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FREDHOLM AND INVERTIBLE n-TUPLES OF OPERATORS. THE DEFORMATION PROBLEM

BY

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ABSTRACT. Using J. L. Taylor's definition of joint spectrum, we study Fredholm and invertible n-tuples of operators on a Hilbert space. We give the foundations for a "several variables" theory, including a natural generalization of Atkinson's theorem and an index which well behaves. We obtain a characterization of joint invertibility in terms of a single operator and study the main examples at length. We then consider the deformation problem and solve it for the class of almost doubly commuting Fredholm pairs with a semi-Fredholm coordinate.

1. Introduction.

1. Let T be a (bounded linear) operator on a Banach space $\mathfrak{X}$. T is said to be invertible if there exists an operator S on $\mathfrak{X}$ such that $TS = ST = 1_{\mathfrak{X}}$, the identity operator on $\mathfrak{X}$. By the Open Mapping Theorem, this is equivalent to $\ker T = (0)$ and $R(T) = \text{range of } T = \mathfrak{X}$. The last formulation does not rely upon the existence of an inverse for T, but rather on the action of the operator T. When T is replaced by an n-tuple of commuting operators, several definitions of nonsingularity exist. J. L. Taylor [19] has obtained one which reflects the actions of the operators, by considering the Koszul complex associated with the n-tuple.

2. In this paper we develop a general "several variables" theory on the basis of Taylor's work and study commuting and almost commuting (= commuting modulo the compacts) n-tuples of operators on a Hilbert space $\mathcal{H}$. We obtain a characterization of joint invertibility in terms of the invertibility of a single operator, which is essential for our approach. From that we get a number of corollaries which generalize nicely the known elementary results in "one variable". At the same time, the referred characterization allows us to define a continuous, invariant under compact perturbations, integer-valued index on the class of Fredholm n-tuples (those almost commuting n-tuples which are invertible in the Calkin algebra). This index extends the classical one for Fredholm operators. We prove that an almost commuting n-tuple of essentially normal operators with all commutators in trace class has index zero ($n \geqslant 2$) and that a natural generalization of Atkinson's theorem holds for n-tuples.

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[1]Some results of this paper are contained in the author's Ph.D. dissertation presented to the State University of New York at Stony Brook and written under the direction of Ronald G. Douglas.

3. It is well known that the invertible operators on a Hilbert space $\mathcal{H}$ form a path-connected set. The analogous question for n-tuples has been studied in [7]. Also, index is the only invariant for the arcwise components of the class of Fredholm operators. The corresponding problem for n-tuples is called the deformation problem. Since our index is continuous, it is certainly an invariant for the path-components of the class of Fredholm n-tuples. In [9], R. G. Douglas has shown that indeed index is the only invariant in the class of essentially normal n-tuples. In the second part of this paper we prove that index is the only invariant for the path-components of the class of almost doubly commuting Fredholm pairs with a semi-Fredholm coordinate. In particular, we show that on $H^2(S^1 \times S^1)$, the pair (W_1, W_2) can be path-connected with (W_1^*, W_2^*) in the Fredholm class, where W_i is the operator of multiplication by the coordinate function z_i $(i = 1, 2)$.

4. The organization of the paper, intended to be expository on the subject, is as follows. Part I is devoted to the study of the basic properties of Fredholm and invertible n-tuples. It comprises §§2–10. Part II deals with the deformation problem and open questions. It includes §§11–16.

In §2 we give a brief summary of notation, the Koszul complex and Taylor's definition and main results. We also include some additional facts on the Koszul complex and obtain a matrix representation for an n-tuple. We devote §3 to state and prove the said characterization of invertibility and to deduce a number of related results. We reserve §4 to study the main examples, multiplication by the coordinates z_i on both $H^2(S^{2n-1})$ and $H^2(S^1 \times \cdots \times S^1)$. In §5 a couple of propositions concerning algebraic manipulations of coordinates are obtained. In §6 we give a natural generalization of the classical theorem of Atkinson. index is presented in §7, along with the proofs of continuity, invariance under compact perturbations and ontoness. An alternative definition, using the Euler characteristic for a chain complex, is also given there. In §8 we calculate indices for the n-tuples in §4 and apply them to find their spectra. We give in §9 a number of propositions that enable us to compute indices of n-tuples related in different ways. We conclude Part I with the theorem on essentially normal n-tuples with all commutators in trace class, done in §10.

Part II begins with a section on general facts on path-connectedness of Fredholm n-tuples. We then give in §12 a detailed proof for the essentially normal case, following the outline in [9]. In §13 we show that $T_z = (T_{z_1}, \ldots, T_{z_n})$ on $H^2(S^{2n-1})$ can be path-connected to $W = (W_1, \ldots, W_n)$ on $H^2(S^1 \times \cdots \times S^1)$ (to be precise, to a copy of W on $H^2(S^{2n-1})$). This result is central to our proof of the deformation problem for the class of almost doubly commuting Fredholm pairs with a semi-Fredholm coordinate, which we give in §14. In §15 we state and prove some additional related results. Finally, §16 is devoted to the concluding remarks and open problems.

I. Fredholm and invertible n-tuples

2. The joint spectrum.

1. Throughout this paper, $\mathcal{H}$ will denote a (complex) Hilbert space, $\mathcal{L}(\mathcal{H})$ the algebra of (bounded linear) operators on $\mathcal{H}$, $\mathcal{K}(\mathcal{H})$ the ideal of compact operators

and $\mathfrak{Q}(\mathcal{H})$ the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, with corresponding Calkin map π: $\mathcal{L}(\mathcal{H}) \to \mathfrak{Q}(\mathcal{H})$. We shall agree to denote the elements of $\mathcal{L}(\mathcal{H})$ by capital letters and those of $\mathfrak{Q}(\mathcal{H})$ by the corresponding small ones; for example, if A and a are in the context, A will denote an operator, a an element of $\mathfrak{Q}(\mathcal{H})$ and $\pi(A) = a$. $H^2(S^{2n-1})$ will denote the Hilbert space of square summable boundary values of holomorphic functions on the interior of the unit ball B^{2n} in $\mathbf{C}^n$, while $H^2(S^1 \times \cdots \times S^1)$ will be the space of square summable boundary values of holomorphic functions on the interior of the polydisc of multiradius 1. There are natural bases for these spaces, namely,

$$e_k = c_k z^k, \qquad k \in \mathbf{Z}_+^n, z^k = z_1^{k_1} \cdot \ldots \cdot z_n^{k_n},$$

$$c_k = \frac{1}{\sqrt{2\pi^n}} \sqrt{\frac{(n + |k| - 1)!}{k!}}\,, \qquad k! = k_1! \cdot \ldots \cdot k_n!, |k| = \sum_{i=1}^{n} k_i$$

for $H^2(S^{2n-1})$ and $f_k = z^k/\sqrt{(2\pi)^n}$, $k \in \mathbf{Z}_+^n$, for $H^2(S^1 \times \cdots \times S^1)$. We shall denote by T_{z_i}, W_i $(i = 1, \ldots, n)$ the operators of multiplication by the coordinate z_i on $H^2(S^{2n-1})$ and $H^2(S^1 \times \cdots \times S^1)$, respectively. Thus, $T_{z_i} e_k = (c_k/c_{k^{(i)}}) e_{k^{(i)}}$, and $W_i f_k = f_{k^{(i)}}$, where $k^{(i)} = (k_1, \ldots, k_i + 1, \ldots, k_n)$.

2. Let E^n be the exterior algebra on n generators, that is, E^n is the complex algebra with identity e generated by indeterminates $e_1, \ldots, e_n$ such that $e_i \wedge e_j = -e_j \wedge e_i$, for all i, j, where $\wedge$ denotes multiplication. E^n is graded, $E^n = \bigoplus_{k=-\infty}^{\infty} E_k^n$, with $E_k^n \wedge E_l^n \subset E_{k+l}^n$. The elements $e_{j_1} \wedge \cdots \wedge e_{j_k}$, $1 \leqslant j_1 < \cdots < j_k \leqslant n$ form a basis for E_k^n $(k > 0)$, while $E_0^n = \mathbf{C}e$ and $E_k^n = (0)$ when $k > n$, $k < 0$. Also $E_n^n = \mathbf{C}(e_1 \wedge \cdots \wedge e_n)$. Moreover, $\dim E_k^n = \binom{n}{k}$, so that, as a vector space over $\mathbf{C}$, E_k^n is isomorphic to $\mathbf{C}^{\binom{n}{k}}$. For $\mathfrak{X}$ a Banach space and $a_1, \ldots, a_n$ a commuting family of (bounded linear) operators on $\mathfrak{X}$, we consider $E_k^n(\mathfrak{X}) = E_k^n \otimes_{\mathbf{C}} \mathfrak{X}$ (notice that, since E_k^n is a finite dimensional vector space, all norms on $E_k^n(\mathfrak{X})$ are equivalent) and define $d_k^{(n)}$: $E_k^n(\mathfrak{X}) \to E_{k-1}^n(\mathfrak{X})$ by

$$d_k^{(n)}(x \otimes e_{j_1} \wedge \cdots \wedge e_{j_k}) = \sum_{i=1}^{k} (-1)^{i+1} a_{j_i} x \otimes e_{j_1} \wedge \cdots \wedge \hat{e}_{j_i} \wedge \cdots \wedge e_{j_k}$$

when $k > 0$ (here $\hat{}$ means deletion), and $d_k^{(n)} = 0$ when $k < 0$, $k > n$.

A straightforward computation shows that $d_k^{(n)} \circ d_{k+1}^{(n)} = 0$ for all k, so that $\{E_k^n(\mathfrak{X}), d_k^{(n)}\}_{k \in \mathbf{Z}}$ is a chain complex, called the Koszul complex for $a = (a_1, \ldots, a_n)$ and denoted $E(\mathfrak{X}, a)$ (cf. [**19**]).

3. We now explain a recursive method to obtain the $d_k^{(n)}$'s. We split the basis of E_k^n into

$$B_1 = \{e_{j_1} \wedge \cdots \wedge e_{j_k} : 1 \leqslant j_1 < \cdots < j_k \leqslant n - 1\}$$

and

$$B_2 = \{e_{j_1} \wedge \cdots \wedge e_{j_{k-1}} \wedge e_n : 1 \leqslant j_1 < \cdots < j_{k-1} \leqslant n - 1\} \quad \text{for } k \geqslant 1, n > 1.$$

We observe that E_k^{n-1} is precisely the subspace of E_k^n generated by B_1 and that a natural isomorphism can be established between E_{k-1}^{n-1} and the subspace of E_k^n

generated by B_2. E_k^n can then be identified in a natural way with $E_k^{n-1} \oplus E_{k-1}^{n-1}$ ($k \geq 1, n > 1$). It is not hard to see that $d_k^{(n)}$ takes the matrix form:

$$d_k^{(n)} = \begin{bmatrix} d_k^{(n-1)} & (-1)^{k+1}\operatorname{diag}(a_n) \\ 0 & d_{k-1}^{(n-1)} \end{bmatrix} \qquad (n > 1, k \geq 1),$$

where $\operatorname{diag}(a_n)$ is meant to be a diagonal matrix with constant diagonal entry a_n. It will often happen that the a_i's belong to an algebra with involution $*$; in that case we define $\hat{a}$ to be

$$\begin{bmatrix} d_1 & & \\ d_2^* & d_3 & \\ & d_4^* & \ddots \end{bmatrix} \in \mathcal{L}(\mathcal{X} \otimes \mathbf{C}^{2^{n-1}}),$$

where $d_i = d_i^{(n)}$, d_i^* is the adjoint matrix of d_i in the obvious way and all entries not explicitly described are zeros. For instance,

$$(a_1, a_2)\hat{} = \begin{pmatrix} a_1 & a_2 \\ -a_2^* & a_1^* \end{pmatrix}.$$

We notice that $\hat{a}$ is invertible if and only if $d_k^* d_k + d_{k+1} d_{k+1}^*$ is invertible (all k). Furthermore, $(a_1, \ldots, a_n)\hat{}$ is, up to permutations of rows and columns, $((a_1, \ldots, a_{n-1})\hat{}, \operatorname{diag}(a_n))\hat{}$. Finally, $(1, 0, \ldots, 0)\hat{} = 1_{\mathcal{X} \otimes \mathbf{C}^{2^{n-1}}}$, so that this n-tuple deserves to be called the identity n-tuple. We shall often denote it by 1.

In **[21]**, Vasilescu gives another way of assigning a matrix to a commuting n-tuple of operators on a Hilbert space which turns out to be selfadjoint, acting on the direct sum of 2^n copies of the space. For our purposes, however, our construction will be more advantageous, especially when studying the index of an almost commuting n-tuple of operators, which will be defined in terms of the index of the corresponding $\hat{}$.

4. We can now give the basic definitions (cf. **[19]**).

DEFINITION 2.1. Let $a = (a_1, \ldots, a_n)$ be a commuting n-tuple of operators on a Banach space $\mathcal{X}$. We define a to be invertible in case its associated Koszul complex $E(\mathcal{X}, a)$ is exact, that is, $\ker d_k^{(n)} = \operatorname{ran} d_{k+1}^{(n)}$ for all k. The spectrum $\operatorname{Sp}(a, \mathcal{X})$ is the set of n-tuples λ of scalars such that $a - \lambda = (a_1 - \lambda_1, \ldots, a_n - \lambda_n)$ is not an invertible n-tuple. In **[19]**, J. L. Taylor showed that, if $\mathcal{X} \neq (0)$, then $\operatorname{Sp}(a, \mathcal{X})$ is a nonempty, compact subset of the polydisc of multiradius $r(a) = (r(a_1), \ldots, r(a_n))$, where $r(a_i)$ is the spectral norm of a_i (see also **[21]** for a different proof). Moreover, if $s\colon \{1, \ldots, j\} \to \{1, \ldots, n\}$ is an injection, $s^*a = (a_{s(1)}, \ldots, a_{s(j)})$ and $s^*z = (z_{s(1)}, \ldots, z_{s(j)})$, then $\operatorname{Sp}(s^*a, \mathcal{X}) = s^* \operatorname{Sp}(a, \mathcal{X})$. In particular, any permutation of an invertible n-tuple is invertible.

Taylor also gave the following criterion for invertibility.

PROPOSITION 2.2. *Let a be as before and $\mathcal{B}$ be some complex algebra containing the a_i's in its center. If there exist $b_1, \ldots, b_n \in \mathcal{B}$ such that $\sum_{i=1}^n a_i b_i = 1$, then a is invertible.*

The preceding sufficient condition actually provides another way of defining invertibility. To be precise, we say that a is invertible with respect to an algebra $\mathfrak{B}$ containing the a_i's in its center if one can find $b_1, \ldots, b_n \in \mathfrak{B}$ satisfying $\sum_{i=1}^n a_i b_i = 1$. The spectrum so obtained is denoted by $\mathrm{Sp}_{\mathfrak{B}}(a)$. Proposition 2.2 then says that $\mathrm{Sp}(a, \mathfrak{X}) \subset \mathrm{Sp}_{\mathfrak{B}}(a)$.

If we denote by $\mathcal{Q}'$ the commutant of the algebra $\mathcal{Q}$ and by (a) the Banach algebra generated by the a_i's, it follows that $\mathrm{Sp}(a, \mathfrak{X}) \subset \mathrm{Sp}_{(a)'}(a) \subset \mathrm{Sp}_{(a)''}(a) \subset \mathrm{Sp}_{(a)}(a)$.

There are easy examples of proper inclusion for all but the first containment, which can also be proper. Taylor gave in **[19]** an example using a 5-tuple. In a written communication to R. G. Douglas, however, he mentioned the fact that (W_1, W_2) on $H^2(\mathbf{D} \times \mathbf{D})$ (W_i standing for multiplication by z_i $(i = 1, 2)$) is an example where proper inclusion also holds.

We now proceed to state the functional calculus.

PROPOSITION 2.3 (THEOREM 4.8 IN **[20]**). *Let $a = (a_1, \ldots, a_n)$ be a commuting n-tuple in $\mathfrak{L}(\mathfrak{X})$, U be a domain containing* $\mathrm{Sp}(a, \mathfrak{X})$ *and $f_1, \ldots, f_m$ be holomorphic on U. Let $f\colon U \to C^m$ be defined by $f(z) = (f_1(z), \ldots, f_m(z))$ and $f(a)$ be the m-tuple $(f_1(a), \ldots, f_m(a))$. Then* $\mathrm{Sp}(f(a), \mathfrak{X}) = f(\mathrm{Sp}(a, \mathfrak{X}))$.

