}) \cup [\pi,\ \frac{8\pi}{7}).$$

It is easy to check that $L^s$is 2π-translation congruent to the set

$$[-\pi,\ -\frac{6\pi}{7}) \cup [-\frac{4\pi}{7},\ -\frac{\pi}{2}) \cup [-\frac{2\pi}{7},\ 0) \cup [0, \frac{2\pi}{7}) \cup [\frac{\pi}{2},\ \frac{4\pi}{7}) \cup [\frac{6\pi}{7},\ \pi),$$

which is a subset of $[-\pi,\ \pi]$. For a frame set G, an elementary computation shows that $\mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}}\chi_G)$ is an MRA frame if and only if G^s is 2π-congruent to a subset of $[-\pi,\ \pi]$. Thus both $\mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}}\chi_E)$ and $\mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}}\chi_L)$ are MRA frame wavelets. Therefore

$$(\mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}}\chi_E),\ \mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}}\chi_L))$$

is an MRA super-wavelets.

It is known (cf. [HW]) that a wavelet ψ is an MRA wavelet if and only if

$$\sum_{j=1}^{\infty}\sum_{l\in\mathbb{Z}}|\hat{\psi}(2^j(s+2l\pi))|^2=\frac{1}{2\pi},\quad a.e.\ s\in\mathbb{R}.$$

So we ask the following problems

Problem E: Characterize all the MRA super-wavelets.

In particular we ask:

Problem E$_1$: Suppose that $(\eta_1,\ \ldots,\ \eta_k)$ is a super-wavelet and suppose that one of the η_i's is an MRA frame wavelet. Does this imply that $(\eta_1,\ \ldots,\ \eta_k)$ is an MRA super-wavelet?

5.6 Interpolation Theory

Von Neumann algebras play an important role in the operator-theoretic approach to wavelet theory in [DL]. If ψ is an orthonormal wavelet then the local commutant $C_\psi(D,T)$ contains many von Neumann algebras as subsets which yield families of wavelets. This led to a new aspect of wavelet analysis – *operator theoretic interpolation theory*, which will be discussed below.

Let $\eta,\ \psi$ be wavelets and let V_ψ^η be the unique unitary operator in $C_\psi(D,T)$ such that $V_\psi^\eta\psi=\eta$. If $\mathcal{F}$ is a family of wavelets such that $V_\psi^\eta(\eta\in\mathcal{F})$ normalizes $\{D,T\}'$ and

$$Group\{V_\psi^\eta:\ \eta\in\mathcal{F}\}\subset C_\psi(D,T),$$

then the von Neumann algebra $\mathcal{M}$ generated by $\{D,T\}'$ and $\{V_\psi^\eta:\eta\in\mathcal{F}\}$ is contained in the local commutant $C_\psi(D,T)$. In this case, for every unitary operator U in this von Neumann algebra, the wavelet $U\psi$ is interpolated from $(\psi,\mathcal{F})$ and we say that $(\psi,\mathcal{F})$ admits operator-interpolation. The most interesting and relatively well-investigated case in [DL] is that when $Group\{V_\psi^\eta:\eta\in\mathcal{F}\}$ is a finite cyclic group and when ψ,η are s-elementary wavelets.

Given wavelet sets E and F. Let $\sigma:E\to F$ be the 1-1, onto map implementing the 2π-translation congruence. Since E and F both generate partitionns of $\mathbb{R}$

under dilation by power of 2, we can extend σ to a 1-1 map of $\mathbb{R}$ onto $\mathbb{R}$ by defining $\sigma(0) = 0$ and

$$\sigma(s) = 2^n \sigma(2^{-n} s)$$

for $s \in 2^n E, n \in \mathbb{Z}$. We adopt the notation σ_E^F from [DL] for this and call it the *interpolation map* for the ordered pair (E, F). This is a *measure-preserving map* and induces a unitary operator U_E^F by

$$(U_E^F f)(s) = f(\sigma_F^E(s))$$

for all $f \in L^2(\mathbb{R})$. It was proved in [DL] that

$$U_E^F \in C_E(\hat{D}, \hat{T}).$$

It was proved by Q. Gu (Interpolation groups of wavelet sets, preprint) that for any finite group $\mathcal{G}$, there exists a family $\mathcal{E}$ of wavelet sets such that

$$\{U_E^F : E, F \in \mathcal{E}\}$$

forms a group which is isomorphic to $\mathcal{G}$ and admits operator-interpolation. In this case the von Nemann algebra generated by this group and $\{\hat{D}, \hat{T}\}'$ is finite. Thus Corollary 3.7 applies to this case.

Let $U = U_E^F$ and $\sigma = \sigma_E^F$. Assume that U has order k and

$$\{U^n : \; n = 0, 1, \; ... \; k-1\}$$

forms an interpolation family, i.e. $U^n \in C_\psi(\hat{D}, \hat{T})$ for all n, where $\psi = \frac{1}{\sqrt{2\pi}} \chi_E$. Then each element in the von Neumann algebra generated by $\{\hat{D}, \hat{T}\}'$ has an expression

$$\sum_{n=0}^{k-1} M_{h_n} U^n,$$

where $h_n \in L^\infty(\mathbb{R})$ with the property that $h_n(2s) = h_n(s), \quad a.e. \; s \in \mathbb{R}$. There exists an *-isomorphism θ from $\mathcal{M}$ to the $k \times k$ function matrix algebra such that $\theta(\sum_{n=0}^{k-1} M_{h_n} U^n) = M$ with

$$M(s) = (\, h_{ij}(s) \,)$$

where $h_{ij}(s) = h_{\alpha(i,j)}(\sigma^{-i}(s))$ and $\alpha(i,j) = (j - i)$ modulo k. So, for instance, if $k = 2$, then

$$M(s) = \begin{pmatrix} h_0(s) & h_1(s) & h_2(s) \\ h_2(\sigma^{-1}(s)) & h_0(\sigma^{-1}(s)) & h_1(\sigma^{-1}(s)) \\ h_1(\sigma^{-2}(s)) & h_2(\sigma^{-2}(s)) & h_0(\sigma^{-2}(s)) \end{pmatrix}.$$

Lemma 5.17. *Let $\eta \in L^2(\mathbb{R})$ such that $\hat{\eta} \in L^\infty(\mathbb{R})$ and the support of $\hat{\eta}$ is contained in the union of $\{(\sigma_E^F)^n(E) : n = 0, 1, \ ..., \ k-1\}$. Then there exists an operator $A \in \mathcal{M}$ such that $\hat{\eta} = A\psi$, where $\psi = \chi_E$.*

Proof. Write $\sigma = \sigma_E^F$ and $U = U_E^F$. By Proposition 2.4 in [GHLL], we know that $\sigma^n(E)$ is wavelet set for all n. We define h_n on

$$K_n := \sigma^n(E) \backslash (E \cup .. \cup \sigma^{n-1}(E))$$

to be $\hat{\eta}$ and zero on $\sigma^n(E)\backslash K_n$. Since $\sigma^n(E)$ is a wavelet set, we can extend h_n uniquely to $\mathbb{R}$ by the relation $h_n(2s) = h_n(s)$. Let

$$A = \sum_{n=0}^{k} M_{h_n} U^n.$$

Then $A \in \mathcal{M}$ and $A\psi = \hat{\eta}$ by the construction of h_n. $\square$

Proposition 5.18. *Suppose that U, $\mathcal{M}$, η, A and h_n be as above. Then the following are equivalent.*

(i) η is a frame wavelet.

(ii) $\{D^n T^m \eta : n, m \in \mathbb{Z}\}$ is a Riesz basis for $L^2(\mathbb{R})$.

(iii) The matrix function $M(s)$ satisfies the condition

$$aI \leq M(s)M(s)^* \leq bI$$

for some constants $a, b > 0$.

