

Fredholm Determinants, Differential Equations and Matrix Models

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Abstract: Orthogonal polynomial random matrix models of $N \times N$ hermitian matrices lead to Fredholm determinants of integral operators with kernel of the form $(\varphi(x)\psi(y) - \psi(x)\varphi(y))/x - y$. This paper is concerned with the Fredholm determinants of integral operators having kernel of this form and where the underlying set is the union of intervals $J = \bigcup_{j=1}^m (a_{2j-1}, a_{2j})$. The emphasis is on the determinants thought of as functions of the end-points a_k .

We show that these Fredholm determinants with kernels of the general form described above are expressible in terms of solutions of systems of PDE's as long as φ and ψ satisfy a certain type of differentiation formula. The (φ, ψ) pairs for the sine, Airy, and Bessel kernels satisfy such relations, as do the pairs which arise in the finite N Hermite, Laguerre and Jacobi ensembles and in matrix models of 2D quantum gravity. Therefore we shall be able to write down the systems of PDE's for these ensembles as special cases of the general system.

An analysis of these equations will lead to explicit representations in terms of Painlevé transcendents for the distribution functions of the largest and smallest eigenvalues in the finite N Hermite and Laguerre ensembles, and for the distribution functions of the largest and smallest singular values of rectangular matrices (of arbitrary dimensions) whose entries are independent identically distributed complex Gaussian variables.

There is also an exponential variant of the kernel in which the denominator is replaced by $e^{bx} - e^{by}$, where b is an arbitrary complex number. We shall find an analogous system of differential equations in this setting. If $b=i$ then we can interpret our operator as acting on (a subset of) the unit circle in the complex plane. As an application of this we shall write down a system of PDE's for Dyson's circular ensemble of $N \times N$ unitary matrices, and then an ODE if J is an arc of the circle.

I. Introduction

It is a fundamental result of Gaudin and Mehta that orthogonal polynomial random matrix models of $N \times N$ hermitian matrices lead to integral operators

whose Fredholm determinants describe the statistics of the spacing of eigenvalues [28, 36]. Precisely, if a weight function $w(x)$ is given, denote by $\{p_k(x)\}$ the sequence of polynomials orthonormal with respect to $w(x)$ and set

$$\varphi_k(x) := p_k(x)w(x)^{1/2}.$$

Then $E(n; J)$, the probability that a matrix from the ensemble associated with $w(x)$ has precisely n eigenvalues in the set J ($n=0, 1, \dots$), is given by the formula

$$E(n; J) = \frac{(-1)^n}{n!} \frac{d^n}{d\lambda^n} \det(I - \lambda K_N)|_{\lambda=1}, \quad (1.1)$$

where K_N is the integral operator on J with kernel

$$K_N(x, y) := \sum_{k=0}^{N-1} \varphi_k(x)\varphi_k(y).$$

It follows from the Christoffel–Darboux formula (cf. (6.3) below) that $\lambda K_N(x, y)$ is a particular case of a kernel of the general form

$$K(x, y) := \frac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x - y}. \quad (1.2)$$

This paper is concerned with the Fredholm determinants of integral operators having kernel of this form and where the underlying set is the union of intervals

$$J := \bigcup_{j=1}^m (a_{2j-1}, a_{2j}).$$

The emphasis is on the determinants thought of as functions of the end-points a_k .

If we denote the operator itself by K then it is well known that

$$\frac{\partial}{\partial a_k} \log \det(I - K) = (-1)^k R(a_k, a_k) \quad (k=1, \dots, 2m), \quad (1.3)$$

where $R(x, y)$, the resolvent kernel, is the kernel of $K(I - K)^{-1}$. This requires only that $\lambda=1$ not be an eigenvalue of K and that $K(x, y)$ be smooth. Jimbo, Miwa, Mōri and Sato [25] showed for the “sine kernel”

$$\frac{\sin(x-y)}{x-y},$$

that if we define

$$r_{k,\pm} := (I - K)^{-1} e^{\pm ix}(a_k),$$

then the $R(a_k, a_k)$ are expressible in terms of the $r_{k,\pm}$ and that these in turn, as functions of the a_1, \dots, a_{2m} , satisfy a completely integrable system of partial differential equations. They deduced from this that in the special case when J is an interval of length s the logarithmic derivative with respect to s of the Fredholm determinant satisfied a Painlevé differential equation. (More precisely, s times this logarithmic derivative satisfied the so-called σ form of P_V of Jimbo–Miwa–Okamoto [24, 33].) We refer the reader to [37] for a derivation of these results in the spirit of the present paper. The discovery that Painlevé transcendents can be used to represent correlation functions in statistical mechanical models first appeared in the 2D Ising model [1, 26, 41].

The sine kernel arises by taking a scaling limit as $N \rightarrow \infty$ in the bulk of the spectrum in a variety of random matrix models of $N \times N$ hermitian matrices. But if we take the Gaussian unitary ensemble (also called the Hermite ensemble; see below), and others as well, and scale at the edge of the spectrum then we are led similarly to the “Airy kernel”

$$\frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y},$$

where $\text{Ai}(x)$ is the Airy function [6, 19, 30, 38]. For this kernel the authors found [38] a completely analogous, although somewhat more complicated, system of PDE's, and showed that for J a semi-infinite interval (s, ∞) there was also a Painlevé equation associated with the determinant – this time P_{II} . Similarly, if we scale the Laguerre ensemble at the left edge of the spectrum or the Jacobi ensemble at either edge (see below for these ensembles also), then we obtain yet another kernel, the “Bessel kernel,” where in (1.2) $\varphi(x) = J_\alpha(\sqrt{x})$, $\psi(x) = x\varphi'(x)$ with J_α the usual Bessel function. Again we found [39] a system of PDE's for general J and, for $J = (0, s)$, a Painlevé equation associated with the Fredholm determinant – this time P_V (actually a special case of P_V which is reducible to P_{III}).

In looking for (and finding) analogous systems of PDE's for finite N matrix ensembles we realized that all we needed were differentiation formulas of a certain form for φ and ψ , namely

$$\begin{aligned} m(x)\varphi'(x) &= A(x)\varphi(x) + B(x)\psi(x), \\ m(x)\psi'(x) &= -C(x)\varphi(x) - A(x)\psi(x), \end{aligned} \tag{1.4}$$

where m, A, B and C are polynomials.

The (φ, ψ) pairs for the sine, Airy, and Bessel kernels satisfy such relations ($m(x) = 1$ for sine and Airy, $m(x) = x$ for Bessel) as do the pairs which arise in the finite N Hermite, Laguerre and Jacobi ensembles ($m(x) = 1$ for Hermite, $m(x) = x$ for Laguerre and $m(x) = 1 - x^2$ for Jacobi) and therefore we shall be able to write down the systems of PDE's for these ensembles at once as special cases of the general system. An analysis of these equations will lead in the cases of the finite N Hermite and Laguerre ensembles to explicit representations in terms of Painlevé transcendents for the distribution functions for the largest and smallest eigenvalue. A consequence of the latter is such a representation for the distribution functions of the largest and smallest singular values of rectangular matrices (of arbitrary dimensions) whose entries are independent identically distributed complex Gaussian variables; for these singular values are the eigenvalues of a matrix from an appropriate Laguerre ensemble [17].

There is also an exponential variant of the kernel (1.2) in which the denominator is replaced by $e^{bx} - e^{by}$ (or equivalently $\sinh \frac{b}{2}(x - y)$), where b is an arbitrary complex number. With an appropriate modification of (1.4) we shall find a completely analogous system of differential equations. Observe that if $b = i$ then we can interpret our operator as acting on (a subset of) the unit circle in the complex plane. As an application of this we shall write down a system of PDE's for Dyson's circular ensemble of $N \times N$ unitary matrices, and then an ODE if J is an arc of the circle. In case b is purely real our results have application to the so-called q -Hermite ensemble [9, 31].

Here, now, is a more detailed description of the contents of the paper.

A. *The Differential Equations.* In Sect. II we derive our general system of partial differential equations. To describe these equations we first define the functions

$$Q := (I - K)^{-1}\varphi, \quad P := (I - K)^{-1}\psi, \quad (1.5)$$

which depend also, of course, on the parameters a_k , and then

$$q_k := Q(a_k), \quad p_k := P(a_k) \quad (k = 1, \dots, 2m), \quad (1.6)$$

$$u_i := (Q(x), x^i \varphi(x)), \quad v_i := (Q(x), x^i \psi(x)), \quad w_i := (P(x), x^i \psi(x)) \quad (i = 0, 1, \dots),$$

where the inner products are taken over J . These are the unknown functions in our system of PDE's. We shall see that for any operator with kernel of the form (1.2) we have for the resolvent kernel the formulas [22]

$$R(a_j, a_k) = \frac{q_j p_k - p_j q_k}{a_j - a_k} \quad (j \neq k), \quad (1.7a)$$

$$R(a_k, a_k) = p_k \frac{\partial q_k}{\partial a_k} - q_k \frac{\partial p_k}{\partial a_k}, \quad (1.7b)$$

for the q_j and p_j the differentiation formulas

$$\frac{\partial q_j}{\partial a_k} = (-1)^k R(a_j, a_k) q_k \quad (j \neq k), \quad (1.8)$$

$$\frac{\partial p_j}{\partial a_k} = (-1)^k R(a_j, a_k) p_k \quad (j \neq k), \quad (1.9)$$

and for the u_j, v_j, w_j differentiation formulas of the form

$$\frac{\partial u_j}{\partial a_k}, \frac{\partial v_j}{\partial a_k}, \frac{\partial w_j}{\partial a_k} = \text{polynomial in } p_k, q_k \text{ and the various } u_i, v_i, w_i.$$

These equations are universal for any kernel of the form (1.2). What depends on (1.4) are the remaining differential formulas

$$m(a_j) \frac{\partial q_j}{\partial a_j} = \text{polynomial in } q_j, p_j \text{ and the } u_i, v_i, w_i \\ - \sum_{k \neq j} (-1)^k R(a_j, a_k) q_k,$$

$$m(a_j) \frac{\partial p_j}{\partial a_j} = \text{polynomial in } q_j, p_j \text{ and the } u_i, v_i, w_i \\ - \sum_{k \neq j} (-1)^k R(a_j, a_k) p_k,$$

and the representation

$$m(a_j) R(a_j, a_j) = \text{polynomial in } q_j, p_j \text{ and the } u_i, v_i, w_i.$$

The polynomials on the right sides are expressed in terms of the coefficients of the polynomials m, A, B, C in (1.4). We mention that in [25] no "extra" quantities $u_i, v_i,$

w_i appear, but this is quite special. In general the number of triples (u_i, v_i, w_i) which occur is at most

$$\max(\deg A, \deg B, \deg C, \deg m - 1),$$

although in practice fewer of these quantities actually appear.

B. The Examples. First in Sect. III we quickly derive, as special cases, the systems of equations for the sine, Airy and Bessel kernels. Then in Sect. IV we derive and investigate the equations for kernels “beyond Airy”. To explain this we replace the variables x, y in (1.2) by λ and μ , think of (a completely new variable) x as a parameter, and observe that for each x

$$K(\lambda, \mu) := \frac{\text{Ai}(x + \lambda)\text{Ai}'(x + \mu) - \text{Ai}'(x + \lambda)\text{Ai}(x + \mu)}{\lambda - \mu} \quad (1.10)$$

has the same properties as the Airy kernel. (In the differentiation formulas (1.4) the variable is now λ and x is a parameter in the coefficients.) Observe also that $\text{Ai}(x + \lambda)$ is, as a function of x , an eigenfunction of the Schrödinger operator with potential $\xi(x) = -x$ corresponding to eigenvalue λ .

In the hermitian matrix models of 2D quantum gravity [8, 7, 13, 12, 21] solutions to the so-called string equation

$$[\mathcal{Q}, \mathcal{P}] = 1$$

determine the functions φ and ψ . In the simplest case of the KdV hierarchy, the operator \mathcal{Q} is the Schrödinger operator (note our convention of sign of D_x^2)

$$\mathcal{Q} = D_x^2 + \xi(x)$$

and the differential operator \mathcal{P} (in x) is

$$\mathcal{P} = (\mathcal{Q}^{(2l-1)/2})_+ \quad (l = 1, 2, \dots).$$

where $(\cdot)_+$ is the differential operator part. The potential ξ then satisfies a differential equation determined by the string equation and $\varphi(\lambda, x)$ is the eigenfunction

$$(D_x^2 + \xi(x))\varphi(\lambda, x) = \lambda\varphi(\lambda, x) \quad (1.11)$$

satisfying

$$\frac{\partial \varphi}{\partial \lambda} = \mathcal{P}\varphi(\lambda, x). \quad (1.12)$$

Setting

$$\psi(\lambda, x) = D_x \varphi(\lambda, x) \quad (1.13)$$

the kernel [6, 30] is then

$$\begin{aligned} K(\lambda, \mu) &= \frac{\varphi(\lambda, x)\psi(\mu, x) - \psi(\lambda, x)\varphi(\mu, x)}{\lambda - \mu} \\ &= \int_x^\infty \varphi(\lambda, y)\varphi(\mu, y)dy. \end{aligned} \quad (1.14)$$

These are the kernels which we say are “beyond Airy” since for $l=1$, $\mathcal{P}=D_x$, $\xi(x)=-x$, (1.14) reduces to the generalized Airy kernel (1.10). From (1.11) and (1.12) it follows that for general l the functions $\varphi(\lambda, x)$ and $\psi(\lambda, x)$ satisfy differentiation formulas (in λ) of the form (1.4). (Again in the differentiation formulas (1.4) the variable is now λ and x is a parameter in the coefficients.) In Sect. IV we illustrate these methods for the case $l=2$.