3. Fredholm and invertible n-tuples. An equivalence.

1. Let $\mathcal{H}$ be a Hilbert space, $\{n_k\}_{k\in\mathbf{Z}}$ be a sequence of nonnegative numbers with $n_k = 0$ for $k < 0$, $\mathcal{H}_k = \mathcal{H} \otimes \mathbf{C}^{n_k}$ and $D_k \in \mathfrak{L}(\mathcal{H}_k, \mathcal{H}_{k-1})$ such that $D_k D_{k+1}$ is compact for all k. We consider the system

$$\cdots \xrightarrow{D_{k+1}} \mathcal{H}_k \xrightarrow{D_k} \mathcal{H}_{k-1} \xrightarrow{D_{k-1}} \cdots \xrightarrow{D_2} \mathcal{H}_1 \xrightarrow{D_1} \mathcal{H}_0 \to 0, \tag{D}$$

and the complex

$$\cdots \xrightarrow{d_{k+1}} \mathfrak{Q}_k \xrightarrow{d_k} \mathfrak{Q}_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_2} \mathfrak{Q}_1 \xrightarrow{d_1} \mathfrak{Q}_0 \to 0, \tag{d}$$

where $\mathfrak{Q}_k = \mathfrak{Q}(\mathcal{H}) \otimes \mathbf{C}^{n_k}$ (n_k copies of the Calkin algebra) and d_k is the matrix associated to D_k in the canonical way (i.e., the entries of d_k are the projections onto $\mathfrak{Q}(\mathcal{H})$ of the entries of D_k).

If $A = (A_1, \ldots, A_n)$ is an almost commuting n-tuple of operators on $\mathcal{H}$ (i.e., $[A_i, A_j] = A_iA_j - A_jA_i \in \mathcal{K}(\mathcal{H})$ for all i, j), the Koszul system $D(A)$ is the one we get by taking $n_k = \binom{n}{k}$ and

$$D_k(x \otimes e_{j_1} \wedge \cdots \wedge e_{j_k}) = \sum_{i=1}^{k} (-1)^{i+1} A_{j_i} x \otimes e_{j_1} \wedge \cdots \wedge \hat{e}_{j_i} \wedge \cdots \wedge e_{j_k},$$

as in §2.2. Although $D_k D_{k+1}$ need not be zero this time, the compactness of the commutators forces it to be compact.

DEFINITION 3.1. A system (D) is said to be Fredholm if the associated complex (d) is exact (that is, $\ker d_k = \operatorname{ran} d_{k+1}$, for all k).

DEFINITION 3.2. An almost commuting n-tuple $A = (A_1, \ldots, A_n)$ is Fredholm (in symbols $A \in \mathfrak{F}$) if the associated Koszul system is Fredholm, i.e., if $a = (a_1, \ldots, a_n)$ is invertible.

DEFINITIONS 3.3. The spectrum $\mathrm{Sp}(A)$ of a commuting n-tuple A is $\mathrm{Sp}(A, \mathcal{H})$. The essential spectrum $\mathrm{Sp}_e(A)$ of an almost commuting n-tuple A is $\mathrm{Sp}(a, \mathcal{Q}(\mathcal{H}))$.

REMARK. Although we have not made any explicit reference to dimension($\mathcal{H}$), we shall always understand it is infinite in case the word compact is in the context.

2. The following proposition is a key result for our work.

PROPOSITION 3.4. *Let $\mathfrak{B}$ be any W^*-algebra or $\mathcal{Q}(\mathcal{H})$ (or $\mathcal{H}$), $0 \leq n_k \in \mathbf{Z}$, $n_k = 0$ for $k < 0$, $\mathfrak{B}_k = \mathfrak{B} \otimes \mathbf{C}^{n_k}$ and $d_k \in \mathcal{L}(\mathfrak{B}_k, \mathfrak{B}_{k-1})$ be an n_{k-1} by n_k matrix over $\mathfrak{B}$ (or $d_k \in \mathcal{L}(\mathcal{H}_k, \mathcal{H}_{k-1})$) with $d_k d_{k+1} = 0$ for all k. Then the complex $\cdots \to \mathfrak{B}_k \xrightarrow{d_k} \mathfrak{B}_{k-1} \to \cdots$ is exact (at every stage) if and only if $l_k = d_k^* d_k + d_{k+1} d_{k+1}^*$ is invertible (all k). (Here d_k^* is the matrix adjoint of d_k.)*

COROLLARY 3.5. *An almost commuting (respectively commuting) n-tuple $A = (A_1, \ldots, A_n)$ is Fredholm (respectively invertible) if and only if $L_k = D_k^* D_k + D_{k+1} D_{k+1}^*$ is Fredholm (respectively invertible) for all k, where $D_k = D_k(A_1, \ldots, A_n)$.*

PROOF. $\pi(L_k) = l_k$.

COROLLARY 3.6. *Let $A = (A_1, \ldots, A_n)$ be an almost commuting (resp. commuting) n-tuple of operators on $\mathcal{H}$. If $A \in \mathcal{F}$ (resp. A is invertible), so are $\sum_{i=1}^n A_i^* A_i$ and $\sum_{i=1}^n A_i A_i^*$.*

PROOF. $\sum_{i=1}^n A_i^* A_i = D_n^* D_n$ and $\sum_{i=1}^n A_i A_i^* = D_1 D_1^*$. But $L_n = D_n^* D_n$ and $L_0 = D_1 D_1^*$.

The statement in parentheses in Corollary 3.6 has been proved by Vasilescu in **[21]**.

PROOF OF THE PROPOSITION. (*Only if*) Since $\mathfrak{B}_{-1} = 0$, we have $d_0 = 0$. By exactness, d_1 is onto. Hence $d_1 d_1^*$ is invertible, or l_0 is invertible. Let us now assume that l_j is invertible for all $j \leq k$ and prove that so is l_{k+1}. We first need a direct sum decomposition of $\mathfrak{B}_{k+1}$ into $\ker d_{k+1} \dotplus \operatorname{ran} d_{k+1}^*$. Clearly $\ker d_{k+1} \cap \operatorname{ran} d_{k+1}^* = 0$. If $b \in \mathfrak{B}_{k+1}$, then $d_{k+1} b \in \mathfrak{B}_k = \operatorname{ran} l_k$, so that there exists $c \in \mathfrak{B}_k$ such that $d_{k+1} b = l_k c = d_k^* d_k c + d_{k+1} d_{k+1}^* c$. Then $d_{k+1}^* d_{k+1} b = d_{k+1}^* d_{k+1} d_{k+1}^* c$, because $d_k d_{k+1} = 0$. Thus $b - d_{k+1}^* c \in \ker d_{k+1}^* d_{k+1} = \ker d_{k+1}$, so that $b \in \ker d_{k+1} + \operatorname{ran} d_{k+1}^*$.

Once we have obtained such a decomposition, we can prove that l_{k+1} is onto (that is, invertible, being selfadjoint). Given $b \in \mathfrak{B}_{k+1}$, there exist $c \in \ker d_{k+1}$ and $d \in \operatorname{ran} d_{k+1}$ such that $b = c + d_{k+1}^* d$. (Notice that since l_{k-1} is invertible, $\mathfrak{B}_k = \ker d_k \dotplus \operatorname{ran} d_k^*$ and $d_{k+1}^* d_k^* = 0$, so that d can be chosen in $\ker d_k = \operatorname{ran} d_{k+1}$.)

Since $c \in \ker d_{k+1}$, exactness implies there is e in $\mathfrak{B}_{k+2}$ such that $c = d_{k+2} e$. Consequently,

$$b = d_{k+2} e + d_{k+1}^* d. \tag{1}$$

But $d = d_{k+1} f$ for some f in $\mathfrak{B}_{k+1}$. Moreover, by polar decomposition, $\operatorname{ran} d_{k+2} \subset \operatorname{ran}(d_{k+2} d_{k+2}^*)^{1/2}$, so that

$$d_{k+2} e = (d_{k+2} d_{k+2}^*)^{1/2} g \quad \text{for some } g \text{ in } \mathfrak{B}_{k+1}. \tag{2}$$

By the direct sum decomposition for $\mathfrak{B}_{k+1}$, $g = g_1 + d^*_{k+1}g_2$ with $g_1 \in \ker d_{k+1}$ and $g_2 \in \mathfrak{B}_k$. But then there is $h \in \mathfrak{B}_{k+2}$: $g_1 = d_{k+2}h \in \operatorname{ran} d_{k+2} \subset \operatorname{ran}(d_{k+2}d^*_{k+2})^{1/2}$, so that $g_1 = (d_{k+2}d^*_{k+2})^{1/2}m$ for some $m \in \mathfrak{B}_{k+1}$. Thus,

$$g = (d_{k+2}d^*_{k+2})^{1/2}m + d^*_{k+1}g_2. \tag{3}$$

Combining (1), (2) and (3) we get: $b = d_{k+2}e + d^*_{k+1}d = (d_{k+2}d^*_{k+2})^{1/2}g + d^*_{k+1}d_{k+1}f = d_{k+2}d^*_{k+2}m + (d_{k+2}d^*_{k+2})^{1/2}d^*_{k+1}g_2 + d^*_{k+1}d_{k+1}f = d_{k+2}d^*_{k+2}m + d^*_{k+1}d_{k+1}f$, since $(d_{k+2}d^*_{k+2})d^*_{k+1} = 0$ and therefore $(d_{k+2}d^*_{k+2})^{1/2}d^*_{k+1} = 0$.

To complete the proof, we observe that m can be chosen in $\ker d_{k+1}$ and f in $\operatorname{ran} d^*_{k+1}$. Thus $l_{k+1}(m + f) = d^*_{k+1}d_{k+1}f + d_{k+2}d^*_{k+2}m = b$, as desired.

(*If*) Assume that $d_kb = 0$. Then $l_kb = d_{k+1}d^*_{k+1}b$. Since l_k is invertible, $b = l_k^{-1}d_{k+1}d^*_{k+1}b$. Observe that l_k and $d_{k+1}d^*_{k+1}$ commute. Therefore $b = d_{k+1}d^*_{k+1}l_k^{-1}b \in \operatorname{ran} d_{k+1}$. Hence $\ker d_k \subset \operatorname{ran} d_{k+1}$. The other inclusion follows from $d_kd_{k+1} = 0$.

REMARK. Although the preceding proof made no distinction between a W^*-algebra or $\mathfrak{Q}(\mathcal{H})$ and a Hilbert space $\mathcal{H}$, it can actually be simplified in the latter case (for instance, the direct sum decomposition needs no proof and is orthogonal, see [7]).

3. We now derive a few more corollaries.

COROLLARY 3.7. *An almost doubly commuting (resp. doubly commuting) n-tuple $A = (A_1, \ldots, A_n)$ (i.e., $[A_i, A_j^*]$ is also compact (resp. zero) for all $i \neq j$) is Fredholm (resp. invertible) if and only if $\sum_{i=1}^n {}^fA_i$ is Fredholm (resp. invertible) for every function f: $\{1, \ldots, n\} \to \{0, 1\}$, where*

$${}^fA_i = \begin{cases} A_i^*A_i, & f(i) = 0, \\ A_iA_i^*, & f(i) = 1. \end{cases}$$

PROOF. A direct calculation shows that in this case $l_k = d_k^*d_k + d_{k+1}d^*_{k+1}$ is a block diagonal matrix of order $\binom{n}{k}$ whose diagonal entries are precisely the $\binom{n}{k}$ different combinations $\sum_{i=1}^n {}^fA_i$, for f: $\{1, \ldots, n\} \to \{0, 1\}$ with $\#\{i: f(i) = 0\} = k$.

COROLLARY 3.8. *An almost doubly commuting (resp. doubly commuting) n-tuple $A = (A_1, \ldots, A_n)$ of essentially hyponormal (resp. hyponormal) operators (i.e., $a_i^*a_i > a_ia_i^*$ (resp. $A_i^*A_i > A_iA_i^*$) for all $i = 1, \ldots, n$) is Fredholm (resp. invertible) if and only if $\sum_{i=1}^n A_iA_i^*$ is Fredholm (resp. invertible).*

PROOF. $\sum_{i=1}^n {}^fa_i > \sum_{i=1}^n a_ia_i^*$ (resp. $\sum_{i=1}^n {}^fA_i > \sum_{i=1}^n A_iA_i^*$) for all f: $\{1, \ldots, n\} \to \{0, 1\}$. Now use Corollary 3.7.

COROLLARY 3.9. *If the A_i's are essentially normal (resp. normal) and they almost commute (resp. commute), then $A = (A_1, \ldots, A_n)$ is Fredholm (resp. invertible) if and only if $\sum_{i=1}^n A_i^*A_i$ is Fredholm (resp. invertible).*

Corollary 3.9 says that for a commuting n-tuple of elements of $\mathfrak{L}(\mathcal{H})$ or $\mathfrak{Q}(\mathcal{H})$, the Koszul complex is exact iff it is exact at any stage, a natural generalization of a well-known "one variable" fact.

COROLLARY 3.10. *Let $A = (A_1, \ldots, A_n)$ be an essentially normal (resp. normal) n-tuple and $\mathfrak{M}$ be the maximal ideal space of the C^*-algebra generated by $a_1, \ldots, a_n$ (resp. $A_1, \ldots, A_n$). Then* $\mathrm{Sp}_e(A) = \mathfrak{M}$ *(resp.* $\mathrm{Sp}(A) = \mathfrak{M}$*), when $\mathfrak{M}$ is regarded as a subset of $\mathbf{C}^n$ under the homeomorphism $\phi \to (\phi(a_1), \ldots, \phi(a_n))$ (resp. $\phi \to (\phi(A_1), \ldots, \phi(A_n))$).*

PROOF. By the preceding corollary, A is Fredholm iff $\sum_{i=1}^n A_i^* A_i$ is Fredholm. Let $\mathfrak{B}$ be the C^*-algebra generated by $a_1, \ldots, a_n$. Then $\mathfrak{B} \cong C(\mathfrak{M})$. Therefore,

$$\lambda \notin \mathrm{Sp}_e(A) \Leftrightarrow A - \lambda \in \mathfrak{F} \Leftrightarrow \sum_{i=1}^n (A_i - \lambda_i)^*(A_i - \lambda_i) \in \mathfrak{F}$$

$$\Leftrightarrow \sum_{i=1}^n (a_i^* - \overline{\lambda_i})(a_i - \lambda_i) \text{ is invertible}$$

$$\Leftrightarrow \phi\left(\sum_{i=1}^n (a_i^* - \overline{\lambda_i})(a_i - \lambda_i)\right) \neq 0 \text{ for all } \phi \in \mathfrak{M}$$

$$\Leftrightarrow \sum_{i=1}^n |z_i - \lambda_i|^2 > 0 \text{ for all } z \in \mathfrak{M} \Leftrightarrow \lambda \notin \mathfrak{M}.$$

The statement in parentheses follows in the same way.

4. The following theorem gives a precise relation between invertibility for an n-tuple a and for its associated $\hat{a}$ (see §2.3).

THEOREM 1. *Let $a = (a_1, \ldots, a_n)$ be a commuting n-tuple of elements of a W^*-algebra $\mathfrak{B}$ (or $\mathfrak{L}(\mathfrak{H})$) acting on $\mathfrak{H}$ or $\mathfrak{B}$ (or on $\mathfrak{L}(\mathfrak{H})$). Then a is invertible if and only if $\hat{a}$ is invertible.*

PROOF. It is well known that $\hat{a}$ is invertible iff so are $\hat{a}^*\hat{a}$ and $\hat{a}\hat{a}^*$. An easy computation shows that $\hat{a}^*\hat{a}$ is a block diagonal matrix whose diagonal entries are the l_k's for odd k's (recall that $l_k = d_k^* d_k + d_{k+1} d_{k+1}^*$). Similarly $\hat{a}\hat{a}^*$ contains those l_k's with even k. The theorem now follows from Proposition 3.4.

We immediately get

COROLLARY 3.11. *An almost commuting (resp. commuting) n-tuple $A = (A_1, \ldots, A_n)$ of operators on $\mathfrak{H}$ is Fredholm (resp. invertible) iff so is $\hat{A} \in \mathfrak{L}(\mathfrak{H} \otimes \mathbf{C}^{2^{n-1}})$.*

COROLLARY 3.12. *Let A be a commuting n-tuple of operators on $\mathfrak{H}$. Then* $\mathrm{Sp}(A, \mathfrak{H}) = \mathrm{Sp}(A, \mathfrak{L}(\mathfrak{H}))$.

PROOF. This corollary states that these two notions of invertibility for A (when the A_i's act on $\mathfrak{H}$ and when they multiply on the left on $\mathfrak{L}(\mathfrak{H})$) are actually the same. It follows easily from Theorem 1 and the fact that it is true for singletons.

COROLLARY 3.13. *Let $\mathfrak{B}$ be a C^*-subalgebra of $\mathfrak{L}(\mathfrak{H})$ (resp. $\mathfrak{L}(\mathfrak{H})$) and $a = (a_1, \ldots, a_n)$ be a commuting n-tuple of elements of $\mathfrak{B}$. Then* $\mathrm{Sp}(a, \mathfrak{B}) \subset \mathrm{Sp}(a, \mathfrak{L}(\mathfrak{H}))$ *(resp.* $\mathrm{Sp}(a, \mathfrak{B}) \subset \mathrm{Sp}(a, \mathfrak{L}(\mathfrak{H}))$*). Consequently, if $\mathfrak{B}$ and $\mathfrak{C}$ are W^*-algebras containing the a_i's, then* $\mathrm{Sp}(a, \mathfrak{C}) = \mathrm{Sp}(a, \mathfrak{B})$ *(spectral permanence for W^*-algebras).*

PROOF. Assume that $\lambda \notin \mathrm{Sp}(a, \mathcal{L}(\mathcal{H}))$, i.e., $a - \lambda$ is invertible (acting on $\mathcal{L}(\mathcal{H})$). By Proposition 3.4, $l_k = d_k^* d_k + d_{k+1} d_{k+1}^*$ is invertible (in $M_{\binom{n}{k}}(\mathcal{L}(\mathcal{H}))$) for all k. By spectral permanence, l_k is then invertible in $M_{\binom{n}{k}}(\mathfrak{B})$ for all k. A look at the "if" part of the proof of Proposition 3.4 shows that $E(\mathfrak{B}, a - \lambda)$ is exact, or $\lambda \notin \mathrm{Sp}(a, \mathfrak{B})$. The statement in parentheses follows in the same way. The rest follows immediately from Theorem 1.

