Fredholm Integral Equation and Splines of the Fifth Order of Approximation

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Abstract: - This paper considers the numerical solution of the Fredholm integral equation of the second kind using local polynomial splines of the fifth order of approximation and the fourth order of approximation (cubic splines). The basis splines in these cases occupy five and four adjacent grid intervals respectively. Different local spline approximations of the fifth (or fourth) order of approximation are used at the beginning of the integration interval, in the middle of the integration interval, and at the end of the integration interval. The construction of the calculation schemes for solving the Fredholm equation of the second kind with these splines is considered. The results of the numerical experiments on the approximation of functions and on the solution of the Fredholm integral equations are presented. The results of the solution of the integral equation which uses the polynomial splines of the fifth order of approximation are compared with ones obtained with cubic splines and with the application of the Simpson's method. Note that in order to achieve a given error using the approximation with quadratic splines, a denser grid of nodes is required than when using the approximation with the cubic splines or splines of the fifth order of approximation.

Key-Words: - Fredholm integral equation of the second kind, polynomial local splines, approximation, fifth order of approximation, fourth order of approximation

Received: June 18, 2021. Revised: March 20, 2022. Accepted: April 19, 2022. Published: May 20, 2022.

1 Introduction

When solving a number of applied problems, researchers have to solve the Fredholm integral equations. One of the classical forms of representation of dynamic systems is integral equations. This representation method is compact and convenient in the case of linear stationary systems, when the spectral characteristics of the input and output of the process associated with the useful signal and noise are known. The integral equations contain the complete statement of the problem together with the initial conditions. Of particular interest is the representation of nonstationary dynamical systems by integral equations. Integral equations are divided into two main classes: linear and non-linear. In this paper, we consider the solution of the linear Fredholm equation of the second kind using local interpolation splines. The solution of the integral equations of Fredholm and Volterra, using local interpolation splines of the second, third and fourth order of approximation, was considered in the author's earlier papers. Here we will focus on the use of local interpolation splines of the fifth order of approximation.

There are many numerical methods for solving the Fredholm integral equation of the second kind. There are various classical methods based on the use of composite quadrature formulas of average rectangles, trapezoids, and the Simpson's method. The properties of these methods such as approximation, stability and convergence are well studied. In some cases, classical methods give a significant error in the solution. Therefore, many researchers are trying to construct new approaches to the numerical solution of integral equations, which can have a smaller error in the solution. It is often convenient to construct a solution to the Fredholm equation based on the use of splines. B-splines are often used for solving Fredholm equations.

Among the papers published over the past 3 years on this topic, we note the papers [1]-[11]. In paper [1], some applications to numerical analysis especially quadrature formulas, differentiation and numerical solutions of linear Fredholm integral equations are given. In paper [2], the isogeometric Galerkin and collocation methods for solving the Fredholm integral eigenvalue problem on arbitrary multipatch domains are introduced. In paper [3], the solution of the nonlinear Fredholm integro-differential equation (NFID) in the complex plane by periodic quasi-wavelets is approximated. In paper [4], a numerical solution of important weakly singular type of Volterra - Fredholm integral equations WSVFIEs using the collocation type quasi-affine biorthogonal method (based on special B-spline tight framelets) is provided. In paper [5], a computational method for solving nonlinear Volterra-Fredholm Hammerstein integral equations is proposed by using compactly supported semiorthogonal cubic B-spline wavelets as the basis functions. In paper [6], spline functions were used to propose a new scheme for solving the linear Volterra–Fredholm integral equations of the second kind. In paper [7], a method to solve the integral equations of the second type with degenerate kernels and shifts, is constructed. In paper [8], an efficient modification of the wavelets method to solve a new class of Fredholm integral equations of the second kind with non symmetric kernel is introduced. In paper [9] a new computational method for solving linear Fredholm integral equations of the second kind, which is based on the use of B-spline quasi-affine tight framelet systems generated by the unitary and oblique extension principles is presented. In paper [10], a new collocation technique for numerical solution of Fredholm, Volterra and mixed Volterra-Fredholm integral equations of the second kind is introduced. In paper [11], Farnoosh and Ebrahimi developed a numerical method based on random sampling for the solution of Fredholm integral equations of the second kind, which was called the Monte Carlo method based on the simulation of a continuous Markov chain.

The theory of constructing approximations using local splines was developed in the works of Prof. Yu.K.Demyanovich and Prof. I.G.Burova. Local polynomial and nonpolynomial splines of the second and third order of approximation were successfully used to construct computational schemes for solving the integral equations of Fredholm and Volterra.

Local polynomial splines of the fifth order of approximation were, in particular, also considered in detail in the works of the author. The features of constructing error estimates of approximations with nonpolynomial splines were discussed in detail in the author's paper [12].

This paper is structured as follows: Section 2 of this paper considers the main properties of polynomial splines of the fifth order of approximation and the splines of the fourth order of approximation. The approximation formulas for different arrangements of the supports of the basis splines, and formulates approximation theorems are given in it. Section 3 considers the application of the splines of the fifth order of approximation to the solution of the Fredholm integral equation of the second kind. Also, in the third section we discuss the construction of calculation schemes. Section 4 presents numerical examples.

As already noted, new numerical methods for solving the Fredholm integral equation of the second kind were considered in papers [10, 11]. In our paper, we will solve the same Fredholm integral equations that were considered in paper [10, 11], but using local splines of the fifth order of approximation instead. In addition, we will compare our results with the results of applying the classical method such as method of Simpson.

2 Approximation with the Local Splines

2.1 Local Approximation with the Splines of the Fifth Order of Approximation

First, we would like to remind the readers of the main details of the local approximation with the splines of the fifth order of approximation.