Proof. The equivalence of (i) and (ii) follows from Corollary 3.7 since $\mathcal{M}$ is finite and $A \in \mathcal{M}$. The equivalence of (i) and (iii) follows from Corollary 3.6 and the fact that θ from $\mathcal{M}$ to the $k \times k$ function matrix algebra is an *-isomorphism. $\square$

We remark that if E and F are two frame set such that they are 2π-translation congruent. Then, like the wavelet sets case, we can similarly define σ_E^F and U_E^F. The unitary operator U_E^F is the unique operator in $C_{\chi_E}(\hat{D}, \ \hat{T})$ such that $U_E^F \chi_E = \chi_F$. Therefore the interpolation theory also works for frame sets. The following observation might be useful in constructing super-wavelets.

Proposition 5.19. *Suppose that (E_1, F_1) is an interpolation pair of frame sets. If both (E_1, E_2) and $(F_1, \ F_2)$ are strong complementary pairs of frame sets, then $(\eta, \ \psi)$ is a super-wavelet for any normalized tight frame wavelets η and ϕ with the property that supp$(\hat{\eta}) \subseteq E_1 \cup E_2$ and supp$(\hat{\psi}) \subseteq F_1 \cup F_2$.*

Proof. By the argument in the proof of Lemma 5.17, there exist unitary operators U and V such that $\hat{\eta} = U\frac{1}{\sqrt{2\pi}}\chi_{E_1}$ and $\hat{\psi} = V\frac{1}{\sqrt{2\pi}}\chi_{F_1}$. Hence the proposition follows. $\square$

Even though we know that for any wavelet ψ the local commutant $C_\psi(D,T)$ contains many von Neumann algebras, it is still open whether it can contain a von Neumann algebra which is not finite. (See [La] Problem C. A von Neumann algebra is called *finite* if it does not contain any proper isometry, that is an isometry which is not unitary). This is an interesting problem from an operator-algebraic point of view. As an application of the results in section 2 and 3, we conclude this section by giving an example to illustrate that for certain ψ, $C_\psi(D,T)$ contains an isometry V, which is not a unitary, for which $V^* \in C_\psi(D,T)$. It is unknown, in fact, whether this example is actually a solution to [La] Problem C.

Example E. Let $E = [-2\pi, -\pi] \cup [\pi, 2\pi]$ and $F = [-\pi, -\frac{\pi}{2} \cup [\frac{\pi}{2}, \pi]$. Then E is a wavelet set and F is frame set by Theorem 5.4. Let $\hat{\psi} = \frac{1}{\sqrt{2\pi}}\chi_E$ and $\hat{\eta} = \frac{1}{\sqrt{2\pi}}\chi_F$. We define V by

$$Vf = \sum_{n,m\in\mathbb{Z}} < f, \hat{D}^n\hat{T}^m\hat{\eta} > \hat{D}^n\hat{T}^m\hat{\psi}, \qquad f \in Ł^2(\mathbb{R}).$$

Then V is an isometry. Also, by the proof of Proposition 2.1, $V^* \in C_{\hat{\psi}}(\hat{D}, \hat{T})$. Moreover we have:

Proposition 5.20. *The operators V^*, VV^*, V^k are contained in $C_{\hat{\psi}}(\hat{D}, \hat{T})$ for all $k \in \mathbb{N}$.*

We will prove this in three lemmas.

Lemma 5.21. *The inclusions*

$$V^n L^2(2^m E) \ \subseteq \ L^2(2^{m+n}E) \quad \textit{and} \quad (V^*)^n L^2(2^m E) \ \subseteq \ L^2(2^{m-n}E)$$

hold for all $m \in \mathbb{Z}$ and all $n \geq 0$.

Proof. Note that $supp(\hat{D}^k\hat{T}^l\hat{\eta}) \subset 2^{k-1}E$. Then for any $f \in L^2(2^m E)$, we have

$$\begin{aligned} < Vf, \hat{D}^k\hat{T}^l\hat{\psi} > &= < f, \hat{D}^k\hat{T}^l V^*\hat{\psi} > \\ &= < f, \hat{D}^k\hat{T}^l\hat{\eta} > = \ 0 \end{aligned}$$

when $k - 1 \neq m$. Thus $VL^2(2^m E) \subseteq L^2(2^{m+1}E)$ and hence $V^n L^2(2^m E) \subseteq L^2(2^{m+n}E)$ for all $n \geq 0$.

For the second inclusion, without loss of generality, we take $f = \hat{D}^m\hat{T}^l\psi$. Then

$$V^* f \ = \ \hat{D}^m\hat{T}^l V^*\hat{\psi} \ = \ \hat{D}^m\hat{T}^l\hat{\eta} \ \subseteq \ L^2(2^{m-1}E).$$

So the inclusion follows. $\square$

Since $V^* \in C_{\hat{\psi}}(\hat{D},\hat{T})$ and $\{\hat{D}\}'$ contains $C_{\hat{\psi}}(\hat{D},\hat{T})$, we have that $p(V,V^*) \in \{\hat{D}\}'$ for all polynomials $p(z,w)$. Thus to show that some particular $p(V,V^*)$ is contained in $C_{\hat{\psi}}(\hat{D},\hat{T})$, it suffices to show that $p(V,V^*)\hat{T}^l\hat{\psi} = \hat{T}^l p(V,V^*)\hat{\psi}$ for all $l \in \mathbb{Z}$. In the proof of the following lemmas we will frequently use the relation: $T^nD^m = D^nT^{2^m n}$ for all $n,m \in \mathbb{Z}$, where T^α is defined by

$$(T^\alpha f)(t) = f(t-\alpha)$$

for all $f \in L^2(\mathbb{R})$.

Lemma 5.22. *The operator V^k is contained in $C_{\hat{\psi}}(\hat{D},\hat{T})$ for all $k \in \mathbb{N}$.*

Proof. By Lemma 2.20, we have

$$\begin{aligned}(I) :&= < V\hat{T}^l\hat{\psi}, \hat{D}^n\hat{T}^j\hat{\psi} > = < \hat{T}^l\hat{\psi}, \hat{D}^n\hat{T}^j\hat{\eta} > \\ &= \begin{cases} 0, & n \neq 1 \\ < \hat{T}^l\hat{\psi}, \hat{D}\hat{T}^j\hat{\eta} >, & n = 1 \end{cases}\end{aligned}$$

and

$$\begin{aligned}(II) :&= < \hat{T}^l V\hat{\psi}, \hat{D}^n\hat{T}^j\hat{\psi} > \\ &= \begin{cases} 0, & n \neq 1 \\ < \hat{T}^l V\hat{\psi}, \hat{D}\hat{T}^j\hat{\psi} >, & n = 1 \end{cases}\end{aligned}$$

When $n = 1$, we get

$$\begin{aligned}(II) &= < V\hat{\psi}, \hat{D}\hat{T}^{-2l+j}\hat{\psi} > = < \hat{\psi}, \hat{D}\hat{T}^{-2l+j}\hat{\eta} > \\ &= < \hat{T}^l\hat{\psi}, \hat{D}\hat{T}^j\hat{\eta} > = (I).\end{aligned}$$

Hence we have $\hat{T}^l V\hat{\psi} = V\hat{T}^l\hat{\psi}$ as required.

Assume that $V^k \in C_\psi(\hat{D},\hat{T})$. We will show that $V^{k+1} \in C_{\hat{\psi}}(\hat{D},\hat{T})$.

