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FREE BANACH SPACES AND THE APPROXIMATION PROPERTIES

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ABSTRACT. We characterize the metric spaces whose free spaces have the bounded approximation property through a Lipschitz analogue of the local reflexivity principle. We show that there exist compact metric spaces whose free spaces fail the approximation property.

1. INTRODUCTION

Let M be a pointed metric space, that is, a metric space equipped with a distinguished point denoted 0. We denote by $\mathrm{Lip}_0(M)$ the Banach space of all real-valued Lipschitz functions defined on M which vanish at 0, equipped with the natural Lipschitz norm

$$\|f\|_L = \sup\Big\{\frac{|f(x)-f(y)|}{\|x-y\|}\ ;\ (x,y)\in M^2,\ x\neq y\Big\}.$$

For all $x \in M$, the Dirac measure $\delta(x)$ defines a continuous linear form on $\mathrm{Lip}_0(M)$. Equicontinuity shows that the closed unit ball of $\mathrm{Lip}_0(M)$ is compact for pointwise convergence on M, and thus the closed linear span of $\{\delta(x)\ ;\ x\in M\}$ in $\mathrm{Lip}_0(M)^*$ is an isometric predual of $\mathrm{Lip}_0(M)$. This predual is called the Arens-Eells space of M in [We] and (when M is a Banach space) the Lipschitz-free space over M in [GK], denoted by $\mathfrak{F}(M)$. We will use this notation and simply call $\mathfrak{F}(M)$ the free space over M. When M is separable, the Banach space $\mathfrak{F}(M)$ is separable as well, since the set $\{\delta(x)\ ;\ x\in M\}$ equipped with the distance induced by $\mathrm{Lip}_0(M)^*$ is isometric to M.

The free spaces over separable metric spaces M constitute a fairly natural family of separable Banach spaces, which are moreover very useful in non-linear geometry of Banach spaces (see [Ka2]). However, they are far from being well-understood at this point, and some basic questions remain unanswered. We recall that a Banach space X has the approximation property (AP) if the identity id_X of X is in the closure of the finite rank operators on X for the topology of uniform convergence on compact sets. The λ-bounded approximation property (λ-BAP) means that there are approximating finite rank operators with norm less than λ, and (1-BAP) is called the metric approximation property (MAP). This note is devoted to the following problem: for which metric spaces M does the space $\mathfrak{F}(M)$ have (AP), or (BAP), or (MAP)? For motivating this query, recall that real-valued Lipschitz functions defined on subsets of metric spaces extend with the same Lipschitz constant

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through the usual inf-convolution formula. However, approximation properties for free spaces are related to the existence of *linear* extension operators for Lipschitz functions defined on subsets (see [Bo], and Proposition 6 below).

It is already known that some free spaces fail AP: indeed one of the main results of [GK] asserts that if X is an arbitrary Banach space and $\lambda \geq 1$, then X has λ-BAP if and only if $\mathfrak{F}(X)$ has λ-BAP. Since moreover any separable Banach space X is isometric to a 1-complemented subspace of $\mathfrak{F}(X)$ ([GK], Theorem 3.1), it follows that $\mathfrak{F}(X)$ fails AP when X does.

This note provides further examples of metric spaces whose free spaces fail AP. We show in particular that some spaces $\mathfrak{F}(K)$, with K compact metric spaces, fail AP although MAP holds for "small" Cantor sets.

Section 2 gives a characterization of the λ-BAP for $\mathfrak{F}(M)$ through weak*-approximation of Lipschitz functions from M into bidual spaces, somewhat similar to the local reflexivity principle. In section 3, a method used in [GK] and localized in [DL] is shown to provide the existence of compact convex sets K with $\mathfrak{F}(K)$ failing AP. Several open questions conclude the note.

2. Lipschitz local reflexivity

For metric spaces M and X, we denote by $\mathrm{Lip}^{\lambda}(M, X)$ the set of λ-Lipschitz maps from M into X. We assume that M is separable and X is complete. Fix a dense sequence $(x_n)_n$ in M and define a metric d on $\mathrm{Lip}^{\lambda}(M, X)$ by

$$d(f,g) = \sum_{n=1}^{\infty} \min\{d(f(x_n), g(x_n)),\, 2^{-n}\}.$$

Then, d is a complete metric on $\mathrm{Lip}^{\lambda}(M, X)$ whose topology coincides with the pointwise convergence topology.