COROLLARY 3.14. *Let $A = (A_1, \ldots, A_n)$ be a Fredholm (resp. invertible) n-tuple, ϕ: $\{1, \ldots, n\} \to \{1, *\}$ be a function and $\phi(A_i) = A_i^{\phi(i)}$. Assume that $[\phi(A_i), \phi(A_j)]$ is compact (resp. zero) for all i, j. Then $\phi(A) = (\phi(A_1), \ldots, \phi(A_n))$ is Fredholm (resp. invertible). Consequently,* $\mathrm{Sp}_e(\phi(A), \mathcal{H}) = \{\phi(\lambda): \lambda \in \mathrm{Sp}_e(A, \mathcal{H})\}$ $(\mathrm{Sp}(\phi(A), \mathcal{H}) = \{\phi(\lambda): \lambda \in \mathrm{Sp}(A, \mathcal{H})\})$.

PROOF. We begin with the following observation: Let $a = (a_1, \ldots, a_n)$ be an n-tuple (not necessarily commuting) and $p \in S_n$ be a permutation. Let p^*a denote the n-tuple $(a_{p(1)}, \ldots, a_{p(n)})$ and d_k, d_k^p be the corresponding Koszul boundary maps. We can form

$$\hat{d} = \begin{bmatrix} d_1 & & \\ d_2^* & d_3 & \\ & & \ddots \end{bmatrix} \quad \text{and} \quad \widehat{d^p} = \begin{bmatrix} d_1^p & & \\ d_2^{p*} & d_3^p & \\ & & \ddots \end{bmatrix}$$

as in the commuting case. Then there exist unitaries U, V: $\mathcal{H} \otimes \mathbf{C}^{2^{n-1}} \to \mathcal{H} \otimes \mathbf{C}^{2^{n-1}}$ such that $\hat{d} = U\widehat{d^p}V$.

For, it is known that there exist unitaries $U_k \in \mathcal{L}(\mathcal{H} \otimes \mathbf{C}^{\binom{n}{k}})$ such that $U_k d_{k+1}^p = d_{k+1} U_{k+1}$ (see **[19]**). Then let

$$V = \begin{bmatrix} U_1^* & & \\ & U_3^* & \\ & & \ddots \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} U_0 & & \\ & U_2 & \\ & & \ddots \end{bmatrix}.$$

We also observe that $\hat{a}^*$ is, up to permutations of rows and columns, $(a_1^*, -a_2, \ldots, -a_n)\hat{}$. A combination of the preceding facts gives the desired conclusion.

COROLLARY 3.15. *Let $A = (A_1, A_2)$ be a doubly commuting pair. If A is invertible, then* $\ker A_1 \perp \ker A_2$.

PROOF. Assume that $A_1 x = 0$. Then $A_1 x + A_2^* \cdot 0 = 0$, so that there exists y such that $x = -A_2^* y$ and $0 = A_1 y$. In particular, x is in $\operatorname{ran} A_2^* \subset (\ker A_2)^\perp$, as desired.

COROLLARY 3.16. *The set of Fredholm (resp. invertible) n-tuples is an open subset of the set of almost commuting (resp. commuting) n-tuples.*

PROOF. The map $(A_1, \ldots, A_n) \to (A_1, \ldots, A_n)\hat{}$ is continuous.

The preceding corollary can be derived in a different way from the results in **[19]**. The continuity of the map $a \to \hat{a}$ can also be used to show that $\mathrm{Sp}(a, \mathfrak{X})$ is a compact subset of the polydisc of multiradius $r(a)$, when $\mathfrak{X}$ is a W^*-algebra, $\mathcal{H}$ or $\mathfrak{L}(\mathcal{H})$, totally independent of Taylor's paper. A straightforward calculation using $\hat{a}$

and $\hat{a}^*$ shows that $\mathrm{Sp}(a, \mathfrak{X}) \supset \sigma_l(a, \mathfrak{X})$, the left spectrum of a on $\mathfrak{X}$, which is nonempty by the results in [4], so that $\mathrm{Sp}(a, \mathfrak{X}) \neq \varnothing$ for $\mathfrak{X}$ as above.

4. Examples.

1. Any almost commuting n-tuple $A = (A_1, \ldots, A_n)$ with one of the A_i's Fredholm is Fredholm.

2. On $H^2(S^1 \times \cdots \times S^1)$, we consider $W = (W_1, \ldots, W_n)$, where W_i is the operator of multiplication by the coordinate z_i. Each W_i is an isometry whose range consists of all those $f \in H^2(S^1 \times \cdots \times S^1)$ such that $f(z) = \sum_{k \in \mathbf{Z}^n; k_i \geq 1} f_k z^k$.

W is a doubly commuting n-tuple of subnormal operators so that, by Corollary 3.8, W will be Fredholm once we show that $\sum_{i=1}^n W_i W_i^*$ is Fredholm. It is not hard to see, however, that $\sum_{i=1}^n W_i W_i^* \geq I - P_0$, where P_0 is the projection onto the constants. Thus, $\sum_{i=1}^n W_i W_i^*$ is Fredholm and, consequently, so is W.

3. We consider $T_z = (T_{z_1}, \ldots, T_{z_n})$ on $H^2(S^{2n-1})$, where T_{z_i} is the Toeplitz operator of multiplication by z_i.

Since $\sum_{i=1}^n T_{z_i}^* T_{z_i} = I$ and each T_{z_i} is essentially normal (see Coburn [5]), Corollary 3.9 implies that T_z is Fredholm.

5. Algebraic perturbations of coordinates.

1. The following propositions will be useful in dealing with the deformation problem.

PROPOSITION 5.1. *Let $\mathfrak{B}$ be a Banach algebra, $\mathfrak{X}$ be a Banach space which is a left $\mathfrak{B}$-module, $a_1, \ldots, a_n$ be commuting elements of $\mathfrak{B}$ and $v \in \mathfrak{B}$ be an invertible element that commutes with $a_2, \ldots, a_n$. Then the following conditions are equivalent*:

(i) *$a = (a_1, \ldots, a_n)$ is invertible.*

(ii) *$va = (va_1, a_2, \ldots, a_n)$ is invertible.*

(iii) *$av = (a_1 v, a_2, \ldots, a_n)$ is invertible.*

PROOF. We shall prove by induction that the Koszul complexes $E(\mathfrak{X}, a)$ and $E(\mathfrak{X}, va)$ are isomorphic, thus establishing (i) $\Leftrightarrow$ (ii). The equivalence of (i) and (iii) follows in the same way.

Assume that $n = 2$ (the result being obvious when $n = 1$); we have

$$E(\mathfrak{X}, a)\colon 0 \to \mathfrak{X} \xrightarrow{d_2} \mathfrak{X} \oplus \mathfrak{X} \xrightarrow{d_1} \mathfrak{X} \to 0$$

and

$$E(\mathfrak{X}, va)\colon 0 \to \mathfrak{X} \xrightarrow{\check{d}_2} \mathfrak{X} \oplus \mathfrak{X} \xrightarrow{\check{d}_1} \mathfrak{X} \to 0,$$

where

$$d_1 = (a_1\ a_2), \quad d_2 = \begin{pmatrix} -a_2 \\ a_1 \end{pmatrix}, \quad \check{d}_1 = (va_1\ a_2) \quad \text{and} \quad \check{d}_2 = \begin{pmatrix} -a_2 \\ va_1 \end{pmatrix}.$$

Define $T_0^{(2)}\colon \mathfrak{X} \to \mathfrak{X}$, $T_1^{(2)}\colon \mathfrak{X} \oplus \mathfrak{X} \to \mathfrak{X} \oplus \mathfrak{X}$ and $T_2^{(2)}\colon \mathfrak{X} \to \mathfrak{X}$ by $x \to vx$, $x \oplus y \to x \oplus vy$ and $x \to x$, respectively. Then

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{X} & \xrightarrow{d_2} & \mathfrak{X} \oplus \mathfrak{X} & \xrightarrow{d_1} & \mathfrak{X} & \to & 0 \\ & & \downarrow T_2^{(2)} & & \downarrow T_1^{(2)} & & \downarrow T_0^{(2)} & & \\ 0 & \to & \mathfrak{X} & \xrightarrow{\check{d}_2} & \mathfrak{X} \oplus \mathfrak{X} & \xrightarrow{\check{d}_1} & \mathfrak{X} & \to & 0 \end{array}$$

is a commutative diagram and $T_k^{(2)}$ is an isomorphism ($k = 0, 1, 2$). Therefore, $E(\mathfrak{X}, a)$ and $E(\mathfrak{X}, va)$ are isomorphic. We now define $T_k^{(m)}: \mathfrak{X}^{\binom{m}{k}} \to \mathfrak{X}^{\binom{m}{k}}$ by

$$T_k^{(m)} = \begin{pmatrix} T_k^{(m-1)} & 0 \\ 0 & T_{k-1}^{(m-1)} \end{pmatrix}$$

with respect to the decomposition $\mathfrak{X}^{\binom{m}{k}} = \mathfrak{X}^{\binom{m-1}{k}} \oplus \mathfrak{X}^{\binom{m-1}{k-1}}$, as we did in §2.3.

Assume that $E(\mathfrak{X}, (a_1, \ldots, a_{n-1}))$ and $E(\mathfrak{X}, (va_1, a_2, \ldots, a_{n-1}))$ are isomorphic with the isomorphism given by the $T_k^{(n-1)}$'s. Consider the following diagram:

$$\begin{array}{ccccccccccc}
0 & \to & \mathfrak{X}^{\binom{n}{n}} & \xrightarrow{d_n^{(n)}} & \cdots & \mathfrak{X}^{\binom{n}{k+1}} & \xrightarrow{d_{k+1}^{(n)}} & \mathfrak{X}^{\binom{n}{k}} & \xrightarrow{d_k^{(n)}} & \cdots & \mathfrak{X}^{\binom{n}{0}} & \to & 0 \\
 & & \downarrow T_n^{(n)} & & & \downarrow T_{k+1}^{(n)} & & \downarrow T_k^{(n)} & & & \downarrow T_0^{(n)} & & \\
0 & \to & \mathfrak{X}^{\binom{n}{n}} & \xrightarrow{\check{d}_n^{(n)}} & \cdots & \mathfrak{X}^{\binom{n}{k+1}} & \xrightarrow{\check{d}_{k+1}^{(n)}} & \mathfrak{X}^{\binom{n}{k}} & \xrightarrow{\check{d}_k^{(n)}} & \cdots & \mathfrak{X}^{\binom{n}{0}} & \to & 0
\end{array}$$

Since the $T_k^{(n)}$'s are clearly isomorphisms (by the way they were constructed), we need only to prove that in the previous diagram all squares commute.

Now, by §2.3,

$$d_k^{(n)} = \begin{bmatrix} d_k^{(n-1)} & (-1)^{k+1} \operatorname{diag}(a_n) \\ 0 & d_{k-1}^{(n-1)} \end{bmatrix}$$

when $n > 1$, $k \geqslant 1$. Therefore, for $k \geqslant 0$ we have

$$T_k^{(n)} d_{k+1}^{(n)} = \begin{pmatrix} T_k^{(n-1)} & 0 \\ 0 & T_{k-1}^{(n-1)} \end{pmatrix} \begin{bmatrix} d_{k+1}^{(n-1)} & (-1)^k \operatorname{diag}(a_n) \\ 0 & d_k^{(n-1)} \end{bmatrix}$$

$$= \begin{bmatrix} T_k^{(n-1)} d_{k+1}^{(n-1)} & (-1)^k T_k^{(n-1)} \operatorname{diag}(a_n) \\ 0 & T_{k-1}^{(n-1)} d_k^{(n-1)} \end{bmatrix}.$$

Since $T_k^{(n-1)}$ is block diagonal and v commutes with a_n, $T_k^{(n-1)} \operatorname{diag}(a_n) = \operatorname{diag}(a_n) T_k^{(n-1)}$.

Furthermore, $T_k^{(n-1)} d_{k+1}^{(n-1)} = \check{d}_{k+1}^{(n-1)} T_{k+1}^{(n-1)}$ by induction hypothesis, and also $T_{k-1}^{(n-1)} d_k^{(n-1)} = \check{d}_k^{(n-1)} T_k^{(n-1)}$. Thus

$$T_k^{(n)} d_{k+1}^{(n)} = \begin{bmatrix} \check{d}_{k+1}^{(n-1)} T_{k+1}^{(n-1)} & (-1)^k \operatorname{diag}(a_n) T_k^{(n-1)} \\ 0 & \check{d}_k^{(n-1)} T_k^{(n-1)} \end{bmatrix}$$

$$= \begin{bmatrix} \check{d}_{k+1}^{(n-1)} & (-1)^k \operatorname{diag}(a_n) \\ 0 & \check{d}_k^{(n-1)} \end{bmatrix} \begin{pmatrix} T_{k+1}^{(n-1)} & 0 \\ 0 & T_k^{(n-1)} \end{pmatrix}$$

$$= \check{d}_{k+1}^{(n)} T_{k+1}^{(n)}.$$

PROPOSITION 5.2. *Let $\mathfrak{B}$, $\mathfrak{X}$, $a_1, \ldots, a_n$ be as before and v be an invertible element of $\mathfrak{B}$ (not necessarily commuting with $a_2, \ldots, a_n$). Then $a = (a_1, \ldots, a_n)$ is invertible iff so is $a_v = (va_1v^{-1}, \ldots, va_nv^{-1})$.*

PROOF. It is easy to verify that $v_k: \mathfrak{X}^{\binom{n}{k}} \to \mathfrak{X}^{\binom{n}{k}}$ given by $v_k = v \oplus \cdots \oplus v$ ($\binom{n}{k}$ times), $k = 0, 1, \ldots, n$, establishes an isomorphism between $E(\mathfrak{X}, a)$ and $E(\mathfrak{X}, a_v)$.

Corollary 5.3. *Let $A = (A_1, \dots, A_n) \in \mathcal{F}$ and V be a Fredholm operator.*

(i) *If $[V, A_k] \in \mathcal{K}(\mathcal{H})$, $k = 2, \dots, n$, then $VA = (VA_1, A_2, \dots, A_n)$ and $AV = (A_1V, A_2, \dots, A_n)$ are Fredholm.*

(ii) *If $\tilde{V}$ denotes any "almost inverse" of V, i.e., $\pi(\tilde{V}) = \pi(V)^{-1}$, then $A_V = (VA_1\tilde{V}, \dots, VA_n\tilde{V})$ is Fredholm.*

6. A generalized Atkinson's theorem.

1. Given a system

$$\cdots \xrightarrow{D_{k+1}} \mathcal{H}_k \xrightarrow{D_k} \mathcal{H}_{k-1} \xrightarrow{D_{k-1}} \cdots \xrightarrow{D_2} \mathcal{H}_1 \xrightarrow{D_1} \mathcal{H}_0 \to 0, \tag{D}$$

as in §3, there is a natural way of getting a complex out of it, without leaving the space $\mathcal{H}$ on which (D) acts. In fact, if P_k is the orthogonal projection in $\mathcal{L}(\mathcal{H}_k)$ onto $\ker D_k$, and $\tilde{D}_k = P_{k-1}D_k$ (all k), then $(\tilde{D})$ is a complex. One is tempted to believe that since D_kD_{k+1} is compact (all k), then D_k and $\tilde{D}_k$ can differ by only a compact operator. The easiest available counterexample is:

$$0 \to \mathcal{H} \xrightarrow{I} \mathcal{H} \xrightarrow{K} \mathcal{H} \to 0, \tag{D}$$

where K is compact and $\ker K = (0)$. Of course, the (D) shown is not Fredholm, so that one might hope that the statement holds in that case. Moreover, if $n_k = 0$ for $k > 3$, it does hold, because $D_1D_1^*$ is Fredholm, so that $\operatorname{ran} D_1$ is closed and therefore there exists $S_1 \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ satisfying $S_1D_1 = P_1^\perp$, so that $D_2 - \tilde{D}_2 = D_2 - P_1D_2 = P_1^\perp D_2 = S_1D_1D_2$, which is compact. Any attempt to extend this proof to the case $n_k > 0$ ($k = 0, 1, 2, 3$) will fail. Consider

$$0 \to \mathcal{H} \xrightarrow{I} \mathcal{H} \xrightarrow{K} \mathcal{H} \xrightarrow{I} \mathcal{H} \to 0, \tag{D}$$

where K is compact and $\ker K = 0$.