In Sect. V we study in some detail the finite N Hermite, Laguerre, Jacobi, and circular ensembles. In orthogonal polynomial ensembles one is given a weight function $w(x)$ and then, for any symmetric function f on \mathbf{R}^N , we have

$$E(f(\lambda_1, \dots, \lambda_N)) = c_N \int \cdots \int f(x_1, \dots, x_N) \prod w(x_i) \prod_{i < j} |x_i - x_j|^2 dx_1 \dots dx_N, \quad (1.15)$$

where “ E ” denotes the expected value, $\lambda_1, \dots, \lambda_N$ are the eigenvalues, and c_N is a constant such that the right side equals one when $f=1$. In the Hermite ensemble $w(x)=e^{-x^2}$ and the integrations are over \mathbf{R} , in the Laguerre ensemble $w(x)=x^\alpha e^{-x}$ and the integrations are over \mathbf{R}^+ , and in the Jacobi ensemble $w(x)=(1-x)^\alpha(1+x)^\beta$ and the integrations are over $(-1, 1)$. In the circular ensemble $w(x)=1$ and the integrations are over the unit circle.

The size parameter N will appear only as a coefficient parameter in the equations we obtain; and we find that the equations for both bulk and edge scaling limits emerge as limiting cases. Our equations also make the study of large N corrections to the scaling limits tractable.

For the Hermite, Laguerre and Jacobi ensembles there are natural intervals depending upon a single parameter s – for Hermite $J=(s, \infty)$ or $(-\infty, s)$, for Laguerre $J=(s, \infty)$ or $(0, s)$, and for Jacobi $J=(s, 1)$ or $J=(-1, s)$ – and in all these cases we shall find an associated ordinary differential equation. For Hermite and Laguerre these will be of Painlevé type. Observe that taking $n=0$ in (1.1) shows that the Fredholm determinant in each of these cases is precisely the distribution function for the largest eigenvalue, or 1 minus the distribution for the smallest eigenvalue.

C. General Matrix Ensembles. In this final section of the paper we show that there are differentiation formulas of the form (1.4) when Hermite, Laguerre, or Jacobi weights are multiplied by $e^{-V(x)}$, where $V(x)$ is an arbitrary polynomial. (Of course it must be of such a form that the resulting integrals are convergent.) It is the finite N matrix models corresponding to certain $V(x)$ which, in an appropriate double scaling limit at the edge of the spectrum, lead to the kernels beyond Airy. (Strictly speaking this is true only for the universality classes $l=1, 3, 5, \dots$ as it is well-known that the cases $l=2, 4, 6, \dots$ require coefficients in $V(x)$ that make $e^{-V(x)}$ unbounded.)

II. The General System of Partial Differential Equations

In this section we derive the system of partial differential equations that determine the functional dependence of the Fredholm determinant $\det(I-K)$ upon the parameters a_k where K has kernel (1.2). After some preliminary definitions and identities in Sect. IIA, in Sec. IIB we derive those equations which are independent of the differentiation formulas (1.4). In Sect. IIC we assume φ and ψ satisfy the

differentiation formulas for the case $m(x)=1$. Then in Sect. IID we indicate the modifications necessary for the general case of polynomial m . Finally, in Sect. IIE we derive the exponential variant of the system of equations.

A. Preliminaries. Our derivation will use, several times, the commutator identity

$$[L, (I-K)^{-1}] = (I-K)^{-1}[L, K](I-K)^{-1}, \quad (2.1)$$

which holds for arbitrary operators K and L , and the differentiation formula

$$\frac{d}{da}(I-K)^{-1} = (I-K)^{-1} \frac{dK}{da} (I-K)^{-1}, \quad (2.2)$$

which holds for an arbitrary operator depending smoothly on a parameter a .

It will be convenient to think of our operator K as acting, not on J , but on a larger natural domain \mathcal{D} and to have kernel

$$K(x, y)\chi_J(y), \quad (2.3)$$

where χ_J is the characteristic function of J and $K(x, y)$ is the kernel (1.2). For example, for the sine and Airy kernel $\mathcal{D}=\mathbf{R}$, for the Bessel kernel $\mathcal{D}=\mathbf{R}^+$, and for the Jacobi kernel $\mathcal{D}=(-1, 1)$. The set J will be a subset of \mathcal{D} . We will continue to denote the resolvent kernel of K by $R(x, y)$ and note that it is smooth in x but discontinuous at $y=a_k$. We will also need the distributional kernel

$$\rho(x, y) = \delta(x-y) + R(x, y)$$

of $(I-K)^{-1}$. The quantities $R(a_j, a_k)$ in Sect. IA are interpreted to mean

$$\lim_{\substack{y \rightarrow a_k \\ y \in J}} R(a_j, y),$$

and similarly for p_j and q_j .

The definitions of u_i , etc. must be modified. Before doing this it will be convenient to introduce

$$Q_j(x) := (I-K)^{-1}x^j\varphi(x), \quad P_j(x) := (I-K)^{-1}x^j\psi(x), \quad (2.4)$$

which for $j=0$ reduce to (1.5) ($Q_0=Q$, $P_0=P$). Then we define

$$u_j := (Q, x^j\varphi\chi_J) = (Q_j, \varphi\chi_J), \quad (2.5a)$$

$$v_j := (Q, x^j\psi\chi_J) = (P_j, \varphi\chi_J), \quad (2.5b)$$

$$\tilde{v}_j := (P, x^j\varphi\chi_J) = (Q_j, \psi\chi_J), \quad (2.5c)$$

$$w_j := (P, x^j\psi\chi_J) = (P_j, \psi\chi_J), \quad (2.5d)$$

where the inner product (\cdot, \cdot) is now over the domain \mathcal{D} . That these definitions for u_j , v_j and w_j are equivalent to those of Sect. IA and the above equalities are left as exercises. They follow from the fact that

$$(I-K^t)^{-1}f\chi_J = (I-K)^{-1}f\chi_J = \begin{cases} (I-K)^{-1}f & \text{on } J \\ 0 & \text{on } J^c \end{cases} \quad (2.6)$$

for any smooth f . Here K' is the transpose of the operator K . (Note that K takes smooth functions to smooth functions while its transpose takes distributions to distributions.) We also observe that $\tilde{v}_0 = v_0$. A final bit of preliminary notation is

$$L \doteq L(x, y),$$

which means the operator L has kernel $L(x, y)$.

B. The Universal Equations. In this subsection M denotes multiplication by the independent variable. If we consider the commutator of M with K and use the representation (2.3), we have immediately

$$[M, K] \doteq (\varphi(x)\psi(y) - \psi(x)\varphi(y))\chi_J(y),$$

and so by (2.1)

$$[M, (I - K)^{-1}] \doteq Q(x)(I - K')^{-1}\psi\chi_J(y) - P(x)(I - K')^{-1}\varphi\chi_J(y). \quad (2.7)$$

(The transpose here arises from the general fact that if $L \doteq U(x)V(y)$ then $T_1 L T_2 \doteq T_1 U(x) T_2 V(y)$.) It follows immediately that

$$[M, (I - K)^{-1}] \doteq (x - y)\rho(x, y) = (x - y)R(x, y),$$

and hence referring to (2.6), we deduce

$$R(x, y) = \frac{Q(x)P(y) - P(x)Q(y)}{x - y} \quad (x, y \in J).$$

In particular we have deduced (1.7a) (recall definitions (1.6)) and the representation

$$R(x, x) = Q'(x)P(x) - P'(x)Q(x) \quad (x \in J). \quad (2.8)$$

We remark that the generality of this expression for $R(x, y)$ was first, as far as the authors are aware, stressed by Its, et al. [22] though it appears, of course, in the context of the sine kernel in the earlier work of JMMS [25].

We have the easy fact that

$$\frac{\partial}{\partial a_k} K \doteq (-1)^k K(x, a_k)\delta(y - a_k),$$

and so by (2.2)

$$\frac{\partial}{\partial a_k} (I - K)^{-1} \doteq (-1)^k R(x, a_k)\rho(y, a_k). \quad (2.9)$$

At this point we use the notations $Q(x, a)$, $P(x, a)$ for $Q(x)$ and $P(x)$, respectively, to remind ourselves that they are functions of a as well as x . We deduce immediately from (2.9) and (1.5) that

$$\frac{\partial}{\partial a_k} Q(x, a) = (-1)^k R(x, a_k)q_k, \quad \frac{\partial}{\partial a_k} P(x, a) = (-1)^k R(x, a_k)p_k. \quad (2.10)$$

Since $q_j = Q(a_j, a)$ and $p_j = P(a_j, a)$ this gives

$$\frac{\partial q_j}{\partial a_k} = (-1)^k R(a_j, a_k)q_k, \quad \frac{\partial p_j}{\partial a_k} = (-1)^k R(a_j, a_k)p_k, \quad (j \neq k).$$

These are Eqs. (1.8) and (1.9). We record for use below

$$\frac{\partial q_j}{\partial a_j} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial a_j} \right) Q(x, a) \Big|_{x=a_j}, \quad \frac{\partial p_j}{\partial a_j} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial a_j} \right) P(x, a_j) \Big|_{x=a_j}. \quad (2.11)$$

To obtain (1.7b) observe that (2.8) gives

$$R(a_k, a_k) = \frac{\partial}{\partial x} [Q(x, a)p_k - P(x, a)q_k] \Big|_{x=a_k}.$$

But the expression in brackets above vanishes identically when $x = a_k$ and so the above is equal to

$$-\frac{\partial}{\partial a_k} [Q(x, a)p_k - P(x, a)q_k] \Big|_{x=a_k}.$$

If we use (2.10) in the computation of this partial derivative, (1.7b) results.

We now show that the \tilde{v}_k can be expressed in terms of the other quantities u_i , v_i and w_i (we could do v_k just as well) and that the P_k and Q_k can be expressed in terms of these quantities and P , Q . From

$$\frac{x^k - y^k}{x - y} = \sum_{\substack{i+j=k-1 \\ i, j \geq 0}} x^i y^j$$

we get

$$[M^k, K] \doteq \sum_{\substack{i+j=k-1 \\ i, j \geq 0}} (x^i \varphi(x) y^j \psi(y) - x^i \psi(x) y^j \varphi(y)) \chi_J(y),$$

and so

$$[M^k, (I - K)^{-1}] \doteq \sum_{\substack{i+j=k-1 \\ i, j \geq 0}} (Q_i(x)(I - K^t)^{-1} y^j \psi(y) \chi_J(y) - P_i(x)(I - K^t)^{-1} y^j \varphi(y) \chi_J(y)).$$

Applying this to φ shows that

$$Q_k(x) = x^k Q(x) - \sum_{\substack{i+j=k-1 \\ i, j \geq 0}} (v_j Q_i(x) - u_j P_i(x)), \quad (2.12)$$

and applying it to ψ shows that

$$P_k(x) = x^k P(x) - \sum_{\substack{i+j=k-1 \\ i, j \geq 0}} (w_j Q_i(x) - \tilde{v}_j P_i(x)). \quad (2.13)$$

These are the recursion relations for Q_k , P_k . Taking the inner product of both sides of the first one with $\psi \chi_J$ gives

$$\tilde{v}_k = v_k - \sum_{\substack{i+j=k-1 \\ i, j \geq 0}} (v_j \tilde{v}_i - u_j w_i), \quad (2.14)$$

recursion formulas which can be used to express the \tilde{v}_k in terms of the u_i , v_i , w_i .

Finally, using the definition of u_j in (2.5a), the fact

$$\frac{\partial}{\partial a_k} \chi_J(y) = (-1)^k \delta(y - a_k),$$

and (2.10) we find that

$$\begin{aligned} \frac{\partial u_j}{\partial a_k} &= (-1)^k (R(x, a_k), x^j \varphi \chi_J) q_k + (-1)^k Q(a_k) a_k^j \varphi(a_k) \\ &= (-1)^k (\rho(x, a_k), x^j \varphi \chi_J) q_k \\ &= (-1)^k Q_j(a_k) q_k \\ &= (-1)^k q_{jk} q_k. \end{aligned} \quad (2.15)$$

where $q_{jk} = Q_j(a_k)$ ($q_{0k} = q_k$). Similarly,

$$\frac{\partial v_j}{\partial a_k} = (-1)^k P_j(a_k) Q(a_k) = (-1)^k p_{jk} q_k, \quad (2.16)$$

$$\frac{\partial \tilde{v}_j}{\partial a_k} = (-1)^k Q_j(a_k) P(a_k) = (-1)^k q_{jk} p_k, \quad (2.17)$$

$$\frac{\partial w_j}{\partial a_k} = (-1)^k P_j(a_k) P(a_k) = (-1)^k p_{jk} p_k, \quad (2.18)$$

where $p_{jk} = P_j(a_k)$ ($p_{0k} = p_k$). From (2.12), (2.13) and (2.14) we recall that q_{jk} and p_{jk} are expressible in terms of the q_j , p_j , u_i , v_i , and w_i .