Let a, b be real and n be integer. Let the values of the function $u(x)$ be known at the nodes of the grid of nodes $\{t_i\}$:

$$\ldots < t_{-1} < a = t_0 < t_1 < \cdots < t_n = b < t_{n+1} < \cdots.$$

Denote $u_i = u(t_i)$. In what follows, we will use the following types of approximations of the function $u(t)$ on interval $[t_i, t_{i+1}]$. At the beginning of the interval $[a, b]$, we apply the approximation with the left splines

$$U_{R5}^i(x) = \sum_{j=i}^{i+4} u_j w_j(x),\ x \in [t_i, t_{i+1}],$$

where u_j, $j = 0, \ldots, n$, are the values of the function in nodes t_j the basis splines $w_i(x)$ are the next:

$$w_i(x) = \frac{(x - t_{i+1})(x - t_{i+2})(x - t_{i+3})(x - t_{i+4})}{(t_i - t_{i+1})(t_i - t_{i+2})(t_i - t_{i+3})(t_i - t_{i+4})},$$

$$w_{i+1}(x) = \frac{(x - t_i)(x - t_{i+2})(x - t_{i+3})(x - t_{i+4})}{(t_{i+1} - t_i)(t_{i+1} - t_{i+2})(t_{i+1} - t_{i+3})(t_{i+1} - t_{i+4})},$$

$$w_{i+2}(x) = \frac{(x - t_i)(x - t_{i+1})(x - t_{i+3})(x - t_{i+4})}{(t_{i+2} - t_i)(t_{i+2} - t_{i+1})(t_{i+2} - t_{i+3})(t_{i+2} - t_{i+4})},$$

$$w_{i+3}(x) = \frac{(x - t_i)(x - t_{i+1})(x - t_{i+2})(x - t_{i+4})}{(t_{i+3} - t_i)(t_{i+3} - t_{i+1})(t_{i+3} - t_{i+2})(t_{i+3} - t_{i+4})},$$

$$w_{i+4}(x) = \frac{(x-t_i)(x-t_{i+1})(x-t_{i+2})(x-t_{i+3})}{(t_{i+4}-t_i)(t_{i+4}-t_{i+1})(t_{i+4}-t_{i+2})(t_{i+4}-t_{i+3})}.$$

The graph of the right basis spline is shown in Fig.1.

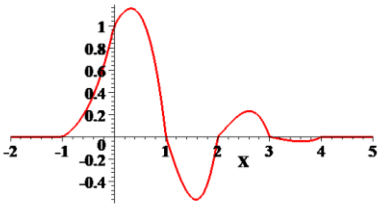

Fig. 1: The graph of the right basis spline

Let us denote $\| u^{(q)} \|_{[a,b]} = \max_{[a,b]} |u^{(q)}(x)|$. On each separate interval $[t_i, t_{i+1}]$, we can estimate the approximation error in the assumption that the function $u(x)$ is 5 times continuously differentiable. We receive the error of approximation from the remainder term of the Lagrange interpolation. If the grid of nodes is such that the nodes are equidistant with a step h, then we can find the error of approximation for $x \in [t_i, t_{i+1}]$ in the form:

$\|U_{R5}^i - u\| \le K \, \|u^{(5)}\|_{[t_i,t_{i+4}]} h^5/5!$, $K = 3.63$.

If the grid nodes are not equidistant, then we denote the length of the maximum grid interval with h.

The next option: In the middle of the interval $[a, b]$, we apply the approximation with the middle splines

$$U_{S5}^i(x) = \sum_{j=i-2}^{i+2} u_j w_j^s(x),\ x \in [t_i, t_{i+1}],$$

where

$$w_{i-2}^s(x) = \frac{(x-t_{i-1})(x-t_i)(x-t_{i+1})(x-t_{i+2})}{(t_{i-2}-t_{i-1})(t_{i-2}-t_i)(t_{i-2}-t_{i+1})(t_{i-2}-t_{i+2})},$$

$$w_{i-1}^s(x) = \frac{(x-t_{i-2})(x-t_i)(x-t_{i+1})(x-t_{i+2})}{(t_{i-1}-t_{i-2})(t_{i-1}-t_i)(t_{i-1}-t_{i+1})(t_{i-1}-t_{i+2})},$$

$$w_i^s(x) = \frac{(x-t_{i-2})(x-t_{i-1})(x-t_{i+1})(x-t_{i+2})}{(t_i-t_{i-2})(t_i-t_{i-1})(t_i-t_{i+1})(t_i-t_{i+2})},$$

$$w_{i+1}^s(x) = \frac{(x-t_{i-2})(x-t_{i-1})(x-t_i)(x-t_{i+2})}{(t_{i+1}-t_{i-2})(t_{i+1}-t_{i-1})(t_{i+1}-t_i)(t_i-t_{i+2})},$$

$$w_{i+2}^s(x) = \frac{(x-t_{i-2})(x-t_{i-1})(x-t_i)(x-t_{i+1})}{(t_{i+2}-t_{i-2})(t_{i+2}-t_{i-1})(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}.$$

If the grid of nodes is such that the nodes are equidistant with step h, then we can find the error of approximation for $x \in [t_i, t_{i+1}]$ in the form:

$\|U_{S5}^i - u\| \le K \, \|u^{(5)}\|_{[t_{i-2},t_{i+2}]} h^5/5!$, $K = 1.42$.

The graph of the middle basis spline (when supp $w_i^{RS} = [t_{i-3}, t_{i+2}]$) is shown in Fig.2.

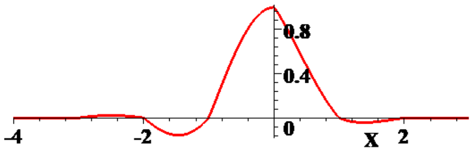

Fig. 2: The graph of the middle basis spline (supp $w_i^{RS} = [\, t_{i-3}, t_{i+2}]$)

The graph of the middle basis spline (when supp $w_i^s = [t_{i-2}, t_{i+3}]$) is shown in Fig.3.