Again by Lemma 2.20, we have

$$< V^{k+1}\hat{T}^l\hat{\psi},\ \hat{D}^n\hat{T}^j\hat{\psi} > = \begin{cases} 0, & n \neq k+1 \\ < V^2\hat{T}^l\hat{\psi},\ \hat{D}^n\hat{T}^j\hat{\psi} >, & n = k+1 \end{cases}$$

and

$$< \hat{T}^l V^{k+1}\hat{\psi},\ \hat{D}^n\hat{T}^j\hat{\psi} >= \begin{cases} 0, & n \neq k+1 \\ < \hat{T}^l V^{k+1}\hat{\psi}, \hat{D}^n\hat{T}^j\hat{\psi} >, & n = k+1 \end{cases}$$

When $n = k+1$, using the assumption $V^k \in C_{\hat{\psi}}(\hat{D}, \hat{T})$, we get

$$\begin{aligned} < \hat{T}^l V^{k+1}\hat{\psi}, \hat{D}^{k+1}\hat{T}^j\hat{\psi} > &= < V^k\hat{\psi}, V^*\hat{D}^{k+1}\hat{T}^{-2^{k+1}l+j}\hat{\psi} > \\ &= < V^k\hat{\psi},\ \hat{D}^{k+1}\hat{T}^{-2^{k+1}l+j}V^*\hat{\psi} > \\ &= < \hat{T}^l V^k\hat{\psi},\ \hat{D}^{k+1}\hat{T}^j V^*\hat{\psi} > \\ &= < V^k\hat{T}^l\hat{\psi},\ V^*\hat{D}^{k+1}\hat{T}^j\hat{\psi} > \\ &= < V^{k+1}\hat{T}^l\hat{\psi},\ \hat{D}^{k+1}\hat{T}^j\hat{\psi} > . \end{aligned}$$

Hence we have $\hat{T}^l V^{k+1}\hat{\psi} = V^{k+1}\hat{T}^l\hat{\psi}$, as required. $\square$

Lemma 5.23. *The operator VV^* is contained in $C_{\hat{\psi}}(\hat{D}, \hat{T})$.*

Proof. . By Lemma 2.20, we have

$$< VV^*\hat{T}^l\hat{\psi},\ \hat{D}^n\hat{T}^j\hat{\psi} > = < \hat{T}^l VV^*\hat{\psi},\ \hat{D}^n\hat{T}^j\hat{\psi} > = 0$$

for all $n \neq 0$. If $n = 0$, then

$$\begin{aligned} < VV^*\hat{T}^l\hat{\psi},\ \hat{T}^j\hat{\psi} > \ &= < V^*\hat{T}^l\hat{\psi},\ V^*\hat{T}^j\hat{\psi} > \\ &= < \hat{T}^l V^*\hat{\psi},\ \hat{T}^j V^*\hat{\psi} >] \\ &= < V^*\hat{\psi},\ \hat{T}^{j-l}V^*\hat{\psi} > \\ &= < V^*\psi,\ V^*\hat{T}^{j-l}\psi > \\ &= < VV^*\hat{\psi},\ \hat{T}^{j-l}\hat{\psi} > \\ &= < \hat{T}^l VV^*\hat{\psi},\ \hat{T}^j\hat{\psi} > . \end{aligned}$$

Hence $VV^*T^l\hat{\psi} = \hat{T}^l VV^*\hat{\psi}$, as required. $\square$

Problem F: In the above notation, is $p(V, V^*)$ contained in $C_{\hat{\psi}}(\hat{D},\ \hat{T})$ for all polynomials in 2 variables $p(z, w)$? Equivalently, is the von Neumann algebra $w^*(V)$ generated by V contained in $C_{\hat{\psi}}(\hat{D},\ \hat{T})$? Note that $w^*(V)$ is the closure of the set of all polynomials $p(V, V^*)$ in the weak operator topology, and is not finite because V is a proper isometry. So a positive answer would answer Problem C in [La].

Chapter 6

Frame Representations for Groups

Let $\mathcal{G}$ be a group. A representation $(\pi, \mathcal{G}, H)$ of $\mathcal{G}$ is called a *frame representation* if $\pi(\mathcal{G})$ has a complete normalized tight frame vector. Two complete normalized tight frame vectors $\eta,\ \xi \in H$ for $\pi(\mathcal{G})$ are said to be *equivalent* if the frames $\{\pi(g)\eta\}_{g\in\mathcal{G}}$ and $\{\pi(g)\xi\}_{g\in\mathcal{G}}$ are unitarily equivalent, or equivalently, if there is a unitary operator $U \in \pi(\mathcal{G})'$ such that $U\eta = \xi$. We will use $[\eta]_\pi$ to denote the equivalent class of complete normalized tight frame vectors represented by η. We say that the frame vector classes $[\eta_1]_\pi,\ .\ .\ .\ ,\ [\eta_k]_\pi$ are strongly disjoint if $\{\pi(g)\eta_1\}_{g\in\mathcal{G}}$, . . . , $\{\pi(g)\eta_k\}_{g\in\mathcal{G}}$ are strongly disjoint. It is clear that this definition is independent of the choices of $\eta_1,\ .\ .\ .\ .\ ,\ \eta_k$. In most cases there are many inequivalent classes of normalized tight frame vectors for one frame representation. The main purpose of this chapter is to study the strongly disjoint classes and their relations with the representations. We also prove that all the complete normalized tight frame vectors for a frame representation can be parameterized by a fixed normalized tight frame vector and the set of all unitary operators in the von Neumann algebra generated by the range of the representation.

6.1 Basics

We recall from ([KR]) that two representations $(\pi, \mathcal{G}, H)$ and $(\sigma, \mathcal{G}, K)$ are said to be *equivalent* if there is a unitary operator $U : H \to K$ such that

$$W\pi(g)W^* = \sigma(g), \qquad g \in \mathcal{G}.$$

We should keep in mind the following simple observation:

Proposition 6.1. *Let $(\pi_1,\ \mathcal{G},\ H_1)$ and $(\pi_2,\ \mathcal{G},\ H_2)$ be two frame representations. Suppose that η_1 and η_2 are complete normalized tight frame vectors for π_1 and π_2, respectively, such that $\{\pi_1(g)\eta_1 :\ g \in \mathcal{G}\}$ and $\{\pi_2(g)\eta_2 :\ g \in \mathcal{G}\}$ are unitarily equivalent as frames with index set $\mathcal{G}$.. Then π_1 and π_2 are unitarily equivalent representations.*

Proof. Assume that there is a unitary transform $W : H_1 \to H_2$ such that

$$W\pi_1(g)\eta_1 = \pi_2(g)\eta_2, \quad g \in \mathcal{G}.$$

Then for any $g, h \in \mathcal{G}$, we have

$$\begin{aligned} W\pi_1(g)\pi_1(h)\eta_1 &= W\pi_1(gh)\eta_1 \\ &= \pi_2(gh)\eta_2 = \pi_2(g)\pi_2(h)\eta_2 \\ &= \pi_2(g)W\pi_1(h)\eta_1. \end{aligned}$$

Since $\{\pi_1(h)\eta_1 : \ h \in \mathcal{G}\}$ generates H_1, we have

$$W\pi_1(g)x = \pi_2(g)Wx$$

for all $x \in H_1$. So π_1 and π_2 are unitarily equivalent representations of $\mathcal{G}$. $\square$

Let $K = l^2(\mathcal{G})$, and let λ be the left regular representation of $\mathcal{G}$. Let $\mathcal{M}$ be the von Neumann algebra generated by $\{\lambda(g) : g \in \mathcal{G}\}$. If $P \in \mathcal{M}'$, then the subrepresentation λ_P is defined by $\lambda_P(g) = \lambda(g)P$ for all $g \in \mathcal{G}$. By Theorem 3.8 or Theorem 3.8$'$, we have

Proposition 6.2. *Every frame representation of $\mathcal{G}$ is unitarily equivalent to a subrepresentation of the left regular representation.*

We also recall that two orthonormal projections P_1, P_2 in a von Neumann algebra $\mathcal{R}$ are called *equivalent* , denoted by $P_1 \sim P_2$, if there is a partial isometry $V \in \mathcal{R}$ such that $VV^* = P_1$ and $V^*V = P_2$. If P_1 is equivalent to a subprojection of P_2, we write $P_1 \preceq P_2$. A projection P in a von Neumann algebra $\mathcal{R}$ is called *finite* if there is no proper subprojection of P equivalent to P.