C. The Case $m(x) = 1$. In this section we derive those partial differential equations that depend upon the differentiation formulas (1.4) in the special case $m(x) \equiv 1$. We let D denote the differentiation operator with respect to the independent variable and recall that if the operator L has distributional kernel $L(x, y)$ then

$$[D, L] \doteq \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) L(x, y). \quad (2.19)$$

Using the differentiation formulas it follows that

$$\begin{aligned} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) K(x, y) &= \frac{A(x) - A(y)}{x - y} (\varphi(x)\psi(y) + \psi(x)\varphi(y)) \\ &\quad + \frac{B(x) - B(y)}{x - y} \psi(x)\psi(y) + \frac{C(x) - C(y)}{x - y} \varphi(x)\varphi(y). \end{aligned} \quad (2.20)$$

Let us write

$$A(x) = \sum \alpha_j x^j, \quad B(x) = \sum \beta_j x^j, \quad C(x) = \sum \gamma_j x^j. \quad (2.21)$$

Then

$$\frac{A(x) - A(y)}{x - y} = \sum_{j, k \geq 0} \alpha_{j+k+1} x^j y^k, \text{ etc.}$$

and we obtain the identity (recall (2.3) and (2.19))

$$\begin{aligned} [D, K] &\doteq \sum_{j, k \geq 0} \alpha_{j+k+1} (x^j \varphi(x) y^k \psi(y) + x^j \psi(x) y^k \varphi(y)) \chi_J(y) \\ &+ \sum_{j, k \geq 0} \beta_{j+k+1} x^j \psi(x) y^k \psi(y) \chi_J(y) \\ &+ \sum_{j, k \geq 0} \gamma_{j+k+1} x^j \varphi(x) y^k \varphi(y) \chi_J(y) - \sum_k (-1)^k K(x, a_k) \delta(y - a_k), \end{aligned}$$

from which it follows that

$$\begin{aligned} [D, (I - K)^{-1}] &\doteq \sum \alpha_{j+k+1} (Q_j(x) (I - K^t)^{-1} y^k \psi(y) \chi_J(y) \\ &+ P_j(x) (I - K^t)^{-1} y^k \varphi(y) \chi_J(y)) \\ &+ \sum \beta_{j+k+1} P_j(x) (I - K^t)^{-1} y^k \psi(y) \chi_J(y) \\ &+ \sum \gamma_{j+k+1} Q_j(x) (I - K^t)^{-1} y^k \varphi(y) \chi_J(y) \\ &- \sum (-1)^k R(x, a_k) \rho(a_k, y). \end{aligned} \quad (2.22)$$

We now use this last commutator to compute $Q'(x)$ and $P'(x)$:

$$\begin{aligned} Q'(x) &= D(I - K)^{-1} \varphi(x) \\ &= (I - K)^{-1} D\varphi(x) + [D, (I - K)^{-1}] \varphi(x) \\ &= (I - K)^{-1} D\varphi(x) + \sum \alpha_{j+k+1} (v_k Q_j(x) + u_k P_j(x)) \\ &+ \sum \beta_{j+k+1} v_k P_j(x) + \sum \gamma_{j+k+1} u_k Q_j(x) \\ &- \sum (-1)^k R(x, a_k) q_k, \end{aligned}$$

and similarly

$$\begin{aligned} P'(x) &= (I - K)^{-1} D\psi(x) + \sum \alpha_{j+k+1} (w_k Q_j(x) + \tilde{v}_k P_j(x)) \\ &+ \sum \beta_{j+k+1} w_k P_j(x) + \sum \gamma_{j+k+1} \tilde{v}_k Q_j(x) \\ &- \sum (-1)^k R(x, a_k) p_k. \end{aligned}$$

Finally we use the differentiation formulas (1.4) and representations (2.21) to deduce

$$\begin{aligned} (I - K)^{-1} D\varphi(x) &= \sum (\alpha_j Q_j(x) + \beta_j P_j(x)), \\ (I - K)^{-1} D\psi(x) &= \sum (-\gamma_j Q_j(x) - \alpha_j P_j(x)), \end{aligned}$$

and so substituting into the above gives

$$\begin{aligned} Q'(x) &= \sum_{j \geq 0} \left(\alpha_j + \sum_{k \geq 0} \alpha_{j+k+1} v_k + \sum_{k \geq 0} \gamma_{j+k+1} u_k \right) Q_j(x) \\ &+ \sum_{j \geq 0} \left(\beta_j + \sum_{k \geq 0} \alpha_{j+k+1} u_k + \sum_{k \geq 0} \beta_{j+k+1} v_k \right) P_j(x) \\ &- \sum_{k=1}^{2m} (-1)^k R(x, a_k) q_k, \end{aligned} \quad (2.23)$$

$$\begin{aligned}
P'(x) = & \sum_{j \geq 0} \left(-\gamma_j + \sum_{k \geq 0} \alpha_{j+k+1} w_k + \sum_{k \geq 0} \gamma_{j+k+1} \tilde{v}_k \right) Q_j(x) \\
& + \sum_{j \geq 0} \left(-\alpha_j + \sum_{k \geq 0} \alpha_{j+k+1} \tilde{v}_k + \sum_{k \geq 0} \beta_{j+k+1} w_k \right) P_j(x) \\
& - \sum_{k=1}^{2m} (-1)^k R(x, a_k) p_k .
\end{aligned} \tag{2.24}$$

From (2.10), (2.11) and these last identities we deduce the equations

$$\begin{aligned}
\frac{\partial q_i}{\partial a_i} = & \sum_{j \geq 0} \left(\alpha_j + \sum_{k \geq 0} \alpha_{j+k+1} v_k + \sum_{k \geq 0} \gamma_{j+k+1} u_k \right) q_{ji} \\
& + \sum_{j \geq 0} \left(\beta_j + \sum_{k \geq 0} \alpha_{j+k+1} u_k + \sum_{k \geq 0} \beta_{j+k+1} v_k \right) p_{ji} \\
& - \sum_{k \neq i} (-1)^k R(a_i, a_k) q_k ,
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
\frac{\partial p_i}{\partial a_i} = & \sum_{j \geq 0} \left(-\gamma_j + \sum_{k \geq 0} \alpha_{j+k+1} w_k + \sum_{k \geq 0} \gamma_{j+k+1} \tilde{v}_k \right) q_{ji} \\
& + \sum_{j \geq 0} \left(-\alpha_j + \sum_{k \geq 0} \alpha_{j+k+1} \tilde{v}_k + \sum_{k \geq 0} \beta_{j+k+1} w_k \right) p_{ji} \\
& - \sum_{i \neq i} (-1)^k R(a_i, a_k) p_k .
\end{aligned} \tag{2.26}$$

Using (2.28), (2.23), and (2.24) we deduce

$$\begin{aligned}
R(a_i, a_i) = & \sum_{j \geq 0} \left(\alpha_j + \sum_{k \geq 0} \alpha_{j+k+1} v_k + \sum_{k \geq 0} \gamma_{j+k+1} u_k \right) q_{ji} p_i \\
& + \sum_{j \geq 0} \left(\beta_j + \sum_{k \geq 0} \alpha_{j+k+1} u_k + \sum_{k \geq 0} \beta_{j+k+1} v_k \right) p_{ji} p_i \\
& + \sum_{j \geq 0} \left(\gamma_j - \sum_{k \geq 0} \alpha_{j+k+1} w_k - \sum_{k \geq 0} \gamma_{j+k+1} \tilde{v}_k \right) q_{ji} q_i \\
& + \sum_{j \geq 0} \left(\alpha_j - \sum_{k \geq 0} \alpha_{j+k+1} \tilde{v}_k - \sum_{k \geq 0} \beta_{j+k+1} w_k \right) p_{ji} q_i \\
& + \sum_{k \neq i} (-1)^k \frac{(q_i p_k - p_i q_k)^2}{a_i - a_k} .
\end{aligned} \tag{2.27}$$

We end this section with two differentiation formulas for $R(a_i, a_i)$. From (2.9) and (2.22) we deduce that for $x, y \in J$,

$$\begin{aligned}
\left(\frac{\partial}{\partial a_i} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) R(x, y) = & \sum \alpha_{j+k+1} (Q_j(x) P_k(y) + P_j(x) Q_k(y)) \\
& + \sum \beta_{j+k+1} P_j(x) P_k(y) + \sum \gamma_{j+k+1} Q_j(x) Q_k(y) \\
& - \sum_{k \neq i} (-1)^k R(x, a_k) R(a_k, y) .
\end{aligned}$$

Hence, using the chain rule,

$$\begin{aligned} \frac{\partial}{\partial a_i} R(a_i, a_i) &= 2 \sum_{j, k \geq 0} \alpha_{j+k+1} q_{ji} p_{ki} + \sum_{j, k \geq 0} \beta_{j+k+1} p_{ji} p_{ki} \\ &+ \sum_{j, k \geq 0} \gamma_{j+k+1} q_{ji} q_{ki} - \sum_{k \neq i} (-1)^k R(a_i, a_k)^2. \end{aligned} \tag{2.28}$$

A variant of this follows from it and (2.9) by the chain rule:

$$\begin{aligned} \frac{d}{dt} R_{tJ}(ta_i, ta_i) &= 2a_i \sum_{j, k \geq 0} \alpha_{j+k+1} q_{ji} p_{ki} + a_i \sum_{j, k \geq 0} \beta_{j+k+1} p_{ji} p_{ki} \\ &+ a_i \sum_{j, k \geq 0} \gamma_{j+k+1} q_{ji} q_{ki} + \sum_{k \neq i} (-1)^k (a_k - a_i) R_{tJ}(ta_i, ta_k)^2. \end{aligned} \tag{2.29}$$

Here the subscripts tJ indicate that this is the underlying interval.

D. The Case of Polynomial m . Now let us see how the above derivation has to be modified if $m(x)$ is an arbitrary polynomial. In this section M denotes multiplication by $m(x)$ and D continues to denote differentiation with respect to the independent variable. In place of the commutator $[D, K]$ we consider the commutator

$$\begin{aligned} [MD, K] &\doteq \left(m(x) \frac{\partial}{\partial x} + m(y) \frac{\partial}{\partial y} + m'(y) \right) K(x, y) \chi_J(y) \\ &- \sum (-1)^k m(a_k) K(x, a_k) \delta(y - a_k), \end{aligned}$$

while using (1.4) we compute that

$$\left(m(x) \frac{\partial}{\partial x} + m(y) \frac{\partial}{\partial y} + \frac{m(x) - m(y)}{x - y} \right) K(x, y) = \text{the right hand side of (2.20)}. \tag{2.30}$$

Therefore if $m(x)$ is linear and we replace D by MD on the left side of (2.22), then the right side has to be changed only by insertion of factors $m(a_k)$ in the last summand. It follows from this that (2.25) and (2.26) require only the following modifications:

Insert on the left sides of (2.25) and (2.26) the factor $m(a_i)$.

Insert in the last summands on the right sides of (2.25) and (2.26)

$$\text{the factors } m(a_k), \tag{2.31}$$

while (2.27) and (2.28) require the following:

Insert on the left sides of (2.27) and (2.28) the factor $m(a_i)$ in front of $R(a_i, a_i)$.

Insert in the last summands of (2.27) and (2.28) the factors $m(a_k)$. \tag{2.32}

For general $m(x)$, if

$$m(x) = \sum \mu_k x^k,$$

then

$$\frac{1}{x - y} \left(m'(y) - \frac{m(x) - m(y)}{x - y} \right) = - \sum_{\substack{j, k \geq 0 \\ j+k \leq \text{deg } m - 2}} (k+1) \mu_{j+k+2} x^j y^k.$$

It follows from this and (2.30) that

$$\left(m(x) \frac{\partial}{\partial x} + m(y) \frac{\partial}{\partial y} + m'(y) \right) K(x, y) = \text{the right hand side of (2.20)} \\ - \sum (k+1) \mu_{j+k+2} x^j y^k (\varphi(x) \psi(y) - \psi(x) \varphi(y)) .$$

So for $m(x)$ of degree greater than 1 the right sides of (2.23) and (2.24) must also be modified by the addition, respectively, of the terms

$$- \sum (k+1) \mu_{j+k+2} (v_k Q_j(x) - u_k P_j(x)) , \\ - \sum (k+1) \mu_{j+k+2} (w_k Q_j(x) - \tilde{v}_k P_j(x)) .$$

The upshot is that in this general case (2.25) and (2.26) require, in addition to (2.31), the following modifications

$$\begin{aligned} & \text{Add to the right side of (2.25)} \\ & - \sum_{\substack{j, k \geq 0 \\ j+k \leq \deg m - 2}} (k+1) \mu_{j+k+2} (v_k q_{ji} - u_k p_{ji}) . \\ & \text{Add to the right side of (2.26)} \\ & - \sum_{\substack{j, k \geq 0 \\ j+k \leq \deg m - 2}} (k+1) \mu_{j+k+2} (w_k q_{ji} - \tilde{v}_k p_{ji}) . \end{aligned} \quad (2.33)$$

And for general $m(x)$ we must modify (2.27) and (2.28), in addition to (2.32), by the following:

$$\begin{aligned} & \text{Add to the right side of (2.27)} \\ & - \sum_{\substack{j, k \geq 0 \\ j+k \leq \deg m - 2}} (k+1) \mu_{j+k+2} (v_k q_{ji} - u_k p_{ji}) p_i \\ & + \sum_{\substack{j, k \geq 0 \\ j+k \leq \deg m - 2}} (k+1) \mu_{j+k+2} (w_k q_{ji} - \tilde{v}_k p_{ji}) q_i . \end{aligned} \quad (2.34)$$

$$\text{Add to the right side of (2.28)} \quad \sum_{\substack{j, k \geq 0 \\ j+k \leq \deg m - 2}} (j-k) \mu_{j+k+2} p_{ji} q_{ki} . \quad (2.35)$$

The identity (2.29) also requires modification if we do not have $m(x) = 1$, but since we shall only use it in this special case there is no need to write down the modification.