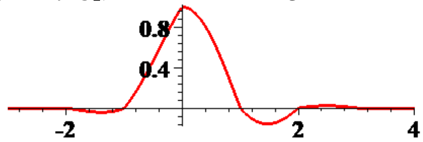

Fig. 3: The graph of the middle basis spline (supp $w_i^s = [t_{i-2}, t_{i+3}]$)

At the end of the interval $[a, b]$, we apply the approximation with the right splines:

$$U_{L5}^i(x) = \sum_{j=i-3}^{i+1} u_j w_j(t),\ t \in [t_i, t_{i+1}],$$

where the basis splines are the following:

$$w_{i-3}(x) = \frac{(x-t_{i-2})(x-t_{i-1})(x-t_i)(x-t_{i+1})}{(t_{i-3}-t_{i-2})(t_{i-3}-t_{i-1})(t_{i-3}-t_i)(t_{i-3}-t_{i+1})},$$

$$w_{i-2}(x) = \frac{(x-t_{i-3})(x-t_{i-1})(x-t_i)(x-t_{i+1})}{(t_{i-2}-t_{i-3})(t_{i-2}-t_{i-1})(t_{i-2}-t_i)(t_{i-2}-t_{i+1})},$$

$$w_{i-1}(x) = \frac{(x-t_{i-3})(x-t_{i-2})(x-t_i)(x-t_{i+1})}{(t_{i-1}-t_{i-3})(t_{i-1}-t_{i-2})(t_{i-1}-t_i)(t_{i-1}-t_{i+1})},$$

$$w_i(x) = \frac{(x-t_{i-3})(x-t_{i-2})(x-t_{i-1})(x-t_{i+1})}{(t_i-t_{i-3})(t_i-t_{i-2})(t_i-t_{i-1})(t_i-t_{i+1})},$$

$$w_{i+1}(x) = \frac{(x-t_{i-3})(x-t_{i-2})(x-t_{i-1})(x-t_i)}{(t_{i+1}-t_{i-3})(t_{i+1}-t_{i-2})(t_{i+1}-t_{i-1})(t_{i+1}-t_i)}.$$

We receive the error of approximation from the remainder term of the Lagrange interpolation. If the grid of nodes is such that the nodes are equidistant with step h, then we can find the error of approximation for $x \in [t_i, t_{i+1}]$ in the form:

$\|U_{L5}^i - u\| \le K \, h^5/5! \; \|u^{(5)}\|_{[t_{i-3},t_{i+1}]}$, $K = 3.63$.

The graph of the left basis spline is shown in Fig.4.

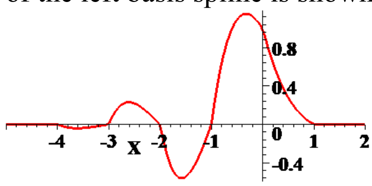

Fig. 4: The graph of the left basis spline

Theorem 1.

Let $u \in C^5[a,b]$. $t_j = a + jh, j = 0,1,\dots,n$, $h = \frac{b-a}{n}, n \geq 4$. To approximate the function $u(x)$, $x \in [t_i, t_{i+1}]$, with the left and right splines, the following inequalities are valid:

$$\left|u(x) - U_{L5}^i(x)\right| \leq Kh^5 \parallel u^{(5)} \parallel_{[t_{i-3},t_{i+1}]}, K = 3.63/5!.$$

$$\left|u(x) - U_{R5}^i(x)\right| \leq Kh^5 \parallel u^{(5)} \parallel_{[t_i,t_{i+4}]}, \ K = 3.63/5!.$$

To approximate the function $u(x)$, $x \in [t_i, t_{i+1}]$, with the middle splines, the following inequality is valid:

$$\left|u(x) - U_{R5}^i(x)\right| \leq Kh^5 \parallel u^{(5)} \parallel_{[t_{i-2},t_{i+2}]}, \ K = 1.42/5!.$$

Proof. It is easy to notice that U_{R5}^i is an interpolation polynomial, and $t_j, t_{j+1}, t_{j+2}, t_{j+3}, t_{j+4}$ are the interpolation nodes,

$$U_{R5}^i(t_i) = u(t_i), \ U_{R5}^i(t_{i+1}) = u(t_{i+1}),$$
$$U_{R5}^i(t_{i+2}) = u(t_{i+2}), U_{R5}^i(t_{i+3}) = u(t_{i+3}),$$
$$U_{R5}^i(t_{i+4}) = u(t_{i+4}).$$

Using the remainder term we get

$$u(x) - U_{R5}^i(x) = \frac{u^{(5)}(\xi)}{5!}(x - t_i)(x - t_{i+1})$$
$$\times (x - t_{i+2})(x - t_{i+3})(x - t_{i+4}), \ \xi \in [t_i, t_{i+4}].$$

We can use $x = t_i + \tau h, \ \tau \in [0,1]$. It can easily be calculated that the next relation is valid:

$$\max_{\tau \in [0,1]} |\tau(\tau-1)(\tau-2)(\tau-3)(\tau-4)| = 3.63.$$

It follows that on the uniform grid with step h we have

$$\max_{x \in [t_i,t_{i+3}]} |u(x) - U_{R5}^i(x)| \leq Kh^5 \max_{[t_i,t_{i+3}]} |u^{(5)}(x)|,$$
$K = 3.63/5!$.

We recall that we constructed an approximation with an error of $O(h^\wedge 5)$ using approximation identities (relations) separately on each grid interval $[t_i, t_{i+1}] \subset [a,b]$. We define the supports of the basis splines as follows: For the middle basis splines $w_i^S(x)$ we use supp $w_i^S = [t_{i-2}, t_{i+3}]$ or we can use supp $w_i^{RS} = [t_{i-3}, t_{i+2}]$. In the case of the middle basis splines, we distinguish two types: supp $w_i^S = [t_{i-2}, t_{i+3}]$ and supp $w_i^{RS} = [t_{i-3}, t_{i+2}]$. For the left basis splines $w_i^L(x)$ we use supp $w_i^L = [t_{i-1}, t_{i+4}]$. For the right basis w_i^R splines we use supp $w_i^R = [t_{i-4}, t_{i+1}]$. Note, that the following relations are valid: $w_i^L(t_i) = 1$, $w_i^R(t_i) = 1$, $w_i^S(t_i) = 1$, $w_i^{RS}(t_i) = 1$, and $w_i^L(t_k) = 0$, $w_i^R(t_k) = 0$, $w_i^S(t_k) = 0$, $w_i^{RS}(t_k) = 0$, when $k \neq i$.

According to the location of the support of the basis splines relative to the root-point (the point at which the basis spline is equal to 1), the four variants of continuous basis splines can be distinguished: the left basis splines, the middle basis splines, and the right basis splines. The interpolation using the right basis splines is used at the beginning of the interpolation interval $[a,b]$. We use the interpolation using the middle basis splines in the middle of the interpolation interval $[a,b]$. We use the interpolation with the use of the left basis splines at the end of the interpolation interval $[a,b]$. So, when approximating with the splines of the fifth order of approximation at the finite interval $[a,b]$, we use the four types of basis splines. We can use only one type of approximation with the fifth-order basis splines, but in this case we have to use the function values that lie outside the bounds of the finite interval $[a,b]$.

