PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 46, Number 1, October 1974

FREE ACTIONS OF CYCLIC GROUPS OF ORDER 2^n ON $S^1 \times S^2$

GERHARD X. RITTER

ABSTRACT. In [4] Y. Tao proved that if h is a fixed point free involution of $S^1 \times S^2$, then $(S^1 \times S^2)/h$ must be homeomorphic to either $M_1 = S^1 \times S^2$, or $M_2 = \mathbf{K}^3$, or $M_3 = S^1 \times \mathbf{P}^2$ or $M_4 = \mathbf{P}^3 \# \mathbf{P}^3$. In this paper we extend this result to free actions of Z_{2n} on $S^1 \times S^2$, showing that, for $n > 1$, $(S^1 \times S^2)/Z_{2n}$ must be homeomorphic to either M_1 or M_2.

1. **Introduction.** A long outstanding problem in topology is the characterization of the manifold M/G, where M is a given compact 3-manifold and G is a finite group acting freely on M. For example, if $M = S^3$, then M/G has only been classified for $G = Z_2$ [1], Z_4 [2], and Z_8 [3]. There is little known if M is a compact manifold other than S^3. In fact, the only other characterization appearing in the literature was given by Tao [4]. Tao proved that if Z_2 acts freely on $S^1 \times S^2$ then $S^1 \times S^2/Z_2$ must either be $S^1 \times S^2$, or $S^1 \times \mathbf{P}^2$, or $\mathbf{K}^3$, or $\mathbf{P}^3 \# \mathbf{P}^3$, the connected sum of two projective spaces. In this paper we show that Tao's results extend without great difficulty to free actions of Z_{2n} on $S^1 \times S^2$.

2. **Notation and preliminary lemmas.** The interior of a topological manifold M will be denoted by $\operatorname{int} M$ and the boundary by ∂M. The n-dimensional sphere, Klein bottle and projective space will be denoted by S^n, $\mathbf{K}^n$ and $\mathbf{P}^n$, respectively.

We shall view $S^1 \times S^2$ as obtained from $[0, 1] \times S^2$ by identifying $0 \times S^2$ with $1 \times S^2$. The next two lemmas are proven in [4].

Lemma 1. *Let D_1, D_2, D_3 be three discs in $S^1 \times S^2$ such that $D_1 \cap D_2 = D_1 \cap D_3 = D_2 \cap D_3 = \partial D_i$, $i = 1, 2$, or 3. If any two of the 2-spheres $S_1 = D_1 \cup D_2$, $S_2 = D_1 \cup D_3$ and $S_3 = D_2 \cup D_3$ separate $S^1 \times S^2$, then the other one also separates $S^1 \times S^2$.*

Received by the editors November 14, 1972 and, in revised form, October 1, 1973.

AMS (MOS) subject classifications (1970). Primary 57A10, 57E25; Secondary 55C35, 54B15.

Key words and phrases. Free actions, piecewise linear, polyhedral, isotopic, manifold.

Lemma 2. *Let S be a 2-sphere in $S^1 \times S^2$ such that $S \cap (0 \times S^2) = \emptyset$, S is a polyhedron in some triangulation of $S^1 \times S^2$ and does not separate $S^1 \times S^2$. Then S is isotopic to $0 \times S^2$ in $S^1 \times S^2$.*

Since we may assume [3] that $S^1 \times S^2$ has a fixed triangulation and that Z_{2n} acts piecewise linearly on this triangulation, all objects in this paper shall henceforth be considered from the polyhedral point of view.

Lemma 3. *Suppose Z_{2n} acts freely on $S^1 \times S^2$, $h \in Z_{2n}$ a generator and $n \geq 2$. Then there is a 2-sphere S in $S^1 \times S^2$ which is isotopic to $0 \times S^2$ and such that $\bigcup h^i S$ is a disjoint collection of 2^n 2-spheres.*

Proof. The proof is inductive. If $n = 2$, then h^2 is a free action of Z_2. By Lemma 3 of [4] we may assume that for the 2-sphere $S = 0 \times S^2$ either $S \cap h^2 S = \emptyset$ or $h^2 S = S$.

We suppose that $S \cap h^2 S = \emptyset$ and $S \cap h^i S \neq \emptyset$ for some integer i with $1 \leq i < 2^n$. Since $\bigcup h^{2i} S$ remains invariant under h^2, we may assume by Proposition 2.2 of [3] that $\mathcal{T} = (\bigcup h^{2i} S) \cap (\bigcup h^{2i+1} S)$ consists of a finite number of simple closed curves. No component of $\mathcal{T}$ remains invariant under h. Hence, for some odd positive integer m (either 1 or 3), there is a simple closed curve J in $S \cap h^m S$ which is innermost on $h^m S$. That is, J bounds a disc D on $h^m S$ such that $D \cap (\bigcup h^{2i} S) = \partial D = J$.

The curve J divides S into two discs D_1 and D_2. By Lemma 1, one of the 2-spheres $S_1 = D \cup D_1$, $S_2 = D \cup D_2$ does not separate $S^1 \times S^2$. We suppose, without loss of generality, that S_1 does not separate $S^1 \times S^2$. Let J_1 be a simple closed curve on D_1, sufficiently close to J, such that the annulus A_1 on D_1, bounded by J and J_1, has the property that $(\bigcup h^{2i+1} S) \cap \operatorname{int} A_1 = \emptyset$. Since h is fixed point free and $\mathcal{T} \cap D = \partial D$, we may choose a disc D', with boundary J_1, sufficiently close to D such that $(\bigcup h^i S) \cap D' = \partial D'$, $D' \cap h^i D' = \emptyset$ if $1 \leq i < 2^n$, and $S' = D' \cup (D_1 - A_1)$ does not separate $S^1 \times S^2$.

