# FREE EXTREME POINTS OF FREE SPECTRAHEDROPS AND GENERALIZED FREE SPECTRAHEDRA

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ABSTRACT. Matrix convexity generalizes convexity to the dimension free setting and has connections to many mathematical and applied pursuits including operator theory, quantum information, noncommutative optimization, and linear control systems. In the setting of classical convex sets, extreme points are central objects which exhibit many important properties. For example, the well-known Minkowski theorem shows that any element of a closed bounded convex set can be expressed as a convex combination of extreme points. Extreme points are also of great interest in the dimension free setting of matrix convex sets; however, here the situation requires more nuance.

In the dimension free setting, there are many different types of extreme points. Of particular importance are free extreme points, a highly restricted type of extreme point that is closely connected to the dilation theoretic Arveson boundary. If free extreme points span a matrix convex set through matrix convex combinations, then they satisfy a strong notion of minimality in doing so. However, not all closed bounded matrix convex sets even have free extreme points. Thus, a major goal is to determine which matrix convex sets are spanned by their free extreme points.

Building on a recent work of J. W. Helton and the author which shows that free spectrahedra, i.e., dimension free solution sets to linear matrix inequalities, are spanned by their free extreme points, we establish two additional classes of matrix convex sets which are the matrix convex hull of their free extreme points. Namely, we show that closed bounded free spectrahedrops, i.e, closed bounded projections of free spectrahedra, are the span of their free extreme points. Furthermore, we show that if one considers linear operator inequalities that have compact operator defining tuples, then the resulting "generalized" free spectrahedra are spanned by their free extreme points.

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## 1. Introduction

A linear matrix inequality (LMI) is an inequality of the form

$$L_A(x) := I - A_1x_1 - \dots A_gx_g \succeq 0$$

where the $A_1, \dots, A_g$ are $d \times d$ symmetric matrices and the $x_1, \dots, x_g$ are real numbers. The solution set of such an LMI, i.e., the set of $x \in \mathbb{R}^g$ such that $L_A(x) \succeq 0$, is a convex set called a spectrahedron. Spectrahedra have connections to many areas including convex analysis, optimization, and linear systems control.

Linear matrix inequalities easily extend to the case where each $X_\ell$ is a $n \times n$ real symmetric matrix whose product with $A_i$ is the Kronecker product. That is,

$$L_A(X) := I_{dn} - A_1 \otimes X_1 - \dots - A_g \otimes X_g \succeq 0.$$

A *free spectrahedron* is the set of all $g$-tuples of symmetric matrices (of any size) such that $L_A(X) \succeq 0$. The term "free" here refers to both the fact that the linear matrix inequality $L_A(X) \succeq 0$ is defined independent of the size of the matrices $X_\ell$, and the fact that the corresponding free spectrahedron contains matrix tuples of all sizes $n \times n$.

Free spectrahedra are prototypical examples of sets that exhibit a dimension-free type of convexity. Namely, free spectrahedra are *matrix convex*, i.e., are closed under *matrix convex combinations* where contraction matrices summing to the identity play the role of the convex coefficients. An important feature of matrix convex combinations is that they allow for combinations of matrix tuples of different sizes. This means that the geometry of any individual level of a matrix convex set is connected to that of all other levels of the set.

Another important example of matrix convex sets are projections of free spectrahedra [34]. Thanks to their association to linear matrix inequalities, these so called free spectrahedrops are more tractable than general matrix convex sets while also being more general than free spectrahedra. Free spectrahedrops and free spectrahedra have for example found use in optimization settings where one would like to determine if a given convex set is "LMI representable", i.e., if it is the projection of a spectrahedron [33]. In addition, free spectrahedrops and free spectrahedra come up in problems related to spectrahedral inclusion [32, 12, 27, 54, 35] and in linear control engineering [36].

Mirroring the role of extreme points in classical convex sets, extreme points play a key role in the understanding of matrix convex sets [24, 28, 45, 22, 14]. However, in the dimension free setting of matrix convexity, there are many types of extreme point. A central goal in the study of matrix convex sets is to determine the most restricted type of extreme point that can recover any element of a given closed bounded matrix convex set through matrix convex combinations. That is, one searches for the strongest possible extension of the classical Minkowski theorem to the dimension free setting.

Of particular note in this pursuit are matrix extreme points [53, 25, 26] and free extreme points [39]. Matrix extreme points are known to span general closed bounded matrix convex sets through matrix convex combinations [40, 31]; however, they are not necessarily a minimal spanning set. Free extreme points, on the other hand, are more restricted than matrix extreme points [24, 20]. Stated informally, a free extreme point of a matrix convex set is an element that can only be expressed via trivial matrix convex combinations of other elements. Free extreme points are of great interest due both to their close connection [24] to the dilation theoretic Arveson boundary [3, 5] and to the fact that they necessarily form a minimal spanning set if they do span. The short coming of free extreme points is that, when restricting to finite dimensions, they can fail to span a given closed bounded matrix convex set. In fact, there are closed bounded matrix convex sets which have no (finite dimensional) free extreme points at all [21]. Thus an important question in the pursuit of a dimension free Minkowski theorem is "which matrix convex sets are the matrix convex hull of their free extreme points".

Recently, [23] showed that in the case of bounded free spectrahedra, free extreme points span. Furthermore, [23] showed that there is a tight dimension bound on the *sum of sizes* of free extreme points needed to express an element of a bounded free spectrahedron as a matrix convex combination of free extreme points. The main contribution of the present article is an extension of this result to two additional classes of matrix convex sets. Namely we show that closed bounded free spectrahedrops and closed bounded "generalized free spectrahedra", i.e., closed bounded solution sets to linear operator inequalities with compact defining tuples, are the matrix convex hull of their free extreme points. An informal statement of this result is as follows.

*Let $K$ be a real closed bounded free spectrahedrop or a real closed bounded generalized free spectrahedron and let $X \in K$ be a $g$-tuple of real symmetric $n \times n$ matrices. Then $X$ is a finite matrix convex combination of free extreme points of $K$ whose sum of sizes is bounded by $n(g+1)$.*

The proof of this result follows the same approach used in [23] which itself was inspired by works such as [1, 19, 13]. Namely, we show the existence of a special type of dilation called a *maximal* 1*-dilation*. We then show that constructing finite sequence of at most $ng$ maximal 1-dilations of $X$ results in a so called *Arveson boundary point* of $K$. Determining the irreducible components of the resulting Arveson boundary point yields an expression of $X$ as a matrix convex combination of free extreme points of $K$. An additional key ingredient which is need when $K$ is a free spectrahedrop is [18, Theorem 3.2] which allows us to express a free spectrahedrop as an intersection of free spectrahedra in a particularly well-behaved manner. A self contained version of this result which has been adapted to our setting is presented as Theorem 2.1.

We point the reader to the upcoming Theorem 2.3 for a formal statement of our main result. Also see Sections 2.2 and 2.3 for formal definitions of generalized free spectrahedra and free spectrahedrops.

1.1. **Related works.** The study of extreme points of matrix convex sets goes back Arveson who, (in the language of completely positive maps on operator systems) conjectured that if one extends to infinite dimensional levels, then infinite dimensional free extreme points span [3, 4]. This infinite dimensional question was studied by a number of authors [30, 1, 43, 19, 5, 39, 44] until it was finally settled in 2015 by Davidson and Kennedy [13].

Matrix convexity is also closely related to the rapidly growing area of free analysis [7, 49, 16, 51, 38, 2, 6, 37]. Here the goal is to extend classical geometric and function theoretic results to the noncommutative setting where one considers functions whose inputs are $g$-tuples of matrices or operators. This study was largely pushed by Voiculescu's introduction of free probability which has since been used to great effect in random matrix theory [50, 42]. Other closely related topics include noncommutative optimization [17, 48, 10, 41, 52] and quantum information and games [9, 8, 29, 15, 47].