Let a grid of nodes $t_0 = a, t_n = b$ be built on the interval $[a,b]$. When approaching with only the middle splines on the interval $[a,b]$, we have to add values at the grid nodes t_{-2}, t_{-1}, t_{n+1}. When approaching with only the left splines on the interval $[a,b]$, we have to add values at the grid nodes t_{-3}, t_{-2}, t_{-1}.When approaching with only the left splines on the interval $[a,b]$, we have to add values at the grid nodes $t_{n+1}, t_{n+2}, t_{n+3}$.

2.2 Approximation of the Functions with the Cubic Polynomial Splines

Now, we recall the features of the approximation of the functions with the cubic polynomial splines near the right end of the interval $[a,b]$, near the left end of the interval $[a,b]$, and at the middle of the interval. Let $\{t_j\}$ be the set of nodes on the interval $[a,b]$. The middle basis splines that form the continuous polynomial approximation in the interval $x \in [t_j, t_{j+1}] \subset [a,b]$ can be written as follows:

$$\omega_j^M(x) = \frac{(x - t_{j+1})(x - t_{j+2})(x - t_{j+3})}{(t_j - t_{j+1})(t_j - t_{j+2})(t_j - t_{j+3})},$$
$$x \in [t_{j+1}, t_{j+2}],$$

$$\omega_j^M(x) = \frac{(x - t_{j+1})(x - t_{j-1})(x - t_{j+2})}{(t_j - t_{j+1})(t_j - t_{j-1})(t_j - t_{j+2})},$$
$$x \in [t_j, t_{j+1}],$$

$$\omega_j^M(x) = \frac{(x - t_{j-1})(x - t_{j-2})(x - t_{j+1})}{(t_j - t_{j-1})(t_j - t_{j-2})(t_j - t_{j+1})},$$
$$x \in [x_{j-1}, x_j],$$

$$\omega_j^M(x) = \frac{(x - t_{j-1})(x - t_{j-2})(x - t_{j-3})}{(t_j - t_{j-1})(t_j - t_{j-2})(t_j - t_{j-3})},$$
$$x \in [t_{j-2}, t_{j-1}],$$

$$\omega_j^M(x) = 0, x \notin [t_{j-2}, t_{j+2}].$$

The approximation with these basis splines can be written in the form:

$$U_j^M(x) = u(t_{j-1})\omega_{j-1}^M(x) + u(t_j)\omega_j^M(x) + u(t_{j+1})\omega_{j+1}^M(x) + u(t_{j+2})\omega_{j+2}^M(x).$$

The continuous polynomial approximation $U_j^R(x)$ near the left end of the interval $[a,b]$ uses the right basis spline $\omega_j^R(x)$ of the form:

$$\omega_j^R(x) = \frac{(x-t_{j+1})(x-t_{j+2})(x-t_{j+3})}{(t_j-t_{j+1})(t_j-t_{j+2})(t_j-t_{j+3})}, \quad x \in [t_j, t_{j+1}],$$

$$\omega_{j+1}^R(x) = \frac{(x-t_j)(x-t_{j+2})(x-t_{j+3})}{(t_{j+1}-t_j)(t_{j+1}-t_{j+2})(t_{j+1}-t_{j+3})}, \quad x \in [t_j, t_{j+1}],$$

$$\omega_{j+2}^R(x) = \frac{(x-t_j)(x-t_{j+1})(x-t_{j+3})}{(t_{j+2}-t_j)(t_{j+2}-t_{j+1})(t_{j+2}-t_{j+3})}, \quad x \in [t_j, t_{j+1}],$$

$$\omega_{j+3}^R(x) = \frac{(x-t_j)(x-t_{j+1})(x-t_{j+2})}{(t_{j+3}-t_j)(t_{j+3}-t_{j+1})(t_{j+3}-t_{j+2})}, \quad x \in [t_j, t_{j+1}].$$

The approximation with these basis splines can be written in the form:

$$U_j^R(x) = u(t_j)\omega_j^R(x) + u(t_{j+1})\omega_{j+1}^R(x) + u(t_{j+2})\omega_{j+2}^R(x) + u(t_{j+3})\omega_{j+3}^R(x).$$

The continuous polynomial approximation $U_j^L(x)$ near the right end of the interval $[a,b]$ uses the left basis spline $\omega_j^L(x)$ of the form:

$$\omega_{j-2}^L(x) = \frac{(x-t_{j-1})(x-t_j)(x-t_{j+1})}{(t_{j-2}-t_{j-1})(t_{j-2}-t_j)(t_{j-2}-t_{j+1})}, \quad x \in [t_j, t_{j+1}],$$

$$\omega_{j-1}^L(x) = \frac{(x-t_{j-2})(x-t_j)(x-t_{j+1})}{(t_{j-1}-t_{j-2})(t_{j-1}-t_j)(t_{j-1}-t_{j+1})}, \quad x \in [t_j, t_{j+1}],$$

$$\omega_j^L(x) = \frac{(x-t_{j-2})(x-t_{j-1})(x-t_{j+1})}{(t_j-t_{j-2})(t_j-t_{j-1})(t_j-t_{j+1})}, \quad x \in [t_j, t_{j+1}],$$

$$\omega_{j+1}^L(x) = \frac{(x-t_{j-2})(x-t_{j-1})(x-t_j)}{(t_{j+1}-t_{j-2})(t_{j+1}-t_{j-1})(t_{j+1}-t_j)}, \quad x \in [t_j, t_{j+1}].$$

The approximation with these basis splines can be written in the form:

$$U_j^L(x) = u(t_{j-2})\omega_{j-2}^L(x) + u(t_{j-1})\omega_{j-1}^L(x) + u(t_j)\omega_j^L(x) + u(t_{j+1})\omega_{j+1}^L(x), \quad x \in [t_j, t_{j+1}].$$

Theorem 2.

