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FREE AREA ESTIMATION IN A PARTIALLY OBSERVED DYNAMIC GERM-GRAIN MODEL[1]

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The estimation problem of the expected local fraction of free area function S for a partially observed dynamic germ-grain model is presented. Properties of the estimators are proved by martingale and product integral methods. Confidence bounds are provided. Furthermore, an estimator of the hazard rate $\alpha(t) = -dS(t)/(S(t)dt)$ is obtained by the kernel function method and asymptotic properties of the estimator are proved and used to find confidence intervals. By a simulated illustrative example, the qualitative behavior of the estimators is shown.

Key words: Dynamic Germ-Grain Model, Fraction of Free Area, Survival Function, Martingale, Product Integral, Kernel Function, Consistency, Asymptotic Normality, Confidence Interval and Bound.

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1. Introduction

In this paper we study the estimation problem for the expected *local fraction of free area* (LFFA) function of a partially observed *dynamic germ-grain model* (DGGM) as defined below.

We start with a model and problem outline. Suppose disks of random area are dropped at random times on the plane $\mathbb{R}^2$ in such a way that their centers are random points of a convex Borel region $\mathbf{C} \subset \mathbb{R}^2$. The time-rate of the dropping-times process at time $t > 0$ is assumed to be of the form $\lambda(t) \cdot \ell(\mathbf{C})$, where λ is a time-dependent parameter and $\ell(\,\cdot\,)$ is the 2-dimensional Lebesgue measure.

For any $t > 0$, $\Theta(t)$ denotes the random set obtained as the union of the disks at time t. A random set such as $\Theta(t)$ is called a germ-grain model (see, for example, [14]). Because of its

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evolution in time, we call the family $\Theta = \{\Theta(t); t > 0\}$ DGGM (see Figure 1, parts A1 and A2).

For a fixed convex Borel set $\mathbf{B} \subset \mathbf{C}$, the LFFA process in $\mathbf{B}$ is the stochastic process $\Delta = \{\Delta(t); t > 0\}$ defined by

$$\Delta(t): = \frac{\ell(\mathbf{B}\backslash\Theta(t))}{\ell(\mathbf{B}}, \; t > 0,$$

where $\mathbf{B}\backslash\Theta(t) = \mathbf{B} \cap \Theta(t)^c$ is the set difference.

Suppose an observer without control of the generation of the DGGM Θ is interested in evaluating the realization of the LFFA process Δ. If the falling times, areas, and center positions of all disks are known at any time, then $\Delta(t)$ is (more or less easily) computable for any $t > 0$.

Now imagine that the region $\mathbf{C}$ is, unfortunately, obscured by a shadowing element so that a survey of the positions of the fallen disks covering $\mathbf{B}$ is not possible. In this case, Δ is no longer computable and thus, an estimation problem naturally arises for the expected LFFA function $S = \{S(t); t > 0\}$ defined by

$$S(t): = E[\Delta(t)], \; t > 0,$$

where E denotes expectation.

We prove (see Proposition 1) that

$$S(t) = e^{-A(t)} = \prod_{s \le t}[1 - dA(s)],$$

where $A(t) = \int_0^t \alpha(s)ds, \; \alpha(s)ds, \; \alpha(s) = \mu(S)\lambda(s), \; \mu(s) = E[V(s)], \; V(s)$ is the random area of a disk dropped at time $s > 0$, and $\prod$ denotes a product integral. (For a survey of product integration, see [1, Sections II.6-II.8] or [9].)

Proposition 1 suggests $S(t)$ may be estimated by

$$\widehat{S}(t) = \prod_{s \le t}[1 - d\widehat{A}(s)],$$

where $\widehat{A}$ denotes the unitary extended area (see (1) below) and is used to estimate A.

We prove properties of the estimators $\widehat{A}$ and $\widehat{S}$ by extensive use of the martingale and product integral theories. Properties of $\widehat{S}$ are proved using properties of $\widehat{A}$.

Note that we suppose the observer is able to record the random areas and times of the falling disks. To use our method, these areas and times must be recorded by the observer as necessary data.

We provide the following example of an application of our method.

Example: (Bombing Problem) Suppose a bombing activity is taking place on a region $\mathbf{C} \subset \mathbb{R}^2$. Bombs of random destructive power are dropped at random times on $\mathbf{C}$. Each bomb will strike a random point in $\mathbf{C}$ and destroy a circular region with its center at the struck point and its area proportional to the bomb's destructive power.

An observer would like to know the fraction of non-destroyed area for a Borel set $\mathbf{B} \subset \mathbf{C}$ that is the realization of Δ. He is able to record the landing times and the destructive power of each bomb. Because of the presence of clouds obscuring $\mathbf{C}$, he cannot observe the point struck by each bomb. So an estimate of S is required. $\square$

Note that for a fixed point $0 \in B$,

$$S(t) = P[0 \notin \Theta(t)], t > 0.$$

(See the proof of Proposition 1.) Thus, if T_0 denotes the time at which the DGGM Θ hits the point 0 for the first time, then

$$S(t) = P[T_0 > t],\ t > 0,$$

where S may be interpreted as a survival function of the point 0.

By analogy with standard results of survival analysis, we observe that

$$\alpha(t) = -\frac{1}{S(t)}\frac{dS(t)}{dt} = \lim_{h \to 0} \frac{1}{h} P[0 \in \Theta(t+h) \mid 0 \notin \Theta(t)],\ t > 0.$$

So α, a hazard rate function of the point 0, is worth estimating in its own right.

In this paper, an estimator $\widehat{\alpha}$ of α is defined as a kernel-function smoothing of $\widehat{A}$ and its asymptotic properties are proved.

An outline of the rest of the paper is as follows. In Section 2, the problem and mathematical model are described in further detail. In Section 3, the statistical problem is described, the estimators $\widehat{A}$ and $\widehat{S}$ are defined, and their preliminary properties are proved. Section 4 contains the asymptotic results for $\widehat{A}$ and $\widehat{S}$. In Section 5, the results of Section 4 are used to find confidence intervals and bounds for A and S. Section 6 is devoted to the estimation of α. Finally, in Section 7, results of numerical simulations are provided.

2. The Model, Notation and Preliminary Results

Let us assign

(i) a convex Borel set $\mathbf{C} \subset \mathbb{R}^2$;

(ii) a Poisson process $N = \{N(t): t > 0\}$ on $(0, +\infty)$ with mean measure m given by

$$m((s,t]) := \ell(\mathbf{C}) \cdot \int_s^t \lambda(u)du,\quad 0 < s \leq t,$$

where $\ell(\cdot)$ is the Lebesgue measure on $\mathbb{R}^2$, and λ is a continuous function bounded below by 0;

(iii) a sequence $X_1, \ldots, X_n, \ldots$ of i.i.d. random variables (r.v.'s) uniformly distributed on $\mathbf{C}$;

(iv) a family $\{V(s): s > 0\}$ of independent $(0, +\infty)$-valued r.v.'s independent of the X_i's and the process N.