In the general case, a sufficient condition is that all $\operatorname{ran} D_k$'s be closed.

Proposition 6.1. *Let (D): $\cdots \to \mathcal{H}_k \xrightarrow{D_k} \mathcal{H}_{k-1} \to \cdots$ be a system and $(\tilde{D})$ be its associated complex. Assume that* $\operatorname{ran} D_k$ *is closed for all k. Then $D_k - \tilde{D}_k$ is compact (all k). In particular, (D) is Fredholm iff $(\tilde{D})$ is Fredholm.*

Proof. By the Open Mapping Theorem, there exists S_k: $\mathcal{H}_{k-1} \to \mathcal{H}_k$ such that $D_kS_k = P_{\operatorname{ran} D_k}$ and $S_kD_k = I - P_{\ker D_k}$. Then $D_{k+1} - \tilde{D}_{k+1} = D_{k+1} - P_kD_{k+1} = P_k^\perp D_{k+1} = S_kD_kD_{k+1} \in \mathcal{K}(\mathcal{H}_{k+1}, \mathcal{H}_k)$.

2. The next result resembles Atkinson's theorem.

Theorem 2. *Let (D): $\cdots \to \mathcal{H}_k \xrightarrow{D_k} \mathcal{H}_{k-1} \to \cdots$ be a system such that $D_k - \tilde{D}_k$ is compact, all k. The following conditions are equivalent:*

(i) *(D) is Fredholm.*

(ii) *$(\tilde{D})$ is Fredholm.*

(iii) $\operatorname{ran} \tilde{D}_k$ *is closed and* $\ker \tilde{D}_k/\operatorname{ran} \tilde{D}_{k+1}$ *is finite dimensional (all k).*

(iv) $\operatorname{ran} D_k$ *is closed and* $\ker D_k \cap (\operatorname{ran} D_{k+1})^\perp$ *is finite dimensional (all k).*

(v) *There exists $S_k \in \mathcal{L}(\mathcal{H}_{k-1}, \mathcal{H}_k)$ $(k \in \mathbb{Z})$ such that $S_kD_k + D_{k+1}S_{k+1} - I$ is compact (all k).*

REMARKS. In case $(D) = (D(A))$ for a commuting n-tuple $A = (A_1, \ldots, A_n)$, (i) $\Rightarrow$ (iv) appears stated (without proof) in a letter of J. L. Taylor to R. G. Douglas. Condition (v) is given as a definition of a Fredholm system in **[18]**.

PROOF OF THE THEOREM. (i) $\Rightarrow$ (ii). Obvious. (ii) $\Rightarrow$ (iii). By Proposition 3.4, $\tilde{L}_k = \tilde{D}_k^* \tilde{D}_k + \tilde{D}_{k+1}\tilde{D}_{k+1}^*$ is Fredholm (all k). Since ran $\tilde{D}_{k+1} \subset \ker \tilde{D}_k$, it follows that ran $\tilde{L}_k =$ ran $\tilde{D}_k^*\tilde{D}_k \oplus$ ran $\tilde{D}_{k+1}\tilde{D}_{k+1}^*$. Since ran $\tilde{L}_k$ is closed, so is ran $\tilde{D}_k^*\tilde{D}_k$. Therefore, ran $\tilde{D}_k$ is closed. Furthermore, $\ker \tilde{L}_k = \ker \tilde{D}_k \cap \ker \tilde{D}_{k+1}^*$. Since $\tilde{L}_k$ is Fredholm, we obtain that $\dim(\ker \tilde{D}_k/\text{ran } \tilde{D}_{k+1}) = \dim(\ker \tilde{D}_k \cap \ker \tilde{D}_{k+1}^*) = \dim \ker \tilde{L}_k$ is finite.

(iii) $\Rightarrow$ (iv). We observe that $\tilde{D}_k|_{(\text{ran } \tilde{D}_{k+1})^\perp}$: $(\text{ran } \tilde{D}_{k+1})^\perp \to \mathcal{H}_{k-1}$ is left semi-Fredholm (closed range and finite dimensional kernel). Since $D_k - \tilde{D}_k$ is compact, we conclude that $D_k|_{(\text{ran } \tilde{D}_{k+1})^\perp}$: $(\text{ran } \tilde{D}_{k+1})^\perp \to \mathcal{H}_{k-1}$ is also left semi-Fredholm. Then ran $D_k = D_k(\text{ran } \tilde{D}_{k+1})^\perp$ is closed (here, we use the fact that ran $\tilde{D}_{k+1} \subset \ker D_k$) and $\ker D_k \cap (\text{ran } \tilde{D}_{k+1})^\perp$ is finite dimensional. Finally, $\ker D_k \cap (\text{ran } \tilde{D}_{k+1})^\perp = \ker D_k \cap \text{ran}(D_{k+1})^\perp$.

(iv) $\Rightarrow$ (iii). $D_k|_{(\text{ran } \tilde{D}_{k+1})^\perp}$: $(\text{ran } \tilde{D}_{k+1})^\perp \to \mathcal{H}_{k-1}$ is left semi-Fredholm. Therefore, $\tilde{D}_k|_{(\text{ran } \tilde{D}_{k+1})^\perp}$: $(\text{ran } \tilde{D}_{k+1})^\perp \to \mathcal{H}_{k-1}$ has closed range and finite dimensional kernel. But $\tilde{D}_k(\text{ran } \tilde{D}_{k+1})^\perp = \text{ran } \tilde{D}_k$ and $\ker \tilde{D}_k|_{(\text{ran } \tilde{D}_{k+1})^\perp} = \ker \tilde{D}_k \cap (\text{ran } \tilde{D}_{k+1})^\perp$.

(iii) $\Rightarrow$ (v). We know that $\tilde{D}_k$ has closed range, so that by the Open Mapping Theorem, we can find $S_k \in \mathfrak{L}(\mathcal{H}_{k-1}, \mathcal{H}_k)$ such that $S_k\tilde{D}_k = P_{(\ker \tilde{D}_k)^\perp}$ and $\tilde{D}_k S_k = P_{\text{ran } \tilde{D}_k}$ and $\ker S_k = (\text{ran } \tilde{D}_k)^\perp$. Thus:

$$S_k\tilde{D}_k + \tilde{D}_{k+1}S_{k+1} = \begin{cases} S_k\tilde{D}_k & \text{on } (\text{ran } \tilde{D}_{k+1})^\perp, \\ \tilde{D}_{k+1}S_{k+1} & \text{on ran } \tilde{D}_{k+1}. \end{cases}$$

Since $\ker \tilde{D}_k/\text{ran } \tilde{D}_{k+1}$ is finite dimensional, we see that $S_k\tilde{D}_k + \tilde{D}_{k+1}S_{k+1} - I$ is compact. But $\tilde{D}_k - D_k \in \mathcal{K}(\mathcal{H}_k, \mathcal{H}_{k-1})$ (all k), so that $S_kD_k + D_{k+1}S_{k+1} - I$ is compact (all k).

(v) $\Rightarrow$ (i). Passing to the Calkin algebra, we have $s_kd_k + d_{k+1}s_{k+1} = 1 \in M_{n_k}(\mathfrak{Q}(\mathcal{H}))$, where $s_k = \pi(S_k)$ and d_k is the kth boundary map of the complex (d).

If $d_ka = 0$, then $d_{k+1}s_{k+1}a = a$, so that $a \in \text{ran } d_{k+1}$, showing that (d) is exact, that is, (D) is Fredholm.

REMARK. (i) $\Leftrightarrow$ (v) can be extended to: Let $\mathfrak{B}$, n_k, d_k be as in Proposition 3.4. Then the complex $\cdots \to \mathfrak{B}_k \xrightarrow{d_k} \mathfrak{B}_{k-1} \to \cdots$ is exact iff there exists $\{s_k\colon \mathfrak{B}_{k-1} \to \mathfrak{B}_k\}_{k\in\mathbf{Z}}$ satisfying $s_kd_k + d_{k+1}s_{k+1} = 1$. Moreover, $s_{k+1}s_k = 0$ for all k.

The "if" part is trivial. For the "only if", use the decomposition $\mathfrak{B}_k = \ker d_k \dotplus \text{ran } d_{k+1}^*$.

COROLLARY 6.2. *Let* (D): $\cdots \to \mathcal{H}_k \xrightarrow{D_k} \mathcal{H}_{k-1} \to \cdots$ *be a complex. Then* (D) *is Fredholm iff* $\ker D_k/\text{ran } D_{k+1}$ *is finite dimensional* (*all* k).

COROLLARY 6.3. *Let* (D): $0 \to \mathcal{H}_2 \xrightarrow{D_2} \mathcal{H}_1 \xrightarrow{D_1} \mathcal{H}_0 \to 0$ *be a system* ($n_k = 0$ *for* $k > 3$). *Then* (D) *is Fredholm iff* ran D_1, ran D_2 *are closed and* $\ker D_2$, $\ker D_1 \cap (\text{ran } D_2)^\perp$ *and* $(\text{ran } D_1)^\perp$ *are finite dimensional.*

PROOF. If (D) is Fedholm, then $D_2 - \tilde{D}_2$ is compact and (i) $\Rightarrow$ (iv) can be used. Conversely, if ran D_1 is closed, then $D_2 - \tilde{D}_2$ is compact, and (iv) $\Rightarrow$ (i) applies.

7. Index of Fredholm n-tuples.

1. We are now ready to introduce the index for a Fredholm n-tuple of almost commuting operators on an infinite dimensional Hilbert space $\mathcal{H}$. As is probably expected, we shall do that by using Corollary 3.11. Naturally, index will be continuous, invariant under compact perturbations and onto $\mathbf{Z}$. We also present in this section an alternative definition, similar to the Euler characteristic of a chain complex.

2. DEFINITION 7.1. Let $A = (A_1, \ldots, A_n)$ be an almost commuting Fredholm n-tuple of operators on $\mathcal{H}$ and $\hat{A} \in \mathcal{L}(\mathcal{H} \otimes \mathbf{C}^{2^{n-1}})$ be as in §2.3. Then index(A) = index$(\hat{A})$.

THEOREM 3. index: $\mathfrak{F} \to \mathbf{Z}$ *is continuous, invariant under compact perturbations and onto* $\mathbf{Z}$. *Consequently*, index *is constant on arcwise components of* $\mathfrak{F}$.

PROOF. Since $A \mapsto \hat{A}$ is continuous, it follows easily that index is continuous. For $K \in \mathcal{K}(\mathcal{H}) \otimes \mathbf{C}^n$, we have $\widehat{A+K} - \hat{A} \in \mathcal{K}(\mathcal{H} \otimes \mathbf{C}^{2^{n-1}})$, so that index is invariant under compact perturbations. We shall see in §8 that

$$\operatorname{index}\left(W_1^{(k)}, W_2, \ldots, W_n\right) = -k \quad \text{for all } k \in \mathbf{Z},$$

where $(W_1, \ldots, W_n)$ is the n-tuple of multiplications by the coordinate functions on $H^2(S^1 \times \cdots \times S^1)$, so that index is onto $\mathbf{Z}$.

3. Suppose now that (D) is a Fredholm Koszul system such that $D_k - \tilde{D}_k$ is compact (all k). According to Theorem 2 of §6, $(\tilde{D})$ is Fredholm. We define index$(\tilde{D})$ to be index$(\tilde{D})^\wedge$.

THEOREM 4. *Let* $(D), (\tilde{D})$ *be as above. Then*

$$\begin{aligned}\operatorname{index}(D) &= \sum_k (-1)^{k+1} \dim(\ker \tilde{D}_k / \operatorname{ran} \tilde{D}_{k+1}) \\ &= \sum_k (-1)^{k+1} \left\{\dim\left(\ker D_k \cap (\operatorname{ran} D_{k+1})^\perp\right) - \dim\left(\operatorname{ran} D_{k+1} \cap (\ker D_k)^\perp\right)\right\}.\end{aligned}$$

PROOF. Since index$(\tilde{D})$ = index$(\tilde{D})^\wedge$ = dim ker$(\tilde{D})^\wedge$ − dim ker$[(\tilde{D})^\wedge]^*$, we shall compute both kernels.

Since $\tilde{D}_k \tilde{D}_{k+1} = 0$ (all k), we get

$$\begin{aligned}\ker(\tilde{D})^\wedge &= \ker[(\tilde{D})^\wedge]^*(\tilde{D})^\wedge \\ &= \bigoplus_{\text{odd } k\text{'s}} \ker(\tilde{D}_k^* \tilde{D}_k + \tilde{D}_{k+1}\tilde{D}_{k+1}^*)\end{aligned}$$

and

$$\begin{aligned}\ker[(\tilde{D})^\wedge]^* &= \ker(\tilde{D})^\wedge[(\tilde{D})^\wedge]^* \\ &= \bigoplus_{\text{even } k\text{'s}} \ker(\tilde{D}_k^* \tilde{D}_k + \tilde{D}_{k+1}\tilde{D}_{k+1}^*).\end{aligned}$$

Now

$$\ker(\tilde{D}_k^* \tilde{D}_k + \tilde{D}_{k+1}\tilde{D}_{k+1}^*) = \ker \tilde{D}_k \cap (\operatorname{ran} \tilde{D}_{k+1})^\perp, \tag{1}$$

for all k. Furthermore, $\ker \tilde{D}_k \supset \ker D_k \supset \operatorname{ran} \tilde{D}_{k+1}$, so that

$$\dim(\ker \tilde{D}_k/\operatorname{ran} \tilde{D}_{k+1}) = \dim(\ker \tilde{D}_k/\ker D_k) + \dim(\ker D_k/\operatorname{ran} \tilde{D}_{k+1}). \quad (2)$$

We now observe that

$$\ker D_k \cap (\operatorname{ran} \tilde{D}_{k+1})^{\perp} = \ker D_k \cap (\operatorname{ran} D_{k+1})^{\perp}, \quad (3)$$

because $\tilde{D}_{k+1} = P_k D_{k+1}$ with P_k the projection onto $\ker D_k$.

Finally, $\ker \tilde{D}_k = \ker P_{k-1} D_k = D_k^{-1}(\ker P_{k-1}) = D_k^{-1}(\ker D_{k-1})^{\perp} = D_k^{-1}((\ker D_{k-1})^{\perp} \cap \operatorname{ran} D_k)$, so that:

$$0 \to \ker D_k \to \ker \tilde{D}_k \to \ker \tilde{D}_k/\ker D_k \to 0$$

and

$$0 \to \ker D_k \to D_k^{-1}\big((\ker D_{k-1})^{\perp} \cap \operatorname{ran} D_k\big) \xrightarrow{D_k} (\ker D_{k-1})^{\perp} \cap \operatorname{ran} D_k \to 0$$

are both exact, from which it is clear that

$$\dim(\ker \tilde{D}_k/\ker D_k) = \dim\big(\operatorname{ran} D_k \cap (\ker D_{k-1})^{\perp}\big). \quad (4)$$

Combining all four equations, the theorem follows.

COROLLARY 7.2. *If (D) is a Fredholm Koszul complex, then* $\operatorname{index}(D) = -\chi(D)$ *where χ denotes Euler characteristic.*

COROLLARY 7.3. *Let $A = (A_1, \ldots, A_n)$ be a doubly commuting Fredholm n-tuple of operators on $\mathcal{H}$. Then $H_k = \ker D_k/\operatorname{ran} D_{k+1}$ is exactly $\bigoplus_{f \in I_k}(\bigcap_{i=1}^{n} \ker {}^fA_i)$, where the sum is orthogonal, $I_k = \{f\colon \{1, \ldots, n\} \to \{0, 1\}/f(i) = 0$ exactly k times$\}$ and ${}^fA_i$, as in Corollary 3.7, is meant to be $A_i^*A_i$ or $A_iA_i^*$ according to $f(i) = 0$ or 1. Therefore*

$$\operatorname{index}(A) = \sum_k (-1)^{k+1} \sum_{f \in I_k} \dim\left(\bigcap_{i=1}^{n} \ker {}^fA_i\right).$$

PROOF. We already know thatxrep $H_k = \ker(D_k^* D_k + D_{k+1} D_{k+1}^*)$. Since A is doubly commuting, $D_k^* D_k + D_{k+1} D_{k+1}^*$ is a block diagonal matrix whose entries are precisely the $\binom{n}{k}$ different combinations $\Sigma_{i=1}^{n} {}^fA_i$ for $f \in I_k$. Since all ${}^fA_i$ are positive operators, we know that $\ker(\Sigma_{i=1}^{n} {}^fA_i) = \bigcap_{i=1}^{n} \ker {}^fA_i$, which completes the proof.