For convenience, we use $H^{(n)}$ to denote the Hilbert space $H \oplus H \oplus ... \oplus H$ (n copies of H) and we use $\pi^{(n)}$ to denote the n-direct sum representation of π. A frame representation (π, H) of $\mathcal{G}$ is said to have *frame-multiplicity* n if n is the supremum of all the natural numbers k with the property that there exist frame vectors $\eta_i (i = 1, 2, ..., k)$ such that

$$\{\pi(g)\eta_1 : g \in \mathcal{G}\}, \ . \ . \ . \ , \qquad \{\pi(g)\eta_k : g \in \mathcal{G}\}$$

are strongly disjoint. We will show (Proposition 6.6) that the frame multiplicity is always finite. Thus a frame representation $(\pi, \mathcal{G}, H)$ has frame-multiplicity n if and only if n is the largest number such that the representation $(\pi^{(n)}, H^{(n)})$ of $\mathcal{G}$ has a complete normalized tight frame vector. Hence the frame multiplicity is the maximal number of strongly disjoint frame vector classses. Therefore the frame multiplicity for a frame representation π is great than or equal to the cardinal number of the set of all equivalence classes of frame vectors for π. In general these two numbers are not equal (see the remark after Proposition 6.9). It is obvious that frame multiplicity and the cardinal number of the inequivalent frame vector classes are invariant under unitary equivalence.

Proposition 3.13 characterizes all the frame representations that have only one equivalent class of normalized tight frame vectors, which can be restated as follows:

Theorem 6.3. *A frame representation* $(\pi,\ \mathcal{G},\ H)$ *has a unique unitary equivalence class of frame vectors if and only if* π *is unitarily equivalent to a subrepresentation* λ_P *of* λ *such that* P *is in the center of* $\mathcal{M}$*.*

6.2 Frame Multiplicity

To prove that the frame multiplicity is always finite for frame representations, we need the following:

Lemma 6.4. *Let* $(\pi, \mathcal{G}, H)$ *be a frame representation. Then* $(\pi^{(n)}, \mathcal{G}, H^{(n)})$ *has a complete normalized tight frame vector if and only if there exist self-adjoint projections* $P_i (i = 1, ..., n)$ *in* $\mathcal{M}'$ *such that* $P_i \sim P$ *and* $P_i K \perp P_j K$ *when* $i \neq j$*.*

Proof. Note that if $(\pi^{(n)}, \mathcal{G}, H^{(n)})$ has a complete normalized tight frame vector, then so does $(\pi^{(m)}, \mathcal{G}, H^{(m)})$ for all $m < n$. Thus we only need to consider the $n = 2$ case. Assume that $\eta_1, \eta_2 \in H$ such that $\{\pi(g)\eta_1 : g \in \mathcal{G}\}$ and $\{\pi(g)\eta_2 : g \in \mathcal{G}\}$ are strongly disjoint. Define V_1 and V_2 by

$$V_i x = \sum_g < x, \pi(g)\eta_i > \chi_g$$

when $x \in H$ and $V_i x = 0$ when $x \in K \ominus H$, where χ_g is the characteristic function at point g. Then, by the proof of Proposition 2.16, $V_1 K \perp V_2 K$. From Proposition 3.1 (i), V_1 and V_2 are partial isometries in $\mathcal{M}'$. Let $P_i = V_i V_i^*$. Then $P_i \sim P$, as required.

Conversely, suppose that $V_i \in \mathcal{M}'$ $(i = 1, 2)$ are partial isometries in $\mathcal{M}'$ such that $V_i^* V_i = P$, $V_1 K \perp V_2 K$. Let $\eta_i = V_i^* \chi_e$, where e is the identity of $\mathcal{G}$. Then, by Proposition 3.1 (i) and Proposition 2.16, $\{\pi(g)\eta_1 : g \in \mathcal{G}\}$ and $\{\pi(g)\eta_2 : g \in \mathcal{G}\}$ are strongly disjoint. $\square$

The proof of Lemma 6.4 also implies that

Corollary 6.5. *Let* p *and* q *be two projections in* $\mathcal{M}'$*, and let* π_p *and* π_q *be the subrepresentations of* λ *restricted to* p *and* q*, respectively. Suppose that* η_p *and* η_q

are complete normalized frame vectors for π_p *and* π_q*, respectively. Then*

$$\{\pi_p(g)\eta_p : \ g \in \mathcal{G}\} \quad and \quad \{\pi_q(g)\eta_q : \ g \in \mathcal{G}\}$$

are strongly disjoint if and only if there exist partial isometries V_P *and* V_q *in* $\mathcal{M}'$ *with the property that* $V_p\chi_e = \eta_p$*,* $V_q\chi_e = \eta_q$*,* $ran(V_p^*) \perp ran(V_q^*)$*,* $V_pV_p^* = p$ *and* $V_qV_q^* = q$*.*

Proposition 6.6. *Let* $(\pi, \mathcal{G}, H)$ *be a frame representation. Then the frame multiplicity of* π *is finite. In particular the representation*

$$(\pi^{(\infty)}, \mathcal{G}, H^{(\infty)})$$

does not have any complete normalized tight frame vector.

Proof. Let $\psi = \chi_e$ be the characteristic function of $\{e\}$. Then ψ is a faithful trace vector for $\mathcal{M}'$ in the sense that

$$< AB\psi, \psi > = < BA\psi, \psi >$$

for all operators $A, B \in \mathcal{M}'$ and if $< S\psi, \psi > = \ 0$ and $S \geq 0$ with $S \in \mathcal{M}'$, then $S = 0$. Let $t =< P\psi, \psi > = \ ||P\psi||^2 > 0$.

Assume, to the contrary, that the frame-multiplicity of π is infinity. Then for any natural number k, $(\pi^{(k)}, \mathcal{G}, H^{(k)})$ has a complete normalized tight frame vector. Thus, from Lemma 6.4, we can find projections $\{P_i\}_{i=1}^k$ in $\mathcal{M}'$ with orthogonal ranges such that $P_i \sim P$ for all i. Let

$$Q \ = \ \sum_i^k P_i.$$

Then $Q \leq I$ and hence $< Q\psi, \psi > \leq \ ||\psi||^2 \ = \ 1$. Since $P_i \sim P$, we have

$$t \ = < P\psi, \psi > = < P_i\psi, \psi >$$

for all i. Thus

$$1 \ \geq < Q\psi, \Psi > = \ \sum_i^k < P_i\psi, \psi > = \ kt,$$

which leads to a contradiction if we let $k \to \infty$. $\square$

Corollary 6.7. *Let π, G, H, P and M be as in Lemma 6.4. Then $(\pi^{(n)}, \mathcal{G}, H^{(n)})$ has a complete normalized tight frame vector if and only if there exist self-adjoint projections $P_i (i = 2, ..., n)$ in $\mathcal{M}'$ such that $P_i \sim P$ and $P_i K \perp P_j K$ when $i \neq j$, where we write $P = P_1$.*

Proof. Let $\{E_a\}_{a\in\mathbb{A}}$ be an orthogonal family of projections in $\mathcal{M}'$ such that $P \in \{E_a\}_{a\in\mathbb{A}}$ and which is maximal with respect to the property that $E_a \sim P$ for all $a \in \mathbb{A}$. Suppose that π has frame multiplicity n, then, by Proposition 6.6 and Lemma 6.4, there is an orthogonal family $\{F_k\}_{k=1}^{n}$ of projections in $\mathcal{M}'$ maximal with respect to the property that $F_k \sim P$ for $k = 1, 2, ..., n$. So, by Theorem 6.3.11 in [KR], $\mathbb{A}$ has cardinal number n. Thus we complete the proof. $\square$

Let M be as in Lemma 6.4. For any projection $P \in \mathcal{M}'$. We write $\pi_P = \lambda|_P$. Let $\psi = \chi_e$ and let $tr(A) =< A\psi, \psi >$ for every $A \in \mathcal{M}'$. So $tr(\cdot)$ is a trace for $\mathcal{M}'$.