Let $\{T_i: i \geq 1\}$ be the sequence of the jump times of the Poisson process N

$$T_i := \inf\{t > 0: N(t) = i\}.$$

An any random time T_i, a closed disk D_i with random center X_i in $\mathbf{C}$ and random area $V_i := V(T_i)$ is dropped on $\mathbb{R}^2$. Note that conditional on $T_i = t$, the random area V_i is distributed as $V(t)$.

For any $t > 0$, we denote by $\Theta(t) = \Theta(t, \omega)$, $\omega \in \Omega$ the random closed set composed of the union of the random disks dropped up to time t,

$$\Theta(t) := \bigcup_{T_i \leq t} D_i.$$

Because of its evolution time, we call the family $\Theta = \{\Theta(t); t > 0\}$ a DGGM.

Formally, all random variables considered in this paper are defined on the same probability space $(\Omega, \mathcal{G}, P)$. The history of the process will be represented by a filtration $\mathcal{F} = \{\mathcal{F}_t; t > 0\}$, where for any $t > 0$, $\mathcal{F}_t$ is the σ-field generated by all the events which have occurred up to time t. That is,

$$\mathcal{F}_t := \sigma\{T_i \leq s, X_i \in A, V_i \in B \mid 0 < s \leq t, A \in \mathcal{B}(\mathbb{R}^2), B \in \mathcal{B}((0, +\infty))\},$$

where $\mathcal{B}(\mathbb{R}^2)$ and $\mathcal{B}((0, +\infty))$ are the Borel σ-fields on $\mathbb{R}^2$ and $(0, +\infty)$, respectively.

We suppose that the following assumption holds for the distributions of the areas.

Assumption A1: For any $t > 0$, $V(t)$ is an absolutely continuous random variable with density $f_V(t,x)$. Furthermore, $0 < h < k$ exist such that

$$supp(f_V(t,x)) := \{x \in \mathbb{R}; f_V(t,x) \neq 0\} \subset (h,k).$$

As a function of t, $f_V(t,x)$ is x-uniformly continuous, that is, for any $\epsilon > 0$, $\delta > 0$ exists such that

$$\mid s - t \mid \, < \delta \;\Rightarrow\; \sup_{x \in (h,k)} \mid f_V(s,x) - f_V(t,x) \mid \, < \epsilon. \qquad \square$$

Remark 1: From Assumption A1, it follows that

$$P(h < V(t) < k) = 1, \text{ for any } t > 0,$$

and that the moments

$$\mu^{(j)}(t) := E[(V(t))^j], \; j \geq 1,$$

are all continuous functions in t. □

From now on, we will suppose a convex Borel set **B** ⊂ **C** is fixed and we will denote by $S = \{S(t); t > 0\}$ the expected LFFA function on **B**, i.e,

$$S(t) := E\left[\frac{\ell(\mathbf{B} \setminus \Theta(t))}{\ell(\mathbf{B})}\right].$$

Furthermore, $\mu(t) := E[V(t)]$ will denote the expected area of a disk dropped at time $t > 0$, and

$$\alpha(t) := \mu(t)\lambda(t).$$

Remark 2: In the sequel, we will study asymptotic properties of estimators for $\mathbf{C} \to \mathbb{R}^2$. So, denoting by $\| \cdot \|_2$ the Euclidean norm, we may assume that

$$\mathbf{B}_{\oplus k} := \{x \in \mathbb{R}^2 : \inf_{y \in \mathbf{B}} \| x - y \|_2 \leq k\} \subset \mathbf{C}.$$

Since, as observed in Remark 1, the areas of the disks are a.s. bounded, there are no edge effects to consider if, as in our case, we are interested only in the region **B**. □

Proposition 1: *The expected LFFA function in* **B** *is given by*

$$S(t) = e^{-A(t)}, t \geq 0$$

where A denotes the deterministic function defined by

$$A(t) := \int_0^t \alpha(s)ds, t > 0.$$

Proof: Note that, using Fubini's Theorem

$$S(t) = \int_\Omega \frac{\ell(\mathbf{B} \backslash \Theta(t,\omega))}{\ell(\mathbf{B})} dP(\omega) = \frac{1}{\ell(\mathbf{B})} \int_\Omega \int_\mathbf{B} I[x \notin \Theta(t,\omega)] d\ell(x) dP(\omega)$$

$$= \frac{1}{\ell(\mathbf{B})} \int_\mathbf{B} P[x \notin \Theta(t)] d\ell(x)$$

where

$$I[x \notin A] = \begin{cases} 0, & x \in A \\ 1, & x \notin A \end{cases},$$

is an indicator function. Given the uniformity of the disk centers, $P[x \notin \Theta(t)]$ does not depend on x, so

$$S(t) = P[0 \notin \Theta(t)],$$

for a fixed point $0 \in \mathbf{B}$.

If, at a time $t > 0$, a disk $D(t)$ drops on $\mathbb{R}^2$ with random center X uniformly distributed on $\mathbf{C}$ and random area $V(t)$ independent of X, then it will cover the point 0 with probability

$$p(t) := P[0 \in D(t)] = P[X \in B(0, \sqrt{V(t)/\pi})]$$

$$= \int_h^k PX \in B(0, \sqrt{v/\pi})] f_{V(t)}(v) dv = \frac{1}{\ell(\mathbf{C})} \int_h^k v f_{V(t)}(v) dv = \frac{\mu(t)}{\ell(\mathbf{C})}.$$

The process $N_0 = \{N_0(t) : t > 0\}$ counting, for any $t > 0$, the number of disks placed in the time interval $(0, t]$ and covering the point 0, is a p-thinning of the Poisson process $N(t)$. Given the continuity of $\mu(t)$ (see Remark 1), N_0 is a Poisson process with mean measure m_0 (see, for example [6]), given by

$$m_0((s,t]) := \ell(\mathbf{C}) \int_s^t p(u) \lambda(u) du = A(t) - A(s), 0 < s < t.$$

It follows that

$$S(t) = P[0 \notin \Theta(t)] = P[N_0(t) = 0] = e^{-A(t)}. \qquad \square$$

3. The Statistical Problem

In the statistical problem solved below, the time rate λ and the distributions of the areas $\{V(t); t > 0\}$ are unknown. The landing times and the area of each disk are the observable data. The centers of the dropped disks (that is, their positions in the plane) are uniformly distributed on **C** but are not observable

Recall that we are looking for an estimator of the expected LFFA function S. An immediate consequence of Proposition 1 is that

$$S(t) = e^{-A(t)} = \prod_{s \leq t}[1 - dA(s)], t > 0,$$

where $\prod$ denotes the product integral.

Therefore, the natural estimator $\widehat{S}(t)$ of $S(t)$ is

$$\widehat{S}(t) \colon = \prod_{s \leq t}[1 - d\widehat{A}(s)],\ t > 0,$$

where $\widehat{A}$ is a suitable estimator of A.