By construction, $(\bigcup h^{2i} S') \cap (\bigcup h^{2i+1} S')$ is a strict subset of $\mathcal{T}$. Since the number of components of $\mathcal{T}$ is finite, it follows that by repeating the above procedure a finite number of times, we can construct a 2-sphere S'' in $S^1 \times S^2$ such that $\bigcup h^i S''$ is a disjoint union of 2-spheres. Furthermore, if $S'' \cap S \neq \emptyset$, then, by our construction of S'', we may use a small deformation of S'' such that $S \cap S'' = \emptyset$ and all other required properties for S'' remain unchanged. By Lemma 2, S'' is isotopic to $0 \times S^2$.

If $h^2 S = S$, then $S \cap (\bigcup h^{2i+1} S) \neq \emptyset$. For otherwise, if x denotes a generator of $H_2(S^1 \times S^2)$, then for the homomorphism h_* induced by h we have $h_*(x) = \pm x$ and $h_*^2(x) = x$. Since $h^2 S = S$ and x is carried by S, the degree of $(h^2|S)\colon S \to S$ is one. Hence h^2 has a fixed point. But this is impossible since h is free.

As before, there must be an integer $m = 1$ or $m = 3$ and a simple closed curve J in $S \cap h^m S$ which is innermost on $h^m S$. If D_1 and D_2 denote the two discs on S with boundary J, then $h^2 D_1 \subset D_2$. Hence, constructing S'' as above, $\bigcup h^i S''$ is a disjoint union of four 2-spheres.

We now proceed by induction, assuming the result to be valid for $n - 1$ with $n \geq 3$. Since h^2 is a free action of $Z_{2^{n-1}}$ we may assume that for the 2-sphere $0 \times S^2$, $\bigcup h^{2i} S$ is a disjoint union of 2^{n-1} 2-spheres. We further assume that $S \cap h^i S \neq \emptyset$ for some integer i with $1 \leq i < 2^n$ and that $\mathcal{T} = (\bigcup h^{2i} S) \cap (\bigcup h^{2i+1} S)$ consists of a finite number of disjoint simple closed curves. Again, there must be an odd positive integer m and a simple closed curve J in $S \cap h^m S$ which is innermost on $h^m S$. Thus, J bounds a disc D on $h^m S$ with $D \cap (\bigcup h^{2i} S) = \partial D = J$ and J divides S into two discs D_1 and D_2. Using these discs we may now adjust S to a 2-sphere S' isotopic to $0 \times S^2$ such that $(\bigcup h^{2i} S') \cap (\bigcup h^{2i+1} S')$ is a strict subset of $\mathcal{T}$ by employing exactly the same argument as given for the case $n = 2$. If $\bigcup h^i S'$ is not a disjoint collection we may repeat the entire argument a finite number of times until the desired result is obtained. This proves Lemma 3.

3. Classifying $S^1 \times S^2/Z_{2^n}$. Let $M_1 = S^1 \times S^2$, $M_2 = \mathbf{K}^3$, $M_3 = S^1 \times \mathbf{P}^2$ and $M_4 = \mathbf{P}^3 \# \mathbf{P}^3$.

Theorem 1. *If Z_{2^n} acts freely on $S^1 \times S^2$ then $S^1 \times S^2/Z_{2^n}$ is homeomorphic to M_i for some $i = 1, 2, 3$ or 4. Furthermore, if $n > 1$ then $(S^1 \times S^2)/Z_{2^n}$ is homeomorphic to either M_1 or M_2.*

Proof. Let $h \in Z_{2^n}$ be a generator. For $n = 1$, this is Theorem 1 of [4]. We suppose that $n > 1$ and that S is a 2-sphere in $S^1 \times S^2$ satisfying Lemma 3. The collection $\bigcup h^i S$ divides $S^1 \times S^2$ into 2^n components $A_1, \cdots, A_{2^n}$, each homeomorphic to $[0, 1] \times S^2$. Since h permutes the components of $\bigcup h^i S$ and $n > 1$, h permutes the spherical shells A_i. Thus, $(S^1 \times S^2)/h$ is obtained from a spherical shell by identifying the boundaries with the homeomorphism h. It follows that $(S^1 \times S^2)/Z_{2^n}$ is homeomorphic to either M_1 or M_2, depending on whether h preserves or reverses orientation.

BIBLIOGRAPHY

1. G. R. Livesay, *Fixed point free involutions on the 3-sphere*, Ann. of Math. (2) 72 (1960), 603–611. MR 22 #7131.

2. P. M. Rice, *Free actions of* Z_4 *on* S^3, Duke Math. J. **36** (1969), 749–751. MR **40** #2064.

3. G. X. Ritter, *Free actions of* Z_8 *on* S^3, Trans. Amer. Math. Soc. **181** (1973), 195–212.

4. Y. Tao, *On fixed point free involutions of* $S^1 \times S^2$, Osaka Math. J. **14** (1962), 145–152. MR **25** #3515.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32601