4. We shall now illustrate Theorem 4 in the case $n = 2$. Here (D) is

$$0 \to \mathcal{H} \xrightarrow{D_2} \mathcal{H} \oplus \mathcal{H} \xrightarrow{D_1} \mathcal{H} \to 0,$$

so that

$$\begin{aligned} \operatorname{index}(D) &= -\dim(\operatorname{ran} D_1)^{\perp} + \dim\big(\ker D_1 \cap (\operatorname{ran} D_2)^{\perp}\big) \\ &\quad - \dim\big(\operatorname{ran} D_2 \cap (\ker D_1)^{\perp}\big) - \dim \ker D_2 \\ &= -\dim \ker D_1^* + \dim(\ker D_1 \cap \ker D_2^*) \\ &\quad - \dim(\operatorname{ran} D_1^* \cap \operatorname{ran} D_2) - \dim \ker D_2. \end{aligned}$$

The term $\dim(\operatorname{ran} D_2 \cap (\ker D_1)^{\perp})$ measures the "lack of complexity" at the middle stage, that is, since $D_1 D_2$ need not be zero, but only a compact operator, there is in general an adjustment in what would be the natural way of computing the index, as negative the Euler characteristic of the complex. The negative sign is required to: (a) fit the unidimensional theory and (b) produce a uniform -1 as index$(W_1, \ldots, W_n)$ on $H^2(S^1 \times \cdots \times S^1)$ (see §8).

Observe that

$$\hat{D} = \begin{pmatrix} D_1 \\ D_2^* \end{pmatrix}$$

is a 2×2 matrix with $\ker \hat{D} = \ker D_1 \cap \ker D_2^*$ and $\ker D_1^*$, $\ker D_2 \subset \ker \hat{D}^*$. The term $\operatorname{ran} D_1^* \cap \operatorname{ran} D_2$ does not directly appear in $\hat{D}$, but an isomorphic image is the piece which $\ker D_1^*$ and $\ker D_2$ need to fill $\ker \hat{D}^*$.

5. *Remarks*. Although we have studied only the Fredholm case, Proposition 3.4 makes possible a reasonable definition of a semi-Fredholm n-tuple, i.e., an almost commuting n-tuple A is semi-Fredholm iff $\hat{A}$ is semi-Fredholm. Consequently, either all even dimensional homology modules are finite dimensional or so are the odd dimensional ones. index is then well defined and Theorems 3 and 4 clearly extend to this case if we restrict attention to the case $D_k - \tilde{D}_k \in \mathcal{K}(\mathcal{H}_k, \mathcal{H}_{k-1})$ (all k) (observe that then $\operatorname{ran} D_{k+1} \cap (\ker D_k)^\perp$ is finite dimensional, because $\operatorname{ran} D_k$ is closed and $D_k(\operatorname{ran} D_{k+1} \cap (\ker D_k)^\perp)$, which is a closed subspace of $\operatorname{ran} D_k D_{k+1}$, is finite dimensional). Using Definition 7.1, we can define the index of a nonsingular n-tuple of elements of the Calkin algebra $\mathcal{Q}(\mathcal{H})$ by lifting it to an almost commuting Fredholm n-tuple of operators on $\mathcal{H}$. A classical result of Bartle-Graves (cf. **[16]**, **[17]**) on cross sections induces immediately a bijection of path-components between $\mathcal{F}$ and $\mathcal{I}(\mathcal{Q}(\mathcal{H}))$ = commuting invertible n-tuples on $\mathcal{Q}(\mathcal{H})$.

The above definition of index was given only for n-tuples of operators (that is, Fredholm Koszul systems), while we could have extended it to more general systems. One approach is to consider the same definition for systems with $\Sigma_{\text{even } k\text{'s}}\, n_k = \Sigma_{\text{odd } k\text{'s}}\, n_k$ in order to get a square matrix $\hat{D}$. Another viewpoint would be to take the content of Theorem 4 as the starting point. We have not pursued this further since our main interest is in Koszul systems.

8. Calculation of indices and applications.

1. In this section we compute the indices of the n-tuples in §4 and then apply them to find their spectra.

(i) Let $A = (A_1, \ldots, A_n) \in \mathcal{F}$ and $(A_{i_1}, \ldots, A_{i_k}) \in \mathcal{F}$ $(k < n)$, where i: $\{1, \ldots, k\} \to \{1, \ldots, n\}$ is injective and $i_j = i(j)$. Then index$(A) = 0$. For, we can assume that $1 \notin i(\{1, \ldots, k\})$ and define γ: $[0, 1] \to \mathcal{F}$ by sending t to $(t + (1-t)A_1, (1-t)A_2, \ldots, (1-t)A_n)$. Since γ and index are continuous, index(A) = index $\gamma(0)$ = index $\gamma(1)$ = index$(I, 0, \ldots, 0)$ = index$(I, 0, \ldots, 0)\hat{}$ = index$(I_{\mathcal{H} \otimes \mathbf{C}^{2^{n-1}}}) = 0$.

(ii) Let $W = (W_1, \ldots, W_n)$ be the n-tuple of multiplications by the coordinate functions on $H^2(S^1 \times \cdots \times S^1)$. Then index$(W) = -1$. More generally, if $k = (k_1, \ldots, k_n) \in \mathbf{Z}^n$, $W_i^{(k_i)}$ is $W_i^{k_i}$ or $(W_i^*)^{-k_i}$ whenever $k_i > 0$ or $k_i < 0$, respectively $(i = 1, \ldots, n)$ and $W^{(k)} = (W_1^{(k_1)}, \ldots, W_n^{(k_n)})$, then index$(W^{(k)}) = -k_1 \cdot \ldots \cdot k_n$.

We shall now give a proof of the first statement. Since the W_i's doubly commute, we can apply Corollary 7.3 to compute index(W), as

$$\sum_{k=0}^{n} (-1)^{k+1} \sum_{f \in I_k} \dim\left(\bigcap_{i=1}^{n} \ker {}^{f}W_i \right).$$

It is clear that the only nonzero terms occur when $f(i) = 1$ for all i, so that index(W) $= -1$. The general statement follows in the same way.

(iii) Let $T_z = (T_{z_1}, \dots, T_{z_n})$ be the n-tuple of multiplications by the coordinates on $H^2(S^{2n-1})$. Then index(T_z) $= -1$. More generally, if $k = (k_1, \dots, k_n) \in \mathbf{Z}^n$, then index($T_z^{(k)}$) $= -k_1 \cdot \dots \cdot k_n$.

One way of proving this is by using a result of Venugopalkrisna **[22]** on the index of a Toeplitz matrix. We shall see in §13, however, that T_z can be connected to a copy of W by a path of Fredholm n-tuples, so that index(T_z) = index(W) $= -1$. A trivial modification of that path will give one from $T_z^{(k)}$ to a copy of $W^{(k)}$ and so index($T_z^{(k)}$) $= -k_1 \cdot \dots \cdot k_n$.

2. We are now ready to calculate the spectra of W and T_z.

THEOREM 5. *Let W and T_z be the n-tuples of multiplications by the coordinate functions in $H^2(S^1 \times \cdots \times S^1)$ and $H^2(S^{2n-1})$, respectively. Let $\mathbf{D}_i$ be the closed unit disc in the ith coordinate space and $\mathbf{B}^{2n}$ the closed unit ball in $\mathbf{C}^n$. Then*

(a) $\mathrm{Sp}(W) = \prod_{i=1}^{n} \mathbf{D}_i$,

(b) $\mathrm{Sp}(T_z) = \mathbf{B}^{2n}$,

(c) $\mathrm{Sp}_e(W) = \mathrm{Fr}(\prod_{i=1}^{n} \mathbf{D}_i) = (\partial\mathbf{D}_1 \times \mathbf{D}_2 \times \cdots \times \mathbf{D}_n) \cup \cdots \cup (\mathbf{D}_1 \times \mathbf{D}_2 \times \cdots \times \partial\mathbf{D}_n)$,

(d) $\mathrm{Sp}_e(T_z) = S^{2n-1}$.

PROOF. (d) Since $\sum_{i=1}^{n} T_{z_i}^* T_{z_i} = I$ and the T_{z_i}'s are essentially normal, we conclude that $\mathrm{Sp}_e(T_z) \subset S^{2n-1}$ (by Corollary 3.10). But index(T_z) $= -1$ and index is continuous, so that $\mathrm{Sp}_e(T_z) = S^{2n-1}$.

(b) Since index is constant on path-components of $\mathfrak{F}$, we conclude that $\mathbf{B}^{2n} \subset \mathrm{Sp}(T_z)$. Moreover, $\mathrm{Sp}(T_z) \subset \mathrm{Sp}_{\mathfrak{B}}(T_z)$, where $\mathfrak{B}$ is the Banach subalgebra of $\mathfrak{L}(H^2(S^{2n-1}))$ generated by $T_{z_1}, \dots, T_{z_n}$, by a result of Taylor's that we stated in Proposition 2.2. Since $\mathfrak{B}$ can be identified with $P(\mathbf{B}^{2n})$, the uniform closure on $C(\mathbf{B}^{2n})$ of the algebra of polynomials in $z_1, \dots, z_n$ and $\mathbf{B}^{2n}$ is polynomially convex, then the maximal ideal space of $\mathfrak{B}$, when seen as a subset of $\mathbf{C}^n$, is $\mathbf{B}^{2n}$ and consequently, $\mathrm{Sp}_{\mathfrak{B}}(T_z) = \mathbf{B}^{2n}$, as needed (see **[11]** for the pertinent results).

(c) Assume that $\lambda \notin \mathrm{Fr}(\prod_{i=1}^{n} \mathbf{D}_i)$. If $|\lambda_1| > 1$, $W_1 - \lambda_1$ is invertible and so is $W - \lambda$ which implies that $\lambda \notin \mathrm{Sp}_e(W)$. If $|\lambda_1| = 1$, then at least one of $\lambda_2, \dots, \lambda_n$ must have modulus greater than one, showing again that $W - \lambda$ is invertible and $\lambda \notin \mathrm{Sp}_e(W)$. If $|\lambda_1| < 1$, then three possibilities occur: $|\lambda_2| > 1$, $|\lambda_2| = 1$ and $|\lambda_2| < 1$. It is again clear that only the case $|\lambda_2| < 1$ deserves further consideration. Continuing this reasoning for the remaining λ_i's, we conclude that only the situation $|\lambda_i| < 1$ ($i = 1, \dots, n$) presents some difficulty. So assume $|\lambda_i| < 1$ for all i. Now $W - \lambda$ is a doubly commuting n-tuple of subnormal operators and by

Corollary 3.8, it will be enough to show that $\Sigma_{i=1}^{n}(W_i-\lambda_i)(W_i-\lambda_i)^*$ is Fredholm, or that D_1 has closed range and finite dimensional cokernel. But ran $D_1 =$ ran$(W_1-\lambda_1)+\cdots+$ran$(W_n-\lambda_n)=\{f\in H^2(S^1\times\cdots\times S^1)\colon \tilde{f}(\lambda_1,\ldots,\lambda_n)=0\}$, where $\tilde{f}$ is the natural extension of f to the interior. Therefore, ran D_1 is closed and dim(ker D_1^*) = 1.

We have thus proved that $\mathrm{Sp}_e(W)\subset \mathrm{Fr}(\Pi_{i=1}^{n}\mathbf{D}_i)$. Since index is continuous and index$(W)=-1$, we must have $\mathrm{Sp}_e(W)=\mathrm{Fr}(\Pi_{i=1}^{n}\mathbf{D}_i)$.

(a) From (c) we obtain: $\Pi_{i=1}^{n}\mathbf{D}_i\subset \mathrm{Sp}(W)$. Moreover if $\lambda\notin\Pi_{i=1}^{n}\mathbf{D}_i$, then for at least one i, $|\lambda_i|>1$. Then $W_i-\lambda_i$ is invertible and so is $W-\lambda$. Thus $\mathrm{Sp}(W)=\Pi_{i=1}^{n}\mathbf{D}_i$.

REMARKS. In case $n=2$, (b) can be derived from index considerations alone. For, it is clear that $\ker(T_{z_1}-\lambda_1)=0$ when $|\lambda|>1$. If we can show that $\mathrm{ran}(T_{z_1}-\lambda_1)+\mathrm{ran}(T_{z_2}-\lambda_2)=H^2(S^3)$ for $|\lambda|>1$ then, since index$(T_z-\lambda)=0$ outside $\mathbf{B}^4$, we must have exactness at the middle stage as well. So let us assume that $f\in H^2(S^3)$ and $T_{z_i}^*f=\bar{\lambda}_i f$ $(i=1,2)$. Recall that $T_{z_1}e_k=(c_k/c_{k'})e_{k'}$ $(k'=(k_1+1,k_2))$ and $T_{z_2}e_k=(c_k/c_{k^+})e_{k^+}$ $(k^+=(k_1,k_2+1))$, where

$$c_k=\frac{1}{\sqrt{2}\,\pi}\sqrt{\frac{(|k|+1)!}{k!}}\ .$$

Then

$$(f,e_{k'})=\frac{c_{k'}}{c_k}(T_{z_1}^*f,e_k)=\bar{\lambda}_1\frac{c_{k'}}{c_k}(f,e_k)$$

and

$$(f,e_{k^+})=\frac{c_{k^+}}{c_k}(T_{z_2}^*f,e_k)=\bar{\lambda}_2\frac{c_{k^+}}{c_k}(f,e_k).$$

Then $(f,e_k)=(c_k/c_{00})\bar{\lambda}_1^{k_1}\bar{\lambda}_2^{k_2}(f,e_{00})$. Therefore

$$\begin{aligned}\|f\|_2^2&=\sum_k|(f,e_k)|^2=\sum_k\frac{c_k^2}{c_{00}^2}|\lambda_1|^{2k_1}|\lambda_2|^{2k_2}|(f,e_{00})|^2\\&=\sum_{l=0}^{\infty}\sum_{|k|=l}\frac{(l+1)!}{k_1!k_2!}|\lambda_1|^{2k_1}|\lambda_2|^{2k_2}|(f,e_{00})|^2\\&=\sum_{l=0}^{\infty}(l+1)|\lambda|^{2l}|(f,e_{00})|^2,\end{aligned}$$

so that, by virtue of the condition $|\lambda|>1$, $(f,e_{00})=0$, or $f=0$.

We also want to mention that Coburn has shown in [5] that $C^*(T_{z_1},\ldots,T_{z_n})/\mathcal{K}(H^2(S^{2n-1}))\simeq C(S^{2n-1})$, from which (d) follows at once.

9. Indices of related Fredholm n-tuples.

1. The following propositions are rather elementary, though useful to find indices of several related Fredholm n-tuples.

PROPOSITION 9.1. *Let $A=(A_1,\ldots,A_n)\in\mathfrak{F}$, $\phi\colon\{1,\ldots,n\}\to\{1,*\}$ be a function and define $\phi(A_i)=A_i^{\phi(i)}$, as in Corollary* 3.14. *Assume that $\phi(A)=(\phi(A_1),\ldots,\phi(A_n))$ is an almost commuting n-tuple. Then $\phi(A)\in\mathfrak{F}$ and* index $\phi(A)=(-1)^{|\phi|}$ index(A), *where $|\phi|=\#\{i\colon\phi(i)=*\}$.*

PROOF. Straightforward from the proof of Corollary 3.14.

COROLLARY 9.2. *If* $A = (A_1, \ldots, A_n) \in \mathfrak{F}$ *and one of the* A_i*'s is essentially selfadjoint, then* $\operatorname{index}(A) = 0$.

PROOF. By the results of [1], an essentially selfadjoint operator is a compact perturbation of a selfadjoint one. We then apply Theorem 3 of §7 and the preceding proposition.

COROLLARY 9.3. *Let* A_1 *and* A_2 *almost doubly commute. If* $(A_1, A_2) \in \mathfrak{F}$, *then so are* (A_1^*, A_2), (A_1, A_2^*) *and* (A_1^*, A_2^*) *and* $\operatorname{index}(A_1, A_2) = \operatorname{index}(A_1^*, A_2^*) = -\operatorname{index}(A_1^*, A_2) = -\operatorname{index}(A_1, A_2^*)$.

PROPOSITION 9.4. *Let* $A = (A_1, \ldots, A_n) \in \mathfrak{F}$, V *be a Fredholm operator such that there exists a path* $\gamma\colon [0, 1] \to \mathfrak{F}$ *with* $\gamma(0) = V$, $\gamma(1) = I$ *and* $[\gamma(t), A_k] \in \mathcal{K}(\mathcal{H})$ *for all* $t \in [0, 1]$, $k > 2$. *Then* $\operatorname{index}(A) = \operatorname{index}(VA) = \operatorname{index}(AV)$, *where* AV *and* VA *are as in Corollary* 5.3.

PROOF. Use continuity of index along with Corollary 5.3.

COROLLARY 9.5. *If* $A = (A_1, \ldots, A_n) \in \mathfrak{F}$ *and* $\lambda \in \mathbf{C} \setminus \{0\}$, *then* $(\lambda A_1, A_2, \ldots, A_n) \in \mathfrak{F}$ *and* $\operatorname{index}(A) = \operatorname{index}(\lambda A_1, A_2, \ldots, A_n)$.

PROPOSITION 9.6. *Let* $A = (A_1, \ldots, A_n) \in \mathfrak{F}$ *and* $p \in S_n$ *be a permutation. Then* $p^*A = (A_{p(1)}, \ldots, A_{p(n)}) \in \mathfrak{F}$ *and* $\operatorname{index}(p^*A) = \operatorname{index}(A)$.