As an estimator of A we choose the *unitary extended area* process $\widehat{A} = \{\widehat{A}(t); t > 0\}$ defined by

$$\widehat{A}(t) \colon = \tfrac{1}{\ell(\mathbf{C})}\int_0^t V(s)dN(s) = \tfrac{1}{\ell(\mathbf{C})} \sum_{T_i \leq t} V(T_i),\ t > 0, \tag{1}$$

where $V(s)$ is the random area of a disk dropped at time s. Note that $\widehat{A}$ is a stochastic jump process adapted to $\{\mathcal{F}_t; t > 0\}$ with random jumps (see Figure 1, A3). In the sequel, we will first obtain results for $\widehat{A}$ and use these results to prove properties of $\widehat{S}$.

Proposition 2: *The process* $\mathcal{M}^A = \{\mathcal{M}^A(t); t > 0\}$, *defined by*

$$\mathcal{M}^A(t) \colon = \sqrt{\ell(\mathbf{C})}(\widehat{A}(t) - A(t)) = \tfrac{1}{\sqrt{\ell(\mathbf{C})}}\int_0^t V(s)dN(s) - \sqrt{\ell(\mathbf{C})}\int_0^t \alpha(s)ds, t > 0,$$

is a zero mean square integrable martingale with predictable variation given by

$$\langle \mathcal{M}^A \rangle (t) = \int_0^t \mu^{(2)}(s)\lambda(s)ds = : v(t),\ \ t > 0. \tag{2}$$

Proof: By definition, $\mathcal{M}^A$ is $\mathcal{F}$-adapted. Moreover, for any $0 < s < t$,

$$E\Big[\widehat{A}(t) - \widehat{A}(s) \mid \mathcal{F}_s\Big] = E\left[\tfrac{1}{\ell(\mathbf{C})}\int_s^t V(u)dN(u)\middle|\mathcal{F}_s\right]$$

$$= \frac{1}{\ell(\mathbf{C})}\int_s^t E[E[V(u)dN(u) \mid \mathcal{F}_{u^-}] \mid \mathcal{F}_s].$$

The assumption of independence on $\{V(s); s > 0\}$, (X_i), and N implies that $V(u)$ is independent of $\mathcal{F}_{u^-}$ and $dN(u)$. Because the increment of a Poisson process $dN(u)$ is independent of $\mathcal{F}_{u^-}$, it follows that

$$E\Big[\widehat{A}(t) - \widehat{A}(s) \mid \mathcal{F}_s\Big] = \frac{1}{\ell(\mathbf{C})}\int_s^t E[E[V(u)dN(u)] \mid \mathcal{F}_s]$$

$$= \int_s^t E[V(u)]\lambda(u)du = A(t) - A(s),$$

so that

$$E[\mathcal{M}^A(t) \mid \mathcal{F}_s] = E[\mathcal{M}^A(t) - \mathcal{M}^A(s) \mid \mathcal{F}_s] + \mathcal{M}^A(s) = \mathcal{M}^A(s).$$

Then $\mathcal{M}^A$ is a zero-mean martingale and is square integrable because $\mu^{(2)}(s) < +\infty$ (see Remark 1).

Reasoning as above, we have the following equalities (a proof of the second equality can be found in [1, p. 54])

$$d\langle\mathcal{M}^A\rangle(t) = E[d(\mathcal{M}^A)^2(t) \mid \mathcal{F}_{t^-}] = E[(d\mathcal{M}^A(t))^2 \mid \mathcal{F}_{t^-}]$$

$$= \frac{1}{\ell(\mathbf{C})}E[V^2(t)dN(t) \mid \mathcal{F}_{t^-}] + \ell(\mathbf{C})(\alpha(t)dt)^2 - 2\alpha(t)dtE[V(t)dN(t) \mid \mathcal{F}_{t^-}]$$

$$= \mu^{(2)}(t)\lambda(t)dt - \ell(\mathbf{C}(\alpha(t))^2(dt)^2.$$

Equation (2) follows by integration. □

In particular, we deduce that $\widehat{A}(t)$ is an unbiased estimator of $A(t)$, i.e., $E[\widehat{A}(t)] = A(t)$, and moreover, $V[\widehat{A}(t)] = E[(\widehat{A} - A(t))^2] = v(t)/\ell(\mathbf{C})$. The deterministic function v, defined in (2), may be estimated by $\ell(\mathbf{C})\widehat{v}$ where

$$\widehat{v}(t) := \frac{1}{(\ell(\mathbf{C}))^2}\int_0^t V^2(s)dN(s) = \frac{1}{(\ell(\mathbf{C}))^2}\sum_{T_i \le t} V^2(T_i), t > 0. \tag{3}$$

This is confirmed by the following result.

Proposition 3: *The process $\ell(\mathbf{C})\widehat{v} - v = \{\ell(\mathbf{C})\widehat{v}(t) - v(t); t > 0\}$ is a zero-mean square integrable martingale with quadratic variation*

$$\langle\ell(\mathbf{C})\widehat{v} - v\rangle(t) = \frac{1}{\ell(\mathbf{C})}\int_0^t \mu^{(4)}(s)\lambda(s)ds, t > 0. \tag{4}$$

Proof: The proof is similar to the proof of Proposition 2 with V replaced by V^2. □

As previously mentioned, for any $t > 0$, we estimate $S(t)$ by

$$\widehat{S}(t) = \prod_{s \le t} [1 - d\widehat{A}(s)] = \prod_{T_i \le t} \Big[1 - \tfrac{V(T_i)}{\ell(\mathbf{C})}\Big].$$

The second equality above holds because $\widehat{A}(t)$ is a step function.

The following result holds.

Proposition 4: *The process $\mathcal{M}^S = \{\mathcal{M}^S(t): t > 0\}$, defined by*

$$\mathcal{M}^S(t) = \sqrt{\ell(\mathbf{C})}\ \ (\widehat{S}(t) - S(t)),\ \ t > 0,$$

is a zero-mean square integrable martingale, and, for any $t > 0$, its predictable variation is given by

$$\langle \mathcal{M}^S \rangle(t) = (S(t))^2 \int_0^t \Big(\tfrac{\widehat{S}(s^-)}{S(s)}\Big)^2 \mu^{(2)}(s)\lambda(s)ds,\ \ t > 0.$$

Proof: By using Duhamel's Equation (see (2.6.5) in [1]),

$$\mathcal{M}^S(t) = \sqrt{\ell(\mathbf{C})} S(t)\Big(\tfrac{\widehat{S}(t)}{S(t)} - 1\Big) = -S(t)\int_0^t \tfrac{\widehat{S}(s^-)}{S(s)} d\mathcal{M}^A(s).$$

From Proposition 2, because

$$\tfrac{\widehat{S}(s^-)}{S(s)} \le \tfrac{1}{S(s)} \le \tfrac{1}{S(T)},$$

we find that $\mathcal{M}^S$ is a zero-mean square integrable martingale with quadratic variation

$$\langle \mathcal{M}^S \rangle(t) = (S(t))^2 \int_0^t \Big(\tfrac{\widehat{S}(s^-)}{S(s)}\Big)^2 d\langle \mathcal{M}^A \rangle(s)$$

$$= (S(t))^2 \int_0^t \Big(\tfrac{\widehat{S}(s^-)}{S(s)}\Big)^2 \mu^{(2)}(s)\lambda(s)ds,\ \ t > 0. \qquad \square$$

It follows that $\widehat{S}(t)$ is an unbiased estimator of $S(t)$, i.e., $E[\widehat{S}(t)] = S(t)$, and

$$Var[\widehat{S}(t)] = E\Big[(\widehat{S}(t) - S(t))^2\Big] = \tfrac{(S(t))^2}{\ell(\mathbf{C})} E\left[\int_0^t \Big(\tfrac{\widehat{S}(s^-)}{S(s)}\Big)^2 \mu^{(2)}(s)\lambda(s)ds\right].$$

Because of the continuity of S, an estimator of the variance of $\widehat{S}(t)$ should be $(\widehat{S}(t))^2\widehat{v}(t)$.