Let Z be a Banach subspace of Y and denote the quotient map by $Q\colon Y \to Y/Z$. We say Z is an M-*ideal with an approximate unit*, or an M-iwau for short, if there are nets of operators $\phi_i\colon Y \to Z$ and $\psi_i\colon Y \to Y$ such that $\phi_i(z) \to z$ for every $z \in Z$, $Q \circ \psi_i = Q$ for all i, $\phi_i + \psi_i \to \mathrm{id}_Y$ pointwise, and $\|\phi_i(x) + \psi_i(y)\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in Y$ and i. We note that $\psi_i \to 0$ on Z and $\|Q(y)\| = \lim \|\psi_i(y)\|$.

Example A. Let X be a separable Banach space and X_n be an increasing sequence of finite-dimensional subspaces whose union is dense. We define

$$Y = \{(x_n)_n \in (\textstyle\prod X_n)_\infty \ ; \text{ the sequence } (x_n)_n \text{ is convergent in } X\}.$$

Then, Y is a Banach space with MAP with the metric surjection $Q\colon Y \to X$ given by the limit. The subspace $\ker Q$ is an M-iwau, with $\phi_k((x_n)_n) = (x_1, \ldots, x_k, 0, 0, \ldots)$.

Example B. Every closed two-sided ideal I in a C*-algebra is an M-iwau.

Lemma 1 (cf. [Ar], Theorem 6). *Let $Z \subset Y$ be an* M*-iwau and M be a separable metric space. Then, for every $\lambda \geq 1$ the set*

$$\{Q \circ f \ ;\ f \in \mathrm{Lip}^{\lambda}(M, Y)\} \subset \mathrm{Lip}^{\lambda}(M, Y/Z)$$

is closed under the pointwise convergence topology.

Proof. Let $(f_n)_n$ be a sequence in $\mathrm{Lip}^{\lambda}(M, Y)$ such that $Q \circ f_n$ converge to $F \in \mathrm{Lip}^{\lambda}(M, Y/Z)$. To prove that F lifts, we may assume that $d(Q \circ f_n, Q \circ f_{n+1}) < 2^{-n}$.

We will recursively construct g_n such that $Q \circ g_n = Q \circ f_n$ and $d(g_n, g_{n+1}) < 2^{-n}$. Then, the sequence $(g_n)_n$ converges and its limit is a lift of F. For g_{n+1}, we define

$$g_{n+1,i} = \phi_i \circ g_n + \psi_i \circ f_{n+1}.$$

Then, $g_{n+1,i} \in \mathrm{Lip}^\lambda(M, Y)$, $Q \circ g_{n+1,i} = Q \circ f_{n+1}$ and

$$\lim_i d(g_n, g_{n+1,i}) = \lim_i d(\psi_i \circ g_n, \psi_i \circ f_{n+1}) = d(Q \circ g_n, Q \circ f_{n+1}) < 2^{-n}.$$

Thus, there is an i such that $g_{n+1} := g_{n+1,i}$ works. □

Theorem 2. *Let M be a separable metric space and $\lambda \geq 1$. Then, the free space $\mathfrak{F}(M)$ has the λ-BAP if and only if M has the following property: For any Banach space Y and any $f \in \mathrm{Lip}^1(M, Y^{**})$, there is a net in $\mathrm{Lip}^\lambda(M, Y)$ which converges to f in the pointwise-weak* topology.*

Proof. Suppose $\mathfrak{F}(M)$ has the λ-BAP and $f \in \mathrm{Lip}^1(M, Y^{**})$ is given. Then, f extends to a linear contraction $\hat{f}\colon \mathfrak{F}(M) \to Y^{**}$. Since $\mathfrak{F}(M)$ has λ-BAP, the local reflexivity principle yields a net of operators $T_i\colon \mathfrak{F}(M) \to Y$ with norm $\leq \lambda$ which weak* converges to $\hat{f}$ pointwise. Restricting it to M, we obtain a desired net.