1.2. **Reader's guide.** Section 2 introduces our definitions and notation and gives a formal statement of our main result, Theorem 2.3. Section 3 introduces the notion of a maximal 1-dilation for free spectrahedrops and shows that maximal 1-dilations in a bounded free spectrahedrop reduce the dimension of the dilation subspace. Section 4 is the equivalent of Section 3 in the case of generalized free spectrahedra. Section 5 discusses the relationship between generalized free spectrahedra and free spectrahedrops and gives an example of a generalized free spectrahedron that is not a free spectrahedrop.

## 2. Definitions, notation, and main results

Throughout the article we let $\mathcal{H}$ denote a real Hilbert space. $B(\mathcal{H})$ and $SA(\mathcal{H})$ respectively denote the sets of bounded operators on $\mathcal{H}$ and bounded self-adjoint operators on $\mathcal{H}$. Additionally let $SA(\mathcal{H})^g$ denote the set of $g$-tuples of the form $X = (X_1, \dots, X_g)$ where each $X_\ell$ is a bounded self-adjoint operator on $\mathcal{H}$. Say tuples $X, Y \in SA(\mathcal{H})^g$ are **unitarily equivalent** if there exists a unitary $U \in B(\mathcal{H})$ such that

$$U^* X U = (U^* X_1 U, \dots, U^* X_g U) = (Y_1, \dots, Y_g) = Y.$$

A bounded operator $B \in B(\mathcal{H})$ is **positive semidefinite** if it is self-adjoint and for any $v \in \mathcal{H}$, one has $\langle Bv, v\rangle \geq 0$. In the case that $B$ is a compact self-adjoint operator, this is equivalent to all of $B$'s eigenvalues being nonnegative. Given two bounded self-adjoint operators $B_1, B_2 \in SA(\mathcal{H})$ let $B_1 \succeq B_2$ denote that $B_1 - B_2$ is positive semidefinite.

While general real Hilbert spaces do play a role in this article, our main interest is matrix convex sets which are restricted to finite dimensions. Thus it is convenient so also have notation for matrix spaces. We let $M_{m\times n}(\mathbb{R})^g$ denote the set of $g$-tuples of $m\times n$ matrices with real entries, and let $M_n(\mathbb{R})^g = M_{n\times n}(\mathbb{R})^g$. Additionally, $SM_n(\mathbb{R})^g$ is the set of all $g$-tuples of real self-adjoint (symmetric) $n \times n$ matrices. Additionally we set $SM(\mathbb{R})^g = \cup_n SM_n(\mathbb{R})^g$. Given a subset $K \subseteq SM(\mathbb{R})^g$, we let $K(n)$ denote the set

$$K(n) := K \cap SM_n(\mathbb{R})^g.$$

That is $K(n)$ is set of $g$-tuples of $n \times n$ matrices that are elements of $K$. The set $K(n)$ is called the **$n$th level of** $K$.

For clarity we emphasize that, although we restrict to real settings, we use the term self-adjoint rather than symmetric to provide consistency in the terminology used for matrices and operators. Similarly, to be consistent with this terminology choice, we use $B^*$ rather than $B^T$ to denote the transpose of a matrix $B$.

2.1. **Linear operator inequalities.** Given a $g$-tuple

$$A = (A_1, \ldots, A_g) \in SA(\mathcal{H})^g$$

of self-adjoint operators on $\mathcal{H}$, a **monic linear pencil** is a sum of the form

$$L_A(x) = I_{\mathcal{H}} - A_1x_1 - A_2x_2 - \cdots - A_gx_g.$$

Given a tuple $X \in SM_n(\mathbb{R})^g$, the **evaluation** of $L_A$ at $X$ is

$$L_A(X) = I_{\mathcal{H}} \otimes I_n - A_1 \otimes X_1 - A_2 \otimes X_2 - \cdots - A_g \otimes X_g$$

where $\otimes$ denotes the Kronecker tensor product. Additionally, let $\Lambda_A(X)$ denote the homogeneous linear part of $L_A(X)$. That is,

$$\Lambda_A(X) = A_1 \otimes X_1 + A_2 \otimes X_2 + \cdots + A_g \otimes X_g,$$

The inequality

$$L_A(X) \succeq 0,$$

is called a **linear operator inequality**. Moreover, if $\mathcal{H}$ is finite dimension, then the inequality $L_A(X) \succeq 0$ is called a **linear matrix inequality**.

2.2. **Free spectrahedra.** Given a $g$-tuple $A \in SA(\mathcal{H})^g$ and a positive integer $n$ define the set $\mathcal{D}_A(n) \subseteq SM_n(\mathbb{R})^g$ by

$$\mathcal{D}_A(n) := \{X \in SM_n(\mathbb{R})^g : L_A(X) \succeq 0\}.$$

That is, $\mathcal{D}_A(n)$ is the set of all $g$-tuples of $n \times n$ real symmetric matrices $X$ such that the evaluation $L_A(X)$ is positive semidefinite. Additionally define the set $\mathcal{D}_A \subseteq SM(\mathbb{R})^g$ to be the union over all $n$ of the the sets $\mathcal{D}_A(n)$. That is,

$$\mathcal{D}_A := \bigcup_{n=1}^{\infty} \mathcal{D}_A(n).$$

If $\mathcal{H}$ is finite dimensional, then $\mathcal{D}_A$ is called a **free spectrahedron** and $\mathcal{D}_A(n)$ is the **free spectrahedron $\mathcal{D}_A$ at level $n$** [24] . In this article we extend to considering the case where $\mathcal{H}$ is infinite dimensional and $A$ is a compact operator. In this case we call $\mathcal{D}_A$ a **generalized free spectrahedron** and we call $\mathcal{D}_A(n)$ a **generalized free spectrahedron at level $n$**. That is, a generalized free spectrahedron $\mathcal{D}_A$ is the solution set of the linear operator inequality

$$L_A(X) \succeq 0$$

where $A$ is a tuple of self-adjoint compact operators on $\mathcal{H}$.

2.3. **Free spectrahedrops.** Central to this article are (coordinate) projections of free spectrahedra, i.e., free spectrahedrops. Given a finite dimensional Hilbert space $\mathcal{H}$ and a $g$-tuple $A \in SA(\mathcal{H})^g$ and an $h$-tuple $B \in SA(\mathcal{H})^h$ of self-adjoint operators on $\mathcal{H}$, let $L_{(A,B)}(x, y)$ denote the monic linear pencil

$$L_{(A,B)}(x, y) = I_{\mathcal{H}} - A_1x_1 - \cdots - A_gx_g - B_1y_1 - \cdots - B_hy_y.$$

Also let $\mathcal{D}_{(A,B)}$ denote the free spectrahedron

$$\mathcal{D}_{(A,B)} = \{(X, Y) \in SM(\mathbb{R})^{g+h} : L_{(A,B)}(X, Y) \succeq 0\}.$$

Define the set $\text{proj}_x \mathcal{D}_{(A,B)}$ by

$$\text{proj}_x \mathcal{D}_{(A,B)} := \{X \in SM(\mathbb{R})^g : \text{ There exists } Y \in SM(\mathbb{R})^h \text{ such that } (X, Y) \in \mathcal{D}_{(A,B)}\}.$$

The set $\text{proj}_x \mathcal{D}_{(A,B)}$ is a called a **free spectrahedrop**.