E. The Exponential Variant. Here we consider kernels of the form

$$K(x, y) \chi_J(y) := \frac{\varphi(x) \psi(y) - \psi(x) \varphi(y)}{e^{bx} - e^{by}} \chi_J(y) , \quad (2.36)$$

where b can be an arbitrary complex number. Because of the different denominator it turns out that the differentiation formulas should now be of the form

$$\begin{aligned} m(x) \varphi'(x) &= \left(A(x) + \frac{b}{2} m(x) \right) \varphi(x) + B(x) \psi(x) , \\ m(x) \psi'(x) &= -C(x) \varphi(x) - \left(A(x) - \frac{b}{2} m(x) \right) \psi(x) , \end{aligned} \quad (2.37)$$

where $A(x)$, $B(x)$, $C(x)$ and $m(x)$ are “exponential polynomials,” finite linear combinations of the exponentials e^{kbx} ($k=0, \pm 1, \pm 2, \dots$). We compute, as the analogue of (2.30), that

$$\left(m(x) \frac{\partial}{\partial x} + m(y) \frac{\partial}{\partial y} + \frac{b}{2} \frac{(m(x) - m(y))(e^{bx} + e^{by})}{e^{bx} - e^{by}} \right) K(x, y) \\ = \text{the right hand side of (2.20) with denominator } x - y \\ \text{replaced by } e^{bx} - e^{by}. \tag{2.38}$$

Now, of course, we write

$$A(x) = \sum_j \alpha_j e^{bjx}, \text{ etc.,}$$

the summations summing over negative and nonnegative indices, and

$$\frac{A(x) - A(y)}{e^{bx} - e^{by}} = \sum_{j, k \geq 0} \alpha_{j+k+1} e^{jbx} e^{kby} - \sum_{j, k \leq -1} \alpha_{j+k+1} e^{jbx} e^{kby}, \text{ etc.} \tag{2.39}$$

What arise now are functions Q_k, \dots, w_k defined, for negative as well as nonnegative values of k , by replacing x^k by e^{kbx} in the earlier definitions. Analogues of (2.12) and (2.13) hold for negative as well as positive values of k so all the Q_k and P_k are expressible in terms of Q_0, P_0 , as well as the \tilde{v}_k in terms of the u_k, v_k, w_k .

Notice that if $m(x)$ is constant then the third term in the large parentheses in (2.38) vanishes and so we obtain in the end the analogue of (2.25) and (2.26); in addition to the change in the range of indices now and the fact that the double sums have two parts, as in (2.39), we must in the single sums over j add $\frac{b}{2} \mu_j$ to the terms α_j and β_j in (2.25) and the terms $-\gamma_j$ and $-\alpha_j$ in (2.26). For general $m(x)$ we must insert factors $m(a_i)$ on the left sides of (2.25) and (2.26) and factors $m(a_k)$ in the last summands on the right, and then add terms coming from the difference

$$\frac{1}{e^{bx} - e^{by}} \left[m'(y) - \frac{b}{2} \frac{(m(x) - m(y))(e^{bx} + e^{by})}{e^{bx} - e^{by}} \right] [\varphi(x)\psi(y) - \psi(x)\varphi(y)], \tag{2.40}$$

as at the end of the preceding section. We shall not write these down since in the only case we consider later we have $m(x)=1$. Two of the equations involving $R(x, y)$ must also be modified. We see first that in (1.7a) the denominator must be replaced by $e^{ba_j} - e^{ba_k}$. Second, (2.8) must have the factor $b e^{bx}$ inserted on the left side, with the result that (2.27) must have the factor $b e^{ba_i}$ inserted on the left side. Note that (2.29) is unchanged.

Remark 1. The product of the first two factors in (2.40) is an exponential polynomial in e^{bx} and e^{by} . It was precisely to achieve this outcome that we required the formulas (2.37) to have the form they do.

Remark 2. In case b is real the change of variable $x \mapsto e^{bx}$ transforms the operator with kernel (1.2) acting on the set e^{bj} to an operator with kernel of the form (2.36) acting on J , with the new (φ, ψ) pair satisfying (2.37). So we see that there is more than simply an analogy between the two situations. In fact we could have allowed the various coefficients in (1.4) to be linear combinations of negative or nonnegative integral powers of x , and then the two situations would have been completely equivalent for real b .

III. Sine, Airy and Bessel

A. *Sine Kernel.* The simplest example is the sine kernel

$$K(x, y) = \frac{\lambda}{\pi} \frac{\sin(x-y)}{x-y},$$

where we take

$$\varphi(x) = \sqrt{\frac{\lambda}{\pi}} \sin x, \quad \psi(x) = \sqrt{\frac{\lambda}{\pi}} \cos x.$$

The differentiation formulas hold with

$$m(x) = 1, \quad A(x) = 0, \quad B(x) = 1, \quad C(x) = 1.$$

(It is useful to incorporate a parameter $\lambda \in [0, 1]$ into K ; cf. formula (1.1).) The partial differentiation equations are (1.8), (1.9) (the universal equations along with universal relation (1.7a)), and the specialization of (2.25) and (2.26) which now read, respectively,

$$\frac{\partial q_i}{\partial a_i} = p_i - \sum_{k \neq i} (-1)^k R(a_i, a_k) q_k, \tag{3.1}$$

$$\frac{\partial p_i}{\partial a_i} = -q_i - \sum_{k \neq i} (-1)^k R(a_i, a_k) p_k, \tag{3.2}$$

along with the specialization of (2.27),

$$R(a_i, a_i) = p_i^2 + q_i^2 + \sum_{k \neq i} (-1)^k \frac{(q_i p_k - p_i q_k)^2}{a_i - a_k}. \tag{3.3}$$

These are the equations of JMMS [25] though they appear here in a slightly different form due to our use of sines and cosines in the definitions of φ and ψ rather than the alternative choice of $e^{\pm ix}$, which we could have taken just as well. (They also appear slightly different in [37] due to out convention here not to put a factor of π into the definition of the sine kernel.)

For the case of a single interval $J = (-t, t)$, $s = 2t$, these equations imply that $\sigma(s; \lambda) := -sR(t, t)$ satisfies the Jimbo–Miwa–Okamoto σ form of Painlevé V. We refer the reader to the literature for a derivation of this, properties of the solution of this equation, and the implications for random matrices [2, 16, 25, 29, 27, 37, 40].

B. *Airy Kernel.* For the Airy kernel we have (again inserting a parameter λ into K)

$$\varphi(x) = \sqrt{\lambda} \text{Ai}(x), \quad \psi(x) = \sqrt{\lambda} \text{Ai}'(x),$$

from which it follows that

$$m(x) = 1, \quad A(x) = 0, \quad B(x) = 1, \quad C(x) = -x,$$

since $\text{Ai}''(x) = x \text{Ai}(x)$. For notational convenience we write $u = u_0$ and $v = v_0$. In addition to the universal relations (1.7a)–(1.9), we have two additional equations

for u and v , viz. (2.15) and (2.16) for $j=0$. Using the recursion relation (2.12) for $k=1$, we deduce that (2.25) and (2.26) reduce to

$$\frac{\partial q_i}{\partial a_i} = -uq_i + p_i - \sum_{k \neq i} (-1)^k R(a_i, a_k) q_k, \quad (3.4)$$

$$\frac{\partial p_i}{\partial a_i} = (a_i - 2v)q_i + up_i - \sum_{k \neq i} (-1)^k R(a_i, a_k) p_k, \quad (3.5)$$

and (2.27) reduces to (again using the recursion relation (2.12) for $k=1$)

$$R(a_i, a_i) = p_i^2 - a_i q_i^2 - 2uq_i p_i + 2vq_i^2 + \sum_{k \neq i} (-1)^k \frac{(q_i p_k - p_i q_k)^2}{a_i - a_k}. \quad (3.6)$$

These are the equations derived in [38]. We mention that in addition to these equations, two first integrals were derived which can be used to represent u and v directly in terms of the q_j and p_j (see (2.18) and (2.19) in [38]). We also remark that in the case $J=(s, \infty)$, the quantity $R(s, \infty)$ was shown to satisfy the second order nonlinear σ DE associated to Painlevé II. Again we refer the reader to [38] for details.

C. Bessel Kernel. For the Bessel kernel

$$\varphi(x) = \sqrt{\lambda} J_\alpha(\sqrt{x}), \quad \psi(x) = x\varphi'(x),$$

from which it follows (using Bessel's equation) that

$$m(x) = x, \quad A(x) = 0, \quad B(x) = 1, \quad C(x) = \frac{1}{4}(x - \alpha^2).$$

Again using the recursion relation (2.12), we deduce that (2.25) and (2.26) become, with the additional insertions (2.31),

$$a_i \frac{\partial q_i}{\partial a_i} = \frac{1}{4} uq_i + p_i - \sum_{k \neq i} (-1)^k a_k R(a_i, a_k) q_k,$$

$$a_i \frac{\partial p_i}{\partial a_i} = \frac{1}{4} (\alpha^2 - a_i + 2v)q_i - \frac{1}{4} up_i - \sum_{k \neq i} (-1)^k a_k R(a_i, a_k) p_k,$$

and (2.27) with insertions (2.32) becomes

$$a_i R(a_i, a_i) = -\frac{1}{4} (\alpha^2 - a_i + 2v)q_i + \frac{1}{2} uq_i p_i + p_i^2 + \sum_{k \neq i} (-1)^k a_k \frac{(q_i p_k - p_i q_k)^2}{a_i - a_k}.$$

(As before, $u=u_0$ and $cv=v_0$.) These are the equations derived in [39]. As was the case for the Airy kernel, two first integrals were derived which can be used to express u and v directly in terms of the q_j and p_j . For the case $J=(0, s)$, the quantity $\sigma(s) = sR(0, s)$ was shown [39] to satisfy the σ DE for Painlevé III [23, 35].

IV. Beyond Airy

In this section we give as an example of our general system of partial differential equations the simplest case “beyond Airy” in the sense discussed in the Introduction.

In the language of 2D quantum gravity matrix models (see [7] for a review), we are considering the case of pure gravity. Thus we take

$$\begin{aligned}\mathcal{Q} &= D_x^2 + \xi(x), \\ \mathcal{P} &= (\mathcal{Q}^{3/2})_+ \\ &= D_x^3 + \frac{3}{2} \xi(x) D_x + \frac{3}{4} \xi'(x),\end{aligned}$$

where D_x is differentiation with respect to x . The string equation implies that $\xi(x)$ satisfies

$$\xi'''(x) + 6\xi(x)\xi'(x) + 4 = 0$$

which when integrated is Painlevé I [8, 13, 21]:

$$\xi''(x) + 3\xi^2(x) + 4x = 0.$$

(Without loss of generality we may set the constant of integration to zero since it corresponds simply to a shift in the variable x . And, of course, the “3” and “4” can be changed by scale transformations to give the canonical form of Painlevé I.) Exactly what solution $\xi(x)$ one chooses for pure gravity is still of some debate (on this point see [11] and references therein). The function $\varphi(\lambda, x)$ satisfies (1.11) and (1.12) which implies that if we define $\psi(\lambda, x)$ by (1.13), then the differentiation formulas are

$$\begin{aligned}m(\lambda) &= 1, \quad A(\lambda) = -\frac{1}{4} \xi'(x), \quad B(\lambda) = \lambda + \frac{1}{2} \xi(x), \\ C(\lambda) &= -\lambda^2 + \frac{1}{2} \xi(x)\lambda + \frac{1}{2} \xi^2(x) + \frac{1}{4} \xi''(x),\end{aligned}$$

where we remind the reader of the change of notation in the independent variable (see Introduction).

Since $C(\lambda)$ is quadratic in λ , the equations will involve u_j, v_j , and w_j for $j=0, 1$. Using the recursion relations (2.12), (2.13), (2.14), Eqs. (2.25) and (2.26) specialize to

$$\frac{\partial}{\partial a_i} \begin{pmatrix} q_i \\ p_i \end{pmatrix} = \mathcal{M}(a_i) \begin{pmatrix} q_i \\ p_i \end{pmatrix} - \sum_{k \neq i} (-1)^k R(a_i, a_k) \begin{pmatrix} q_k \\ p_k \end{pmatrix},$$

where $\mathcal{M}(a_i)$ is the 2×2 matrix whose elements are given by

$$\mathcal{M}_{11}(a_i) = -\frac{1}{4} \xi' - w + \frac{1}{2} \xi u + uv - u_1 - a_i u,$$

$$\mathcal{M}_{12}(a_i) = a_i - u^2 + 2v + \frac{1}{2} \xi,$$

$$\mathcal{M}_{21}(a_i) = a_i^2 - \frac{1}{2} \xi^2 - \frac{1}{4} \xi'' - 2a_i v + 3v^2 - 2v_1 - 2uw - \frac{1}{2} a_i \xi + v \xi,$$

$$\mathcal{M}_{22}(a_i) = -\mathcal{M}_{11}(a_i).$$

Similarly, (2.27) specializes to

$$R(a_i, a_i) = 2\mathcal{M}_{11}(a_i)q_i p_i + \mathcal{M}_{12}(a_i)p_i^2 - \mathcal{M}_{21}(a_i)q_i^2 + \sum_{k \neq i} (-1)^k \frac{(q_i p_k - p_i q_k)^2}{a_i - a_k}.$$

The universal equations are (1.7a)–(1.9) and (2.15)–(2.18).

For the case $J = (s, \infty)$ (this should be compared with the analogous case in Airy [38]), we are able to find two first integrals that allow us to eliminate the quantities u_1 and v_1 . (It is natural to take the boundary condition that all quantities evaluated at ∞ vanish.) We denote by $q = q(s)$, etc. the quantities corresponding to the first endpoint $a_1 = s$. The first relation is quite simple,

$$q^2 - 2uu_1 + v^2 + 2v_1 + \frac{1}{2} \xi u^2 + \xi v - \frac{1}{2} \xi' u = 0,$$

but the second one we found is rather messy,

$$\begin{aligned} & -p^2 + sq^2 + u + 2pqu + xu^2 + q^2u^2 - 2u^3u_1 - u_1^2 - 2xv \\ & - 4q^2v + 6uu_1v + 3u^2v^2 - 8v^3 - 2u^3w - 2u_1w + 6uvw - w^2 \\ & + \xi \left(-q^2 + \frac{1}{2}u^4 + 2uu_1 - 6v^2 + 2uw - \frac{1}{4}\xi u^2 - \frac{3}{2}\xi v \right) \\ & + \xi' \left(-\frac{1}{2}u^3 - \frac{1}{2}u_1 + \frac{3}{2}uv - \frac{1}{2}w + \frac{1}{2}\xi u \right) = 0, \end{aligned}$$

where q, p, u , etc. have argument s and ξ has argument x . (The variable x appears since we used Painlevé I to eliminate $\xi''(x)$ in our equations.)