Conversely, suppose M satisfies the property stated in Theorem 2. We apply the construction described in Example A to $\mathfrak{F}(M)$ and obtain $Q\colon Y \to \mathfrak{F}(M)$. Since $Z = \ker Q$ is an M-ideal, one has a canonical identification $Y^{**} = Z^{**} \oplus_\infty \mathfrak{F}(M)^{**}$. In particular, $M \hookrightarrow Y^{**}$ naturally. By assumption, there is a net $f_i \in \mathrm{Lip}^\lambda(M, Y)$ which approximates the above inclusion. Since $Q \circ f_i \in \mathrm{Lip}^\lambda(M, \mathfrak{F}(M))$ converge to id_M in the point-weak topology, by taking convex combinations if necessary, we may assume that they converge in the point-norm topology. Thus by Lemma 1, $\mathrm{id}_M\colon M \hookrightarrow \mathfrak{F}(M)$ lifts to a function $f \in \mathrm{Lip}^\lambda(M, Y)$. The function f extends to $\hat{f}\colon \mathfrak{F}(M) \to Y$, which is a lift of $\mathrm{id}_{\mathfrak{F}(M)}$. Since Y has MAP, $\mathfrak{F}(M)$ has λ-BAP. □

Corollary 3. *A separable Banach space X has λ-BAP if and only if for any Banach space Y and any $f \in \mathrm{Lip}^1(X, Y^{**})$, there is a net in $\mathrm{Lip}^\lambda(X, Y)$ which converges to f in the pointwise-weak* topology.*

This follows immediately from Theorem 2 since X has λ-BAP if and only if $\mathfrak{F}(X)$ has this same property ([GK], Theorem 5.3). Note that we can replace "Lipschitz maps" by "linear operators" in Corollary 3 and reach the same conclusion. In this case, our argument boils down to a method due to Ando ([An]; also see [HWW], section II.2).

3. Some free spaces failing AP

We first prove:

Theorem 4. *Let X be a separable Banach space, and let C be a closed convex set containing 0 such that $\overline{\mathrm{span}}[C] = X$. Then X is isometric to a 1-complemented subspace of $\mathfrak{F}(C)$.*

Proof. The proof relies on a modification from [DL] (see Lemma 2.1 in that paper) of the proof of ([GK], Theorem 3.1). We first recall that since every real-valued Lipschitz map on C extends a Lipschitz map on X with the same Lipschitz constant, the canonical injection from C into X extends to an isometric injection from $\mathfrak{F}(C)$ into $\mathfrak{F}(X)$ (see [GK], Lemma 2.3). Thus we simply consider $\mathfrak{F}(C)$ as a subspace of $\mathfrak{F}(X)$.

Let $(x_i)_{i\geq 1}$ be a linearly independent sequence of vectors in $C/2$ such that $\overline{\text{span}}[\{x_i\ ;\ i \geq 1\}] = X$ and $\|x_i\| = 2^{-i}$ for all i. We let $E = \text{span}[\{x_i\ ;\ i \geq 1\}]$. We denote by $H = [0,1]^{\mathbb{N}}$ the Hilbert cube, by $t = (t_j)_j$ a generic element of H, and by λ the product of the Lebesgue measures on each factor of H. Of course, λ is a probability measure on H. Moreover, for any $n \in \mathbb{N}$, we denote $H_n = [0,1]^{\mathbb{N}\setminus\{n\}}$ and λ_n the similar probability measure on H_n.