Let $K$ be a free spectrahedrop or generalized free spectrahedron. Say $K$ is **bounded** if there is some real number $C$ so that

$$CI_n - \sum_{i=1}^{g} X_i^2 \succeq 0$$

for all $X = (X_1, X_2, \ldots, X_g) \in K(n)$ and all positive integers $n$. It is not difficult to show that such a set $K$ is bounded if and only if $K(1)$ is bounded.

In addition, we say $K$ is **closed** if $K(n)$ is closed for all $n$. We note that the generalized free spectrahedra considered in this article are, by definition, closed in this sense. However, free spectrahedrops are not necessarily closed, as the projection of an (unbounded) closed set can fail to be closed.

2.4. **Drescher, Netzer, and Thom vs Free spectrahedrops.** Critical to the proof of our main result for free spectrahedrops is a result of Drescher, Netzer, and Thom [18], which affords us a clean membership test for elements of a free spectrahedrop. To present this result, we introduce the following notation.

Given $d$-dimensional Hilbert space $\mathcal{H}$ and an $h$-tuple $B \in SA(\mathcal{H})^h$ of self-adjoint operators on $\mathcal{H}$, let $B^{\oplus n}$ denote the direct sum of $B$ with itself $n$ times. Additionally, define the sets $\mathcal{Z}_B(n)$ and $\mathcal{I}_B(n,r)$ and $\mathcal{I}_B(n)$ by

$$\begin{aligned}
\mathcal{Z}_B(n) &= \{W \in B(\mathcal{H}^{nd}, \mathbb{R}^n) : \ W^*(B^{\oplus nd})W = 0_{n\times n}\} \\
\mathcal{I}_B(n,r) &= \{W \in B(\mathcal{H}^{nd}, \mathbb{R}^r) : \ W^*(B^{\oplus nd})W = 0_{r\times r} \text{ and } W^*W = I_r\} \\
\mathcal{I}_B(n) &= \cup_{r\leq n} \mathcal{I}_B(n,r).
\end{aligned}$$

We will frequently make use of the fact that, for fixed $n$ and $r$, the set $\mathcal{I}_B(n,r)$ is compact.

**Theorem 2.1** ([18, Theorem 3.2, Proposition 3.3])**.** *Let $A \in SM_d(\mathbb{R})^g$ and $B \in SM_d(\mathbb{R})^h$ such that $\operatorname{proj}_x \mathcal{D}_{(A,B)}$ is a closed bounded free spectrahedrop and $X \in SM_n(\mathbb{R})^g$. Then we have the following*

*(1) $X \in \operatorname{proj}_x \mathcal{D}_{(A,B)}(n)$ if and only if for all $W \in \mathcal{Z}_B(n)$ one has*

$$W^*W \otimes I_n - \Lambda_{W^*\left(A^{\oplus nd}\right)W}(X) \succeq 0. \tag{2.1}$$

*(2) $X \in \operatorname{proj}_x \mathcal{D}_{(A,B)}(n)$ if an only if for all $W \in \mathcal{I}_B(n)$ one has*

$$L_{W^*\left(A^{\oplus nd}\right)W}(X) \succeq 0. \tag{2.2}$$

*(3) If $X \in \operatorname{proj}_x \mathcal{D}_{(A,B)}(n)$, then for any $m \in \mathbb{N}$ and any $W \in \mathcal{I}_B(m)$, one has*

$$L_{W^*\left(A^{\oplus md}\right)W}(X) \succeq 0.$$

*Proof.* The proof of Item (1) is essentially the same as that of [18, Theorem 3.2], with minor modifications for the real setting. The proof of Item (2) follows from the same approach used to prove [18, Proposition 3.3]. Finally, the proof of Item (3) quickly follows from Items (1) and (2) together a straightforward argument using techniques that are routine in the study of matrix convex sets. Since this theorem plays a key role in our upcoming proofs, and since the setting and statement here is slightly different than in [18], a self-contained version of the proof is given in the Appendix. ∎

2.5. **Matrix Convex Sets.** Given a finite collection of $g$-tuples $\{X^i\}_{i=1}^{\ell}$ with $X^i \in SM_{n_i}(\mathbb{R})^g$ for each $i = 1, 2, \dots, \ell$, a **matrix convex combination** of $\{X^i\}_{i=1}^{\ell}$ is a sum of the form

$$\sum_{i=1}^{\ell} V_i^* X^i V_i \qquad \text{with} \qquad \sum_{i=1}^{\ell} V_i^* V_i = I_n$$

where $V_i \in M_{n_i \times n}(\mathbb{R})$ and

$$V_i^* X^i V_i = \left(V_i^* X_1^i V_i, V_i^* X_2^i V_i, \dots, V_i^* X_g^i V_i\right) \in SM_n(\mathbb{R})^g$$

for all $i = 1, 2, \dots, \ell$. A key feature of matrix convex combinations is that the tuples $X^i$ need not be the same size.

A set $K \subseteq SM(\mathbb{R})^g$ is **matrix convex** if it is closed under matrix convex combinations and the **matrix convex hull** of $K$, denoted $\mathrm{co}^{\mathrm{mat}}(K)$, is the set of all matrix convex combinations of the elements of $K$. Equivalently, $K$ is matrix convex if and only if $K = \mathrm{co}^{\mathrm{mat}}(K)$. It is straightforward to show that generalized free spectrahedra and free spectrahedrops are matrix convex.

2.6. **Free extreme points of matrix convex sets.** As previously mentioned, matrix convex sets have many different types of extreme points. In this article, we restrict our attention to free extreme points. Given a set $K \subseteq SM(\mathbb{R})^g$, we say a point $X \in K(n)$ is a **free extreme point** of $K$ if whenever $X$ is written as a matrix convex combination

$$X = \sum_{i=1}^{\ell} V_i^* X^i V_i \qquad \text{with} \qquad \sum_{i=1}^{\ell} V_i^* V_i = I_n$$

of points $X^i \in K$ with $V_i \neq 0$ for each $i$, then for all $i = 1, 2, \dots, \ell$ either $V_i \in M_n(\mathbb{R})$ and $X$ is unitarily equivalent to $X^i$ or $V_i \in M_{n_i \times n}(\mathbb{R})$ where $n_i > n$ and there exists $Z^i \in K$ such that $X \oplus Z^i$ is unitarily equivalent to $X^i$. Intuitively, a tuple $X$ is a free extreme point of $K$ if it cannot be written as a nontrivial matrix convex combination of any collection of elements of $K$. We let $\partial^{\mathrm{free}} K$ denote the set of all free extreme points of $K$.

2.7. **Free extreme points, dilations, and the Arveson boundary.** We next discuss the connection between free extreme points and the dilation theoretic Arveson boundary. To do this, we must first introduce the notion of an irreducible tuple of matrices.

Given a matrix $M \in M_n(\mathbb{R})$, a subspace $N \subseteq \mathbb{R}^n$ is a **reducing subspace** if both $N$ and $N^\perp$ are invariant subspaces of $M$, A tuple $X \in SM_n(\mathbb{R})^g$ is **irreducible** (over $\mathbb{R}$) if the matrices $X_1, \dots, X_g$ have no common reducing subspaces in $\mathbb{R}^n$; a tuple is **reducible** (over $\mathbb{R}$) if it is not irreducible. For the remainder of the article, we drop the distinction "over $\mathbb{R}$" when referring to irreducible tuples. However, we warn the reader that a tuple may be reducible over $\mathbb{C}$ even if it is irreducible over $\mathbb{R}$. Thus irreducibility over $\mathbb{R}$ is not equivalent to other well-known definitions of irreducibility which are instead equivalent to irreducibility over $\mathbb{C}$.