Let $u \in C^4[a,b]$. $t_j = a + jh, j = 0,1,\dots,n, h = \frac{b-a}{n}, n \geq 3$. To approximate the function $u(x)$, $x \in [x_j, x_{j+1}]$, with the left and right splines, the following inequalities are valid:

$$|u(x) - U_j^L(x)| \leq Kh^4 \parallel u^{(4)} \parallel_{[t_{j-2},t_{j+1}]}, K = 1.$$

$$|u(x) - U_j^R(x)| \leq Kh^4 \parallel u^{(4)} \parallel_{[t_j,t_{j+3}]}, \ K = 1.$$

To approximate the function $u(x)$, $x \in [t_j, t_{j+1}]$, with the middle splines, the following inequality is valid:

$$|u(x) - U_j^M(x)| \leq Kh^4 \parallel u^{(4)} \parallel_{[t_{j-1},t_{j+2}]}, \quad K = 0.5625.$$

Proof. It is easy to notice that U_j^R is an interpolation polynomial, and $t_j, t_{j+1}, t_{j+2}, t_{j+3}$ are the interpolation nodes,

$$U_j^R(t_j) = u(t_j), \quad U_j^R(t_{j+1}) = u(t_{j+1}), \quad U_j^R(t_{j+2}) = u(t_{j+2}), \quad U_j^R(t_{j+3}) = u(t_{j+3}).$$

Using the remainder term we get

$$u(x) - U_j^R(x) = \frac{u^{(4)}(\xi)}{4!}(x-t_j)(x-t_{j+1})(x-t_{j+2})(x-t_{j+3}), \ \xi \in [t_j, t_{j+3}] \ .$$

We can use $x = x_j + \tau h, \tau \in [0,1]$. It can easily be calculated that

$$\max_{\tau\in[0,1]} |\tau(\tau-1)(\tau-2)(\tau-3)| = 1.$$

It follows that on the uniform grid with step h

$$\max_{x\in[t_j,t_{j+3}]} |u(x) - U_j^R(x)| \leq h^4 \max_{[t_j,t_{j+3}]} |u^{(4)}(x)|.$$

The approximation is constructed separately on each grid interval $[t_j, t_{j+1}]$. When constructing an approximation on the interval $[t_j, t_{j+1}]$ we need the values of the function at several neighboring nodes to the right or left of this interval. Therefore, if the values of the function are given on the grid of nodes, which is constructed on a finite interval $[a,b]$, then we are forced to use the approximation with the right or left splines near points a or b. When constructing an approximation with only the right cubic splines on the interval $[a,b]$, we use the values of the function at the nodes $b = t_n, t_{n+1}, t_{n+2}, t_{n+3}$. When constructing an approximation with only the left cubic splines on the interval $[a,b]$, we use the values of the function at the nodes $t_{-2}, t_{-1}, t_0 = a$. When constructing an approximation with only the right splines of the fifth order of approximation on the interval $[a,b]$, we use the values of the function at the nodes $b = t_n, t_{n+1}, t_{n+2}, t_{n+3}, t_{n+4}$. When constructing an approximation with only the left splines of the fifth order of approximation on the interval $[a, b]$, we use the values of the function at the nodes $t_{-3}, t_{-2}, t_{-1}, t_0 = a$. When constructing an approximation with only the middle splines of the fifth order of approximation on the interval $[a, b]$, we use the values of the function at the nodes $t_{-2}, t_{-1}, t_0 = a, b = t_n, t_{n+1}, t_{n+2}$.

The following Tables show the approximation errors of functions on the interval $[-1,1]$. The actual errors were calculated as follows. At each grid interval $[t_j, t_{j+1}]$, an additional grid D_j of 100 nodes x_{j_i} was constructed. Next, the approximation values of the function at these nodes were calculated. Next, the error maxima were calculated using the formula:

$$\max_{\cup D_j} |u(x_{j_i}) - U(x_{j_i})|.$$

The next tables of theoretical errors contain the maximum deviations of the exact solution from the approximate one on the interval [-1,1] based on the formulas given in the theorems:

$$\max_{x\in[-1,1]} |u(x) - U(x)|.$$

In 1901, Runge established that the interpolation process over equidistant nodes on the interval $[-1,1]$ does not converge with the increasing number of nodes even for a smooth arbitrarily differentiable function $u = \frac{1}{1+25x^2}$. Table 1 presents the actual errors in absolute values of approximation with the polynomial cubic splines $U(x)$ when $h = 0.1$. Table 2 presents the theoretical errors in absolute values of approximation with the polynomial splines of the fifth order of approximation when $h = 0.1$. Table 3 presents the actual errors in absolute values of approximation with the polynomial splines $U(x)$ using the polynomial splines of the fifth order of approximation when $h = 0.1$. Table 4 presents the theoretical errors in absolute values of approximation with the polynomial cubic splines $U(x)$ when $h = 0.1$. Analyzing the information presented in the Tables show that with the same number of interpolation nodes, the approximation using the middle splines gives a smaller error. The results of numerical experiments show that the actual errors of numerical calculations correspond to theoretical errors. Errors when using splines of the fifth order of approximation can be less than when using splines of the fourth order of approximation, if the interpolated function is sufficiently smooth.

Table 1. The results of the approximation using the cubic polynomial splines. Actual errors ($h = 0.1$).

Function $\boldsymbol{u}$	Cubic polynomial splines	
	Left splines	*Middle splines*
$1/(1 + 25x^2)$	0.01388	0.009097
$\sin(3\,x)$	0.0003098	0.0001746
$\sin(5\,x)$	0.002341	0.001327

Table 2. The results of the approximation using the polynomial splines of the fifth order of approximation. Actual errors ($h = 0.1$).

Function $\boldsymbol{u}$	Polynomial splines of the fifth order of approximation	
	Left splines	*Middle splines* $(\boldsymbol{w_j^S})$
$1/(1 + 25x^2)$	0.03372	0.01244
$\sin(3\,x)$	0.00007233	0.00002840
$\sin(5\,x)$	0.0009116	0.0003579

Table 3. The results of the approximation using the cubic polynomial splines. Theoretical errors ($\boldsymbol{h = 0.1}$).

Function $\boldsymbol{u}$	Cubic polynomial splines	
	Left splines	*Middle splines*
$1/(1 + 25x^2)$	0.0625	0.03516
$\sin(3\,x)$	0.0003375	0.0001898
$\sin(5\,x)$	0.002604	0.001465

Table 4. The results of the approximation using the polynomial splines of the fifth order of approximation. Theoretical errors ($h = 0.1$).