Corollary 6.8.

(i) If $(\pi_P, \mathcal{G}, PH)$ has frame multiplicity one, then so does π_Q for any projection $Q \in \mathcal{M}'$ with the property $P \preceq Q$.

(ii) Suppose that $\mathcal{M}$ is a factor von Neumann algebra. Then π_P has frame multiplicity one if and only if $tr(P) > \frac{1}{2}$.

Proof. (i) Let $Q \in \mathcal{M}'$ such that $P \preceq Q$. Note that if $P \sim R$ for some projection in $\mathcal{M}'$, then π_P and π_R are unitarily equivalent. So we can assume that $P < Q$. Suppose that π_Q does not have frame multiplicity one. Then, by Corollary 6.7, we can find a projection $R \leq (I - Q)$ such that $R \sim Q$. Thus, there is a subprojection R_0 of R such that $R_0 \sim P$. Also note that $R_0 \leq Q^{\perp} \leq P^{\perp}$. Hence, by Lemma 6.4, π_P can not have frame multiplicity one, which leads to a contradiction. Therefore π_Q must have frame multiplicity one.

(ii) First assume that $tr(P) \leq \frac{1}{2}$. Since $\mathcal{M}$ is a factor, we have that either $P \preceq P^{\perp}$ or $P^{\perp} \preceq P$. If there is a proper projection R of P such that $R \sim P^{\perp}$, then $1 \geq tr(P^{\perp} + R) = tr(P^{\perp}) + tr(R) = 2tr(P^{\perp}) \geq 1$. Thus $tr(P - R) = 0$, which implies that $P = R$, which is a contradiction. So $P \preceq P^{\perp}$, and thus, by Lemma 6.4, π_P has frame multiplicity at least 2.

Conversely, assume that π_P has frame multiplicity at least 2. Then, by Corollary 6.7, there is a subprojection R of $P^{\perp}$ such that $R \sim P$. Thus

$$2tr(P) = tr(P) + tr(R) = tr(P + R) \leq 1.$$

So $tr(P) \leq 1/2$. $\square$

Proposition 6.9. *Suppose that $(\pi, \mathcal{G}, H)$ is a frame representation and $P \in \mathcal{M} \cap \mathcal{M}'$. Then π has frame-multiplicity one*

Proof. Assume that π is not of frame-multiplicity one. Then $(\pi^{(2)}, H^{(2)})$ has a complete normalized tight frame vector $(\eta_1 \oplus \eta_2)$. By Proposition 3.13, there exists a unitary operator $V \in \pi(\mathcal{G})'$ such that $V\eta_1 = \eta_2$. Let $U = V \oplus I$. Then U is a unitary operator in $\mathcal{M}'$ and thus $U(\eta_1 \oplus \eta_2)$ is a complete normalized tight frame for $\pi^{(2)}$. However

$$\{\pi^{(2)}(U(\eta_1 \oplus \eta_2)) : g \in \mathcal{G}\} = \{\pi(g)\eta_2 \oplus \pi(g)\eta_2 : g \in \mathcal{G}\},$$

which is clearly not a complete normalized tight frame for $H^{(2)}$. Hence $(\pi, \mathcal{G}, H)$ has frame-multiplicity one. □

We note that by Corollary 6.8 (ii), the converse of Proposition 6.9 is not true. Therefore there exists frame multiplicity one representation which has inequivalent frame vectors.

Corollary 6.10. *If $\mathcal{G}$ is an abelian group, then every frame representation has frame-multiplicity one.*

Proposition 6.11. *Suppose that $(\pi,\ \mathcal{G},\ H)$ is a frame representation with a complete normalized tight frame vector η. If η is a trace vector for $w^*(\pi(\mathcal{G}))$, then π has frame multiplicity one.*

Proof. Assume that π has frame multiplicity greater than one. Then there is a complete normalized tight frame vector ξ such that $\{\pi(g)\eta : g \in \mathcal{G}\}$ and $\{\pi(g)\xi : g \in \mathcal{G}\}$ are strongly disjoint. Thus, by Corollary 2.10, for any $x \in H$,

$$\sum_{g\in\mathcal{G}} < x,\ \pi(g)\xi > \pi(g)\eta \ = \ 0.$$

In particular we have

$$\sum_{g\in\mathcal{G}} < \eta,\ \pi(g)\xi >< \pi(g)\eta,\ \pi(h)\eta > = \ 0$$

for all $h \in \mathcal{G}$. Since η is a trace vector for $w^*(\pi(\mathcal{G}))$, we have

$$< \pi(g)\eta,\ \pi(h)\eta > = < \pi(h^{-1})\eta,\ \pi(g^{-1})\eta >,$$

which implies that

$$\begin{aligned} &< \pi(h^{-1})\eta,\ \sum_{g\in\mathcal{G}} \overline{< \eta,\ \pi(g)\xi >}\pi(g^{-1})\eta > \\ &\quad = \sum_{g\in\mathcal{G}} < \eta,\ \pi(g)\xi >< \pi(h^{-1})\eta,\ \pi(g^{-1})\eta > \\ &\quad = \sum_{g\in\mathcal{G}} < \eta,\ \pi(g)\xi >< \pi(g)\eta,\ \pi(h)\eta >= 0 \end{aligned}$$

for all $h \in \mathcal{G}$. Hence

$$\begin{aligned}\xi &= \sum_{g\in\mathcal{G}} < \xi,\ \pi(g^{-1})\eta > \pi(g^{-1}\eta \\ &= \sum_{g\in\mathcal{G}} \overline{< \eta,\ \pi(g)\xi >} \pi(g^{-1})\eta = 0,\end{aligned}$$

which is a contradiction. So π must have frame multiplicity one. $\square$

Corollary 6.12. *Suppose that $(\pi,\ \mathcal{G},\ H)$ is a frame representation with frame multi[plicity greater than one. Then for any complete normalized tight frame vector η, it is a trace vector for $\pi(\mathcal{G})'$ but not a trace vector for $w^*(\pi(\mathcal{G}))$.*

Remark 6.13. The converse of this Proposition 6.11 is false. For example, let π, P and $\mathcal{M}$ be as in (ii) of Corollary 6.8. Since $\mathcal{M}$ is a factor and $tr(P) > \frac{1}{2}$, there is a subprojection $Q < P$ such that $Q \sim P^{\perp}$. Thus, by Lemma 6.4, the frame representation $(\lambda_{P^{\perp}}, \mathcal{G})$ has frame multiplicity greater than one. Therefore, from Proposition, $P^{\perp}\chi_e$ is not a trace vector for $w^*(\lambda_{P^{perp}}(\mathcal{G}))$. Note that χ is a trace vector for $w^*(\lambda(\mathcal{G})$. If $P\chi_e$ is a trace vector for $w^*(\lambda_P(\mathcal{G}))$, then for any $A, B \in w^*(\lambda(\mathcal{G}))$, we have

$$\begin{aligned}< ABP^{\perp}\chi_e,\ P^{\perp}\chi_e > &=< AB\chi_e,\ \chi_e > - < ABP\chi_e,\ P\chi_e > \\ &=< BA\chi_e,\ \chi_e > - < BAP\chi_e,\ P\chi_e > \\ &=< BAP^{\perp}\chi_e,\ P^{\perp}\chi_e >,\end{aligned}$$

which contradicts the fact that $P^{\perp}\chi_e$ is not a trace vector for $w^*(\lambda_{p^{\perp}}(\mathcal{G}))$.