PROOF. By the first observation in the proof of Corollary 3.14, $\hat{A}$ and $\widehat{p^*A}$ are unitarily related, so that $\operatorname{index}(p^*A) = \operatorname{index}(\widehat{p^*A}) = \operatorname{index}(\hat{A}) = \operatorname{index}(A)$.

10. Index of an essentially normal n-tuple with trace class commutators.

1. Although it is easy to see that a normal n-tuple $N = (N_1, \ldots, N_n)$ (i.e., $N_iN_j = N_jN_i$ and $N_iN_j^* = N_j^*N_i$ for all $i, j = 1, \ldots, n$) which is Fredholm will have necessarily index zero (because its asssociated $\hat{N}$ is normal), it is not trivial that the same will hold for essentially normal n-tuples ($n > 2$) with all commutators in trace class (and in fact it is false when $n = 1$).

THEOREM 6. *Let* $A = (A_1, \ldots, A_n)$ ($n > 2$) *be an essentially normal n-tuple* (*that is*, $[A_i, A_j]$, $[A_i^*, A_j] \in \mathcal{K}(\mathcal{H})$ *for all* i, j) *with all commutators in trace class. Assume that* A *is Fredholm. Then* $\operatorname{index}(A) = 0$.

We shall need the following lemma, which appears in [15].

LEMMA 10.1. *Let* $T = (T_{ij}) \in L(\mathcal{H}^N)$ *be a Fredholm operator and* $[T_{ik}, T_{lm}] \in \mathcal{C}_1$ (*all* $i, k, l, m = 1, \ldots, N$), *i.e., all commutators are in trace class. Then* $\det(T)$ *is well defined,* $\det(T)$ *is Fredholm and* $\operatorname{index}(\det(T)) = \operatorname{index}(T)$.

PROOF OF THE THEOREM. We apply the preceding lemma to $\hat{A}$ and thus conclude that $\operatorname{index}(\hat{A}) = \operatorname{index}(\det(\hat{A}))$. An easy calculation shows that $\det(\hat{A}) - (\sum_{i=1}^n A_i^*A_i)^{n-1}$ is compact. Therefore, $\operatorname{index}(\det(\hat{A})) = (n-1)\operatorname{index}(\sum_{i=1}^n A_i^*A_i) = 0$, since the last operator is positive.

2. *Remarks*. We wish to point out that a doubly commuting Fredholm n-tuple with a normal coordinate has also index 0, which follows from the fact that for a

doubly commuting n-tuple A, $\hat{A}$ is normal iff A_1 is normal. When $n < 3$, the same holds without assuming double commutativity.

The preceding theorem has certain points of contact with a result of Helton and Howe [13, Part II, Theorem 2].

II. The deformation problem

11. Preliminaries.

1. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and $A = (A_1, \ldots, A_n)$ be an almost commuting n-tuple of operators on $\mathcal{H}$. If A is Fredholm, index(A) is a well-defined integer; by Theorem 3 of §7, index is an invariant for the path-components of $\mathfrak{F}$. In [9], R. G. Douglas raised the following question: is it the only invariant? In other words, given two n-tuples $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ in $\mathfrak{F}$ with the same index, is it always possible to find a continuous path γ: $[0, 1] \to \mathfrak{F}$ such that $\gamma(0) = A$ and $\gamma(1) = B$? This is the deformation problem. For $n = 1$ the answer is known to be yes (cf. [8]) and for the case A, B essentially normal, Douglas himself gave a proof in [9], using techniques from extension theory ([1], [2] and [3]). We shall give a detailed exposition of this fact in §12. We then consider again $(W_1, \ldots, W_n)$ and $(T_{z_1}, \ldots, T_{z_n})$ and show that they lie in the same path-connected component. As a consequence, we obtain the nonobvious fact that $(W_1, \ldots, W_n)$ can be connected to $(W_1^*, \ldots, W_n^*)$ for n even. This is done in §13. We present in §14 the affirmative answer to the deformation problem for the class of almost doubly commuting Fredholm pairs with a semi-Fredholm coordinate. In §15 we give a number of additional facts on Fredholm and invertible n-tuples. Finally, §16 is devoted to the concluding remarks and open problems. The rest of §11 deals with a basic result on connectedness of Fredholm n-tuples.

2. *Notation.* If $\mathfrak{x} \subset \mathcal{L}(\mathcal{H}) \otimes \mathbf{C}^n$, $A = (A_1, \ldots, A_n)$, $B = (B_1, \ldots, B_n) \in \mathfrak{x}$ and there exists a continuous γ: $[0, 1] \to \mathfrak{x}$ such that $\gamma(0) = A$ and $\gamma(1) = B$, we write $A \overset{\mathfrak{x}}{\sim} B$. Also, we denote by I the n-tuple $(I, 0, \ldots, 0)$.

Proposition 11.1. *Let $A = (A_1, \ldots, A_n) \in \mathfrak{F}$ $(n > 2)$ and assume that A_i is Fredholm for some i. Then $A \overset{\mathfrak{F}}{\sim} I$. In particular,* index$(A) = 0$. *More generally, if $(A_{i_1}, \ldots, A_{i_k}) \in \mathfrak{F}$ $(k < n)$, where i: $\{1, \ldots, k\} \to \{1, \ldots, n\}$ is an injection, then $A \overset{\mathfrak{F}}{\sim} I$ and* index$(A) = 0$.

Proof. See §8.1(i).

12. The essentially normal case.

1. In this section we give a detailed exposition of Douglas' affirmative answer to the deformation problem for essentially normal Fredholm n-tuples, following the outline in [9].

Lemma 12.1. *Let $a = (a_1, \ldots, a_n)$ be a doubly commuting n-tuple on a C^*-algebra $\mathfrak{B}$. Then $\hat{a} \in M_{2^{n-1}}(\mathfrak{B})$ is normal iff a_1 is normal.*

Proof. A straightforward computation shows that $\hat{a}^*\hat{a} - \hat{a}\hat{a}^*$ is a block diagonal matrix whose diagonal entries are either $a_1^*a_1 - a_1a_1^*$ or $a_1a_1^* - a_1^*a_1$.

REMARK. The assertion: (a_1, a_2) is invertible iff so is (a_2, a_1) was proved establishing an isomorphism between the Koszul complexes. Since we have an associated matrix

$$\begin{pmatrix} a_1 & a_2 \\ -a_2^* & a_1^* \end{pmatrix},$$

one might expect that for some unitary U,

$$U\hat{a}U^* = \begin{pmatrix} a_2 & a_1 \\ -a_1^* & a_2^* \end{pmatrix}.$$

The preceding lemma says, however, that this is not always possible.

2.

LEMMA 12.2. *Let $a = (a_1, \ldots, a_n)$ be an invertible normal n-tuple on $\mathcal{Q}(\mathcal{H})$ and $\hat{a}$ be its associated 2^{n-1} by 2^{n-1} matrix over $\mathcal{Q}(\mathcal{H})$. Then, if $\hat{a} = vp$ is the polar decomposition of $\hat{a}$, there exist $u_1, \ldots, u_n$, $q \in \mathcal{Q}(\mathcal{H})$ such that $q > 0$, $u = (u_1, \ldots, u_n)$ is a commuting normal n-tuple and $(q, 0, \ldots, 0)\hat{} = p$, $\hat{u} = v$.*

PROOF. We first notice that since $\hat{a}$ is invertible, it has a polar decomposition $\hat{a} = vp$ with v unitary and $p > 0$ $(v, p \in M_{2^{n-1}}(\mathcal{Q}(\mathcal{H})))$. Let $q = (\sum_{i=1}^n a_i^* a_i)^{1/2}$. It is almost obvious that $\hat{a}^*\hat{a} = [(q, 0, \ldots, 0)\hat{}]^2 = p^2$. Since p is invertible, $v = \hat{a}p^{-1} = \hat{a}(q^{-1}, 0, \ldots, 0)\hat{}$. Observe that $(q^{-1}, 0, \ldots, 0)\hat{}$ is diagonal, so that

$$\hat{a}(q^{-1}, 0, \ldots, 0)\hat{} = \left(a_1 q^{-1}, \ldots, a_n q^{-1}\right)\hat{}.$$

Let $u_i = a_i q^{-1}$. Then $u = (u_1, \ldots, u_n)$ is a commuting normal n-tuple and $\hat{u} = v$.

DEFINITIONS 12.3. We shall denote by $\mathcal{E}\mathcal{N}$ the class of essentially normal n-tuples on $\mathcal{H}$. Also, $\mathcal{E}\mathcal{N}\mathcal{F} = \mathcal{E}\mathcal{N} \cap \mathcal{F}$. An n-tuple $A = (A_1, \ldots, A_n)$ is essentially unitary (in symbols, $A \in \mathcal{E}\mathcal{U}$) iff $A \in \mathcal{E}\mathcal{N}$ and $\sum_{i=1}^n A_i^* A_i - I \in \mathcal{K}(\mathcal{H})$ (i.e., $a = (a_1, \ldots, a_n)$ is normal and $\sum_{i=1}^n a_i^* a_i = 1$). Notice that if $A \in \mathcal{E}\mathcal{U}$, then $\mathrm{Sp}_e(A) \subset S^{2n-1}$.

The following fact is an easy consequence of Lemma 12.2.

LEMMA 12.4. *Let $A = (A_1, \ldots, A_n) \in \mathcal{E}\mathcal{N}$. Then there exist $U_1, \ldots, U_n \in \mathcal{L}(\mathcal{H})$ such that $U = (U_1, \ldots, U_n) \in \mathcal{E}\mathcal{U}$ and $A \overset{\mathcal{E}\mathcal{N}\mathcal{F}}{\sim} U$.*

PROOF. By Lemma 12.2, $\hat{a} = (u_1 q, \ldots, u_n q)\hat{}$, where $q = (\sum_{i=1}^n a_i^* a_i)^{1/2}$ and $u = (u_1, \ldots, u_n)$ is a normal n-tuple with $\sum_{i=1}^n u_i^* u_i = 1$. Let $q_t = (1 - t)q + t$, $t \in [0, 1]$, and let $\gamma(t) = (U_1 Q_t, \ldots, U_n Q_t)$. Then γ is continuous, $\gamma(t) \in \mathcal{E}\mathcal{N}\mathcal{F}$, $\gamma(0) = A$ and $\gamma(1) = U$.

LEMMA 12.5. *Let $A = (A_1, \ldots, A_n) \in \mathcal{E}\mathcal{U}$ and assume that $\mathrm{Sp}_e(A) \subsetneq S^{2n-1}$. Then $\mathrm{index}(A) = 0$ and in fact $A \overset{\mathcal{E}\mathcal{N}\mathcal{F}}{\sim} I$.*

PROOF. Since index is continuous and $\mathbf{C}^n \setminus \mathrm{Sp}_e(A)$ is connected, we see that $\mathrm{index}(A) = 0$. Let $z = (z_1, \ldots, z_n) \in S^{2n-1} \setminus \mathrm{Sp}_e(A)$. Let i be such that $z_i \neq 0$ and $C > \|A_i\| / |z_i|$. We define γ: $[0, 1] \to \mathcal{L}(\mathcal{H}) \otimes \mathbf{C}^n$ by $\gamma(t) = (A_1 - Ctz_1, \ldots, A_n - Ctz_n)$. Clearly $\gamma(t) \in \mathcal{E}\mathcal{N}\mathcal{F}$ and $\gamma(0) = A$. Now, $\gamma(1)_i = A_i - Cz_i$ is invertible, so that, by Proposition 11.1, $\gamma(1) \overset{\mathcal{E}\mathcal{N}\mathcal{F}}{\sim} I$.

3.

THEOREM 7. *Let* $A = (A_1, \dots, A_n)$, $B = (B_1, \dots, B_n) \in \mathcal{E}\mathfrak{N}\mathfrak{F}$ *and assume that* $\operatorname{index}(A) = \operatorname{index}(B)$. *Then* $A \overset{\mathcal{E}\mathfrak{N}\mathfrak{F}}{\sim} B$.

PROOF. By Lemma 12.4, we can assume that $A, B \in \mathcal{E}\mathfrak{U}$. Suppose now that $\operatorname{Sp}_e(A) = \operatorname{Sp}_e(B) = S^{2n-1}$. Since $C^*(a_1, \dots, a_n) \cong C(S^{2n-1})$ (Corollary 3.10), we see that A induces an element τ_A of $\operatorname{Ext}(S^{2n-1})$ (for a complete exposition on Ext see [1], [2] and [3]). It is known that $\operatorname{Ext}(S^{2n-1}) \cong \mathbf{Z}$, so that τ_A is equivalent to τ_k for some $k \in \mathbf{Z}$, where, for $k \neq 0$, τ_k is the extension generated by $(T_{z_1}^{(-k)}, T_{z_2}, \dots, T_{z_n})$, conveniently normalized so as to have essential spectrum S^{2n-1}, and τ_0 is the extension generated by any commuting n-tuple of normal operators whose essential spectrum is S^{2n-1} (take for instance a sequence $\{\lambda^{(j)}\}$ dense in S^{2n-1} and define N_i as $\lambda_i^{(1)} \cdot I_{\mathcal{H}} \oplus \lambda_i^{(2)} \cdot I_{\mathcal{H}} \oplus \cdots$). Since τ_A and τ_k are equivalent, there exist an isometric isomorphism $U \in \mathcal{L}(\mathcal{H}, H^2(S^{2n-1}))$ and compact operators $K_1, \dots, K_n$ such that $(A_1, \dots, A_n) = (U^*T_{z_1}^{(-k)}U + K_1, U^*T_{z_2}U + K_2, \dots, U^*T_{z_n}U + K_n)$ (or $(A_1, \dots, A_n) = (U^*N_1U + K_1, \dots, U^*N_nU + K_n)$). Therefore, $\operatorname{index}(A) = \operatorname{index}(T_{z_1}^{(-k)}, T_{z_2}, \dots, T_{z_n}) = k$ (or $\operatorname{index}(A) = 0$). Similarly, B induces an extension τ_B which is equivalent to τ_l for some $l \in \mathbf{Z}$, so that $\operatorname{index}(B) = l$. Thus, $k = l$, which implies that τ_A and τ_B are equivalent. Consequently, there exist a unitary $V \in \mathcal{L}(\mathcal{H})$ and compact operators $L_1, \dots, L_n$ such that

$$A_i = V^*B_iV + L_i \qquad (i = 1, \dots, n).$$

Since the set of unitaries is arcwise connected, there is a path unitaries V_t $(0 \leq t \leq 1)$ such that $V_0 = V$ and $V_1 = I$. Then

$$\gamma(t) = (V_t^*B_1V_t + (1-t)L_1, \dots, V_t^*B_nV_t + (1-t)L_n)$$

defines a path of essentially unitary n-tuples from A to B.

Suppose now that $\operatorname{Sp}_e(A) = S^{2n-1}$ and $\operatorname{Sp}_e(B) \subsetneqq S^{2n-1}$. By Lemma 12.5, $\operatorname{index}(B) = 0$ and $B \overset{\mathcal{E}\mathfrak{N}\mathfrak{F}}{\sim} I$. Therefore τ_A is equivalent to τ_0, so that A can be connected to a commuting n-tuple of normal operators $N = (N_1, \dots, N_n)$ by a path of essentially unitary n-tuples. But N can be joined to I by a path of commuting n-tuples of normal operators (see for instance [7, Theorem 3]), so that $A \overset{\mathcal{E}\mathfrak{N}\mathfrak{F}}{\sim} B$. Finally, if $\operatorname{Sp}_e(A) \subsetneqq S^{2n-1}$ and $\operatorname{Sp}_e(B) \subsetneqq S^{2n-1}$ we have $A \overset{\mathcal{E}\mathfrak{N}\mathfrak{F}}{\sim} I$ and $B \overset{\mathcal{E}\mathfrak{N}\mathfrak{F}}{\sim} I$ (by Lemma 12.5). Thus $A \overset{\mathcal{E}\mathfrak{N}\mathfrak{F}}{\sim} B$.

13. A path from T_z to W.

1. We consider $W = (W_1, \dots, W_n)$ on $H^2(S^1 \times \cdots \times S^1)$ and $T_z = (T_{z_1}, \dots, T_{z_n})$ on $H^2(S^{2n-1})$. We already know that $\operatorname{index}(W) = \operatorname{index}(T_z) = -1$. In this section we show that a copy of W on $H^2(S^{2n-1})$ and T_z can be joined by a path of almost doubly commuting Fredholm n-tuples.

Let us define S_i on $H^2(S^{2n-1})$ $(i = 1, \dots, n)$ by $S_ie_k = e_{k^{(i)}}$, $k^{(i)} = (k_1, \dots, k_i + 1, \dots, k_n)$, where $\{e_k\}_{k \in \mathbf{Z}_+^n}$ is the standard basis for $H^2(S^{2n-1})$. It is obvious that S_i is unitarily equivalent to W_i $(i = 1, \dots, n)$, so that by Corollary 5.3, $S = (S_1, \dots, S_n) \in \mathfrak{D}\mathfrak{F}$, the class of almost doubly commuting Fredholm n-tuples.

THEOREM 8. $T_z \overset{\mathfrak{D}\mathfrak{F}}{\sim} S$.