Using the first integrals to eliminate u_1 and v_1 (note that w also drops out) we obtain the system of equations:

$$\begin{aligned} q'' &= \left(xs + s^3 + \frac{1}{2}x\xi + \frac{1}{8}\xi^3 + \frac{1}{16}(\xi')^2 \right) q + p + (4s - 4v - \xi)q^3 + 4uq^2p - 2qp^2, \\ p'' &= \left(2s - 2v - \frac{1}{2}\xi \right) q + \left(xs + s^3 + \frac{1}{2}x\xi + \frac{1}{8}\xi^3 + \frac{1}{16}(\xi')^2 + 2u \right) p \\ &+ (4s - 4v - \xi)pq^2 + 4uqp^2 - 2p^3, \end{aligned}$$

and, of course, we still have the universal equations

$$u' = -q^2, \quad v' = -qp.$$

Letting $R(s) = R(s, \infty)$ we find from (2.28) and (2.35) that

$$R' = \left(-2s + 2v + \frac{1}{2}\xi \right) q^2 + p^2 - 2uqp.$$

Clearly, further analysis of these equations is needed to be able to analyze the associated Fredholm determinant. For example, can one derive a differential equation for R itself?

V. Finite N Hermite, Laguerre, Jacobi and Circular

A. Hermite Kernel

1. *The partial differential equations and a first integral.* It follows from the Christoffel–Darboux formula that the kernel $\lambda K_N(x, y)$ for the finite N Gaussian Unitary Ensemble (GUE) is of the form (1.2) provided we choose

$$\varphi(x) = \lambda^{1/2} \left(\frac{N}{2}\right)^{1/4} \varphi_N(x), \quad \psi(x) = \lambda^{1/2} \left(\frac{N}{2}\right)^{1/4} \varphi_{N-1}(x)$$

with $\varphi_k(x)$ the harmonic oscillator wave functions

$$\varphi_k(x) = \frac{1}{\sqrt{2^k k!} \sqrt{\pi}} e^{-x^2/2} H_k(x), \quad k=0, 1, \dots,$$

and $H_k(x)$ are the Hermite polynomials [18]. This is well-known and we refer the reader to [28] for details. From the differentiation and recurrence formulas for Hermite polynomials it follows that the differentiation formulas for φ and ψ hold with

$$m(x) = 1, \quad A(x) = -x, \quad B(x) = C(x) = \sqrt{2N}.$$

We can therefore immediately write down the equations

$$\frac{\partial q_j}{\partial a_j} = -a_j q_j + (\sqrt{2N} - 2u) p_j - \sum_{k \neq j} (-1)^k \frac{q_j p_k - p_j q_k}{a_j - a_k} q_k, \quad (5.1)$$

$$\frac{\partial p_j}{\partial a_j} = a_j p_j - (\sqrt{2N} + 2w) q_j - \sum_{k \neq j} (-1)^k \frac{q_j p_k - p_j q_k}{a_j - a_k} p_k \quad (5.2)$$

along with

$$\begin{aligned} R(a_j, a_j) &= -2a_j p_j q_j + (\sqrt{2N} - 2u) p_j^2 + (\sqrt{2N} + 2w) q_j^2 \\ &\quad + \sum_{k \neq j} (-1)^k \frac{(q_j p_k - p_j q_k)^2}{a_j - a_k}, \end{aligned} \quad (5.3)$$

$$\frac{\partial}{\partial a_j} R(a_j, a_j) = -2p_j q_j - \sum_{k \neq j} (-1)^k R(a_j, a_k)^2, \quad (5.4)$$

$$\frac{d}{dt} R_{t,j}(ta_j, ta_j) = -2a_j p_j q_j + \sum_{k \neq j} (-1)^k (a_k - a_j) R_{t,j}(ta_j, ta_k)^2. \quad (5.5)$$

These follow from formulas (2.25)–(2.29).

We now derive a first integral involving u , w , p_j and q_j ; namely we show

$$\sqrt{2N}(u - w) + 2uw = - \sum_j (-1)^j p_j q_j. \quad (5.6)$$

Observe that

$$-\left(\sum_k \frac{\partial}{\partial a_k}\right) p_j q_j = (\sqrt{2N} + 2w) q_j^2 - (\sqrt{2N} - 2u) p_j^2$$

and

$$\frac{\partial}{\partial a_j} (\sqrt{2N}(u - w) + 2uw) = (-1)^j (\sqrt{2N} + 2w) q_j^2 - (-1)^j (\sqrt{2N} - 2u) p_j^2 .$$

Multiplying the first equation by $(-1)^j$ and summing both equations over j , results in

$$\left(\sum_k \frac{\partial}{\partial a_k}\right) \left(-\sum_j (-1)^j p_j q_j\right) = \left(\sum_k \frac{\partial}{\partial a_k}\right) (\sqrt{2N}(u - w) + 2uw) .$$

It follows that the two sides of (5.6) differ by a function of (a_1, \dots, a_{2m}) which is invariant under translation by any vector (s, \dots, s) . Since, clearly, both sides tend to zero as all $a_i \rightarrow \infty$, their difference must be identically zero.

2. *Bulk scaling limit of finite N equations.* We now show how (5.1)–(5.3) reduce to the sine kernel equations (3.1)–(3.3) in the “bulk scaling limit.” For a fixed point z , i.e. independent of N , the density $\rho(z)$ in the GUE is asymptotic to $\sqrt{2N}/\pi$ as $N \rightarrow \infty$. The bulk scaling limit corresponds to measuring fluctuations about this fixed point z on a stretched length scale proportional to $\sqrt{2N}$ and then taking $N \rightarrow \infty$. Denoting for the moment the bulk quantities with a superscript B , this means we set

$$a_j = z + \frac{a_j^B}{\sqrt{2N}}$$

and consider the limit $N \rightarrow \infty$, $a_j \rightarrow z$ such that a_j^B is fixed and $O(1)$. In this limit we deduce from the asymptotics of the harmonic oscillator wave functions (see, e.g., Appendix 10 in [28]) that both φ and ψ are $O(1)$ quantities in the bulk scaling limit. From this and the fact that it is $K(x, y)dy$ which is $O(1)$, we deduce that both q_j and p_j are $O(1)$ quantities in the bulk scaling limit. An examination of the inner products defining both u and v shows that these too are $O(1)$ quantities. Thus if we formally replace

$$a_j \rightarrow z + \frac{a_j^B}{\sqrt{2N}}, \quad q_j \rightarrow q_j^B, \quad p_j \rightarrow p_j^B, \quad R(a_j, a_j) \rightarrow \sqrt{2N} R^B(a_j^B, a_j^B)$$

in (5.1)–(5.3) (and replace all derivatives by derivatives with respect to a_j^B), take $N \rightarrow \infty$, we obtain (3.1)–(3.3).

3. *Semi-infinite interval and Painlevé IV.* In this section we specialize the finite N GUE equations to the case of $m=1$, $a_1=s$ and $a_2=\infty$, i.e. $J=(s, \infty)$. We write $q(s)$, $p(s)$, and $R(s)$ for q_1 , p_1 , and $R(a_1, a_1)$, respectively, of the previous section.

The differential equations reduce to ($' = d/ds$)

$$q' = -sq + (\sqrt{2N} - 2u)p, \quad (5.7)$$

$$p' = sp - (\sqrt{2N} + 2w)q, \quad (5.8)$$

$$u' = -q^2, \quad (5.9)$$

$$w' = -p^2, \quad (5.10)$$

(5.3) reduces to

$$R(s) = -2spq + (\sqrt{2N} - 2u)p^2 + (\sqrt{2N} + 2w)q^2, \quad (5.11)$$

and the first integral is now

$$\sqrt{2N}(u - w) + 2uw = pq. \quad (5.12)$$

We proceed to derive a second order differential equation for $R(s)$ and show that it is a special case of the Jimbo–Miwa–Okamoto σ form of Painlevé IV [24, 34]. Relation (5.4) is now

$$R' = -2pq, \quad (5.13)$$

while (5.7) and (5.8) give

$$(pq)' = (\sqrt{2N} - 2u)p^2 - (\sqrt{2N} + 2w)q^2.$$

Differentiating one more time gives

$$\begin{aligned} (pq)'' &= 2s(\sqrt{2N} - 2u)p^2 + (\sqrt{2N} + 2w)q^2 - 8pq(N + \sqrt{2N}(w - u) - 2uw) + 4p^2q^2 \\ &= 2s\{(\sqrt{2N} - 2u)p^2 + (\sqrt{2N} + 2w)q^2\} - 8Npq + 12p^2q^2, \end{aligned}$$

where we used the first integral (5.12) to obtain the second equality. Referring back to (5.11) we see that the term in curly brackets in the last expression is $R + 2spq$. Using (5.13) to eliminate all terms involving pq in the last equation we find

$$R''' = -4s(R - sR') - 8NR' - 6(R')^2.$$

This third order equation can be integrated (the constant of integration is zero) to give

$$(R'')^2 + 4(R')^2(R' + 2N) - 4(sR' - R)^2 = 0. \quad (5.14)$$

Comparing this with (C.37) of [24] (see also [34]), we see immediately that this is the σ version of Painlevé IV with parameters (in notation of [24]) $v_1 = 0$ and $v_2 = 2N$. Explicitly in terms of the Painlevé IV transcendent $y = y(s)$ we have

$$R(s) = Ny - \frac{s^2}{2}y - \frac{s}{2}y^2 - \frac{1}{8}y^3 + \frac{1}{8y}(y')^2 \quad (5.15)$$

with P_{IV} parameters $\alpha = 2N - 1$ and $\beta = 0$. Recall that $w = w(z)$ is a Painlevé IV transcendent with parameters α and β if it satisfies the P_{IV} equation

$$\frac{d^2w}{dz^2} = \frac{1}{2w} \left(\frac{dw}{dz} \right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}.$$

We are, of course, interested in the family of solutions that vanish as $s \rightarrow +\infty$.

This particular P_{IV} has been studied by Bassom, et al. [3] (see also [10]). To make contact with their notation define

$$\eta(\xi) := 2^{-3/4} \sqrt{y(s)}, \quad \xi := \sqrt{2s}$$

(y is the above P_{IV} transcendent) so that η satisfies

$$\frac{d^2 \eta}{d\xi^2} = 3\eta^5 + 2\xi\eta^3 + \left(\frac{\xi^2}{4} - v - \frac{1}{2}\right)\eta,$$

with $v = N - 1$. They analyze the one-parameter family of solutions $\eta_k(\xi; v)$ satisfying the boundary condition $\eta(\infty) = 0$. The parameter k is defined uniquely by the asymptotic condition:

$$\eta_k(\xi; v) \sim k\xi^v \exp\left(-\frac{\xi^2}{4}\right) \text{ as } \xi \rightarrow \infty.$$

In terms of our parameter λ we have

$$k^2 = \frac{\lambda}{2^{3/2}(N-1)! \sqrt{\pi}}.$$

(This identity is derived by examining the large positive s asymptotics of $R(s)$, which is easy because of the rapid decrease of the kernel as $s \rightarrow +\infty$.) These authors prove that for all positive integers $N > 1$ the solution $\eta_k(\xi, N - 1)$ exists for all ξ whenever $\lambda < 1$, and that $\eta_k(\xi, N - 1)$ blows up for a finite ξ whenever $\lambda > 1$. These results are in complete agreement with what one expects from the spectral theory of the Fredholm determinant. There are formal, but not rigorous, results that solve the connection problem for the asymptotics as $\xi \rightarrow -\infty$; in particular, for $\lambda = 1$,

$$\eta_k(\xi; N - 1) \sim \left(-\frac{\xi}{2}\right)^{1/2} \text{ as } \xi \rightarrow -\infty. \tag{5.16}$$

Using (5.16) (and computing higher order terms by using the differential equation) we find that

$$R(s) = -2Ns - \frac{N}{s} + \frac{N^2}{s^3} - \frac{N^2(1 + 9N^2)}{4s^5} + \frac{N^3(10 + 27N^2)}{4s^7} + \dots \text{ as } s \rightarrow -\infty. \tag{5.17}$$

4. *Distribution functions for λ_{\max} and λ_{\min} .* If we denote the smallest and largest eigenvalues of a matrix from the GUE by λ_{\min} and λ_{\max} , respectively, then in the notation of (1.1) we have

$$P(\lambda_{\max} < s) = E(0; (s, \infty)) = \det(I - K).$$

Thus, using (1.3) we deduce the representation

$$P(\lambda_{\max} < s) = \exp\left\{-\int_s^\infty R(t; 1) dt\right\},$$

where $R(s; \lambda)$ denotes the function $R(s)$ of the preceding section with parameter value λ . This is our representation of the distribution function for λ_{\max} in terms of a Painlevé transcendent. There is of course a similar representation for the distribution function for λ_{\min} .

The authors of [3, 10] give an algorithm to compute the quantities $\eta_k(\xi, \nu)$, $\nu =$ positive integer, of the last section exactly in terms of the error function

$$I(\xi) = 2 \int_{\xi}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx .$$

That such elementary solutions of the P_{IV} transcendent exist, at least for the case $\lambda = 1$, is now clear from the random matrices point of view since $E(0; (s, \infty))$ is expressible in terms of integrals of the form

$$\int_{-\infty}^s x^j e^{-x^2} dx .$$

This follows from (1.15) with f the characteristic function of $(-\infty, s)$.

5. *Edge scaling limit from Painlevé IV equation.* The edge scaling limit [38] corresponds to the replacements

$$s \rightarrow \sqrt{2N} + \frac{2}{\sqrt{2N}^{1/6}} \quad \text{and} \quad R \rightarrow \sqrt{2N}^{1/6} R$$

in (5.14) and retaining only the leading order term as $N \rightarrow \infty$. The result of doing this is

$$(R'')^2 + 4(R')^3 + 4R'(R - sR') = 0 \tag{5.18}$$

which is the equation derived in [38]. We remark that (5.18) is the σ form for Painlevé II, see (C.17) in [24] and Proposition 1.1 in [34].