6.3 Parameterizations of Frame Vectors

Given a frame representation $(\pi,\ H)$ for a group $\mathcal{G}$ with a complete normalized tight frame vector η. As pointed out in Proposition 3.13 that in general not every complete normalized tight frame vector can be obtained by applying a unitary operator in $\pi(\mathcal{G})'$ to η. However we will prove in Theorem 6.17 that the set all of complete normalized tight frame vectors for $\pi(\mathcal{G})$ is equal to the set $\{U\eta :\ U \in \mathbb{U}(w^*(\pi(\mathcal{G})))\}$, where $\mathbb{U}(\mathcal{S})$ denotes the set of all unitary operators in $\mathcal{S}$. Thus we have

$$\mathbb{U}(\pi(\mathcal{G})')\eta \subseteq \mathbb{U}(w^*(\pi(\mathcal{G})))\eta.$$

When realizing π as a subrepresentation λ_P of the left regular representation λ for some projection P in $\lambda(\mathcal{G})'$, then, from Proposition 3.13, the equality holds if and only if P is in the center of the von Neumann algebra $w^*(\lambda(\mathcal{G}))$.

Lemma 6.14. *Let π be a subrepresentation of the left regular representation λ of a group $\mathcal{G}$. Then for every complete normalized tight frame η for $\pi(\mathcal{G})$, there is a vector $\xi \in l^2(\mathcal{G})$ with the property that $\eta + \xi$ is a complete wandering vector for $\lambda(\mathcal{G})$.*

Proof. Let $\pi = \lambda_P$ for some projection in $\lambda(\mathcal{G})'$ and let $\psi = \chi_e$. Suppose that η is complete normalized tight frame vector for $\pi(\mathcal{G})$. Then, by Proposition 3.1, there is a partial isometry $V \in \lambda(\mathcal{G})'$ such that $V\psi = \eta$ and $VV^* = P$. Write $V^*V = Q$. Then $P \sim Q$. Since P and Q are finite projections, we have that $P^\perp \sim Q^\perp$ (cf [KR]). Let W be the partial isometry in $\lambda(\mathcal{G})'$ such that $WW^* = P^\perp$ and $W^*W = Q^\perp$. Write $U = V + W$. Then U is a unitary operator in $\lambda(\mathcal{G})'$, and so $U\psi$ is a complete wandering vector for $\lambda(\mathcal{G})$ (cf [DL], Proposition 1.3). Note that

$$UQ \;=\; (V+W)Q \;=\; VQ = \; VV^*V$$

and

$$PV \;=\; P(V+W) \;=\; PV = \; VV^*V.$$

Thus $UQ = PU$, and therefore

$$PU\psi \;=\; UQ\psi \;=\; VQ\psi \;=\; VV^*V\psi \;=\; P\eta \;=\; \eta.$$

Let $\xi = P^\perp U\psi$. Then ξ will satisfy our requirement. $\square$

Corollary 6.15. *Let $(\pi,\ H)$ be a frame representation of $\mathcal{G}$ and let π_Q be a subrepresentation of π. Then a vector $\eta \in QH$ is a complete normalized tight frame vector for $\pi_Q(\mathcal{G})$ if and only if $\eta = Q\xi$ for some complete normalized tight frame vector ξ of $\pi(\mathcal{G})$.*

Proof. By Theorem 3. 8, we can assume that $\pi = \lambda_P$ for some projection $P \in \lambda(\mathcal{G})'$ and $Q \in \lambda(\mathcal{G})'$. From Lemma 6.14, there is a complete wandering vector ϕ for $\lambda(\mathcal{G})$ such that $\eta = Q\phi$. Let $\xi = P\phi$. Then ξ is a complete normalized tight frame vector for $\pi(\mathcal{G})$ and $Q\xi = \eta$.

Lemma 6.16. *Let $\mathcal{R}$ be a von Neumann algebra on a Hilbert space H and let $P \in \mathcal{R}'$ be a projection. Suppose that $U \in \mathcal{R}|_{PH}$ is a unitary operator. Then there is a unitary operator $W \in \mathcal{R}$ such that $U = W|_{PH}$.*

Proof. Let $S \in \mathcal{R}|_{PH}$ be a self-adjoint operator such that $U = e^{iS}$. Then there is a operator $A \in \mathcal{R}$ such that $A|_{PH} = S$. Let $B = \frac{1}{2}(A + A^*)$. Note that $A^*|_{PH} = S^* = S$. It follows that $B|_{PH} = S$. Let $W = e^{iB}$. Then $W \in \mathcal{R}$ is unitary and $W|_{PH} = U$. $\square$

Theorem 6.17. *Let $(\pi,\ \mathcal{G},\ H)$ be a frame representation with a complete normalized tight frame vector η. Then the set of all complete normalized tight frame vectors for $\pi(\mathcal{G})$ equals $\mathbb{U}(w^*(\pi(\mathcal{G})))\eta$.*

Proof. By Theorem 3.6, we assume that $\pi = \lambda_P$ for some projection $P \in \lambda(\mathcal{G})'$ and $\eta = P\psi$, where $\psi = \chi_e$. For convenience, write $\mathcal{R} = w^*(\pi(\mathcal{G}))$ and $\mathcal{M} = w^*(\lambda(\mathcal{G}))$.

Let $J : l^2(\mathcal{G}) \to l^2(\mathcal{G})$ be defined by

$$JA\psi \ = \ A^*\psi$$

for all $A \in \mathcal{M}$. Then it is well-known (cf [KR]) that $\alpha : A \to JAJ$ is conjugate linear isomorphism from $\mathcal{M}$ onto $\mathcal{M}'$. It is also obvious that $J^2 = I$ and $J\psi = \psi$. Let $B = JAJ \in \mathcal{M}'$ for $A \in \mathcal{M}$. Then

$$\begin{aligned} JB\psi \ &= \ J(JAJ)\psi \ = A\psi = JA^*\psi \\ &= \ JA^*J\psi \ = \ B^*\psi. \end{aligned}$$

Thus $JB\psi = B^*\psi$ for all $B \in \mathcal{M}'$.

Let us first assume that $V \in \mathbb{U}(\mathcal{R})$ is a unitary operator. Then, by Lemma 6.16, there is a unitary operator $W \in \mathcal{M}$ such that $V = W|_H$. Since $W\psi = JW^*\psi = JW^*J\psi = \alpha(W^*)\psi$ and since $\alpha(W^*)$ is a unitary operator in $\mathcal{M}'$, we have that $W\psi$ is a complete wandering vector for $\lambda(\mathcal{G})$. Note that

$$U\eta = W\eta = WP\psi = PW\psi.$$

Thus $U\eta$ is complete normalized tight frame vector for $\pi(\mathcal{G})$.

Conversely, suppose that ξ is a complete normalized tight frame vector for $\pi(\mathcal{G})$. Then, by Lemma 6.11, there is a vector x in the range of $P^{\perp}$ such that $\xi + x$ is a complete wandering vector for $\lambda(\mathcal{G})$. Thus there is a unitary operator $U \in \mathcal{M}'$ such that $\xi = PU\psi$. Let $A = PJU^*JP$. Then $A \in \mathcal{R}$ is a unitary operator and

$$\begin{aligned} A\eta \ &= \ AP\psi \ = \ PJU^*JP\psi \ = \ PJU^*J\psi \\ &= \ PJU^*\psi \ = \ PU\psi \ = \ \xi. \end{aligned}$$