PROOF. We first notice that S_i is the partial isometry in the polar decomposition for $T_{z_i} = S_i P_i$, where $P_i e_k = (c_k / c_{k^{(i)}}) e_k$ (recall that $e_k = c_k z^k$).

We now define γ: $[0, 1] \to \mathfrak{L}(H^2(S^{2n-1})) \otimes \mathbf{C}^n$ by $\gamma(t)_i = S_i[(1-t)P_i + t]$ ($i = 1, \ldots, n$). γ is certainly continuous, so we need to prove that $\gamma(t) \in \mathfrak{D}\mathfrak{F}$ (all $t \in [0, 1]$).

First of all, we have to verify almost double commutativity. This amounts to showing that $[S_i, T_{z_j}]$, $[S_i, T_{z_j}^*] \in \mathcal{K}(H^2(S^{2n-1}))$ ($i \neq j$). Now

$$[S_i, T_{z_j}] = S_i S_j P_j - S_j P_j S_i = S_j(S_i P_j - P_j S_i),$$

and

$$S_i P_j^2 e_k = \frac{c_k^2}{c_{k^{(j)}}^2} e_{k^{(i)}}, \qquad P_j^2 S_i e_k = \frac{c_{k^{(i)}}^2}{c_{k^{(i)(j)}}^2} e_{k^{(i)}}$$

where

$$c_k = \frac{1}{\sqrt{2\pi^n}} \sqrt{\frac{(n + |k| - 1)!}{k!}}.$$

Thus

$$(S_i P_j^2 - P_j^2 S_i) e_k = \left(\frac{k_j + 1}{n + |k|} - \frac{k_j + 1}{n + |k| + 1} \right) e_{k^{(i)}} = \frac{k_j + 1}{(n + |k|)(n + |k| + 1)} e_{k^{(i)}}$$

so that $[S_i, P_j^2] \in \mathcal{K}(H^2(S^{2n-1}))$. Then $[S_i, P_j]$ is compact and so is $[S_i, T_{z_j}]$, for all i, j.

Similarly, $[S_i, T_{z_j}^*] = S_i P_j S_j^* - P_j S_j^* S_i = (S_i P_j - P_j S_i) S_j^*$, so that $[S_i, T_{z_j}^*] \in \mathcal{K}(H^2(S^{2n-1}))$ (all $i \neq j$). We now show that $\gamma(t) \in \mathfrak{F}$. Since $S \in \mathfrak{F}$ and $[P_1, S_j] \in \mathcal{K}(H^2(S^{2n-1}))$ ($j \geq 2$), Corollary 5.3 implies that $(S_1[(1-t)P_1 + t], S_2, \ldots, S_n) \in \mathfrak{F}$ when $t > 0$, or $(\gamma(t)_1, S_2, \ldots, S_n) \in \mathfrak{F}$ ($t > 0$). By the same argument (and the fact that $[P_1, P_2]$ is compact) we get: $(\gamma(t)_1, \gamma(t)_2, S_3, \ldots, S_n) \in \mathfrak{F}$ ($t > 0$), and finally $\gamma(t) \in \mathfrak{F}$. This completes the proof.

COROLLARY 13.1. $T_z^{(k)} \overset{\mathfrak{D}\mathfrak{F}}{\sim} S^{(k)}$, $k \in \mathbf{Z}^n$.

PROOF. By the spectral mapping theorem for n-tuples (Proposition 2.3) and Corollary 3.14, we conclude that $\gamma(t)^{(k)} = (\gamma(t)_1^{(k_1)}, \ldots, \gamma(t)_n^{(k_n)}) \in \mathfrak{D}\mathfrak{F}$, where γ is the path in the preceding theorem. Thus $\gamma^{(k)}$: $T_z^{(k)} \overset{\mathfrak{D}\mathfrak{F}}{\sim} S^{(k)}$.

14. The deformation problem in $\mathfrak{D}\mathfrak{F}$.

1. Throughout this section, we shall restrict attention to a separable infinite dimensional Hilbert space $\mathcal{K}$ and almost doubly commuting Fredholm pairs. Our goal is to prove that in $\mathfrak{D}$ = almost doubly commuting Fredholm pairs on $\mathcal{K}$ with a semi-Fredholm coordinate, the deformation problem has an affirmative answer. Our proof is built on a series of results that reduce the situation to $W^{(k)}$ on $H^2(S^1 \times S^1)$.

2.

PROPOSITION 14.1. *Let* $A = (A_1, \ldots, A_n) \in \mathfrak{F}$, $[A_1, A_k^*] \in \mathcal{K}(\mathcal{H})$ $(k \geq 2)$ *and assume that* ran A_1 *is closed. Let* $A_1 = VP$ *be the polar decomposition for* A_1. *Then* $[V, A_k^*] \in \mathcal{K}(\mathcal{H})$, $(V, A_2, \ldots, A_n) \in \mathfrak{F}$ *and* $A \overset{\mathfrak{F}}{\sim} (V, A_2, \ldots, A_n)$ *by a path* $\gamma(t)$ *satisfying* $[\gamma(t)_1, A_k^*] \in \mathcal{K}(\mathcal{H})$ $(k \geq 2)$.

We shall need the following

LEMMA 14.2. *Let* $S, T \in \mathfrak{L}(\mathcal{H})$, $[S, T] \in \mathcal{K}(\mathcal{H})$, $[S, T^*] \in \mathcal{K}(\mathcal{H})$ *and* $T = VP$ *be the polar decomposition for* T. *Assume that* ran T *is closed. Then* $[V, S]$, $[V, S^*] \in \mathcal{K}(\mathcal{H})$.

PROOF. We know that $\ker T = \ker V = \ker P$. Consider $\mathcal{H} = \ker T \oplus \operatorname{ran} T^*$. Then

$$T = \begin{pmatrix} 0 & T_1 \\ 0 & T_2 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & V_1 \\ 0 & V_2 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ 0 & P_2 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} S_1 & K_1 \\ K_2 & S_2 \end{pmatrix}.$$

Since ran T is closed, an application of the Open Mapping Theorem shows that K_1 and K_2 are compact. Moreover, P_2 is invertible, $T_1 = V_1P_2$, $T_2 = V_2P_2$, $T_1S_2 - S_1T_1 \in \mathcal{K}(\operatorname{ran} T^*, \ker T)$ and $T_2S_2 - S_2T_2 \in \mathcal{K}(\operatorname{ran} T^*)$, or $V_1P_2S_2 - S_1V_1P_2$, $V_2P_2S_2 - S_2V_2P_2$ compact. But $P \in C^*(T)$, and T almost doubly commutes with S, so that $[P, S] \in \mathcal{K}(\mathcal{H})$, or $[P_2, S_2] \in \mathcal{K}(\operatorname{ran} T^*)$. Thus,

$$(V_1S_2 - S_1V_1)P_2 \in \mathcal{K}(\operatorname{ran} T^*, \ker T).$$

and

$$(V_2S_2 - S_2V_2)P_2 \in \mathcal{K}(\operatorname{ran} T^*).$$

Since P_2 is invertible, we conclude that $V_1S_2 - S_1V_1$ and $V_2S_2 - S_2V_2$ are compact, which implies that $[V, S] \in \mathcal{K}(\mathcal{H})$. Similarly, $[V, S^*] \in \mathcal{K}(\mathcal{H})$ (this time using the fact that $[P, S^*] \in \mathcal{K}(\mathcal{H})$).

PROOF OF THE PROPOSITION. By the lemma, we know that $(V, A_2, \ldots, A_n)$ is an almost commuting n-tuple with $[V, A_k^*] \in \mathcal{K}(\mathcal{H})$ $(k \geq 2)$. Since $\mathfrak{F}$ is an open subset of the set of almost commuting n-tuples (Corollary 3.16), there exists $\varepsilon > 0$ such that $(A_1 + \lambda V, A_2, \ldots, A_n) \in \mathfrak{F}$ whenever $|\lambda| < \varepsilon$. Now $A_1 + \varepsilon V = VP + \varepsilon V = V(P + \varepsilon)$. By Corollary 5.3, $(V, A_2, \ldots, A_n) \in \mathfrak{F}$. It is now clear that $\gamma(t) = (V[(1 - t)P + t], A_2, \ldots, A_n)$ defines a path in $\mathfrak{F}$ from A to $(V, A_2, \ldots, A_n)$ satisfying the condition $[\gamma(t)_1, A_k^*] \in \mathcal{K}(\mathcal{H})$ $(k \geq 2)$.

REMARK. The preceding proposition is not obvious, since in general the partial isometry lies only in the von Neumann algebra generated by T.

3. We now turn to study those $A = (V, A_2) \in \mathfrak{F}$, where V is a partial isometry with finite dimensional kernel or cokernel and $[V, A_2^*] \in \mathcal{K}(\mathcal{H})$. By Proposition 11.1, we can restrict attention to the case $V \notin \mathfrak{F}$.

PROPOSITION 14.3. *Let* A *be as above and* dim(ker V) *be finite. Then* $A \overset{\mathfrak{D}\mathfrak{F}}{\sim} (S, T)$, *where* S *is a unilateral shift of infinite multiplicity.*

PROOF. By taking a compact perturbation, if necessary, we can assume that V is an isometry. By the Wold decomposition, $V = U \oplus S$, where U is unitary and S is a shift of multiplicity equal to dim ker V^*. Now S can be written as a direct sum of shifts of multiplicity 1. By Corollary 2.3 of [1], the first summand "absorbs" U up to unitary equivalence modulo the compacts, so that $U \oplus S$ is unitarily equivalent to a compact perturbation of S. Corollary 5.3 and the connectedness of the unitary group complete the proof.

4. We shall need the following lemma in dealing with the (S, T) situation.

LEMMA 14.4. *Let $\mathfrak{B}$ be a C^*-algebra, $s \in \mathfrak{B}$ be an isometry and $a_2 \in \mathfrak{B}$ be such that $sa_2 = a_2 s$ and $sa_2^* = a_2^* s$. Then (s, a_2) is invertible if and only if* $\ker s^* \cap \ker a_2 = 0$ *and* $\operatorname{ran} s + \operatorname{ran} a_2 = \mathfrak{B}$ (ker *and* ran *understood to be the kernel and range of the left multiplications induced by s and a_2).*

PROOF. The "only if" part is trivial. For the "if" part we need to prove exactness of the Koszul complex for (s, a_2) at stages 2 and 1. Since s is an isometry, $\ker s = 0$ and stage 2 is done. Assume that $sa + a_2 b = 0$. Then $a = -s^* a_2 b = -a_2 s^* b$. Let $c = s^* b$. Then $s^*(b - sc) = s^* b - s^* sc = c - c = 0$ and $a_2(b - sc) = a_2 b - a_2 sc = -(sa + sa_2 c) = -(sa + s(a_2 s^* b)) = -(sa - sa) = 0$. Thus $b - sc \in \ker s^* \cap \ker a_2 = 0$, or $b = sc$, as desired.

Let $\mathfrak{M}$ be a Hilbert space and $\mathfrak{N} = \mathfrak{M} \oplus \mathfrak{M} \oplus \cdots = l^2(\mathfrak{M})$. For $T \in \mathfrak{L}(\mathfrak{M})$ define $\hat{T} \in \mathfrak{L}(\mathfrak{N})$ by $\hat{T} = T \oplus T \oplus \cdots = T \otimes 1_{l^2}$.

PROPOSITION 14.5. *Let $(S, T) \in \mathfrak{D}\mathfrak{F}$, where S is a unilateral shift of infinite multiplicity acting on $\mathfrak{N} = l^2(\mathfrak{M})$. Let T_{00} be the $(0, 0)$-entry of T. Then $(S, T) \overset{\mathfrak{D}\mathfrak{F}}{\sim} (S, \widehat{T_{00}})$.*

PROOF. Since $SS^* + T^*T$ and $SS^* + TT^*$ are both Fredholm (Corollary 3.7), $\ker S^* = \mathfrak{M} \oplus 0 \oplus 0 \oplus \cdots$ and $(T_{01} T_{02} T_{03} \cdots)$, $(T_{10}^* T_{20}^* T_{30}^* \cdots)$ are compact, we conclude that T_{00} is Fredholm. Consequently,

$$\operatorname{ran} s + \operatorname{ran} t_\lambda = \mathfrak{Q}(\mathfrak{N}), \tag{1}$$

where, as usual, small letters are used for the projections in the Calkin algebra and $t_\lambda = (1 - \lambda)t + \lambda \widehat{t_{00}}$, $\lambda \in \mathbf{C}$. Suppose that $s^* a = t_\lambda a = 0$. Then

$$\begin{bmatrix} A_{10} & A_{11} & A_{12} & \\ A_{20} & A_{21} & A_{22} & \\ A_{30} & A_{31} & A_{32} & \\ & & & \ddots \end{bmatrix}$$

is compact, so that A can be chosen as

$$\begin{bmatrix} A_{00} & A_{01} & A_{02} & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ & & & \ddots \end{bmatrix}.$$

Since $t_\lambda a = 0$,

$$\begin{bmatrix} T_{00}A_{00} & T_{00}A_{01} & T_{00}A_{02} & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ & & & \ddots \end{bmatrix}$$

is compact. But then

$$\begin{bmatrix} T_{00} & T_{01} & T_{02} & \\ T_{10} & T_{11} & T_{12} & \\ T_{20} & T_{21} & T_{22} & \\ & & & \ddots \end{bmatrix}\begin{bmatrix} A_{00} & A_{01} & A_{02} & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ & & & \ddots \end{bmatrix}$$

is compact, or $ta = 0$. Since (s, t) is invertible and $s^*a = ta = 0$, we have $a = 0$. We have thus proved:

$$\ker s^* \cap \ker t_\lambda = 0. \tag{2}$$

Combining (1), (2) and Lemma 14.4, we obtain that (s, t_λ) is invertible for every λ. Taking $\lambda \in [0, 1]$, we have a path from (s, t) to $(s, \widehat{t_{00}})$.

The next proposition gives a characterization of the pairs $(S, \hat{C})$, where S is a unilateral shift and $\hat{C} = C \oplus C \oplus \cdots$.

PROPOSITION 14.6. *Let S be a unilateral shift on $\mathfrak{N} = \mathfrak{M} \oplus \mathfrak{M} \oplus \cdots$ and $C \in \mathcal{L}(\mathfrak{M})$. Then $(S, \hat{C}) \in \mathfrak{D}\mathfrak{F}$ iff C is Fredholm. In that case,* index$(S, \hat{C}) =$ index(C).

PROOF. "*If*". Clearly $[S, \hat{C}] = [S^*, \hat{C}] = 0$. If $s^*a = \hat{c}a = 0$, the argument in the preceding lemma again shows that $a = 0$. Similarly, ran s + ran $\hat{c} = \mathfrak{Q}(\mathfrak{N})$. By Lemma 14.4, $(s, \hat{c})$ is invertible.

"*Only if*". ran $\hat{C}^*\hat{C} + SS^*$ closed $\Rightarrow$ ran C closed. Furthermore $\ker \hat{C} = \ker C \oplus \ker C \oplus \cdots$ and $\ker \hat{C}^* \cap \ker S^*$, $\ker \hat{C} \cap \ker S^*$ are both finite dimensional (Corollary 7.3). Thus $\ker C \oplus 0 \oplus 0 \oplus \cdots$ and $\ker C^* \oplus 0 \oplus 0 \oplus \cdots$ are finite dimensional, which completes the proof of the Fredholmness of C.

Now, by Corollary 7.3, we know that: index$(S, \hat{C}) = \dim(\ker S^* \cap \ker \hat{C}) - \dim(\ker S^* \cap \ker \hat{C}^*) = \dim \ker C - \dim \ker C^* =$ index(C).

5. We need one more result before we can prove our theorem on the deformation problem.

PROPOSITION 14.7. *On $H^2(S^1 \times S^1)$, $(W_1, W_2) \overset{\mathfrak{D}\mathfrak{F}}{\sim} (W_1^*, W_2^*)$. More generally, for $k = (k_1, k_2) \in \mathbf{Z} \times \mathbf{Z}$, $(W_1^{(k_1)}, W_2^{(k_2)}) \overset{\mathfrak{D}\mathfrak{F}}{\sim} (W_1^{(-k_1)}, W_2^{(-k_2)})$.*

PROOF. By Theorem 8 of §13, $(S_1, S_2) \overset{\mathfrak{D}\mathfrak{F}}{\sim} (T_{z_1}, T_{z_2})$ and by Corollary 13.1, $(S_1^*, S_2^*) \overset{\mathfrak{D}\mathfrak{F}}{\sim} (T_{z_1}^*, T_{z_2}^*)$. Since (T_{z_1}, T_{z_2}) and $(T_{\bar{z}_1}, T_{\bar{z}_2})$ produce equivalent extensions of S^3, we know that there exists a unitary $U \in \mathcal{L}(H^2(S^3))$ such that $T_{z_i} = U^*T_{z_i}^*U + K_i$, $K_i \in \mathcal{K}(H^2(S^3))$ $(i = 1, 2)$. It is now clear that $(T_{z_1}, T_{z_2}) \overset{\mathfrak{D}\mathfrak{F}}{\sim} (T_{z_1}^*, T_{z_2}^*)$. Therefore, $(S_1, S_2) \overset{\mathfrak{D}\mathfrak{F}}{\sim} (T_{z_1}, T_{z_2}) \overset{\mathfrak{D}\mathfrak{F}}{\sim} (T_{z_1}^*, T_{z_2}^*) \overset{\mathfrak{D}\mathfrak{F}}{\sim} (S_1^*, S_2^*)$. The general statement follows in the same way.