We denote $R\colon E \to \mathfrak{F}(X)$ as the unique linear map which satisfies for all $n \geq 1$ and all $f \in \text{Lip}_0(X)$,

$$R(x_n)(f) = \int_{H_n} \big[f(x_n + \sum_{j\neq n} t_j x_j) - f(\sum_{j\neq n} t_j x_j)\big]\, d\lambda_n(t).$$

It is clear that the map R actually takes its values in the subspace $\mathfrak{F}(C)$ of $\mathfrak{F}(X)$. If f is Gâteaux-differentiable, then Fubini's theorem shows that

$$R(x)(f) = \int_H \langle \{\nabla f\}(\sum_j t_j x_j), x\rangle\, d\lambda(t),$$

and thus $|R(x)(f)| \leq \|x\|\,\|f\|_L$. Since the subset of the unit ball of $\text{Lip}_0(X)$ consisting of functions which are Gâteaux-differentiable is uniformly dense in this unit ball (see [BL], Corollary 6.43), it follows that $\|R\| \leq 1$. Since E is dense in X, the map R extends to a linear operator of norm 1 from X to $\mathfrak{F}(C)$, which we still denote by R.

If β denotes the canonical quotient map from $\mathfrak{F}(X)$ onto X (see [GK], Lemma 2.4), we have $\beta R = Id_X$, and thus $R(X)$ is a subspace of $\mathfrak{F}(C)$ isometric to X and 1-complemented by the projection $R\beta$. □

The main corollary of this result is the following.

Corollary 5. *There exists a compact metric space K such that $\mathfrak{F}(K)$ fails AP.*

Proof. Let X be a separable Banach space failing AP. It is classical and easily seen that there is a compact convex set K containing 0 such that $\overline{\text{span}}[K] = X$. By Theorem 4, the space $\mathfrak{F}(K)$ contains a complemented subspace failing AP, and thus $\mathfrak{F}(K)$ itself fails AP. □

Example C. This result emphasizes the need to decide for which metric spaces M - and in particular for which compact metric spaces - the corresponding free space has AP. It is well-known that MAP holds when K is an interval of the real line since then $\mathfrak{F}(K)$ is isometric to L^1 and, more generally, if M is any subset of the real line since then $\mathfrak{F}(M)$ is 1-complemented in L^1. If C is a closed convex subset of the Hilbert space ℓ_2, then $\mathfrak{F}(C)$ has MAP. Indeed C is a 1-Lipschitz retract of ℓ_2, and thus $\mathfrak{F}(C)$ is 1-complemented in $\mathfrak{F}(\ell_2)$ which has MAP by ([GK], Theorem 5.3). A metric space M is isometric to a subset of a metric tree T if and only if $\mathfrak{F}(M)$ embeds isometrically into L^1 ([Go]). It follows from [Mat] that for any such M the space $\mathfrak{F}(M)$ has BAP. Finally, it is shown in [LP] among other things that for any $n \geq 1$ the space $\mathfrak{F}(\mathbf{R}^n)$ has a basis and that $\mathfrak{F}(M)$ has (BAP) for any doubling metric space M.

We now observe that "small" Cantor sets yield to free spaces with MAP.

Proposition 6. *Let K be a compact metric space such that there exist a sequence $(\epsilon_n)_n$ tending to 0, a real number $\rho < 1/2$ and finite ϵ_n-separated subsets N_n of K which are $\rho\epsilon_n$-dense in K. Then $\mathfrak{F}(K)$ has MAP.*

Proof. It follows from ([Bo], Theorem 4) that if M is a separable metric space and $(M_n)_n$ is an increasing sequence of finite subsets of M whose union is dense in M, then $\mathfrak{F}(M)$ has BAP if and only if there is a uniformly bounded sequence of linear operators $E_n\colon \mathrm{Lip}(M_n) \to \mathrm{Lip}(M)$ such that if R_n denotes the restriction operator to M_n, then for every $f \in \mathrm{Lip}(M)$ the sequence $f_n = E_n R_n(f)$ converges pointwise to f. Our assumptions imply the existence of λ-Lipschitz retractions P_n from K onto N_n, with $\lambda = (1-2\rho)^{-1}$, and then $E_n(f) = f \circ P_n$ shows that $\mathfrak{F}(K)$ has BAP. To conclude the proof, we observe that in the notation of ([We], Definition 3.2.1), the little Lipschitz space $\mathrm{lip}_0(K)$ uniformly separates the points in K (use the characteristic functions of the balls of radius $\rho\epsilon_n$ centered at points in N_n), and thus by ([We], Theorem 3.3.3) the space $\mathfrak{F}(K)$ is isometric to the dual space of $\mathrm{lip}_0(K)$. Now Grothendieck's theorem shows that $\mathfrak{F}(K)$ has MAP since it is a separable dual with BAP. □