Thus $\xi \in \mathbb{U}(\mathcal{R})\eta$, as required. $\square$

6.4 Disjoint Group Representations

Suppose that $(\pi_1, \mathcal{G}, H_1)$ and $(\pi_2, \mathcal{G}, H_2)$ are frame representations with normalized tight frame vectors η and ξ, respectively. If $\{\pi_1(g)\eta\}_{g\in\mathcal{G}}$ and $\{\pi_2\xi\}_{g\in\mathcal{G}}$ are unitarily equivalent, then by Proposition 6.1, this equivalence induces the usual equivalence relation for the representations. It is possible that there exist normalized tight frame vectors η_1, ξ_1 for $(\pi_1, \mathcal{G}, H_1)$ and η_2, ξ_2 for $(\pi_2, \mathcal{G}, H_2)$, respectively, such that $\{\pi_1(g)\eta_1\}_{g\in\mathcal{G}}$ and $\{\pi_2(g)\eta_2\}_{g\in\mathcal{G}}$ are strongly disjoint, but $\{\pi_1(g)\xi_1\}_{g\in\mathcal{G}}$ and $\{\pi_2(g)\xi_2\}_{g\in\mathcal{G}}$ are unitarily equivalent (and hence are not strongly disjoint). For instance, let $\mathcal{G}$ be a group such that the corresponding left regular representation von Neumann algebra $\mathcal{M}$ is a factor. Choose a projection $P \in \mathcal{M}'$ such that $P \sim P^{\perp}$. Let $\pi_1 = \lambda|_P$ and $\pi_2 = \lambda|_{P^{\perp}}$. Also let $\eta_1 = P\chi_e$ and $\eta_2 = P^{\perp}\chi_e$. Then, clearly $\{\pi_1(g)\eta_1\}_{g\in\mathcal{G}}$ and $\{\pi_2(g)\eta_2\}_{g\in\mathcal{G}}$ are strongly disjoint normalized tight frames. Let $V \in \mathcal{M}'$ be the partial isometry such that $VV^* = P^{\perp}$ and $V^*V = P$. Let $\xi = V\chi_e = VP\chi_e$. Then ξ is a normalized tight frame vector for $\{\pi_2(g) : g \in \mathcal{G}\}$. But V induces a unitary equivalence between $\{\pi_1(g)\eta_1\}_{g\in\mathcal{G}}$ and $\{\pi_2(g)\xi\}_{g\in\mathcal{G}}$.

We recall from ([KR]) that two representations $(\pi, \mathcal{G}, H)$ and $(\sigma, \mathcal{G}, K)$ are said to be *disjoint* if no subrepresentation of π is equivalent to a subrepresentation of σ. Also recall that for a projection P in a von Neumann algebra $\mathcal{R}$ acting on a Hilbert space K, the *central carrier* C_P is the projection from K onto $[\mathcal{R}P(K)]$, where $[\cdot]$ denotes the norm closure. We conclude this chapter with the following characterizations for disjoint group representations in terms of disjointness of frame vectors.

Theorem 6.18. *Let $P, Q \in \mathcal{M}'$ be projections and let*

$$\pi : g \to \lambda(g)P, \qquad \sigma : g \to \lambda(g)Q$$

be the corresponding subrepresentations of λ. Then the following are equivalent:

(i) π and σ are disjoint,

(ii) for any complete normalized tight frame vectors η and ξ for $\{\pi(g) : g \in \mathcal{G}\}$ and $\{\sigma(g) : g \in \mathcal{G}\}$, respectively, $\{\pi(g)\eta\}_{g\in\mathcal{G}}$ and $\{\sigma(g)\xi\}_{g\in\mathcal{G}}$ are strongly disjoint,

(iii) for any complete normalized tight frame vectors η and ξ for $\{\pi(g) : g \in \mathcal{G}\}$ and $\{\sigma(g) : g \in \mathcal{G}\}$, respectively, $\{\pi(g)\eta\}_{g\in\mathcal{G}}$ and $\{\sigma(g)\xi\}_{g\in\mathcal{G}}$ are disjoint,

(iv) for any complete normalized tight frame vectors η and ξ for $\{\pi(g) : g \in \mathcal{G}\}$ and $\{\sigma(g) : g \in \mathcal{G}\}$, respectively, $\{\pi(g)\eta\}_{g\in\mathcal{G}}$ and $\{\sigma(g)\xi\}_{g\in\mathcal{G}}$ are weakly disjoint.

Proof. We only need to prove $(iv) \Rightarrow (i) \Rightarrow (ii)$. For $(iv) \Rightarrow (i)$, suppose that π and σ are not disjoint. Then, by Theorem 10.3.3 and Proposition 6.1.8 in [KR], there exist nonzero subprojections $E < P$ and $F < Q$ in $\mathcal{M}'$ such that $E \sim F$, where $\mathcal{M} = w^*(\lambda(\mathcal{G}))$. Let $V : EH \to FH$ be the partial isometry inducing the equivalence of E and F. Let $\eta = P\chi_e$. Then η is a complete normalized tight frame vector for $\{\pi(g) : g \in \mathcal{G}\}$. Since $VE\chi_e$ is a complete normalized tight frame vector for $\lambda_F(\mathcal{G})$, it follows, by Corollary 6.15, that there is a complete normalized tight frame vector ξ for $\sigma(\mathcal{G})$ such that $VE\chi_e = F\xi$. Note that

$$E\pi(g)\eta = \pi(g)E\eta \;=\; \pi(g)E\chi_e$$

and

$$F\sigma(g)\xi = \sigma(g)F\xi = \sigma(g)VE\eta \;=\; \sigma(g)VE\chi_e.$$

Thus $\{E\pi(g)\eta\}_{g\in\mathcal{G}}$ and $\{F\sigma(g)\xi\}_{g\in\mathcal{G}}$ are unitarily equivalent frames, which contradicts (iv). Hence $(iv) \Rightarrow (i)$.

For $(i) \Rightarrow (ii)$, let η and ξ be complete normalized tight frame vectors for $\{\pi(g) : g \in \mathcal{G}\}$ and $\{\sigma(g) : g \in \mathcal{G}\}$, respectively. Define V and U by

$$Vx = \sum_g < x, \pi(g)\eta > \chi_g$$

and

$$Uy = \sum_g < y, \sigma(g)\xi > \chi_g,$$

where $x \in PH$, $y \in QH$ and χ_g is the characteristic function at point g. Let E and F be the projections onto $ran(V)$ and $ran(U)$, respectively. Then $E \sim P$ and $F \sim Q$ in $\mathcal{M}'$. By the proof of Lemma 6.4, $\{\pi(g)\eta\}_{g\in\mathcal{G}}$ and $\{\sigma(g)\xi\}_{g\in\mathcal{G}}$ are strongly disjoint if and only if $E \perp F$. Suppose that E is not orthogonal to F. Then $C_E C_F \neq 0$. Thus $C_P C_Q \neq 0$ since $C_P = C_E$ and $C_F = C_Q$. By Theorem 10.3.3 (iii), π and σ are not disjoint. Therefore $(i) \Rightarrow (ii)$. $\square$

In the case that $\mathcal{G}$ is an abelian group, we know from Proposition 3.18 that there is only one unitary equivalence class of normalized tight frame vectors. Thus Theorem 6.18 implies the following:

Corollary 6.19. *Let $\mathcal{G}$ be an abelian group, and let π and σ be two frame representations with normalized tight frame vectors η and ξ, respectively. Then the following are equivalent:*

(i) $\{\pi(g)\eta\}_{g\in\mathcal{G}}$ and $\{\sigma(g)\xi\}_{g\in\mathcal{G}}$ are strongly disjoint,

(ii) $\{\pi(g)\eta\}_{g\in\mathcal{G}}$ and $\{\sigma(g)\xi\}_{g\in\mathcal{G}}$ are disjoint,

(iii) $\{\pi(g)\eta\}_{g\in\mathcal{G}}$ and $\{\sigma(g)\xi\}_{g\in\mathcal{G}}$ are weakly disjoint.

Chapter 7

Concluding Remarks

7.1. Spectral families of frames:

As in Example A_2, if $(\Omega,\ \mu)$ is a measurable space and $\{f_n\}$ is an orthonormal basis for $L^2(\Omega,\mu)$, then for each measurable subset E of Ω, $\{P_E f_n\}$ is a normalized tight frame for $L^2(E)$, where P_E is the projection from $L^2(\Omega,\mu)$ onto $L^2(E)$. We have pointed out in Corollary 3.10 that every normalized tight frame induced by a unitary and a frame vector is unitarily equivalent to $\{e^{ins}|_E\}$ for some measurable subset E of $\mathbb{T}$. This is generic for arbitrary normalized tight frames, and in fact for commutative normalized tight frame families.