REMARKS. An obvious extension of the preceding proof shows that $(S_1, S_2, \ldots, S_n) \overset{\mathfrak{D}\mathfrak{F}}{\sim} (S_1^*, S_2^*, \ldots, S_n^*)$ iff n is even (recall that $\operatorname{index}(S_1^*, \ldots, S_n^*) = (-1)^{n+1}$).

A combination of all the stated facts shows that $S^{(k)} = (S_1^{(k_1)}, \ldots, S_n^{(k_n)}) \overset{\mathfrak{D}\mathfrak{F}}{\sim} S^{(m)} = (S_1^{(m_1)}, \ldots, S_n^{(m_n)})$ iff $k_1 \cdot \ldots \cdot k_n = m_1 \cdot \ldots \cdot m_n$.

6.

THEOREM 9. *Let* $A = (A_1, A_2)$, $B = (B_1, B_2) \in \mathfrak{D}$, *the class of almost doubly commuting Fredholm pairs with a semi-Fredholm coordinate. Assume that* $\operatorname{index}(A) = \operatorname{index}(B)$. *Then* $A \overset{\mathfrak{D}}{\sim} B$.

PROOF. By Proposition 14.1, we can assume that $A_1 = V$, $B_1 = W$ are semi-Fredholm partial isometries. Also, by Proposition 11.1, we need only to consider the case $V, W \notin \mathfrak{F}$. If $\dim\ker V$ is finite then, by Proposition 14.3, $A \overset{\mathfrak{D}\mathfrak{F}}{\sim} (S, T)$. If $\dim\ker V^*$ is finite, then $(V^*, A_2) \overset{\mathfrak{D}\mathfrak{F}}{\sim} (S, T)$, so that $(V, A_2) \overset{\mathfrak{D}\mathfrak{F}}{\sim} (S^*, T)$. Similarly, $B \overset{\mathfrak{D}\mathfrak{F}}{\sim} (S_1, T_1)$ or $B \overset{\mathfrak{D}\mathfrak{F}}{\sim} (S_1^*, T_1)$. (Here S, S_1 are unilateral shifts of infinite multiplicity.) Since $\mathfrak{H}$ is separable, any two unilateral shifts of infinite multiplicity are unitarily equivalent. By Corollary 5.3 and the connectedness of the unitary group, we can assume $S = S_1$.

Thus, without loss of generality, our situation is $\mathfrak{H} = H^2(S^1 \times S^1) = H^2(S^1) \otimes H^2(S^1)$, $S = W_1$, $A = (W_1, \hat{T})$ or $A = (W_1^*, \hat{T})$ and $B = (W_1, \hat{R})$ or $B = (W_1^*, \hat{R})$. Four possibilities arise:

(i) $A = (W_1, \hat{T})$ and $B = (W_1, \hat{R})$,
(ii) $A = (W_1, \hat{T})$ and $B = (W_1^*, \hat{R})$,
(iii) $A = (W_1^*, \hat{T})$ and $B = (W_1, \hat{R})$,
(iv) $A = (W_1^*, \hat{T})$ and $B = (W_1^*, \hat{R})$.

Case (i). $\operatorname{index}(T) = \operatorname{index}(A) = \operatorname{index}(B) = \operatorname{index}(R)$, by Proposition 14.6. Consequently, there is a path of Fredholm operators joining T and R. Using the "if" part of Proposition 14.6, $A \overset{\mathfrak{D}\mathfrak{F}}{\sim} B$.

Case (iii). Let $m = \operatorname{index}(A) = -\operatorname{index}(T)$. Then $T \overset{\mathfrak{F}}{\sim} U_+^{(m)}$, where U_+ is the unilateral shift of multiplicity one on $H^2(S^1)$. Thus, $A \overset{\mathfrak{D}\mathfrak{F}}{\sim} (W_1^*, \widehat{U_+^{(m)}})$, since $(W_1, \widehat{T^*}) \overset{\mathfrak{D}\mathfrak{F}}{\sim} (W_1, \widehat{U_+^{(-m)}})$. Similarly, $\operatorname{index}(R) = \operatorname{index}(B) = m$ implies $R \overset{\mathfrak{F}}{\sim} U_+^{(-m)}$, so that $B \overset{\mathfrak{D}\mathfrak{F}}{\sim} (W_1, \widehat{U_+^{(-m)}})$. It is easy to see that $\widehat{U_+} = W_2$, so that we actually have

$$A \overset{\mathfrak{D}\mathfrak{F}}{\sim} (W_1^*, W_2^{(m)}) \quad \text{and} \quad B \overset{\mathfrak{D}\mathfrak{F}}{\sim} (W_1, W_2^{(-m)}).$$

By Proposition 14.7, $(W_1^*, W_2^{(m)}) \overset{\mathfrak{D}\mathfrak{F}}{\sim} (W_1, W_2^{(-m)})$, so that $A \overset{\mathfrak{D}\mathfrak{F}}{\sim} B$, as desired.

Case (ii) is completely analogous to (iii).

Case (iv). Consider $(W_1, \hat{T})$, $(W_1, \hat{R})$, use (i) to find a path in $\mathfrak{D}\mathfrak{F}$ and then use Corollary 3.14 to take adjoints in the first coordinate.

REMARKS. The separability of $\mathfrak{H}$ was only used in the proof of Theorem 9. The condition $n = 2$ was needed to invoke the one dimensional situation (see the treatment of cases (i) and (iii)).

15. Some additional results.

1. The following is a characterization of invertibility when $\mathcal{H}$ is finite dimensional.

PROPOSITION 15.1. *Let $A = (A_1, A_2)$ be a commuting pair on a finite dimensional Hilbert space $\mathcal{H}$. Then the following conditions are equivalent.*

(i) *A is invertible.*

(ii) $\ker A_1 \cap \ker A_2 = (0)$.

(iii) $\ker D_1 = \operatorname{ran} D_2$, *where D is the Koszul complex for A.*

(iv) $\ker A_1^* \cap \ker A_2^* = (0)$.

We shall need the following lemma, whose proof can be found in [**12**, Problem 56].

LEMMA 15.2 (J. SCHUR). *Let $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a matrix on a finite dimensional vector space, with $CD = DC$. Then*

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC).$$

In particular, $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is invertible iff $AD - BC$ is invertible.

PROOF OF THE PROPOSITION. (iv) $\Rightarrow$ (i). We are assuming that D_1 is onto, so that $A_1A_1^* + A_2A_2^*$ is invertible. By the lemma, so is

$$\hat{A} = \begin{pmatrix} A_1 & A_2 \\ -A_2^* & A_1^* \end{pmatrix},$$

that is, A is invertible (Corollary 3.11).

(ii) $\Rightarrow$ (i). $\ker A_1 \cap \ker A_2 = (0)$ implies that $A_1^*A_1 + A_2^*A_2$ is invertible. Therefore, so is

$$\widehat{A^*} = \begin{pmatrix} A_1^* & A_2^* \\ -A_2 & A_1 \end{pmatrix},$$

that is, $A^* = (A_1^*, A_2^*)$ is invertible. By Corollary 3.14, A is invertible.

(iii) $\Rightarrow$ (i). Since $\ker \hat{A} = \ker D_1 \cap (\operatorname{ran} D_2)^\perp = (0)$, we see that $\hat{A}$ is one-to-one. Since $\dim(\mathcal{H})$ is finite, we conclude that $\hat{A}$ is invertible, so that A is invertible.

(i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv) follow trivially.

COROLLARY 15.3. *Let $A = (A_1, A_2)$ be a commuting pair on $\mathcal{H}$ and* $\dim \mathcal{H} < \infty$. *Assume that* $\ker A_1 \cap \ker A_2 = (0)$. *Then there exist polynomials $p, q \in \mathbf{C}[z_1, z_2]$ such that $A_1p(A_1, A_2) + A_2q(A_1, A_2) = I$.*

PROOF. By Proposition 15.1, A is invertible. Since $\operatorname{Sp}(A)$ is finite, we have $\operatorname{Sp}(A) = \operatorname{Sp}_{(A)}(A)$, where (A) is the algebra of polynomials in A_1, A_2, by Theorem 5.5 of [**20**]. The conclusion then follows.

2. We now consider the pairs $(T_\phi \otimes I, I \otimes T_\psi)$ on $H^2(S^1 \times S^1)$, where $\phi, \psi \in C(S^1)$ and T_ϕ, T_ψ are their associated Toeplitz operators.

PROPOSITION 15.4. *Let $\phi, \psi \in C(S^1)$ and assume that neither T_ϕ nor T_ψ is invertible. Then $(T_\phi \otimes I, I \otimes T_\psi) \in \mathfrak{F}$ iff T_ϕ and T_ψ are Fredholm, so that* $\mathrm{Sp}_e(T_\phi \otimes I, I \otimes T_\psi) = \mathrm{Sp}_e(T_\phi) \times \mathrm{Sp}(T_\psi) \cup \mathrm{Sp}(T_\phi) \times \mathrm{Sp}_e(T_\psi)$. *If $\phi_1, \psi_1, \phi_2, \psi_2 \in C(S^1)$ and* $\mathrm{index}(T_{\phi_1} \otimes I, I \otimes T_{\psi_1}) = \mathrm{index}(T_{\phi_2} \otimes I, I \otimes T_{\psi_2})$, *there is a path of Fredholm pairs joining $(T_{\phi_1} \otimes I, I \otimes T_{\psi_1})$ and $(T_{\phi_2} \otimes I, I \otimes T_{\psi_2})$; also,*

$$\mathrm{index}(T_\phi \otimes I, I \otimes T_\psi) = -\mathrm{index}(T_\phi) \cdot \mathrm{index}(T_\psi).$$

PROOF. Let $\Phi(z_1, z_2) = \phi(z_1)$ and $\Psi(z_1, z_2) = \psi(z_2)$. Then $(T_\phi \otimes I, I \otimes T_\psi)$ is (T_Φ, T_Ψ). By the Corollary to Theorem 4 in **[10]**, we know that $(T_\Phi, T_\Psi) \in \mathfrak{F}$ iff $(T_{\Phi(z_1,\cdot)}, T_{\Psi(z_1,\cdot)})$ and $(T_{\Phi(\cdot,z_2)}, T_{\Psi(\cdot,z_2)})$ are invertible for all $(z_1, z_2) \in S^1 \times S^1$. A moment's thought shows that this is equivalent to $\phi \neq 0$, $\psi \neq 0$, which in turn is equivalent to T_ϕ, T_ψ both Fredholm. The rest follows easily from this.

3.

PROPOSITION 15.5. *Let $A = (A_1, \ldots, A_n) \in \mathfrak{D}\mathfrak{F}$ $(n > 2)$, where A_1 is an essentially normal operator with closed range. Then* $\mathrm{index}(A) = 0$ *and indeed $A \overset{\mathfrak{D}\mathfrak{F}}{\sim} I$, while keeping the first coordinate essentially normal with closed range.*

PROOF. Consider $\mathcal{H} = \ker A_1 \oplus \operatorname{ran} A_1^*$. Then

$$A_1 = \begin{pmatrix} 0 & B_1 \\ 0 & C_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} D_2 & B_2 \\ E_2 & C_2 \end{pmatrix}, \ldots, \quad A_n = \begin{pmatrix} D_n & B_n \\ E_n & C_n \end{pmatrix}.$$

Since $A \in \mathfrak{D}\mathfrak{F}$, a direct calculation using the Open Mapping Theorem for A_1 shows that $B_2, \ldots, B_n, E_2, \ldots, E_n$ are compact. By Corollary 3.7, it follows at once that $(D_2, \ldots, D_n)$ is a Fredholm $(n-1)$-tuple. (We should notice at this point that, in case $\ker A_1$ or $\operatorname{ran} A_1^*$ is finite dimensional, the conclusion follows immediately, because A_1 is either Fredholm or finite rank (forcing $(A_2, \ldots, A_n) \in \mathfrak{F}$).) We now claim that B_1 is compact, C_1 is essentially normal and C_1 is Fredholm.

From $A_1^*A_1 - A_1A_1^* \in \mathcal{K}(\mathcal{H})$ we get $B_1B_1^* \in \mathcal{K}(\ker A_1)$ and $B_1^*B_1 + C_1^*C_1 - C_1C_1^* \in \mathcal{K}(\operatorname{ran} A_1^*)$. Therefore, B_1 is compact and $[C_1^*, C_1] \in \mathcal{K}(\operatorname{ran} A_1^*)$. Finally, since $\ker A_1 = \ker A_1^*A_1$ and $\operatorname{ran} A_1$ is closed, we see that $B_1^*B_1 + C_1^*C_1$ is invertible. Then $C_1^*C_1$ is Fredholm and, since C_1 is essentially normal, C_1 is Fredholm.

We now connect A to

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & C_1 \end{pmatrix}, \begin{pmatrix} D_2 & 0 \\ 0 & C_2 \end{pmatrix}, \ldots, \begin{pmatrix} D_n & 0 \\ 0 & C_n \end{pmatrix} \right)$$

(by the line segment) and then use the proof of Proposition 11.1 to obtain the desired conclusion.

REMARK. We have seen in Proposition 14.1 that if $A = (A_1, \ldots, A_n) \in \mathfrak{F}$, $[A_1, A_k^*] \in \mathcal{K}(\mathcal{H})$ $(k > 2)$ and $\operatorname{ran} A_1$ is closed, then $A \overset{\mathfrak{F}}{\sim} (V, A_2, \ldots, A_n)$, where V is the partial isometry in the polar decomposition $A_1 = VP$. One might expect that a slight perturbation of an n-tuple $A \in \mathfrak{F}$ would provide one with first (or any other) coordinate having closed range. It is clear that a compact perturbation will not do it. Proposition 15.5 tells us that, unless $\mathrm{index}(A) = 0$ or we can afford to lose important algebraic properties (like A_1 being essentially normal), we shall not succeed.

16. Concluding remarks and open problems.

1. We have seen in Corollary 3.13 that spectral permanence for n-tuples holds when we consider W^*-algebras; in other words, if $\mathcal{B}$ is a W^*-subalgebra of the W^*-algebra $\mathcal{C}$ and $a = (a_1, \ldots, a_n)$ is a commuting n-tuple of elements of $\mathcal{B}$, then $\mathrm{Sp}(a, \mathcal{B}) = \mathrm{Sp}(a, \mathcal{C})$. The author does not know whether this is true for general C^*-algebras. It holds for $n = 2$, as a slight variant of the proof of Proposition 3.4 shows.

2. The extra condition in Theorem 9 (that at least one coordinate must be semi-Fredholm) might involve a second invariant, needed for a complete description of the path-components of $\mathcal{F}(\mathcal{H})$.

3. For the classes studied in §§12, 13 and 14, the formula

$$\operatorname{index} A^{(k)} = \operatorname{index}\left(A_1^{(k_1)}, \ldots, A_n^{(k_n)}\right) = k_1 \cdot \ldots \cdot k_n \operatorname{index}(A)$$

for an n-tuple A such that both A and $A^{(k)}$ belong to $\mathcal{F}$, where $k = (k_1, \ldots, k_n) \in \mathbf{Z}^n$, holds. Whether it holds in general is unknown to us. An affirmative answer to the deformation problem will immediately settle this issue, so that it can be used as a test for that problem.

4. We define an n-tuple $S = (S_1, \ldots, S_n)$ to be subnormal in case there exists a commuting family of normal operators on $\mathcal{K} \supset \mathcal{H}$ such that $N_i\mathcal{H} \subset \mathcal{H}$ and $N_i|_{\mathcal{H}} = S_i(1, \ldots, n)$. There is then a minimal normal extension, which is unique up to isometric isomorphism. For $n = 1$, it is known that, if N is minimal, then $\mathrm{Sp}(S) \supset \mathrm{Sp}(N)$ and $\mathrm{Sp}(S)$ can be obtained from $\mathrm{Sp}(N)$ by "filling in holes".

For $n > 1$, J. Janas [**14**] has shown that $\mathrm{Sp}_{\mathcal{Q}}(S) \supset \mathrm{Sp}(N)$, when $\mathcal{Q}$ is a maximal abelian algebra containing the S_i's. (We recall that $\mathrm{Sp}_{\mathcal{Q}}(S) \supset \mathrm{Sp}(S, \mathcal{H})$.)

It would be interesting to know if a spectral inclusion exists for Taylor spectrum and, if so, how $\mathrm{Sp}(S)$ can be obtained from $\mathrm{Sp}(N)$.

ADDED IN PROOF. We have recently shown that the answer to question 1 is affirmative (R. E. Curto, *Spectral permanence for joint spectra*, Proc. Sympos. Pure Math. (to appear)). Also, we have a proof of the spectral inclusion (question 4 above) when S is doubly commuting (R. E. Curto, *Spectral inclusion for doubly commuting subnormal n-tuples*, preprint).

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