Function $\boldsymbol{u}$	Polynomial splines of the fifth order of approximation	
	Left splines	*Middle splines* $\boldsymbol{w_j^S}$
$1/(1 + 25x^2)$	0.09496	0.03715
$\sin(3\,x)$	0.00007351	0.00002876
$\sin(5\,x)$	0.0009453	0.0003698

Note that if the derivatives of the solution grow rapidly, then the approximation by cubic splines may turn out to be more profitable than the approximation by splines of the fifth order of approximation Let the Runge function be given at the nodes of a uniform grid with a step of $h = 0.1$ on the interval $[-1, 1]$. The approximation error of the approximation of the Runge function obtained with the cubic polynomial splines is given in Fig.5. The maximum of the error in absolute error is 0.009097. The approximation error in absolute value (of the approximation of the Runge function, $h = 0.1$) obtained with the right polynomial splines of the fifth order of approximation is given in Fig.6.

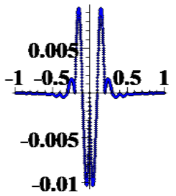

Fig. 5: The approximation error obtained with the middle cubic polynomial splines

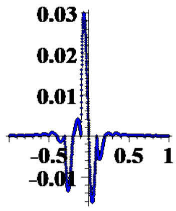

Fig. 6: The approximation error obtained with the right cubic polynomial splines

The approximation error in absolute value (of the approximation of the Runge function, $h = 0.1$) obtained with the middle polynomial splines of the fifth order of approximation (with w_j^S) is given in Fig.7 (the maximum of the error in absolute error is 0.012438).

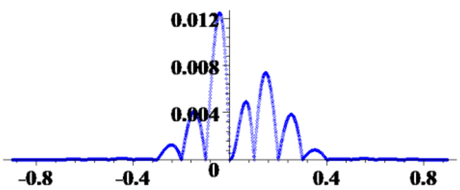

Fig. 7: The approximation error (in absolute value) obtained with the middle polynomial splines (w_j^S)of the fifth order of approximation

Fig. 7 confirms the theoretical estimate (Theorem 2) that the approximation with the middle splines give lesser error than the approximation with the left or right splines.

3 Applying splines to the Solution of the Fredholm Equation of the Second Kind

Consider the Fredholm equation of the second kind

$$\varphi(x) - \int_a^b K(x,s)\,\varphi(s)\,ds = f(x). \qquad (1)$$

We construct an approximate solution of the integral equation by applying the polynomial splines of the fourth order of approximation as follows. Let $\{t_i\}$ be a grid of nodes on the interval $[a,b]$. Divide the interval $[a,b]$ into n parts, $n \geq 4$.

$$\int_a^b K(x,t)\,\varphi(t)\,dt = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} K(x,t)\,\varphi(t)dt.$$

Let $s + r = 4, s \geq 1, r \geq 1$.On each grid interval $[t_i, t_{i+1}]$, we apply a formula of the form:

$$\varphi(t) = \sum_{j=i-s}^{i+r} C_j w_j(t),\ t \in [t_i, t_{i+1}],$$
$$i = 0, \ldots, n-1.$$

Here $w_j(t)$ are the basis splines that are discussed above, and C_j are unknowns (values of the solution of the equation at grid points, $C_i \approx \varphi(t_i)$) to be found. Now we transform the following expression

$$\int_{t_i}^{t_{i+1}} K(x,t)\,\varphi(t)dt$$

$$\approx \int_{t_i}^{t_{i+1}} K(x,t) \sum_{j=i-s}^{i+r} C_j w_j(t)dt$$

$$= \sum_{j=i-s}^{i+r} C_j \int_{t_i}^{t_{i+1}} K(x,t)\ w_j(t)dt.$$

At the beginning of the interval $[a,b]$ at $i = 0,1,2,$ the values of the parameters $s = 0,\ r = 4$ should be taken. At the end of the interval $[a,b]$ at $i = n-1, n-2,$ the values of the parameters $s = i-3, r = 1$ should be taken.

Denote by $\alpha_{ij}(x)$ the integral

$$\alpha_{ij}(x) = \int_{t_i}^{t_{i+1}} K(x,t)\ w_j(t)dt.$$

Now we take $= t_i$. The problem of solving the integral equation is reduced to solving the system of linear algebraic equations. When $i = 0,1$ we have the equations

$$C_i - \sum_{i=0}^{1} \sum_{j=i}^{i+4} C_j \alpha_{ij}(t_i) - \sum_{i=2}^{n-2} \sum_{j=i-2}^{i+2} C_j \alpha_{ij}(t_i)$$
$$- \sum_{i=n-1}^{n-1} \sum_{j=i-3}^{i+1} C_j \alpha_{ij}(t_i) = f(t_i),$$
$$i = 0, 1, 2, \ldots, n.$$

When $i = 3,4, \ldots, n-3$ we have the equations

$$C_i - \sum_{i=0}^{2} \sum_{j=i}^{i+4} C_i \alpha_{ij}(t_i) - \sum_{i=3}^{n-3} \sum_{j=i-2}^{i+2} C_i \alpha_{ij}(t_i)$$
$$- \sum_{i=n-2}^{n-1} \sum_{j=i-3}^{i+1} C_i \alpha_{ij}(t_i) = f(t_i),$$
$$i = 3,4, \ldots, n-3,$$

When $i = n-2, n-1$ we have the equations

$$C_i - \sum_{i=0}^{2} \sum_{j=i}^{i+4} C_i \alpha_{ij}(t_i) - \sum_{i=3}^{n-3} \sum_{j=i-2}^{i+2} C_i \alpha_{ij}(t_i)$$

$$-\sum_{i=n-2}^{n-1}\sum_{j=i-3}^{i+1} C_i\alpha_{ij}(t_i) = f(t_i)\,,$$
$$i = n-2, n-1\,.$$

The unknowns in the system of equations are C_i, $i = 0,1,\dots,n-2,n-1$.

4 Results of the Numerical Experiments

In this section, we present the results of the numerical experiments.

Problem 1. Now we take the next Fredholm integral equation:

$$u(x) + \int_0^1 \exp(x+t)\,u(t)dt$$
$$= exp(-x) + exp(x), x \in [0,1].$$

The exact solution of the integral equation is $u(x) = exp(-x)$. Figs. 8, 9 show the errors of the solution of Problem 1 with polynomial splines of the fifth order of approximation when $n = 6$, 16, $Digits = 20$. Figs. 10, 11 show the errors of the solution of Problem 1 with cubic polynomial splines when $n = 16$, 32 ($Digits = 20$). In the Figures, grid nodes are marked along the X axis at the interval $[0,1]$.