Remark 3: Using terminology of the General Theory of Stochastic Processes, it follows, from Propositions 2 and 4, that A and S are, respectively, the compensators of $\widehat{A}$ and $\widehat{S}$.

4. Asymptotic Properties of the Estimators $\widehat{A}$ and $\widehat{S}$

The objective in this section is to show that if the region **C** is large enough, then the computed estimators $\widehat{A}$ and $\widehat{S}$ have good properties. For this reason, as is usual in spatial statistics, we fix the time interval for observations to be $[0,T]$ and study asymptotic properties of the estimators where limits in the region **C** expands to fill the space $\mathbb{R}^2$.

To do so, let us consider a convex averaging sequence $\{\mathbf{C}_n; n \geq 1\}$, as defined in [6, p. 332]. That is,

(i) $\mathbf{C}_n \subset \mathbb{R}^2$ is a convex Borel set,

(ii) $\mathbf{C}_n \subset \mathbf{C}_{n+1}$, for $n = 1, 2, \ldots,$

(iii) $r(\mathbf{C}_n) \to \infty$, $n \to \infty$, where $r(A){:} = \sup\{r > 0; A \text{ contains a ball of radius } r\}$.

Note that $\ell(\mathbf{C}_n) \to \infty$, $n \to \infty$.

From here on, N_n will denote the process of the landing times of disks on $\mathbf{C}_n$. It is assumed that N_n is a Poisson process with mean measure

$$m_n((s,t]){:} = \ell(\mathbf{C}_n) \cdot \int_s^t \lambda(u)du, \;\; 0 < s \leq t.$$

Analogously, the following notations will be used in the sequel

$$\widehat{A}_n(t){:} = \tfrac{1}{\ell(\mathbf{C}_n)} \int_0^t V(s)dN_n(s), \;\; \widehat{S}_n(t){:} = \prod_{s \leq t}[1 - d\widehat{A}_n(s)], \tag{5}$$

$$\mathcal{M}_n^A(t){:} = \sqrt{\ell(\mathbf{C}_n)}(\widehat{A}_n(t) - A(t)), \mathcal{M}_n^S(t){:} = \sqrt{\ell(\mathbf{C}_n)}(\widehat{S}_n(t) - S(t)),$$

$$\widehat{v}_n(t){:} = \tfrac{1}{(\ell(\mathbf{C}_n))^2} \int_0^t V^2(s)dN_n(s).$$

The following uniform consistency and asymptotic normality results hold for the estimators $\widehat{A}_n$, $\widehat{S}_n$ and $\ell(\mathbf{C}_n)\widehat{v}_n$.

Theorem 1: (Uniform Consistency)

(i) *$\widehat{A}_n$ is a uniformly consistence estimator of A in $[0,T]$, that is,*

$$\sup_{t \in [0,T]} \mid \widehat{A}_n(t) - A(t) \mid \xrightarrow{P} 0, \;\; n \to \infty, \tag{6}$$

(ii) *$\widehat{S}_n$ is a uniformly consistent estimator of S in $[0,T]$, that is,*

$$\sup_{t \in [0,T]} \mid \widehat{S}_n(t) - S(t) \mid \xrightarrow{P} 0, \;\; n \to \infty, \qquad 7)$$

(iii) *$\ell(\mathbf{C}_n)\widehat{v}_n$ is a uniformly consistent estimator of v in $[0,T]$, that is,*

$$\sup_{t \in [0,T]} \mid \ell(\mathbf{C}_n)\widehat{v}_n(t) - v(t) \mid \overset{P}{\rightarrow} 0, \quad n \rightarrow \infty. \tag{8}$$

Proof: (i) By using Lenglart's inequality (see [1, Equation (2.5.18)]) and Proposition 2, for any $\epsilon, \delta > 0$, we obtain

$$P\left[\sup_{t \leq T} \mid \widehat{A}_n(t) - A(t) \mid \; > \epsilon\right] \leq \frac{\delta}{\epsilon^2} + P\Big[\Big(\frac{1}{\ell(\mathbf{C}_n)}\langle \mathcal{M}_n^a\rangle(T)\Big) > \delta\Big]$$

$$= \frac{\delta}{\epsilon^2} + P\left[\frac{1}{\ell(\mathbf{C}_n)}\int_0^T \mu^{(2)}(s)\lambda(s)ds > \delta\right] \rightarrow \frac{\delta}{\epsilon^2}, \; n \rightarrow \infty.$$

Equation (6) follows by letting $\delta \rightarrow 0$.

(ii) The result follows from (i) and the continuity result for product integration (see [1, p. 114]).

(iii) By using Lenglart's inequality and Proposition 3,

$$P\left[\sup_{t \leq t} \mid \ell(\mathbf{C}_n)\widehat{v}_n(t) - v(t) \mid \; > \epsilon\right] \leq \frac{\delta}{\epsilon^2} + P[(\langle \ell(\mathbf{C}_n)\widehat{v}_n - v\rangle(T)) > \delta]$$

$$\leq \frac{\delta}{\epsilon^2} + P\left[\frac{1}{\ell(\mathbf{C}_n)}\int_0^T \mu^{(4)}(s)\lambda(s)ds > \delta\right] \rightarrow \frac{\delta}{\epsilon^2}, \; n \rightarrow \infty.$$

Equation (8) follows by letting $\delta \rightarrow 0$. □

Theorem 2: (Asymptotic Normality)

(i) *The process $\mathcal{M}_n^A$ converges on the Skorokhod function space $D(0,T)$ to a Gaussian martingale $\mathcal{M}$ with $\mathcal{M}(0) = 0$ and variance function $v = \{v(t): \; t > 0\}$ defined by* (2)

$$\mathcal{M}_n^A \overset{D}{\rightarrow} \mathcal{M}, \; n \rightarrow \infty. \tag{9}$$

(ii) *The process $\mathcal{M}_n^S$ converges on the Skorokhod function space $D(0,T)$ to $-S \cdot \mathcal{M}$, where $\mathcal{M}$ is as in* (i)

$$\mathcal{M}_n^S \overset{D}{\rightarrow} -S \cdot \mathcal{M}, \; n \rightarrow \infty. \tag{10}$$

The proof of Theorem 2 requires the following Martingale Central Limit Theorem (MCLT) (see [11]).