6. *Symmetric single interval case.* In this section we specialize the finite N GUE equations to the case of $m = 1$, $a_1 = -t$ and $a_2 = t$, i.e. $J = (-t, t)$. We denote by $q(t)$ and $p(t)$ the quantities q_2 and p_2 , respectively. Since $\varphi_N(-x) = (-1)^N \varphi_N(x)$ and $K(-x, -y) = K(x, y)$, we have $q_1 = (-1)^N q$ and $p_1 = -(-1)^N p$. We further set

$$R(t) := R(t, t) = R(-t, -t), \quad \tilde{R}(t) := (-1)^N R(-t, t) = (-1)^N R(t, -t)$$

and record that

$$\frac{d \log D(J; \lambda)}{dt} = -2R(t) .$$

Now φ is even or odd depending on whether N is even or odd, with ψ having the opposite parity. It follows from this fact, and our choice of sign in the definition of \tilde{R} , that (1.7a) specializes in this case to

$$\tilde{R}(t) = -\frac{qp}{t} , \tag{5.19}$$

while (5.3) and (5.5) specialize to

$$R(t) = -2tpq + (\sqrt{2N} - 2u)p^2 + (\sqrt{2N} + 2w)q^2 - \frac{2}{t}p^2q^2, \quad (5.20)$$

$$\frac{dR}{dt} = 2t\tilde{R} + 2\tilde{R}^2. \quad (5.21)$$

The last is the finite N analogue of the Gaudin relation. (See, e.g. [28, 37].) The differential equations specialize to

$$\frac{dq}{dt} = -\frac{\partial q_2}{\partial a_1} + \frac{\partial q_2}{\partial a_2} = 2\tilde{R}q - tq + (\sqrt{2N} - 2u)p, \quad (5.22)$$

$$\frac{dp}{dt} = -2\tilde{R}p + tp - (\sqrt{2N} + 2w)q, \quad (5.23)$$

$$\frac{du}{dt} = 2q^2, \quad (5.24)$$

$$\frac{dw}{dt} = 2p^2. \quad (5.25)$$

And the first integral (5.6) is now

$$\sqrt{2N}(u - w) + 2uw = -2pq. \quad (5.26)$$

7. *Differential equations for R and \tilde{R} .* It follows easily from (5.19), (5.20) and (5.21) that

$$\frac{d}{dt}(t\tilde{R}) = \Re(r^2), \quad (5.27)$$

$$\frac{d}{dt}(tR) = |r|^2 + 4t^2\tilde{R}, \quad (5.28)$$

where

$$r = q\sqrt{\sqrt{2N} + 2w} + ip\sqrt{\sqrt{2N} - 2u}.$$

Equations (5.27) and (5.28) are the finite N analogue of those derived by Mehta [29]. (See also discussion in [37].)

We now eliminate the quantity r . For this derivation only, we write $a(t) := tR(t)$ and $b(t) := t\tilde{R}(t)$. We begin with the obvious

$$|r|^4 = \Re(r^2)^2 + \Im(r^2)^2. \quad (5.29)$$

Now

$$\begin{aligned} \Im(r^2)^2 &= 4p^2q^2(\sqrt{2N} - 2u)(\sqrt{2N} + 2w) \\ &= 4p^2q^2(2N + 4pq) \\ &= 4b^2(2N - 4b). \end{aligned} \quad (5.30)$$

We now use (5.27) and (5.28) to obtain expressions for $\Re(r^2)^2$ and $|r|^4$, respectively, and the above identity for $\Im(r^2)^2$. These expressions, when used in (5.29), give an equation for a, b , and their first derivatives. If we use the generalized Gaudin relation (5.21) to eliminate the appearance of da/dt (the one appearing to the first power), we obtain

$$\left(\frac{da}{dt}\right)^2 = 8ab + 8Nb^2 + \left(\frac{db}{dt}\right)^2 \quad (5.31)$$

and together with (5.21), which in the a and b variables reads

$$t\left(\frac{da}{dt}\right) = a + 2b^2 + 2t^2b, \quad (5.32)$$

we have two differential equations for R and \tilde{R} .

Eliminating a , we obtain a single second order equation for b and therefore \tilde{R} :

$$(t\tilde{R}'' + 2\tilde{R}' - 24t^2\tilde{R}^2 + 8Nt\tilde{R})^2 - 4(2\tilde{R} - t)^2(8t^2\tilde{R}^2(N - 2t\tilde{R}) + (\tilde{R} + t\tilde{R}')^2) = 0. \quad (5.33)$$

This last equation is the finite N analogue of (1.18) of Mahoux and Mehta [27]. We could, in a similar way, eliminate b and so obtain a second order equation for R , but the result is messy and we shall not write it down.

8. *Small t expansions of R and \tilde{R} .* The boundary conditions at $t=0$ for (5.21) and (5.33) follow from an examination of the Neuman expansion of the resolvent kernel. Setting $\rho_0 := K(0, 0)$, the density of eigenvalues at 0, we find

$$\begin{aligned} R(t) = & \rho_0 + 2\rho_0^2 t + \rho_0(4\rho_0^2 + (-1)^N)t^2 + \frac{8\rho_0^2}{9}(9\rho_0^2 - 2N + (-1)^N)t^3 \\ & + \frac{\rho_0}{18}(288\rho_0^4 + 40\rho_0^2((-1)^N - 2N) - 12(-1)^N N - 3)t^4 + O(t^5) \end{aligned} \quad (5.34)$$

and

$$\begin{aligned} (-1)^N \tilde{R}(t) = & \rho_0 + 2\rho_0^2 t + \frac{\rho_0}{3}(12\rho_0^2 - 4N - (-1)^N)t^2 + \frac{8\rho_0^2}{9}(9\rho_0^2 - 2N + (-1)^N)t^3 \\ & + \frac{\rho_0}{90}(1440\rho_0^4 + 200\rho_0^2((-1)^N - 2N) + 48N^2 \\ & + 12(-1)^N N + 9)t^4 + O(t^5). \end{aligned} \quad (5.35)$$

9. *Level spacing probability density $p_N(t)$.* For $m=1$ if we let $E_N(0; a_1, a_2)$ denote the probability that no eigenvalues lie in the interval (a_1, a_2) and $p_N(0; a_1, a_2)da_2$ the conditional probability that given an eigenvalue at a_1 the next one lies between a_2 and $a_2 + da_2$, then the two quantities are related by

$$p_N(0; a_1, a_2) = \frac{1}{\rho(a_1)} \frac{\partial^2 E_N(0; a_1, a_2)}{\partial a_1 \partial a_2},$$

where $\rho(a_1)$ is the density of eigenvalues at a_1 . From the expression for the logarithmic derivative of the determinant (with $\lambda=1$), we have

$$\frac{\partial^2 E_N(0; a_1, a_2)}{\partial a_1 \partial a_2} = \frac{\partial R(a_1, a_1)}{\partial a_2} E_N - R(a_1, a_1) R(a_2, a_2) E_N.$$

Differentiating (5.3) (with $m=1$ and $j=1$) with respect to a_2 we obtain

$$\frac{\partial R(a_1, a_1)}{\partial a_2} = \left(\frac{q_1 p_2 - p_1 q_2}{a_1 - a_2} \right)^2$$

which when evaluated at $a_1 = -t$, $a_2 = t$ becomes

$$\left. \frac{\partial R(a_1, a_1)}{\partial a_2} \right|_{a_1 = -t, a_2 = t} = (\tilde{R})^2.$$

Calling $p_N(t) := p_N(0; -t, t)$ we thus obtain

$$p_N(t) = \frac{1}{\rho(t)} (\tilde{R}^2(t) - R^2(t)) E_N(t),$$

where $E_N(t) = E_N(0; -t, t)$ (we used also $\rho(-t) = \rho(t)$).

Using the expansions of R , \tilde{R} and $E_N(t)$ we find

$$\begin{aligned} p_N(t) &= \frac{8}{3} (N + (-1)^N) \rho_0 t^2 - \frac{8}{45} (16N^2 + 29(-1)^N N + 13) \rho_0 t^4 \\ &\quad + \frac{4}{315} (128N^3 + 452(-1)^N N^2 + 529N + (-1)^N 205) \rho_0 t^6 + \dots \end{aligned}$$

Not only does this hold for fixed N and t , but it also holds uniformly in N and t as long as $t = O(N^{-1/2})$. The reason is that in this range of the parameters the operator K has norm less than a constant which is less than 1 and has bounded Hilbert–Schmidt norm. Thus the Neumann series for the resolvent kernel converges to trace norm.

To compare with the bulk scaling limit we replace t by t/ρ_0 , and deduce (recall $\rho_0 \sim \sqrt{2N/\pi}$ as $N \rightarrow \infty$) that

$$p(t) := \lim_{N \rightarrow \infty} \frac{1}{\rho_0} p_N \left(\frac{t}{\rho_0} \right) = \frac{\pi^2}{3} t^2 - \frac{2\pi^4}{45} t^4 + \frac{\pi^6}{315} t^6 + \dots$$

which is the well-known result [28]. Observe that the large N corrections to these limiting coefficients are $O(1/N)$. (Note that we inserted a factor of π in our definition of the new t variable so as to have the same normalization as in [28].)

B. Laguerre Kernel

1. *The partial differential equations.* Again by the Christoffel–Darboux formula it follows that the kernel for the finite N Laguerre Ensemble of $N \times N$ hermitian matrices is of the form (1.2) provided we choose

$$\varphi(x) = \sqrt{\lambda(N(N+\alpha))^{1/4}} \varphi_{N-1}(x), \quad \psi(x) = \sqrt{\lambda(N(N+\alpha))^{1/4}} \varphi_N(x) \quad (5.36)$$

(a_N in (6.3) is negative now) where

$$\varphi_k(x) = \sqrt{\frac{k!}{\Gamma(k+\alpha+1)}} x^{\alpha/2} e^{-x/2} L_k^\alpha(x),$$

and $L_k^\alpha(x)$ are the (generalized) Laguerre polynomials [18]. See Chap. 19 of [28] and [32] for further details and references. From the differentiation and recurrence formulas for Laguerre polynomials it follows that we have differentiation formulas (1.4) for φ and ψ with

$$m(x) = x, \quad A(x) = \frac{1}{2}x - \frac{\alpha}{2} - N, \quad B(x) = C(x) = \sqrt{N(N+\alpha)}.$$

We therefore have the equations

$$\begin{aligned} a_j \frac{\partial q_j}{\partial a_j} &= \left(\frac{1}{2} a_j - \frac{\alpha}{2} - N \right) q_j + (\sqrt{N(N+\alpha)} + u) p_j \\ &\quad - \sum_{k \neq j} (-1)^k a_k R(a_j, a_k) q_k, \end{aligned} \quad (5.37)$$

$$\begin{aligned} a_j \frac{\partial p_j}{\partial a_j} &= -(\sqrt{N(N+\alpha)} - w) q_j - \left(\frac{1}{2} a_j - \frac{\alpha}{2} - N \right) p_j \\ &\quad - \sum_{i \neq j} (-1)^k a_k R(a_j, a_k) p_k, \end{aligned} \quad (5.38)$$

$$\begin{aligned} a_j R(a_j, a_j) &= (a_j - \alpha - 2N) q_j p_j + (\sqrt{N(N+\alpha)} + u) p_j^2 + (\sqrt{N(N+\alpha)} - w) q_j^2 \\ &\quad + \sum_{k \neq j} (-1)^k a_k \frac{(q_j p_k - p_j q_k)^2}{a_j - a_k} \end{aligned} \quad (5.39)$$

$$\frac{\partial}{\partial a_j} a_j R(a_j, a_j) = q_j p_j - \sum_{k \neq j} (-1)^k a_k R(a_j, a_k)^2. \quad (5.40)$$

These follow from formulas (2.25)–(2.28) as modified by (2.31) and (2.32).

2. *Single Interval Cases* $(0, s)$ and (s, ∞) . We consider first the interval $(0, s)$. We set $a_1 = 0$, $a_2 = s$, $q_2 = q$, $p_2 = p$, $R(s, s) = R(s)$ and find that Eqs. (5.37)–(5.40) with $j=2$ specialize to

$$s q' = \left(\frac{1}{2} s - \frac{\alpha}{2} - N \right) q + (\sqrt{N(N+\alpha)} + u) p, \quad (5.41)$$

$$s p' = -(\sqrt{N(N+\alpha)} - w) q - \left(\frac{1}{2} s - \frac{\alpha}{2} - N \right) p, \quad (5.42)$$

$$s R(s) = (s - \alpha - 2N) q p + (\sqrt{N(N+\alpha)} + u) p^2 \quad (5.43)$$

$$+ (\sqrt{N(N+\alpha)} - w) q^2, \quad (5.44)$$

$$(s R(s))' = q p \quad (5.45)$$

(notice that the terms corresponding to $k=1$ in the double sums on the right sides of (5.37)–(5.40) are equal to zero), while (2.15) and (2.18) specialize to

$$u' = q^2, \quad w' = p^2. \quad (5.46)$$

Tedious but straightforward computation using (5.41)–(5.46) gives

$$s(pq)' = (\sqrt{N(N+\alpha)} + u)p^2 - (\sqrt{N(N+\alpha)} - w)q^2, \quad (5.47)$$

$$\begin{aligned} s^2(pq)'' &= (2N + \alpha - s) \{ (\sqrt{N(N+\alpha)} + u)p^2 + (\sqrt{N(N+\alpha)} - w)q^2 \} \\ &\quad - \{ (\sqrt{N(N+\alpha)} + u)p^2 - (\sqrt{N(N+\alpha)} - w)q^2 \} + 2sp^2q^2 \\ &\quad - 4N(N+\alpha)pq + \{ uw + \sqrt{N(N+\alpha)}(w-u) \} 4pq. \end{aligned} \quad (5.48)$$

Now it follows from (5.45) and (5.47) that

$$sR(s) - sqp \quad (5.49)$$

has derivative

$$(\sqrt{N(N+\alpha)} + u)p^2 - (\sqrt{N(N+\alpha)} - w)q^2.$$

But it follows from (5.46) that

$$uw + \sqrt{N(N+\alpha)}(w-u)$$

has exactly the same derivative. Hence the two must differ by a constant. This constant must be 0 since (5.49) clearly vanishes when $s=0$, and so do u and w . Thus we have derived the identity

$$uw + \sqrt{N(N+\alpha)}(w-u) = sR(s) - sqp. \quad (5.50)$$

Now we can see that that every term in (5.48) can be expressed in terms of $R(s)$ and its derivatives (up to order 3). By (5.45) this is clear for all products pq and its derivatives. This is true of the first expression in curly brackets in (5.48) by what we just said and (5.44), of the first expression in curly brackets by (5.47), and of the last expression in curly brackets by (5.50).