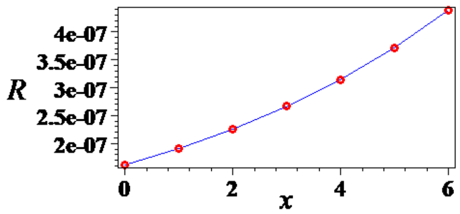

Fig. 8: The plot of the errors of the solution of Problem 1 with polynomial splines of the fifth order of approximation, n = 6

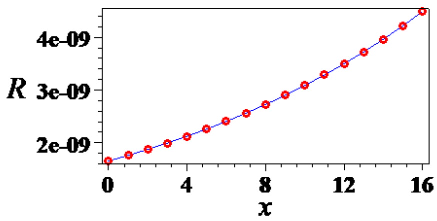

Fig. 9: The plot of the errors of the solution of Problem 1 with polynomial splines of the fifth order of approximation, n = 16

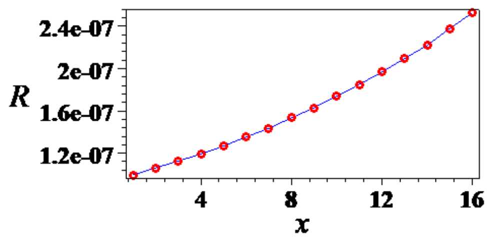

Fig. 10: The plot of the errors of the solution of Problem 1 with polynomial cubic splines, n = 16

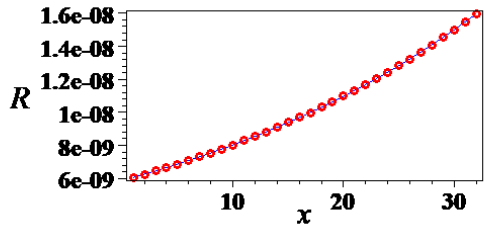

Fig. 11: The plot of the errors of the solution of Problem 1 with polynomial cubic splines, n = 32

Now we present the result of applying Simpson's rule to solving Problem 1. Figure 12 shows the graph of the error of problem 1 when the Simpson's method was used.

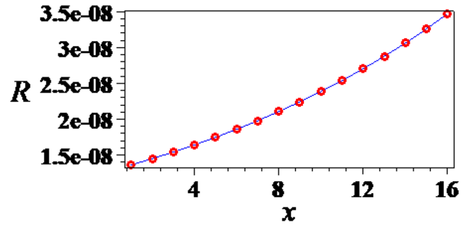

Fig. 12: The graph of the error of problem 1 when the Simpson's method was used.

The maximum of the error in absolute value is about $0.3471 \cdot 10^{-7}$ These results show that the use of splines of the fifth order of approximation contributes to a significant reduction in the number of grid nodes. However, it must be remembered that the use of splines of the fifth order of approximation assumes that the solution of the equation and the kernel are five times differentiable functions.

Problem 2. Let the right side of the system of equations be constructed under the assumption that the solution of the integral equation is $u = 1/(1+25x^2)$. We leave the kernel of the equation the same. Figure 13 shows the graph of the error of Problem 2 ($n = 64$).

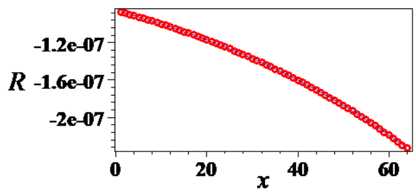

Fig. 13: The plot of the errors of the solution when the polynomial cubic splines were used and $u = 1/(1+25\,x^2)$, n = 64

At the same time, when using splines of the fifth order of approximation, we obtain the error of the solution, the graph of which is shown in Fig.14, $n = 64$.

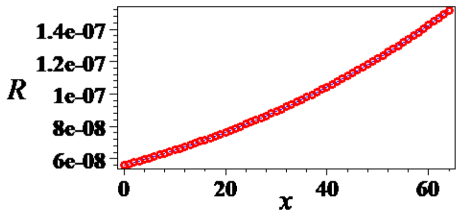

Fig. 14: The plot of the errors of the solution when the polynomial splines of the fifth order of approximation were used and $u = 1/(1 + 25\,x^2)$, $n = 64$

The errors presented in the last two graphs show that if the derivatives of the solution grow rapidly, then it is enough to use traditional methods for solving integral equations. In this case, it is often advisable to use an uneven grid of nodes

Problem 3. Now, we again take the Fredholm integral equation from paper [10]:

$$u(x) - \int_0^1 x^2 \exp(t(x-1))\, u(t)dt = x + (1-x)\, exp(x), x \in [0,1].$$

The exact solution of the integral equation is $u(x) = exp(x)$. Figs. 15,16 show the errors of the solution of Problem 1 with polynomial splines of the fifth order of approximation, $n = 8, n = 5$.

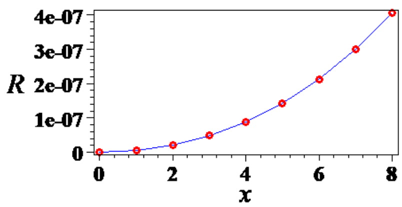

Fig. 15: The plot of the errors of the solution of Problem 1 with polynomial splines of the fifth order of approximation, $n = 8$

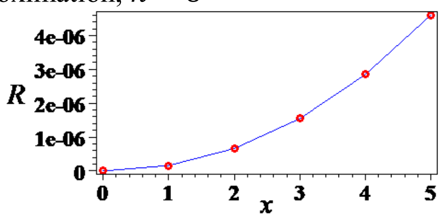

Fig. 16: The plot of the errors of the solution of Problem 1 with polynomial splines of the fifth order of approximation, $n = 5$

Now consider the solution of the following problem.

Problem 3.

$$u(x) + \int_0^1 \exp(-x\,t)\, u(t)dt = F(x), \quad x \in [0,1],$$

where the right side of $F(x)$ is constructed according to the known solution $u = \sin(3x)\sin(x-1)$.