MCLT: *Let $(\mathcal{M}_n)$ be a sequence of zero-mean square integrable martingales. For any n and $\epsilon > 0$, let $(\mathcal{M}_{n,\epsilon})$ be a square integrable martingale containing all the jumps of $\mathcal{M}_n$ larger in absolute value than ϵ. Finally, let v be a continuous, nondecreasing function on $[0,T]$ with $v(0) = 0$.*

Suppose that

(*i*) $\langle \mathcal{M}_n \rangle(t) \xrightarrow{P} v(t)$, *$n \to \infty$, for all $t \in [0,T]$,*

(*ii*) $\langle \mathcal{M}_{n,\epsilon} \rangle(t) \xrightarrow{P} 0$, *and $n \to \infty$, for all $t \in [0,T]$ and $\epsilon > 0$.*

Then there exists a continuous Gaussian martingale $\mathcal{M}$, with $\langle \mathcal{M} \rangle(t) = v(t)$ for all t, such that

$$\mathcal{M}_n \xrightarrow{D} \mathcal{M}, \ n \to \infty.$$

If conditions (i) and (ii) hold for $t = T$, then

$$\mathcal{M}_n(T) \xrightarrow{D} \mathcal{N}(0, v(T)), \ n \to \infty. \qquad \square$$

Proof of Theorem 2: We apply the MCLT. According to Proposition 2, $\langle \mathcal{M}_n^A \rangle(t) = v(t)$ and hence, condition (i) of the MCLT is trivially verified. To prove (ii), let us define for any $t > 0$

$$\mathcal{M}_{n,\epsilon}^A(t) := \frac{1}{\sqrt{\ell(\mathbf{C}_n)}} \int_0^t H_{n,\epsilon}(s) dN^{(n)}(s) - \sqrt{\ell(\mathbf{C}_n)} \int_0^t E[H_{n,\epsilon}(s)] \lambda(s) ds,$$

where $H_{n,\epsilon}(s) := V(s) I[V(s) > \epsilon \sqrt{\ell(\mathbf{C}_n)}]$.

Similarly, as in the proof of Proposition 2, it can be proved that $\mathcal{M}_{n,\epsilon}^A$ is a square integrable martingale, and that

$$\langle \mathcal{M}_{n,\epsilon}^A \rangle(t) = \int_0^t E[V^2(s) I[V(s) > \epsilon \sqrt{\ell(\mathbf{C}_n)}]] \lambda(s) ds.$$

Furthermore, $\mathcal{M}_{n,\epsilon}^A$ contains all the jumps of $\mathcal{M}_n^A$ larger in absolute value than ϵ. From Remark 1, for n large enough, $\langle \mathcal{M}_{n,\epsilon}^A \rangle(t) = 0$, and therefore condition(ii) of the MCLT is also proved.

Because of the asymptotic equivalence result for product integration, (ii) follows from (i) (see [1, p. 111]). $\square$

5. Confidence Intervals and Bounds for A and S

From Theorems 1 and 2 in the previous section, for any $t \in [0,T]$,

$$\left(\frac{1}{\widehat{v}_n(t)}\right)^{1/2} (\widehat{A}_n(t) - A(t)) \text{ and } \left(\frac{1}{\widehat{v}_n(t)}\right)^{1/2} \frac{\widehat{S}_n(t) - S(t)}{\widehat{S}_n(t)}$$

both asymptotically have a standard normal distribution as $n \to \infty$.

Therefore, the asymptotic $100(1-\alpha)\%$ confidence intervals for $A(t)$ and $S(t)$ are, respectively,

$$\left[\widehat{A}_n(t) - \sqrt{\widehat{v}_n(t)} z_{\alpha/2}, \widehat{A}_n(t) + \sqrt{\widehat{v}_n(t)} z_{\alpha/2}\right],$$

and

$$\Big[\widehat{S}_n(t) - \sqrt{\widehat{v}_n(t)}\widehat{S}_n(t)z_{\alpha/2}, \widehat{S}_n(t) + \sqrt{\widehat{v}_n(t)}\widehat{S}_n(t)z_{\alpha/2}\Big],$$

where $z_{\alpha/2}$ is the upper $(\alpha/2)$-quantile of the standard normal distribution.

To find the confidence bounds for A and S consider the following convergence results using the same notation and assumptions as in the previous sections.

Theorem 3:

$$\sup_{t \leq T} \left| \tfrac{\sqrt{\ell(\mathbf{C}_n)}}{1+\ell(\mathbf{C}_n)\widehat{v}_n(t)} (\widehat{A}_n(t) - A(t)) \right| \overset{D}{\rightarrow} \sup_{x \in [0,c]} \mid W^0(x) \mid, n \rightarrow \infty,$$

and

$$\sup_{t \leq T} \left| \tfrac{\sqrt{\ell(\mathbf{C}_n)}}{1+\ell(\mathbf{C}_n)\widehat{v}_n(t)} \; \tfrac{\widehat{S}_n(t) - S(t))}{\widehat{S}_n(t)} \right| \overset{D}{\rightarrow} \sup_{x \in [0,c]} \mid W^0(x) \mid, n \rightarrow \infty,$$

where W^0 is a standard Brownian bridge and $c{:} = v(T)/(1+v(T))$.

Proof: From Theorems 1 and 2,

$$\tfrac{\sqrt{\ell(\mathbf{C}_n)}}{1+\ell(\mathbf{C}_n)\widehat{v}_n} (\widehat{A}_n - A) = \tfrac{\mathcal{M}_n^A}{1+\ell(\mathbf{C}_n)\widehat{v}_n} \overset{D}{\rightarrow} \tfrac{W(v)}{1+v}, n \rightarrow \infty$$

and

$$\tfrac{\sqrt{\ell(\mathbf{C}_n)}}{1+\ell(\mathbf{C}_n)\widehat{v}_n} \; \tfrac{\widehat{S}_n - S}{\widehat{S}_n} = \tfrac{\mathcal{M}_n^S}{(1+\ell(\mathbf{C}_n)\widehat{v}_n)\cdot \widehat{S}_n} \overset{D}{\rightarrow} \tfrac{W(v)}{1+v}, n \rightarrow \infty$$

where W denotes standard Brownian motion.