2.7.1. *Dilations.* Let $K \subseteq SM(\mathbb{R})^g$ be a matrix convex set and let $X \in K(n)$. If there exists a positive integer $\ell \in \mathbb{N}$ and $g$-tuples $\beta \in M_{n\times\ell}(\mathbb{R})^g$ and $\gamma \in SM_\ell(\mathbb{R})^g$ such that

$$Y = \begin{pmatrix} X & \beta \\ \beta^* & \gamma \end{pmatrix} = \left( \begin{pmatrix} X_1 & \beta_1 \\ \beta_1^* & \gamma_1 \end{pmatrix}, \dots, \begin{pmatrix} X_g & \beta_g \\ \beta_g^* & \gamma_g \end{pmatrix} \right) \in K,$$

then we say $Y$ **is an $\ell$-dilation of** $X$. The tuple $Y$ is a **trivial dilation** of $X$ if $\beta = 0$. A key connection between matrix convex combinations and dilations is the following. If $Y$ is a dilation of $X$ and $V^* = \begin{pmatrix} I_n & 0 \end{pmatrix}$, then $X = V^*YV$ with $V^*V = I_n$. That is, if $Y$ is a dilation of $X$, then $X$ can be expressed as a matrix convex combination of $Y$. We note that this matrix convex combination is non-trivial if the dilation itself is non-trivial.

Given a matrix convex set $K$ and an element $X \in K$, we define the **dilation subspace of** $K$ **at** $X$, denoted $\mathfrak{D}_X(K)$, to be

$$\mathfrak{D}_X(K) = \operatorname{span}\left( \left\{ \beta \in M_{n\times 1}(\mathbb{R})^g : \text{there exists a } \gamma \in \mathbb{R}^g \text{ s.t. } \begin{pmatrix} X & \beta \\ \beta^* & \gamma \end{pmatrix} \in K \right\} \right)$$

See Lemma 3.1 and Lemma 4.4 for further discussion of the dilation subspace.

In the case that $K$ is a free spectrahedron, the definition of the dilation subspace given here is equivalent to the definition given in [23], though the presentation is different, see [23, Lemma 2.1]. We present the definition in this form so that we can use single definition for both free spectrahedrops and generalized free spectrahedra.

2.7.2. *Arveson extreme points.* A tuple $X \in K$ is an **Arveson extreme point** of $K$ if $K$ does not contain a nontrivial dilation of $X$. More precisely, $X \in K$ is an Arveson extreme point of $K$ if and only if, if

$$\begin{pmatrix} X & \beta \\ \beta^* & \gamma \end{pmatrix} \in K(n+\ell) \tag{2.3}$$

for some tuples $\beta \in M_{n\times\ell}(\mathbb{R})^g$ and $\gamma \in SM_\ell(\mathbb{R})^g$, then $\beta = 0$. Equivalently, $X$ is an Arveson extreme point of $K$ if and only if $\dim \mathfrak{D}_X(K) = 0$. If $Y$ is an Arveson extreme point of $K$ and $Y$ is an ($\ell$-)dilation of $X \in K$, then we will say $Y$ is an **Arveson ($\ell$-)dilation** of $X$.

The following theorem gives the connection between free and Arveson extreme points.

**Theorem 2.2** ([24, Theorem 1.1 (3)])**.** *Let $K$ be a matrix convex set. Then $X$ is a free extreme point of $K$ if and only if $X$ is an Arveson extreme point of $K$ and is irreducible.*

This theorem has the consequence that the irreducible components of an Arveson extreme point are all free extreme points. This means that expressing an element $X$ of a matrix convex set as a matrix convex combination of free extreme points can be accomplished by finding an Arveson dilation of $X$.

2.8. **Main result.** We are now in position to give a formal statement of our main result.

**Theorem 2.3.** *Let $K \subseteq SM(\mathbb{R})^g$ be a closed bounded free spectrahedrop or a closed bounded generalized free spectrahedron. Given a tuple $X \in K(n)$, there exists an integer $k$ satisfying*

$$k \leq \dim \mathfrak{D}_X(K) \leq ng \tag{2.4}$$

*and a $k$-dilation $Y \in K(n+k)$ of $X$ such that $Y$ is an Arveson extreme point of $K$.*

*As an consequence, $X$ can be written as a matrix convex combination*

$$X = \sum_{i=1}^{k} V_i^* X^i V_i \qquad \text{s.t.} \qquad \sum_{i=1}^{k} V_i^* V_i = I$$

*of free extreme points $X^i \in K(n_i)$ of $K$ where $\sum_{i=1}^k n_i \leq n+k \leq n(g+1)$. Thus, $K$ is the matrix convex hull of its free extreme points.*

*Proof.* The proof of the first part of Theorem 2.3 quickly follows from Theorems 3.7 and 4.6. In particular, let $X \in K$, and set $\ell = \dim \mathfrak{D}_X(K)$. Assume that $X$ is not an Arveson extreme point of $K$, i.e., that $\ell > 0$. Also note that $\ell \leq ng$ since $\mathfrak{D}_X(K)$ is a subspace of $M_{n\times 1}(\mathbb{R})^g$.

Using Theorem 3.7 when $K$ is a bounded free spectrahedrop and Theorem 4.6 when $K$ is a bounded generalized free spectrahedron shows that there exists an integer $k \leq \ell$ and a collection of tuples $\{Y^i\}_{i=0}^k \subseteq K$ such that the following hold:

(1) $Y^0 = X$.
(2) For each $i = 1, \dots, k-1$, the tuple $Y^{i+1}$ is a 1-dilation of $Y^i$ and

$$\dim \mathfrak{D}_{Y^{i+1}}(K) < \dim \mathfrak{D}_{Y^i}(K).$$

(3) $Y^k$ satisfies $\dim \mathfrak{D}_{Y^k}(K) = 0$.

It follows from Lemmas 3.1 (3) and 4.4 (3) that $Y$ is an Arveson extreme point of $K$. Thus $Y^k \in K(n+k)$ is an Arveson $k$-dilation of $X$.

The proof of the second part of the theorem quickly follows the first part together with routine dilation theoretic arguments. In particular, one can use the same argument as is used to prove the corresponding statement in [23, Theorem 1.3]. ∎

Theorem 2.3 has important implications for another open question in the study of free spectrahedrops. Namely, is there a necessary and sufficient condition for a matrix convex set $K$ to be a closed bounded free spectrahedrop? From [34, Theorem 4.11], if $K \subseteq SM(\mathbb{R})^g$ is a closed bounded free spectraherop, then the **polar dual** of $K$ is also a closed bounded free spectrahedrop. Here the polar dual of $K$, denoted $K^\circ$ is defined by

$$K^\circ = \{A \in SM(\mathbb{R})^g : L_X(A) \succeq 0 \text{ for all } X \in K\}.$$

**Corollary 2.4.** *If $K$ is a closed bounded free spectrahedrop, then both $K$ and $K^\circ$ are the matrix convex hull of their free extreme points.*

*Proof.* Immediate from Theorem 2.3 together with [34, Theorem 4.11]. ∎

In the upcoming Theorem 5.1 and Proposition 5.2 we give an example of a closed bounded matrix convex set $K$ such that $K$ is the matrix convex hull of its free extreme points while $K^\circ$ is not. This illustrates that Corollary 2.4 provides nontrivial restrictions that a matrix convex set $K$ must satisfy to be a closed bounded free spectrahedrop.

2.8.1. *Free extreme points over the complexes.* One can naturally extend the sets $\mathcal{D}_A$ and $\text{proj}_x \mathcal{D}_{(A,B)}$ to include tuples of complex self-adjoint matrices. Theorem 2.3 can be extended to this setting using the same approach as in [23] provided that $A$ and $B$ themselves are tuples of real matrices. Note that the dimension bound given in equation 2.4 changes from $\dim \mathfrak{D}_X(K)$ to $2\dim \mathfrak{D}_X(K)$ in this complex setting. Considering this extension introduces additional notational burden as the fact that the result holds over the reals is important in the proof over the complexes. Since the proof of this extension is identical to the proof in [23], we have omitted details so as to simplify exposition.