Thus we have derived a third-order differential equation for $R(s)$. In terms of $\sigma(s) := sR(s)$ it reads

$$s^2\sigma''' = (2N + \alpha - s)\sigma + (\alpha^2 + s^2 - 4Ns - 2\alpha s)\sigma' - s\sigma'' + 6s(\sigma')^2 - 4\sigma\sigma'. \quad (5.51)$$

It follows from this that the two sides of a purported identity

$$\begin{aligned} (s\sigma'')^2 &= 4s(\sigma')^3 + \sigma^2 + (2\alpha + 4N - 2s)\sigma\sigma' \\ &\quad + (\alpha^2 - 2\alpha s - 4Ns + s^2)(\sigma')^2 - 4\sigma(\sigma')^2 \end{aligned} \quad (5.52)$$

differ by a constant. (The third-order equation is equivalent to the two sides' here having the same derivative.) Now it is clear that if α is sufficiently large then σ is twice continuously differentiable up to $s=0$ and $\sigma(0) = \sigma'(0) = 0$. Hence both sides of (5.52) vanish at $s=0$ and so the difference in question equals 0. Thus the identity is established for α large. But both sides of the identity are (for $s>0$) real-analytic for $\alpha > -1$ and so if they agree for large α they must agree for all α .

Comparing (5.52) this with (C.45) of [24] we see that $-\sigma(s)$ satisfies the σ version of Painlevé V with parameters $v_0 = v_1 = 0$, $v_2 = N$ and $v_3 = N + \alpha$.

The boundary condition at $s=0$ for σ depends, of course, on the parameter λ in (5.36), and we write $\sigma(s; \lambda)$ instead of $\sigma(s)$ to display this dependence. With the help of the Neumann series for the resolvent kernel, we compute the small s expansion

$$\begin{aligned} \sigma(s; \lambda) &= \lambda c_0 s^{\alpha+1} \left(1 - \frac{\alpha+2N}{2+\alpha} s + \dots \right) \\ &+ \lambda^2 \frac{c_0^2}{1+\alpha} s^{2\alpha+2} \left(1 - \frac{(2\alpha+3)(\alpha+2N)}{(2+\alpha)^2} s + \dots \right) \\ &+ \lambda^3 \frac{c_0^3}{(1+\alpha)^2} s^{3\alpha+3} (1 + \dots) \\ &+ \dots, \end{aligned}$$

where

$$c_0 = \frac{\Gamma(N+\alpha+1)}{\Gamma(N)\Gamma(\alpha+1)\Gamma(\alpha+2)}.$$

And now, as in the case of the GUE, we have a representation

$$P(\lambda_{\min} > s) = \exp \left\{ - \int_0^s \frac{\sigma(t; 1)}{t} dt \right\}. \quad (5.53)$$

There are only minor changes required in the above analysis when we take $J=(s, \infty)$ and this leads to an analogous representation for the distribution function of λ_{\max} .

3. Singular values of rectangular matrices. If A is an $N \times M$ rectangular matrix ($N \leq M$) whose entries are independent identically distributed complex Gaussian variables with mean 0 and variance 1 then the $N \times N$ matrix AA^* (whose eigenvalues are the squares of the singular values of A) belongs to the orthogonal polynomial ensemble associated with the weight function

$$w(x) = x^{M-N} e^{-x/2}.$$

(See [17], Cor. 3.1.) It follows that the distribution of the smallest singular value of A is given by the right side of (5.53) with s replaced by $\sqrt{s}/2$ and, of course, $\alpha = M - N$. There is a similar representation for the distribution function of the largest singular value of A .

C. Jacobi Kernel. The situation here is so similar to the preceding that we shall only indicate the main points. For the finite N Jacobi ensemble

$$w(x) = (1-x)^\alpha (1+x)^\beta$$

with $\alpha, \beta > -1$ and we must take in (1.2),

$$\varphi(x) = \sqrt{\lambda a_N} \varphi_N(x), \quad \psi(x) = \sqrt{\lambda a_N} \varphi_{N-1}(x),$$

where

$$a_N = \frac{1}{N + (\alpha + \beta)/2} \sqrt{\frac{N(N+\alpha)(N+\beta)(N+\alpha+\beta)}{(2N+\alpha+\beta-1)(2N+\alpha+\beta+1)}}.$$

From the differentiation and recurrence formulas for the Jacobi polynomials we have the differentiation formulas (1.4) with

$$m(x) = 1 - x^2, \quad A(x) = \alpha_0 + \alpha_1 x, \quad B(x) = \beta_0, \quad C(x) = \gamma_0,$$

where

$$\alpha_0 = \frac{\beta^2 - \alpha^2}{2(2N + \alpha + \beta)},$$

$$\alpha_1 = -\left(N + \frac{\alpha + \beta}{2}\right),$$

$$\beta_0 = \frac{2}{2N + \alpha + \beta} \sqrt{N(N + \alpha)(N + \beta)(N + \alpha + \beta)} \sqrt{\frac{2N + \alpha + \beta + 1}{2N + \alpha + \beta - 1}},$$

$$\gamma_0 = \frac{2}{2N + \alpha + \beta} \sqrt{N(N + \alpha)(N + \beta)(N + \alpha + \beta)} \sqrt{\frac{2N + \alpha + \beta - 1}{2N + \alpha + \beta + 1}}.$$

Using these the interested reader could without difficulty write down the Jacobi analogues of the general equations (5.37)–(5.40) in the Laguerre ensemble. We shall restrict ourselves here to the case $J = (-1, s)$, the analogue of the interval $(0, s)$ in Laguerre, and find the following Jacobi analogues of (5.41)–(5.46):

$$(1 - s^2)q' = (\alpha_0 + \alpha_1 s + v)q + (\beta_0 + (2\alpha_1 - 1)u)p, \quad (5.54)$$

$$(1 - s^2)p' = (-\gamma_0 + (2\alpha_1 + 1)w)q - (\alpha_0 + \alpha_1 s + v)p, \quad (5.55)$$

$$(1 - s^2)R(s) = 2(\alpha_0 + \alpha_1 s + v)pq + \mathcal{A} + \mathcal{B}, \quad (5.56)$$

$$((1 - s^2)R(s))' = 2\alpha_1 pq, \quad (5.57)$$

$$v' = pq, \quad (5.58)$$

where

$$\mathcal{A} = (\beta_0 + (2\alpha_1 - 1)u)p^2, \quad \mathcal{B} = (\gamma_0 - (2\alpha_1 + 1)w)q^2.$$

From (5.57) and (5.58) we deduce

$$(1 - s^2)R(s) = 2\alpha_1 v \quad (5.59)$$

(note that both sides here vanish when $s = -1$) and (5.54) and (5.55) give

$$(1 - s^2)(pq)' = \mathcal{A} - \mathcal{B}. \quad (5.60)$$

Another differentiation gives

$$\begin{aligned} (1 - s^2)(pq)'' &= -2(\alpha_0 + \alpha_1 s + v)(\mathcal{A} + \mathcal{B}) + 2s(\mathcal{A} - \mathcal{B}) \\ &\quad + 4\alpha_1(1 - s^2)p^2q^2 - \frac{4}{pq}\mathcal{A}\mathcal{B}. \end{aligned} \quad (5.61)$$

Now (5.57) and (5.59) show that pq and v are expressible in terms of R and its first derivative. And so, using (5.56) and (5.60), we deduce that \mathcal{A} and \mathcal{B} are expressible in terms of R and its two derivatives. Finally, using (5.61), we obtain a third order

equation for R . Instead of this we write down the analogue of (5.51). If we define now $\sigma(s) := (1 - s^2)R(s)$, then we have

$$(1 - s^2)^2 \sigma''' = (1 - s^2)^2 \frac{(\sigma'')^2}{\sigma'} - 2s(1 - s^2)\sigma'' - 2(1 - s^2)(\sigma')^2 - 2 \left(1 - \frac{2\alpha_1^2}{\sigma'}\right) \sigma^2 - 4\alpha_1(\alpha_0 + \alpha_1 s)\sigma. \quad (5.62)$$

For the boundary condition at $s = -1$ we compute the small $s + 1$ expansion to be (with an obvious notation)

$$\begin{aligned} \sigma(s; \lambda) &= \lambda c_0 (1 + s)^{1 + \beta} \left(1 + \frac{\beta(1 - \alpha) - 2N(\alpha + \beta + N)}{2(2 + \beta)} (1 + s) + \dots \right) \\ &+ \frac{\lambda^2 c_0^2}{2(1 + \beta)} (1 + s)^{2 + 2\beta} \\ &\times \left(1 + \frac{2 + 6\beta + 3\beta^2 - (3 + 2\beta)(2N^2 + 2N\alpha + 2N\beta + \alpha\beta)}{2(2 + \beta)^2} (1 + s) + \dots \right) \\ &+ \frac{\lambda^3 c_0^3}{4(1 + \beta)^2} (1 + s)^{3 + 3\beta} (1 + \dots) + \dots, \end{aligned}$$

where

$$c_0 = \frac{\Gamma(N + \alpha + \beta + 1)\Gamma(N + \beta + 1)}{2^\beta \Gamma(\beta + 1)\Gamma(\beta + 2)\Gamma(N)\Gamma(N + \alpha)}.$$

We have not been able to find a first integral for (5.62), in other words a second order equation which is analogous to (5.52).

D. The Circular Ensemble.

1. *The partial differential equations.* If $\mathcal{U}(N)$ denotes the group of $N \times N$ unitary matrices, then the finite N circular ensemble of unitary matrices (sometimes denoted CUE) is this set $\mathcal{U}(N)$ together with the normalized Haar measure. Just as for the orthogonal polynomial hermitian matrix ensembles, the level spacing distributions are expressed in terms of a Fredholm determinant of an integral operator defined now on the unit circle. All of this is well-known and we refer the reader to either Dyson [14] or Mehta [28] for details.

The integral operator for the finite N CUE is

$$K \doteq \frac{\lambda}{2\pi} \frac{\sin \frac{N}{2}(x - y)}{\sin \frac{1}{2}(x - y)} \chi_J(y) = \frac{\phi(x)\psi(y) - \psi(x)\phi(y)}{e^{ix} - e^{iy}} \chi_J(y),$$

where

$$\phi(x) = \sqrt{\frac{\lambda}{2\pi}} e^{i\frac{N+1}{2}x}, \quad \psi(x) = \sqrt{\frac{\lambda}{2\pi}} e^{-i\frac{N-1}{2}x}.$$

Thus the differentiation formulas (2.37) hold with $b = i$ and

$$m(x) = 1, \quad A(x) = \frac{i}{2}N, \quad B(x) = C(x) = 0,$$

and so from the considerations in Sect. III E we deduce that

$$R(a_j, a_k) = \frac{q_j p_k - p_j q_k}{e^{ia_j} - e^{ia_k}} \quad (j \neq k), \quad (5.63)$$

$$ie^{ia_j} R(a_j, a_j) = iN p_j q_j - \sum_{k \neq j} (-1)^k R(a_j, a_k) (q_k p_j - p_k q_j), \quad (5.64)$$

and

$$\begin{aligned} \frac{\partial q_j}{\partial a_j} &= i \frac{N+1}{2} q_j - \sum_{k \neq j} (-1)^k R(a_j, a_k) q_k, \\ \frac{\partial p_j}{\partial a_j} &= -i \frac{N-1}{2} p_j - \sum_{k \neq j} (-1)^k R(a_j, a_k) p_k. \end{aligned} \quad (5.65)$$

2. *Single interval case.* Now we specialize to $J = (-t, t)$ and take $a_1 = -t, a_2 = t$. Write p, q for p_2, q_2 and note that since $\varphi(x) = \varphi(-x)$ and $K(x, y)$ is even we have $p_1 = \bar{p}, q_1 = \bar{q}$. Let's also write

$$R(t) := R(t, t), \quad \tilde{R}(t) := R(-t, t) = R(t, -t).$$

Then (2.29) with $J = (-1, 1)$ and $i = 2$ gives

$$R' = 2\tilde{R}^2 \quad \left(' = \frac{d}{dt} \right), \quad (5.66)$$

(5.64) gives

$$e^{it} R(t) = Npq - i\tilde{R}(t)(\bar{q}p - \bar{p}q), \quad (5.67)$$

whereas (5.63) with $j = 2, k = 1$ gives

$$q\bar{p} - p\bar{q} = 2i \sin t \tilde{R}. \quad (5.68)$$

Substituting (5.68) into (5.67) and using (5.66) give

$$e^{it} R(t) = Npq - \sin t R'(t). \quad (5.69)$$

Now

$$q' = i \frac{N+1}{2} q + \tilde{R}\bar{q}, \quad p' = -i \frac{N-1}{2} p + \tilde{R}\bar{p}.$$