Therefore, we only have to observe that $\frac{W(v)}{1+v}$ and $W^0\big(\frac{v}{1+v}\big)$ have the same distribution. □

From Theorem 3, it follows that, for any $y > 0$, as $n \rightarrow \infty$,

$$P \sup_{t \leq T} \Big[\ell(\mathbf{C}_n) 1 + \ell(\mathbf{C}_n)\widehat{v}_n(t) \; (\widehat{A}_n(t) - A_{ex}(t)) \leq y\Big] \; \rightarrow \; P \left[\sup_{x \in [0,c]} \mid W^0(x) \mid \; \leq y \right]$$

and

$$P \sup_{t \leq T} \Big[\ell(\mathbf{C}_n) 1 + \ell(\mathbf{C}_n)\widehat{v}_n(t) \; \tfrac{\widehat{S}_n(t) - S(t))}{\widehat{S}_n(t)} \leq y\Big] \; \rightarrow \; P \left[\sup_{x \in [0,c]} \mid W^0(x) \mid \; \leq y \right].$$

Then, the asymptotic $100(1-\alpha)\%$ confidence bounds for A and S in $[0,T]$ are, respectively,

$$\Big[\widehat{A}_n(t) - \tfrac{1+\ell(\mathbf{C}_n)\widehat{v}_n(t)}{\sqrt{\ell(\mathbf{C}_n)}} e_{\alpha/2}(c), \widehat{A}_n(t) + \tfrac{1+\ell(\mathbf{C}_n)\widehat{v}_n(t)}{\sqrt{\ell(\mathbf{C}_n)}} e_{\alpha/2}(c)\Big], t \in [0,T], \qquad (11)$$

$$\left[\widehat{S}_n(t)\Big(1-\ \frac{1+\ell(\mathbf{C}_n)\widehat{v}_n(t)}{\sqrt{\ell(\mathbf{C}_n)}}e_{\alpha/2}(c)\Big),\,\widehat{S}_n(t)\Big(1+\frac{1+\ell(\mathbf{C}_n)\widehat{v}_n(t)}{\sqrt{\ell(\mathbf{C}_n)}}e_{\alpha/2}(c)\Big)\right],$$

$$t\in[0,T], \tag{12}$$

where $e_{\alpha/2}(c)$ denotes the upper $(\alpha/2)$-quantile of the distribution of $\sup_{x\in[0,c]}\mid W^0(x)\mid$.

6. Estimation of α

As pointed out in the introduction, the estimation of $\alpha(t)=\mu(t)\lambda(t)$ is important. In this section we obtain the estimator $\widehat{\alpha}(t)$ of $\alpha(t)$ by smoothing $\widehat{A}$ with a kernel function $\mathcal{K}$ (see (13)). In the following, $\mathcal{K}$ is a function of bounded variation that vanishes outside $[-1,1]$, such that

$$\int_{-1}^{1}\mathcal{K}(u)du=1,$$

and $(b_n)=(b(\ell(\mathbf{C}_n)))$ is a sequence such that as $n\to\infty$

$$\ell(\mathbf{C}_n)b_n^2\to\infty \text{ and } \ell(\mathbf{C}_n)b_n^3\to 0.$$

For example, we could take $b_n:=(\ell(\mathbf{C}_n))^{-2/5}$.

From here on, the interval of estimation $[t_1,t_2]\subset(0,T)$ will be fixed and n will be so large that $[t_1-b_n,t_2+b_n]\subset[0,T]$.

We define, for any $t\in[t_1,t_2]$, the estimator $\widehat{\alpha}_n(t)$ by

$$\widehat{\alpha}_n(t):=\frac{1}{b_n}\int_0^T\mathcal{K}\Big(\frac{t-s}{b_n}\Big)d\widehat{A}_n(s)=\frac{1}{\ell(\mathbf{C}_n)b_n}\sum_{t-b_n<T_i\le t+b_n}\mathcal{K}\Big(\frac{t-T_i}{b_n}\Big)V(T_i). \tag{13}$$

Because of Proposition 2, for any $t\in[t_1,t_2]$, the process $\overline{\mathcal{M}}_n^A(t,\cdot)=\{\overline{\mathcal{M}}_n^A(t,u);\ u\in[0,T]\}$, defined by

$$\overline{\mathcal{M}}_n^A(t,u):=\frac{1}{b_n}\int_0^u\mathcal{K}\Big(\frac{t-s}{b_n}\Big)d\mathcal{M}_n^A(s),\ u\in[0,T], \tag{14}$$

is a zero-mean square integrable martingale, and

$$\langle\overline{\mathcal{M}}_n^A(t,u)\rangle=\frac{1}{b_n^2}\int_0^u\mathcal{K}^2\Big(\frac{t-s}{b_n}\Big)d\langle\mathcal{M}_n^A\rangle(s)$$

$$=\frac{1}{b_n^2}\int_0^u\mathcal{K}^2\Big(\frac{t-s}{b_n}\Big)dv(s),\ u\in[0,T].$$

It follows that

$$E[\widehat{\alpha}_n(t)] = E\left[\frac{\bar{\mathcal{M}}_n^A(t,T)}{\sqrt{\ell(\mathbf{C}_n)}} + \frac{1}{b_n}\int_0^T \mathcal{K}\left(\frac{t-s}{b_n}\right)dA(s)\right] = \frac{1}{b_n}\int_0^T \mathcal{K}\left(\frac{t-s}{b_n}\right)dA(s),$$

so

$$\widehat{\alpha}_n(t) - E[\widehat{\alpha}_n(t)] = \frac{\bar{\mathcal{M}}_n^A(t,T)}{\sqrt{\ell(\mathbf{C}_n)}} = \frac{1}{\sqrt{\ell(\mathbf{C}_n)}b_n}\int_0^T \mathcal{K}\left(\frac{t-s}{b_n}\right)d\mathcal{M}_n^A(s).$$

Furthermore,

$$Var[\widehat{\alpha}_n(t)] = E[(\widehat{\alpha}_n(t) - E[\widehat{\alpha}_n(t)])^2] = \frac{\langle\bar{\mathcal{M}}_n^A(t,T)\rangle}{\ell(\mathbf{C}_n)} = \frac{1}{\ell(\mathbf{C}_n)b_n^2}\int_0^T \mathcal{K}^2\left(\frac{t-s}{b_n}\right)dv(s).$$

We define, for any $t \in [t_1, t_2]$

$$\widehat{\tau}_n^2(t) := \frac{1}{b_n^2}\int_0^T \mathcal{K}^2\left(\frac{t-s}{b_n}\right)d\widehat{v}_n(s).$$

Because

$$\widehat{\tau}_n^2(t) - V[\widehat{\alpha}_n(t)] = \frac{1}{\ell(\mathbf{C}_n)b_n^2}\int_0^T \mathcal{K}^2\left(\frac{t-s}{b_n}\right)d(\ell(\mathbf{C}_n)\widehat{v} - v)(s),$$

it follows from Proposition 3 that $\widehat{\tau}_n^2(t)$ is an unbiased estimator of $Var[\widehat{\alpha}_n(t)]$, that is, $E[\widehat{\tau}_n^2(t)] = Var[\widehat{\alpha}_n(t)]$.

The following result holds.