We warn the reader that the situation can, however, be markedly different if $A$ and $B$ have complex-valued entries. In fact, [44] gives an example of a tuple of complex self-adjoint matrices $A$ such that if $\mathcal{D}_A$ is allowed to contain complex self-adjoint matrices, then $\mathcal{D}_A$ is a closed bounded complex free spectrahedron which is not the matrix convex hull of its free extreme points.

## 3. Arveson extreme points and free spectrahedrops

This section introduces maximal 1-dilations for free spectrahedrops and presents Theorem 3.7 which shows that maximal 1-dilations in a free spectrahedrop reduce the dimension of the dilation subspace. Since we only consider finite dimensional spaces $\mathcal{H}$ when working with free spectrahedrops, throughout the section we set $\mathcal{H} = \mathbb{R}^d$.

We begin by establishing an alternative definition of the dilation subspace for free spectrahedrops. Let $A \in SM_d(\mathbb{R})^g$ and $B \in SM_d(\mathbb{R})^h$ such that $\text{proj}_x \mathcal{D}_{(A,B)}$ is a closed bounded free spectrahedrop. Given a tuple $X \in \operatorname{proj}_x \mathcal{D}_{(A,B)}$, define the subspace $\mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)}) \subseteq M_{n\times 1}(\mathbb{R})^g$ by

(3.1)
$$\mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)}) = \cap_{W\in \mathcal{I}_B(n+1)}\{\beta \in M_{n\times 1}(\mathbb{R})^g : \ \ker L_{W^*(A^{\oplus n+1})W}(X) \subseteq \ker \Lambda_{W^*(A^{\oplus n+1})W}(\beta^*)\}.$$

Note that $\mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)})$ is indeed a subspace of $M_{n\times 1}(\mathbb{R})^g$ since the set

$$\{\beta \in M_{n\times 1}(\mathbb{R})^g : \ \ker L_{W^*(A^{\oplus n+1})W}(X) \subseteq \ker \Lambda_{W^*(A^{\oplus n+1})W}(\beta^*)\} \tag{3.2}$$

is a subspace of $M_{n\times 1}(\mathbb{R})^g$ for each fixed $W \in \mathcal{I}_B(n+1)$.

The following lemma shows that $\mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)})$ is equal to the dilation subspace of $\operatorname{proj}_x \mathcal{D}_{(A,B)}$ at $X$ and explains the connection between $\mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)})$ and dilations of $X$.

**Lemma 3.1.** *Let $\operatorname{proj}_x \mathcal{D}_{(A,B)}$ be a closed bounded free spectrahedrop and let $X \in \operatorname{proj}_x \mathcal{D}_{(A,B)}(n)$.*

*(1) If $\beta \in M_{n\times 1}(\mathbb{R})^g$ and*
$$Y = \begin{pmatrix} X & \beta \\ \beta^* & \gamma \end{pmatrix} \in \operatorname{proj}_x \mathcal{D}_{(A,B)}(n+1)$$
*is a 1-dilation of $X$, then $\beta \in \mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)})$.*

*(2) Let $\beta \in M_{n\times 1}(\mathbb{R})^g$. Then $\beta \in \mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)})$ if and only if there is a tuple $\gamma \in \operatorname{proj}_x \mathcal{D}_{(A,B)}(1)$ and a real number $c_\gamma > 0$ such that*
$$\begin{pmatrix} X & c_\gamma\beta \\ c_\gamma\beta^* & \gamma \end{pmatrix} \in \operatorname{proj}_x \mathcal{D}_{(A,B)}(n+1).$$
*In particular, one may take $\gamma = 0 \in \mathbb{R}^g$.*

*(3) One has*
$$\mathfrak{D}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)}) = \mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)}).$$
*As a consequence, $X$ is an Arveson extreme point of $\operatorname{proj}_x \mathcal{D}_{(A,B)}$ if and only if*
$$\dim \mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)}) = 0.$$

*Proof.* Item (1) follows from considering the Schur complement of $L_{W^*(A^{\oplus n+1})W}(Y)$ for a dilation
$$Y = \begin{pmatrix} X & \beta \\ \beta^* & \gamma \end{pmatrix} \in \operatorname{proj}_x \mathcal{D}_{(A,B)}(n+1)$$
of $X$. Indeed, multiplying $L_{W^*(A^{\oplus n+1})W}(Y)$ by permutation matrices, sometimes called canonical shuffles, see [46, Chapter 8], shows

$$L_{W^*(A^{\oplus n+1})W}(Y) \succeq 0 \quad \text{if and only if} \quad \begin{pmatrix} L_{W^*(A^{\oplus n+1})W}(X) & \Lambda_{W^*(A^{\oplus n+1})W}(\beta) \\ \Lambda_{W^*(A^{\oplus n+1})W}(\beta^*) & L_{W^*(A^{\oplus n+1})W}(\gamma) \end{pmatrix} \succeq 0. \tag{3.3}$$

Taking the appropriate Schur complement then shows that $L_{W^*(A^{\oplus n+1})W}(Y) \succeq 0$ if and only if

$$\begin{aligned} &L_{W^*(A^{\oplus n+1})W}(\gamma) \succeq 0, \\ \text{and } &L_{W^*(A^{\oplus n+1})W}(X) - \Lambda_{W^*(A^{\oplus n+1})W}(\beta) L_{W^*(A^{\oplus n+1})W}(\gamma)^{\dagger} \Lambda_{W^*(A^{\oplus n+1})W}(\beta^*) \succeq 0, \end{aligned} \tag{3.4}$$

where $\dagger$ denotes the Moore-Penrose pseudoinverse. It follows that if $L_{W^*(A^{\oplus n+1})W}(Y) \succeq 0$, then

$$\ker L_{W^*(A^{\oplus n+1})W}(X) \subseteq \ker \Lambda_{W^*(A^{\oplus n+1})W}(\beta^*).$$

Using Theorem 2.1 (2) shows that $Y \in \operatorname{proj}_x \mathcal{D}_{(A,B)}(n+1)$ if and only if $L_{W^*(A^{\oplus n+1})W}(Y) \succeq 0$ for all $W \in \mathcal{I}_B(n+1)$. Combining this with the above, we conclude that if $Y \in \operatorname{proj}_x \mathcal{D}_{(A,B)}(n+1)$, then

$$\ker L_{W^*(A^{\oplus n+1})W}(X) \subseteq \ker \Lambda_{W^*(A^{\oplus n+1})W}(\beta^*)$$

for all $W \in \mathcal{I}_B(n+1)$.

We now prove Item (2). We first show that for each fixed $W \in \mathcal{I}_B(n+1)$ there exists some constant $c_W > 0$ such that

$$L_{W^*(A^{\oplus n+1})W} \begin{pmatrix} X & c_W \beta \\ c_W \beta^* & 0 \end{pmatrix} \succeq 0. \tag{3.5}$$

Note that $L_{W^*(A^{\oplus n+1})W}(0) = I$, so similar to before using the Schur complement shows the above inequality holds if and only if

$$L_{W^*(A^{\oplus n+1})W}(X) - c_W^2 \Lambda_{W^*(A^{\oplus n+1})W}(\beta) \Lambda_{W^*(A^{\oplus n+1})W}(\beta^*) \succeq 0. \tag{3.6}$$

If $\beta \in \mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)})$ then

$$\ker L_{W^*(A^{\oplus n+1})W}(X) \subseteq \ker \Lambda_{W^*(A^{\oplus n+1})W}(\beta) \Lambda_{W^*(A^{\oplus n+1})W}(\beta^*),$$

so picking $c_W > 0$ small enough so that $\|c_W^2 \Lambda_A(\beta) \Lambda_A(\beta^*)\|_2$ is less than the smallest nonzero eigenvalue of $L_{W^*(A^{\oplus n+1})W}(X)(X)$ guarantees that inequality (3.6) holds, hence inequality (3.5) holds.