We refer to [Ka1] for more on little Lipschitz spaces and the "snowflaking" operation. On the other hand, Corollary 5 provides a negative result. It should be noted that the existence of finite nested metric spaces with no "good" extension operator for Lipschitz functions is known (see Lemma 10.5 in [BB]). This is obtained below from Corollary 5 by abstract nonsense. Conversely, it would be interesting to exhibit spaces failing (AP) from combinatorial considerations on finite metric spaces.

Proposition 7. *For any $\lambda \geq 1$, there exist a finite metric space H_λ and a subset G_λ of H_λ such that if $E\colon \mathrm{Lip}(G_\lambda) \to \mathrm{Lip}(H_\lambda)$ is a linear operator such that $RE = \mathrm{id}_{\mathrm{Lip}(G_\lambda)}$ (where R is the operator of restriction to G_λ), then $\|E\| \geq \lambda$.*

Proof. Let K be a compact metric space such that $\mathfrak{F}(K)$ fails AP and let $(G_n)_n$ be an increasing sequence of finite subsets of K whose union is dense in K. Assume that Proposition 7 fails for some $\lambda_0 \in \mathbb{R}$ and thus that extension operators with norm bounded by λ_0 exist for all pairs (G, H) of finite metric spaces with $G \subset H$. For any given n, we can apply this to (G_n, G_k) with $k \geq n$ and use a diagonal argument to get an operator $E_n\colon \mathrm{Lip}(G_n) \to \mathrm{Lip}(K)$ with $R_n E_n = \mathrm{id}_{\mathrm{Lip}(G_n)}$ (where R_n is the operator of restriction to G_n) and $\|E_n\| \leq \lambda_0$. The operator E_n is conjugate to a projection from $\mathfrak{F}(K)$ onto $\mathfrak{F}(G_n)$, and it follows that $\mathfrak{F}(K)$ has λ_0-BAP (with a sequence of projections), contradicting our assumption on K. □

Our work leads to a number of natural questions. We conclude this note by stating some of them. The first one is due to N. J. Kalton (see [Ka3], Problem 1):

Question 1. Let M be an arbitrary uniformly discrete metric space; that is, there exists $\theta > 0$ such that $d(x, y) \geq \theta$ for all $x \neq y$ in M. Does $\mathfrak{F}(M)$ have the BAP? Note that AP holds by ([Ka1], Proposition 4.4). Proposition 7 shows that a simple step-by-step approach could not suffice. A positive answer to Question 1 would imply that every separable Banach space X is approximable; that is, the identity id_X is pointwise limit of an equi-uniformly continuous sequence of maps with relatively compact range. Note that by ([Ka3], Theorem 4.6) it is indeed so for X and X^* when X^* is separable. On the other hand, a negative answer to Question 1 would provide an equivalent norm on ℓ_1 failing MAP, and this would solve two classical problems in approximation theory ([Ca], Problems 3.12 and 3.8).

Question 2. Is there a countable compact space K such that $\mathfrak{F}(K)$ fails (AP)?

Question 3. Let X be a separable Banach space. Does there exist a compact convex subset K of X containing 0 such that $\overline{\text{span}}[K] = X$ and moreover K is a Lipschitz retract of X? Note that when it is so, X has BAP if and only if $\mathfrak{F}(K)$ has BAP. The answer to this question is positive when X has an unconditional basis: indeed if all the coordinates of $x \in X$ are strictly positive, the order interval $[-x, x] = K$ works since truncation by x shows that K is a Lipschitz retract.

Question 4. According to ([GK], Definition 5.2), a separable Banach space X has λ-Lipschitz BAP if id_X is the pointwise limit of a sequence F_n of λ-Lipschitz maps *with finite-dimensional range*, and this property is shown in [GK] to be equivalent with the usual λ-BAP. Is it possible to dispense with the assumption that the F_n's have finite-dimensional range and still reach the conclusion? Corollary 5 suggests that this improvement should not be straightforward.

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