Proposition 5:

(i) *$\widehat{\alpha}_n$ is an asymptotically uniformly-unbiased estimator of α in $[t_1, t_2]$, that is,*

$$\sup_{t \in [t_1, t_2]} \mid E[\widehat{\alpha}_n(t)] - \alpha(t) \mid \to 0,\ n \to \infty.$$

(ii) *$\ell(\boldsymbol{C}_n)b_n\widehat{\tau}_n^2$ is an asymptotically uniformly-unbiased estimator in $[t_1, t_2]$ of the function τ^2 defined by*

$$\tau^2(t) := \mu^{(2)}(t)\lambda(t)\int_{-1}^{+1} \mathcal{K}^2(u)du,$$

that is,

$$\sup_{t \in [t_1, t_2]} \mid \ell(\boldsymbol{C}_n)b_nE[\widehat{\tau}_n^2(t)] - \tau^2(t) \mid \to 0, \quad n \to \infty.$$

Proof: Because $\mathcal{K}$ is bounded and μ and λ are uniformly continuous in $[0, T]$, we only have to observe that as $n \to \infty$,

$$\sup_{t \in [t_1, t_2]} \mid E[\widehat{\alpha}_n(t)] - \alpha(t) \mid \ \leq \ \sup_{t \in [t_1, t_2]} \int_{-1}^{+1} \mid \mathcal{K}(u) \mid \ \mid \alpha(t - b_n u) - \alpha(t) \mid du \to 0,$$

and

$$\sup_{t \in [t_1, t_2]} \mid \ell(\mathbf{C}_n) b_n E[\widehat{\tau}_n^2(t)] - \tau^2(t) \mid$$

$$\leq (\pi k^2)^2 \sup_{t \in [t_1, t_2]} \int_{-1}^{+1} \mid \mathcal{K}^2(u) \mid \ \mid \lambda(t - b_n u) - \lambda(t) \mid du \to 0. \qquad \square$$

We are now ready to prove the following asymptotic results.

Theorem 4: (Uniform consistency)

(i) $\widehat{\alpha}_n$ *is a uniformly consistent estimator of* α *in* $[t_1, t_2]$*, that is,*

$$sup_{t \in [t_1, t_2]} \mid \widehat{\alpha}_n(t) - \alpha(t) \mid \ \xrightarrow{P} 0, \ n \to \infty,$$

(ii) $\ell(\mathbf{C}_n) b_n \widehat{\tau}_n^2(t)$ *is a uniformly consistent estimator of* $\tau^2(t)$ *in* $[t_1, t_2]$*, that is,*

$$sup_{t \in [t_1, t_2]} \mid \ell(\mathbf{C}_n) b_n \widehat{\tau}_n^2(t) - \tau^2(t) \mid \ \xrightarrow{P} 0, n \to \infty.$$

Proof: (i) Because of Proposition 5, we have to prove that

$$\sup_{t \in [t_1, t_2]} \mid \widehat{\alpha}_n(t) - E[\widehat{\alpha}_n(t)] \mid \ \xrightarrow{P} 0, \ n \to \infty.$$

Because $\mathcal{K}$ is of bounded variation, then for any $t \in [t_1, t_2]$,

$$\mid \widehat{\alpha}_n(t) - E[\widehat{\alpha}_n(t)] \mid \ = \left| \frac{1}{\sqrt{\ell(\mathbf{C}_n) b_n}} \int_0^T \mathcal{K}\left(\frac{t-u}{b_n}\right) d\mathcal{M}_n^A(u) \right|$$

$$\leq \frac{2\| V(\mathcal{K}) \|}{\sqrt{\ell(\mathbf{C}_n) b_n}} \sup_{u \in [0, T]} \mid \mathcal{M}_n^A(u) \mid ,$$

where $\| V(\mathcal{K}) \|$ denotes the variation of $\mathcal{K}$.

Furthermore, by using Lenglart's inequality and (ii), for any $\epsilon, \delta > 0$, we obtain

$$P\left[\frac{1}{\sqrt{\ell(\mathbf{C}_n)}b_n}\sup_{u\in[0,T]}|\mathcal{M}_n^A(u)|\geq\epsilon\right]\leq\frac{\delta}{\epsilon^2}+P\left[\frac{1}{\ell(\mathbf{C}_n)b_n^2}\langle\mathcal{M}_n^2\rangle(T)\geq\delta\right]$$

$$=\frac{\delta}{\epsilon^2}+P\left[\frac{1}{\ell(\mathbf{C}_n)b_n^2}\int_0^T\mu^{(2)}(s)\lambda(s)ds\geq\delta\right]\to\frac{\delta}{\epsilon^2},\ n\to\infty,$$

because $\ell(\mathbf{C}_n)b_n^2\to\infty$. By letting $\delta\to 0$, we deduce that

$$\frac{1}{\sqrt{\ell(\mathbf{C}_n)}b_n}\sup_{u\in[0,T]}|\mathcal{M}_n^A(u)|\xrightarrow{P}0,\ n\to\infty,$$

and the result follows.

(ii) Because of Proposition 5, we have to prove that

$$\sup_{t\in[t_1,t_2]}|\ell(\mathbf{C}_n)b_n\widehat{\tau}_n^2(t)-\ell(\mathbf{C}_n)b_nE[\widehat{\tau}_n^2(t)]|\xrightarrow{P}0,\ n\to\infty.$$

Because the variation of $\mathcal{K}$ (and thus $\mathcal{K}^2$) is bounded, then for any $t\in[t_1,t_2]$,

$$|\ell(\mathbf{C}_n)b_n\widehat{\tau}_n^2(t)-\ell(\mathbf{C}_n)b_nE[\widehat{\tau}_n^2(t)]|=\left|\frac{1}{b_n}\int_0^T\mathcal{K}^2\left(\frac{t-u}{b_n}\right)d(\ell(\mathbf{C}_n)\widehat{v}_n-v)(u)\right|$$

$$\leq\frac{2\|V(\mathcal{K}^2)\|}{b_n}\sup_{u\in[0,T]}|\ell(\mathbf{C}_n)\widehat{v}_n(u)-v(u)|.$$

Furthermore, by using Lenglart's inequality and (4), for any $\epsilon,\delta>0$, we have

$$P\left[\frac{1}{b_n}\sup_{u\in[0,T]}|\ell(\mathbf{C}_n)\widehat{v}_n(u)-v(u)|\geq\epsilon\right]\leq\frac{\delta}{\epsilon^2}+P\left[\frac{1}{b_n^2}\langle\ell(\mathbf{C}_n)\widehat{v}_n-v\rangle(t)\geq\delta\right]$$

$$=\frac{\delta}{\epsilon^2}+P\left[\frac{1}{\ell(\mathbf{C}_n)b_n^2}\int_0^T\mu^{(4)}(s)\lambda(s)ds\geq\delta\right]\to\frac{\delta}{\epsilon^2},\ n\to\infty,$$

because $\ell(\mathbf{C}_n)b_n^2\to\infty$. By letting $\delta\to 0$, we deduce that

$$\frac{1}{b_n}\sup_{u\in[0,T]}|\ell(\mathbf{C}_n)\widehat{v}_n(u)-v(u)|\xrightarrow{P}0,\ n\to\infty.$$

The proof is complete. □

Theorem 5: (Asymptotic Normality) *Let $t \in [0, T]$ and suppose that α has a bounded derivative in a neighborhood of t, that is, numbers $\epsilon > 0$ and $c > 0$ exist such that*

$$\sup_{s \in (t-\epsilon, t+\epsilon)} \mid \alpha'(s) \mid \; \leq c.$$

Then

$$\sqrt{\ell(\mathbf{C}_n) b_n}(\widehat{\alpha}_n(t) - \alpha(t)) \xrightarrow{D} \mathcal{N}(0, \tau^2(t)), \; n \to \infty.$$

Proof: Let $t \in [0, T]$ be fixed. Note that

$$\sqrt{\ell(\mathbf{C}_n) b_n}(\widehat{\alpha}_n(t) - E[\widehat{\alpha}_n(t)]) = \sqrt{b_n}\, \bar{\mathcal{M}}_n^A(t, T),$$

where $\bar{\mathcal{M}}_n^A(t, \cdot)$ is the zero-mean square integrable martingale defined by (14).