Now for each $W \in \mathcal{I}_B(n+1)$, set

$$\begin{aligned} \tilde{c}_W := \quad &\sup_{c_W \in \mathbb{R}} \quad c_W \\ &\text{s.t.} \quad L_{W^*(A^{\oplus n+1})W} \begin{pmatrix} X & c_W \beta \\ c_W \beta^* & 0 \end{pmatrix} \succeq 0. \end{aligned}$$

Note that our previous argument shows that $\tilde{c}_W > 0$ for all $W \in \mathcal{I}_B(n+1)$. Additionally, again considering the Schur complement shows that for each fixed $W$ one has

$$L_{W^*(A^{\oplus n+1})W}\begin{pmatrix} X & \alpha\beta \\ \alpha\beta^* & 0 \end{pmatrix} \succeq 0$$

for all $\alpha \in [0, \tilde{c}_W]$.

Now set

$$\tilde{c} = \inf_{W \in \mathcal{I}_B(n+1)} \tilde{c}_W.$$

If $\tilde{c} = \infty$, then we have that

$$L_{W^*(A^{\oplus n+1})W}\begin{pmatrix} X & \beta \\ \beta^* & 0 \end{pmatrix} \succeq 0$$

for all $W \in \mathcal{I}_B(n+1)$, in which case the result follows. On the other hand, if $\tilde{c} < \infty$, then since $\mathcal{I}_B(n+1, r)$ is compact for each $r \leq n$ and since $\mathcal{I}_B(n+1) = \cup_{r \leq n+1} \mathcal{I}_B(n+1, r)$, a straightforward argument shows that there is some $W' \in \mathcal{I}_B(n+1)$ such that

$$\tilde{c} = \tilde{c}_{W'} \qquad \text{hence} \qquad \tilde{c} > 0.$$

Since $0 < \tilde{c} \leq \tilde{c}_W$ for all $W \in \mathcal{I}_B(n+1)$ we conclude that

$$L_{W^*(A^{\oplus n+1})W}\begin{pmatrix} X & \tilde{c}\beta \\ \tilde{c}\beta^* & 0 \end{pmatrix} \succeq 0$$

for all $W \in \mathcal{I}_B(n+1)$. Using Theorem 2.1 then shows that

$$\begin{pmatrix} X & \tilde{c}\beta \\ \tilde{c}\beta^* & 0 \end{pmatrix} \in \operatorname{proj}_x \mathcal{D}_{(A,B)},$$

as claimed. The reverse direction is a consequence of Item (1).

Item (3) follows from Items (1) and (2). ■

**Remark 3.2.** *As in [23], the ability to take $\gamma = 0 \in \mathbb{R}^g$ in Lemma 3.1 (2) helps simplify the NC LDL$^*$ calculations used in the upcoming proof of Theorem 3.7.*

We will soon define maximal 1-dilations for a given element of a free spectrahedrop. Before doing so, we give two lemmas which together imply this upcoming notion is well-defined.

**Lemma 3.3.** *Let $A \in SM_d(\mathbb{R})^g$ and $B \in SM_d(\mathbb{R})^h$ and assume $\operatorname{proj}_x \mathcal{D}_{(A,B)}$ is a closed bounded free spectrahedrop. Given a tuples $X \in \operatorname{proj}_x \mathcal{D}_{(A,B)}(n)$ and $\beta \in M_{n\times 1}(\mathbb{R})^g$, the* *optimization problem*

$$\begin{aligned} \tilde{\alpha} := \quad & \sup_{\alpha \in \mathbb{R}, \gamma \in \mathbb{R}^g} \quad \alpha \\ \text{s.t.} \quad & L_{W^*(A^{\oplus n+1})W} \begin{pmatrix} X & \alpha\beta \\ \alpha\beta^* & \gamma \end{pmatrix} \succeq 0 \\ \text{for all} \quad & W \in \mathcal{I}_B(n+1) \end{aligned} \tag{3.7}$$

*achieves its maximum.*

*Proof.* Straightforward from the assumption that $\operatorname{proj}_x \mathcal{D}_{(A,B)}$ is compact. ■

Before presenting our next lemma, we introduce some notation. Given a matrix convex set $K$ and tuples $X \in SM_n(\mathbb{R})^g$ and $\beta \in M_{n \times 1}(\mathbb{R})^g$, define

$$\Gamma_{X,\beta}(K) := \left\{ \gamma \in \mathbb{R}^g : \begin{pmatrix} X & \beta \\ \beta^* & \gamma \end{pmatrix} \in K \right\}. \tag{3.8}$$

**Lemma 3.4.** *Let $K$ be a closed bounded matrix convex set. Fix $X \in SM_n(\mathbb{R})^g$ and $\beta \in M_{n\times 1}(\mathbb{R})^g$. Then the set $\Gamma_{X,\beta}(K)$ is a closed bounded convex set.*

*Proof.* If $\Gamma_{X,\beta}(K)$ is empty, then the result trivially holds. When $\Gamma_{X,\beta}(K)$ is nonempty, the fact that $\Gamma_{X,\beta}(K)$ is closed and bounded is immediate from the fact that $K$ is closed and bounded.

We now show that $\Gamma_{X,\beta}(K)$ is convex. To this end, let $\{\gamma_1, \dots, \gamma_k\} \subseteq \Gamma_{X,\beta}(K)$ and let $\{c_1, \dots, c_k\}$ be nonnegative constants such that $\sum_{i=1}^k c_i = 1$. For each $i = 1 \dots, k$ set $V_i = \sqrt{c_i} I_{n+1}$. Then we have

$$\begin{pmatrix} X & \beta \\ \beta^* & \sum_{i=1}^k c_i \gamma^i \end{pmatrix} = \sum_{i=1}^k V_i^* \begin{pmatrix} X & \beta \\ \beta^* & \gamma^i \end{pmatrix} V_i \qquad \text{and} \qquad \sum_{i=1}^k V_i^* V_i = 1.$$

From the definition of $\Gamma_{X,\beta}(K)$, the above is a matrix convex combination of elements of $K$. Since $K$ is matrix convex, it follows that

$$\begin{pmatrix} X & \beta \\ \beta^* & \sum_{i=1}^k c_i \gamma^i \end{pmatrix} \in K,$$

from which we obtain $\sum_{i=1}^k c_i \gamma^i \in \Gamma_{X,\beta}(K)$. That is, $\Gamma_{X,\beta}$ is convex. ■

3.1. **Maximal 1-dilations in free spectrahedrops.** We now present our definition of maximal 1-dilations for elements of free spectradrops. We mention that our definition is a generalization of the definition of maximal 1-dilations given in [23]. The definition used in [23] was itself inspired by works such as [19], [5], and [13].