Using these and (5.68) we get

$$\begin{aligned} (2i \sin t \tilde{R})' &= \left(i \frac{N+1}{2} q + \tilde{R}\bar{q} \right) \bar{p} + q \left(i \frac{N-1}{2} \bar{p} + \tilde{R}p \right) \\ &\quad - \left(-i \frac{N-1}{2} p + \tilde{R}\bar{p} \right) \bar{q} - p \left(-i \frac{N+1}{2} \bar{q} + \tilde{R}q \right) \\ &= iN(q\bar{p} - p\bar{q}). \end{aligned}$$

This and (5.68) may be written

$$\begin{aligned}(\sin t \tilde{R})' &= N\Re(q\bar{p}), \\ \sin t \tilde{R} &= \Im(q\bar{p}),\end{aligned}$$

and (5.69) may be written

$$e^{it}R(t) + \sin t R'(t) = Npq.$$

Thus (taking the square of the absolute value of both sides)

$$(\cos t R(t) + \sin t R'(t))^2 + \sin^2 t R(t)^2 = (\sin t \tilde{R})'^2 + N^2 \sin^2 t \tilde{R}^2.$$

Using

$$\tilde{R}^2 = \frac{1}{2} R', \quad \tilde{R}'\tilde{R} = \frac{1}{4} R'', \quad \tilde{R}'^2 = \frac{1}{8} \frac{R''^2}{R'},$$

which follow from (5.66), we get the second order equation

$$\begin{aligned}R(t)^2 + 2\sin t \cos t R(t)R'(t) + \sin^2 t R'(t)^2 \\ = \frac{1}{2} \left(\frac{1}{4} \sin^2 t \frac{R''(t)^2}{R'(t)} + \sin t \cos t R''(t) + (\cos^2 t + N^2 \sin^2 t) R'(t) \right).\end{aligned}\quad (5.70)$$

From the Neumann expansion of the resolvent kernel $R(t; \lambda) = R(t)$ we obtain the expansion

$$R(t, \lambda) = \rho_0 + 2\rho_0^2 t + 4\rho_0^3 t^2 + \frac{2}{9} (1 - N^2 + 36\rho_0^2) \rho_0^2 t^3 + \cdots, \quad (5.71)$$

where

$$\rho_0 = \frac{\lambda N}{2\pi}.$$

If we denote by $E_N(0; s)$ the probability that an interval (of the unit circle) of length s contains no eigenvalues (modifying here the notation of (1.1)), then

$$E_N(0; s) = \exp \left\{ -2 \int_0^t R(x, 1) dx \right\} \quad (s = 2t).$$

Using the expansion (5.71) with $\lambda = 1$ (and additional terms computed from the differential equation (5.70)) we find that

$$\begin{aligned}E_N \left(0; \frac{2\pi}{N} s \right) &= 1 - s + \frac{\pi^2}{36} \left(1 - \frac{1}{N^2} \right) s^4 - \frac{\pi^4}{675} \left(1 - \frac{5}{2N^2} + \frac{3}{2N^4} \right) s^6 \\ &+ \frac{\pi^6}{17640} \left(1 - \frac{14}{3N^2} + \frac{7}{N^4} - \frac{10}{3N^6} \right) s^8 + \cdots \quad (s \rightarrow 0),\end{aligned}$$

where we have replaced s by $\frac{2\pi}{N} s$ so that the $N \rightarrow \infty$ limit is clear. This converges uniformly for all N and bounded s . Observe that the corrections to the limiting coefficients are $O(1/N^2)$ as $N \rightarrow \infty$.

VI. Generalizations of Hermite, Laguerre and Jacobi

In this final section we shall show that there are differentiation formulas of the form (1.4) for very general orthogonal polynomial ensembles, and that if the weight function is the standard Hermite, Laguerre, or Jacobi weight function multiplied by the exponential of an arbitrary polynomial then the coefficients $m(x)$, $A(x)$, $B(x)$ and $C(x)$ in (1.4) are themselves polynomials. Some, but not all, of our derivation can be found in the orthogonal polynomial literature [4, 5] but our presentation will be self-contained.

Throughout, we shall write our weight function as

$$w(x) = e^{-V(x)} .$$

As stated in the Introduction, the polynomials orthonormal with respect to $w(x)$ are denoted $p_k(x)$ ($k=0, 1, \dots$), and we set $\varphi_k(x) = p_k(x)w(x)^{1/2}$ so that $\{\varphi_k\}$ is orthonormal with respect to Lebesgue measure. The underlying domain \mathcal{D} of all these functions we take to be a finite or infinite interval. We are interested in differentiation formulas (1.4) when, up to constant factors,

$$\varphi(x) = \varphi_N(x), \quad \psi(x) = \varphi_{N-1}(x) . \quad (6.1)$$

It is well-known that if k_N denotes the highest coefficient in $p_N(x)$, and if

$$a_N = \frac{k_{N-1}}{k_N} ,$$

then there is a recursion formula

$$xp_N(x) = a_{N+1}p_{N+1}(x) + b_Np_N(x) + a_Np_{N-1}(x) \quad (6.2)$$

as well as the Christoffel–Darboux formula

$$\sum_{k=0}^{N-1} p_k(x)p_k(y) = a_N \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x-y} . \quad (6.3)$$

(See Chap. 10 in [18].)

We shall always assume that our weight function satisfies

$$x^k w(x) \text{ is bounded for each } k=0, 1, \dots ,$$

and that $V(x)$ is continuously differentiable in the interior of \mathcal{D} . And we assume that

$$\lim_{x \rightarrow \partial \mathcal{D}} w(x) = 0 \quad (6.4)$$

although this will be relaxed later. We define [4, 5]

$$U(x, y) = \frac{V'(x) - V'(y)}{x - y} ,$$

$$A_N(x) = a_n \int_{\mathcal{D}} \varphi_N(y) \varphi_{N-1}(y) U(x, y) dy, \quad B_N(x) = a_n \int_{\mathcal{D}} \varphi_N(y)^2 U(x, y) dy .$$

Proposition. *If (6.4) holds and φ and ψ are given by (6.1) then we have (1.4) with*

$$m(x)=1, \quad A(x)=-A_N(x)-\frac{1}{2}V'(x),$$

$$B(x)=B_N(x), \quad C(x)=\frac{a_N}{a_{N-1}}B_{N-1}(x).$$

Proof. If

$$\tilde{K}(x, y):=\sum_{k=0}^{N-1} p_k(x)p_k(y),$$

then for any polynomial $\pi(x)$ of degree at most $N-1$,

$$\pi(x)=\int \tilde{K}(x, y)\pi(y)w(y)dy.$$

Apply this to $\pi(x)=p'_N(x)$ and integrate by parts, using (6.4) to eliminate any boundary terms. We obtain

$$p'_N(x)=-\int \frac{\partial \tilde{K}(x, y)}{\partial y} p_N(y)w(y)dy + \int \tilde{K}(x, y)V'(y)p_N(x)w(y)dy.$$

Now both $\partial \tilde{K}/\partial y$ and \tilde{K} , as polynomials in y , are orthogonal (with respect to w) to $p_N(y)$ since they have degree at most $N-1$. It follows that the first integral above vanishes and that we can write the resulting identity as

$$p'_N(x)=\int \tilde{K}(x, y)(V'(y)-V'(x))p_N(y)w(y)dy$$

$$=-a_N \int [p_N(x)p_{N-1}(y)-p_{N-1}(x)p_N(y)]p_N(y)U(x, y)w(y)dy$$

by (6.3) and the definition of $U(x, y)$. We have shown

$$p'_N(x)=-A_N(x)p_N(x)+B_N(x)p_{N-1}(x). \quad (6.5)$$

It follows from this, of course, that

$$p'_{N-1}(x)=-A_{N-1}(x)p_{N-1}(x)+B_{N-1}(x)p_{N-2}(x).$$

We use the recursion formula (6.2) (which holds also if each p_N is replaced by φ_N) and find that this is equal to

$$-\int \varphi_{N-1}(y)[y\varphi_{N-1}(y)-a_N\varphi_N(y)-b_{N-1}\varphi_{N-1}(y)]U(x, y)dy p_{N-1}(x)$$

$$+\int \varphi_{N-1}^2(y)U(x, y)dy [xp_{N-1}(x)-a_Np_N(x)-b_{N-1}p_{N-1}(x)]$$

$$=\int \varphi_{N-1}^2(y)(x-y)U(x, y)dy p_{N-1}(x)+A_N(x)p_{N-1}(x)-\frac{a_N}{a_{N-1}}B_{N-1}(x)p_N(x).$$

The last integral on the right side equals

$$\int p_{N-1}^2(y)(V'(x)-V'(y))w(y)dy=V'(x)+\int p_{N-1}^2(y)w'(y)dy.$$

The last integral vanishes, as we see by integrating by parts and noting that p'_{N-1} is orthogonal to p_{N-1} . Thus we have shown

$$p'_{N-1}(x)=[A_N(x)+V'(x)]p_{N-1}(x)-\frac{a_N}{a_{N-1}}B_{N-1}(x)p_N(x). \quad (6.6)$$

The statement of the proposition now follows from (6.5) and (6.6) if we use the fact that

$$\varphi'_N = p'_N w^{1/2} + \frac{1}{2} \frac{w'}{w} p_N w^{1/2},$$

and similarly for φ'_{N-1} . ■

Remark. The assumption (6.4) is not just a technical requirement. The conclusion of the proposition is false without it. (Consider, for example, the Legendre polynomials on $(-1, 1)$, where $V(x)=0$ and the conclusion of the proposition reads $\varphi'_n(x)=0$.) Nevertheless, we shall be able to handle some cases where (6.4) fails.

Example 1 (generalized Hermite). Here $V(x)$ is a polynomial of even degree (at least 2) with positive leading coefficient and $\mathcal{D} = (-\infty, \infty)$. The conclusion of the proposition holds and so we have (1.4) with $m(x)=1$, with $A(x)$ a polynomial of degree at most $\deg V-1$, and with $B(x)$ and $C(x)$ polynomials of degree at most $\deg V-2$.

Example 2 (generalized Laguerre). Here

$$w(x) = x^\alpha e^{-W(x)},$$

where $\alpha > -1$ and W is a polynomial of degree at least 1 with positive leading coefficient, and $\mathcal{D} = (0, \infty)$. In this case

$$U(x, y) = \frac{\alpha}{xy} + \frac{W'(x) - W'(y)}{x - y}. \tag{6.7}$$

Now (6.4) is satisfied if $\alpha > 0$ and the proposition tells us that in this case we have (1.4) with

$$\begin{aligned} m(x) &= x, & A(x) &= -x A_N(x) + \frac{\alpha}{2} - \frac{x}{2} W'(x), \\ B(x) &= x B_N(x), & C(x) &= \frac{a_N}{a_{N-1}} x B_{N-1}(x); \end{aligned} \tag{6.8}$$

now $A(x)$ is a polynomial of degree at most $\deg W$ while $B(x)$ and $C(x)$ are polynomials of degree at most $\deg W-1$.

To extend this to all $\alpha > -1$ we see that there are problems in the integrals defining $A_N(x)$ and $B_N(x)$ arising from the term α/xy in (6.7). The contribution of this term to the integral defining $A_N(x)$, say, equals (we assume now $\alpha > 0$)

$$a_N \frac{\alpha}{x} \int_0^\infty p_N(y) p_{N-1}(y) e^{-W(y)} y^{\alpha-1} dy.$$

Integration by parts shows that this equals

$$-\frac{a_N}{x} \int_0^\infty (p_N(y) p_{N-1}(y) e^{-W(y)})' y^\alpha dy.$$

This expression is well-defined for all $\alpha > -1$ and in fact represents a function of α which is real-analytic there. (The coefficients of the p_N are clearly real-analytic functions of α .)

This argument shows that both sides of (1.4), with the coefficient polynomials given by (6.8), are (or extend to be) real-analytic for $\alpha > -1$. Since they agree for $\alpha > 0$ they must also agree for $\alpha > -1$.

Example 3 (generalized Jacobi). Here

$$w(x) = (1-x)^\alpha (1+x)^\beta e^{-W(x)},$$

where $\alpha, \beta > -1$ and W is a polynomial, $\mathcal{D} = (-1, 1)$. In this case

$$U(x, y) = \frac{\alpha(1+x)(1+y) + \beta(1-x)(1-y)}{(1-x^2)(1-y^2)} + \frac{W'(x) - W'(y)}{x-y},$$

and the proposition tells us that for $\alpha, \beta > 0$ we have (1.4) with

$$m(x) = 1 - x^2, \quad A(x) = -(1-x^2)A_N(x) - \frac{\alpha}{2}(1+x) + \frac{\beta}{2}(1-x) - \frac{1-x^2}{2}W'(x),$$

$$B(x) = (1-x^2)B_N(x), \quad C(x) = \frac{a_N}{a_{N-1}}(1-x^2)B_{N-1}(x);$$

now $A(x)$ is a polynomial of degree at most $\deg W + 1$ while $B(x)$ and $C(x)$ are polynomials of degree at most $\deg W$. The identity can be extended to $\alpha, \beta > -1$ as in Example 2. (It is convenient to express the integrals defining $A_N(x)$ and $B_N(x)$ as sums of integrals by using a representation

$$1 = u(x) + (1-u(x)),$$

where $u = 1$ in a neighborhood of $x = -1$ and $u = 0$ in a neighborhood of $x = 1$; this separates the difficulties at the two end-points. The details are left to the reader.)

Remark. It is clear that the last examples can be generalized to any weight function of the form

$$\prod |x - a_i|^{\alpha_i} e^{-W(x)},$$

where $W(x)$ is a polynomial and for each a_i which is in the closure of \mathcal{D} we have $\alpha_i > -1$.

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