First, we apply the calculation scheme constructed using the cubic polynomial splines. Using the set of nodes $\{x_j\}$ with $n = 16$ we obtain the approximate solution u_j in the nodes (see Fig.18). Fig.17 shows the error of the solution obtained in the nodes. It can be calculated that the following relation is valid:

$$\max_{\{x_j\}} |u(x_j) - u_j| = 0.8753 \cdot 10^{-5}.$$

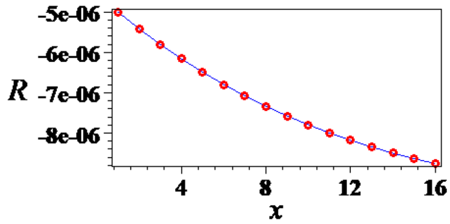

Fig. 17: The plot of the error of the approximate solution of Problem 3 ($n = 16$).

Then we apply the numerical method with the splines of the fifth order of approximation. Fig.19 shows the error of the solution of Problem 3 when $n = 16$. We have $\max_{\{x_j\}} |u(x_j) - u_j| = 0.4416 \cdot 10^{-5}$.

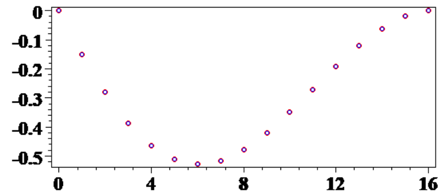

Fig. 18: The plot of the approximate solution of Problem 3

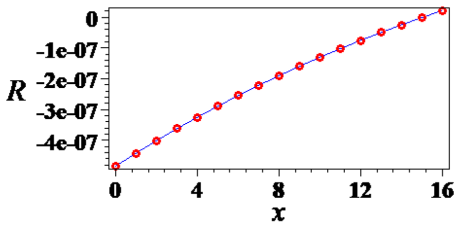

Fig. 19: The plot of the error of the solution of Problem 3 when splines of the fifth order of approximation were used ($n = 16$).

Problem 4.

As is known, in the internal Dirichlet problem of potential theory, it is required to find a function $u(x, y)$ that is harmonic in domain D and takes given values on the boundary γ of this domain D (for example, see Kollatz [13]). Let the boundary functions be given by the equations

$$x = \xi(t), y = \eta(t).$$

The solution of the internal Dirichlet problem can be written as:

$$u(x,y) = \int_\gamma \mu(t)\frac{d\theta}{dt}dt,$$

where the angle theta is calculated by the formula

$$\theta = arctg\frac{\eta(t) - y}{\xi(t) - x}$$

and function $\mu(s)$ satisfies the Fredholm integral equation of the second kind.

$$\pi\mu(s) + \int_0^{2\pi} K(s,t)\ \mu(t)dt = g(s),$$

where

$$K(s,t) = \frac{\partial}{\partial t}arctg\frac{\eta(t) - \eta(s)}{\xi(t) - \xi(s)}.$$

If area D is an ellipse: $x = a\cos t,\ y = b\cos t$ with semi-axes $a = 2, b = 1$, then the kernel of the integral equation can be written as:

$$K(s,t) = \frac{ab}{a^2 + b^2 - (a^2 - b^2)\cos(a+b)}.$$

Let us find the function $\mu(s)$. To do this, we solve the integral equation

$$\pi\mu(s) + \int_0^{2\pi}\frac{2\mu(t)dt}{5-3\cos(s+t)} = \begin{cases}\sin(s), & 0 \le s \le \pi,\\ 0, & -\pi \le s \le 0\end{cases}.$$

Let us choose $n = 8$. Having solved the integral equation, we obtain the following values:
μ_0=-0.04290; μ_1:= -0.1066; μ_2:=-0.1360;
μ_3:=-0.1045; μ_4:=-0.04017; μ_5:= 0.2330;
μ_6:=0.3427; μ_7:=0.2322; μ_8:=-0.04290.

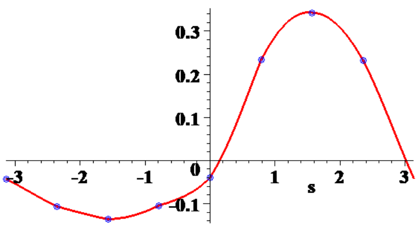

Fig. 20: The plot of the solution $\mu(s)$ $(n = 8)$

Then we choose $n = 16$. Further, the obtained solution can be represented using splines of the fifth order of approximation in the following form. The graph of the solution is shown in Fig. 20. In Figures 20, 21, the values of s are plotted along the x-axis, and the calculated values $\mu(s_j)$ at the points s_j are connected using splines of the fifth order of approximation.

Now we can calculate the temperature on the axis of the cylinder: $u(0,0)$. We can use Simpson's method and the trapezium method. After the calculations, we get $u(0,0)$≈0.43.

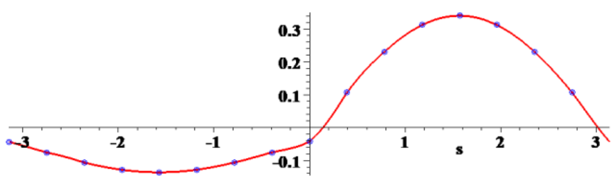

Fig. 21: The plot of the solution $\mu(s)$ $(n = 16)$

5 Conclusion

In this paper, we study the numerical solution of the Fredholm integral equation of the second kind using polynomial splines of the fifth order of approximation. Here, a comparison is also made with the results of applying cubic splines to the solution of the Fredholm equation. The paper also gives approximation theorems for polynomial cubic splines of the fourth order of approximation and polynomial splines of the fifth order of approximation. As is known, theorems on the solution of an integral equation follow from approximation theorems. However, when using cubic splines and splines of the fourth order of approximation, assumptions are required about the sufficient smoothness of the kernel of the integral equation, the solution of the integral equation and its right side. If the smoothness is insufficient, then the desired error reduction cannot be achieved. In this case, it is preferable to use splines of the second or third order of approximation or the classical methods of trapezoids or middle rectangles.

Thus, we summarize the results obtained. If the kernel of the integral equation is represented by a delta function, then we almost immediately obtain a solution. If the kernel of the integral equation and the solution of the integral equation have sufficient smoothness, then the method proposed in this paper will give a good result when we use a small number of grid nodes.

In the author's next studies, new numerical methods for solving the nonlinear Volterra and Fredholm equations using spline approximations will be considered.

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Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself
The author is highly and gratefully indebted to St. Petersburg University for financial supporting the publication of the paper (Pure ID 93852135, 92424538)