**Definition 3.5.** *Given a closed bounded free spectrahedrop* $\text{proj}_x \mathcal{D}_{(A,B)}$ *and a tuple* $X \in \text{proj}_x \mathcal{D}_{(A,B)}(n)$*, say the dilation*

$$\hat{Y} = \begin{pmatrix} X & \hat{\beta} \\ \hat{\beta}^* & \hat{\gamma} \end{pmatrix} \in \text{proj}_x \mathcal{D}_{(A,B)}(n+1)$$

*is a* **maximal** 1**-dilation** *of* $X$ *if* $\hat{\beta} \in M_{n\times 1}(\mathbb{R})^g$ *is nonzero and the following two conditions hold:*

*(1) The real number* 1 *satisfies*

$$\begin{aligned} 1 = & \max_{\alpha\in\mathbb{R},\gamma\in\mathbb{R}^g} \quad \alpha \\ \text{s.t.} \quad & L_{W^*(A^{\oplus n+1})W} \begin{pmatrix} X & \alpha\hat{\beta} \\ \alpha\hat{\beta}^* & \gamma \end{pmatrix} \succeq 0 \\ \text{for all} \quad & W \in \mathcal{I}_B(n+1) \end{aligned}$$

*(2)* $\hat{\gamma}$ *is an extreme point of the closed bounded convex set* $\Gamma_{X,\hat{\beta}}(\text{proj}_x \mathcal{D}_{(A,B)})$.

**Proposition 3.6.** *Let* $\text{proj}_x \mathcal{D}_{(A,B)}$ *be a closed bounded free spectrahedrop and let* $X \in \text{proj}_x \mathcal{D}_{(A,B)}(n)$*. If* $X$ *is not an Arveson extreme point of* $\text{proj}_x \mathcal{D}_{(A,B)}$*, then there exists a*

$$\hat{Y} \in \text{proj}_x \mathcal{D}_{(A,B)}(n+1)$$

*such that* $\hat{Y}$ *is a maximal* 1*-dilation of* $X$*.*

*Proof.* The existence of maximal 1-dilations follows from Lemma 3.1 together with Lemmas 3.3 and 3.4. In particular, Lemma 3.1 shows that if $X$ is not Arveson extreme, then there exists some nonzero $\beta \in \mathfrak{K}_X(\text{proj}_x \mathcal{D}_{(A,B)})$. Furthermore, Lemmas 3.1 and 3.3 together show that if one takes $\tilde{\alpha}$ to as defined in equation (3.7) and sets $\hat{\beta} = \tilde{\alpha}\beta$, then $\hat{\beta}$ is nonzero and Condition 4.5 (1) is satisfied. Combining this with Lemma 3.4 then shows that then $\Gamma_{X,\hat{\beta}}(\text{proj}_x \mathcal{D}_{(A,B)})$ is a closed bounded convex set. Furthermore, Lemma 3.1 shows that $\Gamma_{X,\hat{\beta}}(\text{proj}_x \mathcal{D}_{(A,B)})$ is nonempty. Thus, if one takes $\hat{\gamma}$ to be an extreme point of $\Gamma_{X,\hat{\beta}}(\text{proj}_x \mathcal{D}_{(A,B)})$ then Condition 4.5 (2) is satisfied. It follows that

$$\hat{Y} = \begin{pmatrix} X & \hat{\beta} \\ \hat{\beta}^* & \hat{\gamma} \end{pmatrix},$$

is a maximal 1-dilation of $X$. ∎

Our next theorem shows that maximal 1-dilations of an element of a bounded free spectrahedrop reduce the dimension of the dilation subspace.

**Theorem 3.7.** *Let $A \in SM_d(\mathbb{R})^g$ and $B \in SM_d(\mathbb{R})^h$ such that $\operatorname{proj}_x \mathcal{D}_{(A,B)}$ is a closed bounded real free spectrahedrop and let $X \in \operatorname{proj}_x \mathcal{D}_{(A,B)}(n)$. Assume $X$ is not an Arveson extreme point of $\operatorname{proj}_x \mathcal{D}_{(A,B)}$. Then there exists a maximal 1-dilation $\hat{Y} \in \operatorname{proj}_x \mathcal{D}_{(A,B)}(n+1)$ of $X$. Furthermore, any such $\hat{Y}$ satisfies*

$$\dim \mathfrak{D}_{\hat{Y}}(\operatorname{proj}_x \mathcal{D}_{(A,B)}) < \dim \mathfrak{D}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)}).$$

*Proof.* Before proceeding, we mention that the proof of this result follows the same flow as the proof of [23, Theorem 2.4]. Modifications are made to handle the fact that the kernel containment appearing in equation (3.2) must hold for all $W \in \mathcal{I}_B(n+1)$ for a tuple $\beta$ to be an element of $\mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)})$.

The existence of a maximal 1-dilation of $X$ is proved in Proposition 3.6. Now, let

$$\hat{Y} = \begin{pmatrix} X & \hat{\beta} \\ \hat{\beta}^* & \hat{\gamma} \end{pmatrix}$$

be a maximal 1-dilation of $X$. Using Lemma 3.1 (3), it is sufficient to show that

$$\dim \mathfrak{K}_{\hat{Y}}(\operatorname{proj}_x \mathcal{D}_{(A,B)}) < \dim \mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)}).$$

First consider the subspace

$$\mathfrak{E}_{\hat{Y}}(\operatorname{proj}_x \mathcal{D}_{(A,B)}) := \left\{ \eta \in M_{n\times 1}(\mathbb{R})^g \ : \ \exists\, \sigma \in \mathbb{R}^g \text{ s.t. } \begin{pmatrix} \eta \\ \sigma \end{pmatrix} \in \mathfrak{K}_{\hat{Y}}(\operatorname{proj}_x \mathcal{D}_{(A,B)}) \right\}.$$

In other words $\mathfrak{E}_{\hat{Y}}(\operatorname{proj}_x \mathcal{D}_{(A,B)})$ is the projection $\iota$ of $\mathfrak{K}_Y(\operatorname{proj}_x \mathcal{D}_{(A,B)})$ defined by

$$\mathfrak{E}_Y(\operatorname{proj}_x \mathcal{D}_{(A,B)}) := \iota(\mathfrak{K}_{\hat{Y}}(\operatorname{proj}_x \mathcal{D}_{(A,B)})) \text{ where } \iota \begin{pmatrix} \eta \\ \sigma \end{pmatrix} = \eta$$

for $\eta \in M_{n\times 1}(\mathbb{R})^g$ and $\sigma \in \mathbb{R}^g$. We will show

$$\dim \mathfrak{E}_{\hat{Y}}(\operatorname{proj}_x \mathcal{D}_{(A,B)}) < \dim \mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)}).$$

If $\eta \in \mathfrak{E}_{\hat{Y}}(\operatorname{proj}_x \mathcal{D}_{(A,B)})$, then there exists a tuple $\tilde{\sigma} \in \mathbb{R}^g$ such that

$$\begin{pmatrix} \eta^* & \tilde{\sigma} \end{pmatrix} \in \mathfrak{K}_{\hat{Y}}(\operatorname{proj}_x \mathcal{D}_{(A,B)}).$$

From Lemma 3.1 (2), it follows that there is a real number $c > 0$ so that setting $\sigma = c\tilde{\sigma}$ gives

$$\begin{pmatrix} X & \hat{\beta} & c\eta \\ \hat{\beta}^* & \hat{\gamma} & \sigma \\ c\eta^* & \sigma^* & 0 \end{pmatrix} \in \operatorname{proj}_x \mathcal{D}_{(A,B)}.$$

Since $\operatorname{proj}_x \mathcal{D}_{(A,B)}$ is matrix convex it follows that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X & \hat{\beta} & c\eta \\ \hat{\beta}^* & \hat{\gamma} & \sigma \\ c\eta^* & \sigma^* & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} X & c\eta \\ c\eta^* & 0 \end{pmatrix} \in \operatorname{proj}_x \mathcal{D}_{(A,B)},$$

so Lemma 3.1 (1) shows $\eta \in \mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)})$. In particular this shows

$$\mathfrak{E}_{\hat{Y}}(\operatorname{proj}_x \mathcal{D}_{(A,B)}) \subseteq \mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)}), \tag{3.9}$$

hence

$$\dim \mathfrak{E}_{\hat{Y}}(\operatorname{proj}_x \mathcal{D}_{(A,B)}) \leq \dim \mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)}). \tag{3.10}$$