From Proposition 5, as $n \to \infty$,

$$\langle \sqrt{b_n}\, \bar{\mathcal{M}}_n^A(t, T) \rangle = \ell(\mathbf{C}_n) b_n V[\widehat{\alpha}_n(t)] = \ell(\mathbf{C}_n) b_n E[\widehat{\tau}_n^2(t)] \to \tau^2(t),$$

and condition (i) of the MCLT is verified by $\sqrt{b_n}\, \bar{\mathcal{M}}_n^A(t, \cdot)$ in T.

To prove condition (ii), define for any $u \in [0, T]$,

$$\bar{\mathcal{M}}_{n,\epsilon}^A(t, u) \colon = \frac{1}{\sqrt{\ell(\mathbf{C}_n)}} \int_0^u H_{n,\epsilon}(t, s) dN^{(n)}(s) - \sqrt{\ell(\mathbf{C}_n)} \int_0^u E[H_{n,\epsilon}(t, s)] \lambda(s) ds,$$

where

$$H_{n,\epsilon}(t, s) \colon = \frac{V(s)}{\sqrt{b_n}} \mathcal{K}\left(\frac{t-s}{b_n}\right) I\left[\frac{V(s)}{\sqrt{\ell(\mathbf{C}_n) b_n}} \mathcal{K}\left(\frac{t-s}{b_n}\right) > \epsilon\right].$$

Similarly as in the proof of Proposition 2, it can be proved that $\bar{\mathcal{M}}_{n,\epsilon}^A(t, \cdot)$ is a zero-mean square integrable martingale, and that

$$\langle \bar{\mathcal{M}}_{n,\epsilon}^A(t, u) \rangle = \int_0^u E[(H_{n,\epsilon}(t, s))^2] \lambda(s) ds.$$

Furthermore, $\bar{\mathcal{M}}_{n,\epsilon}^A(t, \cdot)$ contains all the jumps of $\sqrt{b_n}\, \bar{\mathcal{M}}_n^A(t, \cdot)$ larger in absolute value than ϵ.

From Remark 1, for n large enough, $\langle \bar{\mathcal{M}}_{n,\epsilon}^A(t, T) \rangle = 0$, and hence condition (ii) of the MCLT is also verified. It follows that

$$\sqrt{\ell(\mathbf{C}_n) b_n}(\widehat{\alpha}_n(t) - E[\widehat{\alpha}_n(t)]) = \sqrt{b_n}\, \bar{\mathcal{M}}_n^A(t, T) \xrightarrow{D} \mathcal{N}(0, \tau^2(t)), n \to \infty.$$

Furthermore, by hypothesis

$$\sqrt{\ell(\mathbf{C}_n) b_n} \mid E[\widehat{\alpha}_n(t)] - \alpha(t) \mid \; \leq \sqrt{\ell(\mathbf{C}_n) b_n^3}\, c \int_{-1}^{+1} \mid \mathcal{K}(u) u \mid du \to 0, \; n \to \infty$$

because $\ell(\mathbf{C}_n)b_n^3 \to 0$. The proof is complete by observing that

$$\sqrt{\ell(\mathbf{C}_n)b_n}\,(\widehat{\alpha}_n(t)-\alpha(t)) = \sqrt{\ell(\mathbf{C}_n)b_n}(\widehat{\alpha}_n(t)-E[\widehat{\alpha}_n(t)]) + \sqrt{\ell(\mathbf{C}_n b_n}(E[\widehat{\alpha}_n(t)]-\alpha(t)).$$

□

From Theorem 5 and Theorem 4, part (ii) it follows that for any $t \in [t_1, t_2]$,

$$\tfrac{\widehat{\alpha}_n(t)-\alpha(t)}{\widehat{\tau}_n(t)} \overset{D}{\to} \mathcal{N}(0,1),\ n \to \infty.$$

Thus, the asymptotic $100(1-\alpha)\%$ confidence interval for $\alpha(t)$ is

$$[\widehat{\alpha}_n(t) - \widehat{\tau}_n(t) z_{\alpha/2}, \widehat{\alpha}_n(t) + \widehat{\tau}_n(t) z_{\alpha/2}], \tag{15}$$

where $z_{\alpha/2}$ is the upper $(\alpha/2)$-quantile of the standard normal distribution.

7. Numerical Simulations

To show the evolution in time of the DGGM Θ, we have simulated it in a time interval $[0, 5]$. As an example, we have assumed that

$$V(t){:}\ = \pi(R_0 \sin(t) + 0.4)^2 \text{ and } \lambda(t){:}\ = \tfrac{10}{\sqrt{t+1}},\ t \in [0,5],$$

where R_0 is a continuous random variable having uniform distribution on $(0.1, 0.3)$.

In Figure 1, parts A1 and A2, we have plotted the state of Θ at two different times. In part A3 of Figure 1, the related realization of the unitary extended area process $\widehat{A}$ is shown.

To illustrate the qualitative asymptotic behavior of the estimators, we have also simulated the DGGM on regions $\mathbf{C}_n$ of different Lebesgue measures. We then computed the estimators $\widehat{A}_n(t)$ and $\widehat{S}_n(t)$ defined by (5). The 95% confidence bounds on the time interval $[0, 5]$ have also been computed following (11) and (12). We have taken the value of $e_\alpha(c) = e_{0.05}(c)$ from the table in [13] after estimating the unknown c by

$$\widehat{c}{:}\ = \tfrac{\widehat{v}_n(T)}{1+\widehat{v}_n(T)},$$

where $T = 5$. In Figure 2, we have plotted the true function A, the computed estimator $\widehat{A}_n$ and the confidence bounds for A. In Figure 3, we have plotted the true function S, the estimator $\widehat{S}_n$ and the related confidence bounds.

With the simulated data, the estimator $\widehat{\alpha}_n(t)$ has been obtained using (13). The related confidence intervals have been calculated following (15). The results obtained are plotted in Figure 4.

A1 and A2: An example of realization of the dynamic germ-grain model Θ observed at two different times.

A3: Plot of the related realization of the unitary extended area $\widehat{A}$

Figure 1

Figure 2: Comparison between A (continuous line), its estimator $\widehat{A}_n$, and the 95% confidence bounds for $\ell(\mathbf{C}_n) = 30$ and $\ell(\mathbf{C}_n) = 70$.

Figure 3: Comparison between S (continuous line), its estimator $\widehat{S}_n$, and the 95% confidence bounds for $\ell(\mathbf{C}_n) = 30$ and $\ell(\mathbf{C}_n) = 70$.

Figure 4: Comparison between α (continuous line), its estimator $\widehat{\alpha}_n$, and the 95% confidence bounds for $\ell(\mathbf{C}_n) = 50$ and $\ell(\mathbf{C}_n) = 1000$.

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