We recall that a family of normalized tight normalized tight frames is said to be commutative if the family of the projections for the ranges of their frame transforms is commutative. We have the following *Spectral Theorem.*

Theorem 7.1. *Suppose that $\{\{x_{in}\}_{n\in\mathbb{J}} : i \in \mathcal{I}\}$ is a commutative family of normalized tight frames. Then there exists a locally compact space Ω, a Borel measure μ on Ω, a fixed orthonormal basis $\{f_n\}_{n\in\mathbb{J}}$ for $L^2(\Omega,\mu)$, and a family $\{E_i : i \in \mathbb{I}\}$ of Borel subsets of Ω such that for each $i \in \mathbb{I}$, $\{x_{in}\}_{n\in\mathbb{J}}$ is (separately) unitarily equivalent to the normalized tight frame $\{P_{E_i} f_n\}_{n\in\mathbb{J}}$.*

Proof. Let θ_i be the frame transform for $\{x_{in}\}_{n\in\mathbb{J}}$ and let Q_i be the orthogonal projection from $l^2(\mathbb{J})$ onto the range of θ_i. Then $\{Q_i :\ i \in \mathcal{I}\}$ is commutative family of projections. Suppose that $\mathcal{M}$ is the maximal von Neumann algebra containing $\{Q_i\}$. Then it is well known that there exists a locally compact space Ω and a Borel measure μ on Ω such that there is a unitary transform $W\colon\ l^2(\mathbb{J}) \to L^2(\Omega,\mu)$ with the property that $W\mathcal{M}W^* = \{M_f :\ f \in L^\infty(\Omega,\mu)$, where M_f is the multiplication operator multiplied by f. In particular WQ_iW^* is a projection in $\{M_f :\ f \in L^\infty(\Omega,\mu)$. Therefore there is a measurable subset E_i of Ω such that $WQ_iW^* = P_{E_i}$.

Let $\{e_n\}$ be the standard orthonormal basis for $l^2(\mathbb{J})$. Then each normalized tight frame $\{x_{in}\}_{n\in\mathbb{J}}$ is unitarily equivalent (by the frame transform) to $\{Q_ie_n\}$, which in turns is unitarily equivalent to $\{P_{E_i}f_n\}$, where $\{f_n\} = \{We_n\}$ is an orthonormal basis. Thus $\{x_{in}\}_{n\in\mathbb{J}}$ is unitarily equivalent to $\{P_{E_i}f_n\}$, as required. $\square$

We note that although every normalized tight frame in the commutative family is unitarily equivalent to some $\{P_E f_n\}$, the unitary operators implementing the equivalence can be different from each other. So we pose the following problem:

Problem G: Is there an abstract characterization of those commutative families, as in Theorem 7.1, for which all the unitary equivalences, as above, can be implemented by a *single* unitary operator.

7.2. A Joint Project with Pete Casazza

We comment on a subsequent project that was motivated to a large extent by Remark 1.8 in the present manuscript. In Chapter 1 we pointed out that any frame can be dilated to a Riesz basis for some larger Hilbert space. This result can be extended considerably. Given a frame $\{x_n\}$ and one of its alternate duals $\{y_n\}$ on a Hilbert space H, a natural question is: Can we dilate $\{x_n\}$ to a Riesz basis $\{z_n\}$ for some Hibert space $K \supseteq H$ such that $x_n = Pz_n$ and $y_n = P(z_n^*)$ for all n, where P is the projection from K onto H and $\{z_n^*\}$ is the unique (canonical) dual of the Riesz basis $\{z_n\}$? By using the properties of disjoint frames we can prove this is true and in fact in a joint work with Casazza, which will appear elsewhere, we have proven this result even for an appropriate notion of Banach space frames. Moreover, it turns out that a sequence $\{x_n\}_{n\in\mathbb{J}}$ in an infinite dimensional Hilbert space H can fail to be a frame for H in the (Hilbertian) sense of definition (1) in Chapter 1, and yet there may be a Banach space M which is not a Hilbert space and a sequence $\{y_n\}_{n\in\mathbb{J}}$ in M such that the inner direct sum $\{x_n \oplus y_n : n \in \mathbb{J}\}$ is a bounded unconditional basis for the direct sum Banach space $H \oplus M$. (Since all the direct sum norms on $H \oplus M$ are equivalent it does not matter which one we take.) Such sequence is then a *non-Hilbertian frame*, in the sense of Remark 1.8, for the Hilbert space H. It turns out that many of the generalized frames for the Hilbert space $L^2(\mathbb{R}^n)$ investigated in [FGWW], in particular, are actually non-Hilbertian frames in this sense: It can be proven that they are inner direct summands of bounded unconditional bases. There are also connections of our "direct summand of bases" interpretation of frames with the established theory in the literature of Banach frames and atomic decompositions (c.f. [CH]). Because this is a separate project we will not go into any details on this in the present article.

7.3. A Matrix Completion Characterization of Frames

Suppose that $H_n := \mathbb{C}^n$ is a finite dimensional Hilbert space with standard orthonormal basis $\{e_1, \ . \ . \ . \ , e_n\}$, where e_i is the column vector which has i-th coordinate vector 1 and other coordinates 0. If $l \geq n$ and if $\{x_1, \ . \ . \ . \ , x_l\} \subset H_n$, then Theorem 1.7 implies that $\{x_i\}_{i=1}^{l}$ is a frame for H_n if and only if the $n \times l$ matrix whose column vectors are $x_1, \ . \ , \ . \ , x_l$ constitutes the first n rows of an $l \times l$ nonsingular matrix. Moreover, $\{x_i\}_{i=1}^{l}$ is a normalized tight frame if and only if the nonsingular matrix can be taken to be unitary. This is true because an $l \times l$ matrix is nonsingular if and only if its column vectors form a basis, and is unitary if and only if the basis is an orthonormal basis. Viewed like this, the question of whether a given l-tuple of vectors is a frame is really a matrix-completion problem of a particularly elementary nature. In fact, it is clear that this matricial characterization of frames is valid for infinite dimensional Hilbert space as well, and perhaps adds some addtional perspective to the methods and results in this manuscript.

7.4 Some Acknowledgements

(a) After this manuscript was nearly completely written it was pointed out to us by P. Casazza that James R. Holub also made the observation in [Ho] that a general frame sequence indexed by $\mathbb{N}$ is isomorphic to $\{Pe_n\}$, where $\{e_n\}$ is the standard orthonormal basis for $l^2(\mathbb{N})$ and P is some projection in $B(l^2(\mathbb{N}))$. However, he used it in [Ho] for completely different purposes than we have used it in this paper. In particular, our notions of complementary frame and alternate duals seem to be new in our paper and was not observed in [Ho], and the fact (Corollary 2.7) that the similarity classes (unitary equivalence classes) of frames (normalized tight frames) indexed by $\mathbb{J}$ is in $1-1$ *correspondence* with the set of orthogonal projections in $B(l^2(\mathbb{J}))$ seems to be new.

(b) In September 1997, after this manuscript was complete, we learned in conversations with Ingrid Daubechies and Michael Lacey at the Wabash Mini-Conference that some other researchers have observed the dilation point of view for frames we independently observed and utilized in Chapter 1, and we thank both of them for providing us with this information. This is not at all surprising especially in view of the simplicity of the concept. These ideas have apparently not yet surfaced in the literature-at least we were unaware of them. In particular, we learned that there is a certain degree of overlap between some parts of Chapters 1, 2 and 5 in the present manuscript and some parts of the thesis work of a current student of Daubechies, Radu Balan, who has independently worked with the notions of "disjointness" or " orthogonality" of frames concerning potential application properties of frame wavelet n-tuples, and in fact apparently along the same lines we outlined in our Remark 2.27. Except for these items, we know of no overlap between other

aspects of the work we present in this manuscript and the work of other researchers in frame theory. We would not be surprised to hear of more instances. However, many of the types of problems we have addressed are quite different from those usually addressed by applications-oriented researchers.

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