Now, assume towards a contradiction that

$$\dim \mathfrak{K}_{\hat{Y}}(\operatorname{proj}_x \mathcal{D}_{(A,B)}) \geq \dim \mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)}).$$

We next show that this implies that there is a real number $c$ and a tuple $\sigma \in \mathbb{R}^g$ so that

$$L_{W^*\left(A^{\oplus d(n+2)}\right)W} \begin{pmatrix} X & \hat{\beta} & c\hat{\beta} \\ \hat{\beta}^* & \hat{\gamma} & \sigma \\ c\hat{\beta}^* & \sigma & 0 \end{pmatrix} \succeq 0 \qquad \text{for all } W \in \mathcal{I}_B(n+2). \tag{3.11}$$

and such that either $c \neq 0$ or $\sigma \neq 0$. To see this, observe that equation (3.9) implies that if $\dim \mathfrak{E}_{\hat{Y}}(\operatorname{proj}_x \mathcal{D}_{(A,B)}) = \dim \mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)})$, then we must have

$$\mathfrak{E}_{\hat{Y}}(\operatorname{proj}_x \mathcal{D}_{(A,B)}) = \mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)}).$$

In this case, using Lemma 3.1 (2) together with Theorem 2.1 (2) shows that there is a nonzero $c \in \mathbb{R}$ and some (possibly zero) $\sigma \in \mathbb{R}^g$ such that inequality (3.11) holds. On the other hand, if $\dim \mathfrak{E}_{\hat{Y}}(\operatorname{proj}_x \mathcal{D}_{(A,B)}) < \dim \mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)})$, then there must exist tuples $\eta \in M_{n\times 1}(\mathbb{R})^g$ and $\sigma^1, \sigma^2 \in \mathbb{R}^g$ such that $\sigma^1 \neq \sigma^2$ and so

$$\begin{pmatrix} \eta \\ \sigma^1 \end{pmatrix}, \begin{pmatrix} \eta \\ \sigma^2 \end{pmatrix} \in \mathfrak{K}_Y(\operatorname{proj}_x \mathcal{D}_{(A,B)}), \qquad \text{hence} \qquad \begin{pmatrix} 0 \\ \sigma^1 - \sigma^2 \end{pmatrix} \in \mathfrak{K}_Y(\operatorname{proj}_x \mathcal{D}_{(A,B)}). \tag{3.12}$$

In this case, setting $\sigma = \alpha(\sigma^1 - \sigma^2)$ for an appropriately chosen constant $\alpha \in \mathbb{R}$ and again using using Lemma 3.1 (2) together with Theorem 2.1 (2) shows that there is a (possibly zero) real number $c$ and a nonzero tuple $\sigma$ such that inequality (3.11) holds. Thus we have proved our claim that there is either a nonzero $c \in \mathbb{R}$ or a nonzero $\sigma \in \mathbb{R}^g$ such that inequality (3.11) holds.

We now use inequality (3.11) together with the NC LDL$^*$-decomposition to show that $\hat{Y}$ cannot be a maximal $1-$dilation of $X$, a contradiction to our definition of $\hat{Y}$. Applying the

NC LDL$^*$-decomposition (up to canonical shuffles) shows that, for each fixed $W \in \mathcal{I}_B(n+2)$, inequality (3.11) holds if and only if $L_{W^*\left(A^{\oplus d(n+2)}\right)W}(X) \succeq 0$ and the Schur complements

$$I_{n+2} - c^2 Q_W \succeq 0 \tag{3.13}$$

and

$$\begin{aligned} &L_{W^*\left(A^{\oplus d(n+2)}\right)W}(\hat{\gamma}) - Q_W \\ &- \left(\Lambda_{W^*\left(A^{\oplus d(n+2)}\right)W}(\sigma) - cQ_W\right)^* \left(I - c^2 Q_W\right)^\dagger \left(\Lambda_{W^*\left(A^{\oplus d(n+2)}\right)W}(\sigma) - cQ_W\right) \succeq 0 \end{aligned} \tag{3.14}$$

where

$$Q_W := \Lambda_{W^*\left(A^{\oplus d(n+2)}\right)W}(\hat{\beta}^*) L_{W^*\left(A^{\oplus d(n+2)}\right)W}(X)^\dagger \Lambda_{W^*\left(A^{\oplus d(n+2)}\right)W}(\hat{\beta}). \tag{3.15}$$

It follows that

$$L_{W^*\left(A^{\oplus d(n+2)}\right)W}(\hat{\gamma}) - Q_W \succeq 0 \tag{3.16}$$

and

$$\ker[L_{W^*\left(A^{\oplus d(n+2)}\right)W}(\hat{\gamma}) - Q_W] \subseteq \ker[\Lambda_{W^*\left(A^{\oplus d(n+2)}\right)W}(\sigma) - cQ_W]. \tag{3.17}$$

Inequalities (3.16) and (3.17) imply that for each fixed $W \in \mathcal{I}_B(n+2)$ there exists a real number $\alpha_W > 0$ such that $0 < \alpha \leq \alpha_W$ implies

$$L_{W^*\left(A^{\oplus d(n+2)}\right)W}(\hat{\gamma}) - Q_W \ \pm \ \alpha \left(\Lambda_{W^*\left(A^{\oplus d(n+2)}\right)W}(\sigma) - cQ_W\right) \succeq 0. \tag{3.18}$$

Our next goal is to show that there is some real number $\tilde{\alpha} > 0$ which is independent of $W \in \mathcal{I}_B(n+2)$ such that equation (3.18) holds for all $0 < \alpha \leq \tilde{\alpha}$ and all $W \in \mathcal{I}_B(n+2)$. We note that the proof of this fact is similar to the proof of Lemma 3.1 (2), however, we include the details for the sake of completeness.

For each fixed $W$, set $\tilde{\alpha}_W$ to be the largest real number such that equation (3.18) holds for this fixed $W$ and for all $0 < \alpha \leq \tilde{\alpha}_W$. Additionally define

$$\tilde{\alpha} = \min_{r \leq n+2} \ \inf_{W \in \mathcal{I}_B(n+2,r)} \ \tilde{\alpha}_W. \tag{3.19}$$

A routine compactness argument shows that for each fixed $r \in \{1, 2, \dots, n+2\}$ there is some $\tilde{W}_r \in \mathcal{I}_B(n+2, r)$ such that

$$\tilde{\alpha}_{\tilde{W}_r} = \inf_{W \in \mathcal{I}_B(n+2,r)} \ \tilde{\alpha}_W.$$

From this we can conclude that there is some some $\tilde{r} \in \{1, 2, \dots, n+2\}$ such that $\tilde{\alpha} = \tilde{\alpha}_{\tilde{W}_{\tilde{r}}}$. Recall from the previous part of the proof that $\tilde{\alpha}_{\tilde{W}_{\tilde{r}}} > 0$. We conclude that if one sets $\tilde{\alpha}$ as in equation (3.19), then one indeed has that $\tilde{\alpha} > 0$ and that that equation (3.18) holds for all $0 < \alpha \leq \tilde{\alpha}$ and all $W \in \mathcal{I}_B(n+2)$.

After simplifying equation (3.18), it follows from the above that

$$L_{W^*\left(A^{\oplus d(n+2)}\right)W}(\hat{\gamma} \pm \tilde{\alpha}\sigma) - (1 \pm c\tilde{\alpha})Q_W \succeq 0 \tag{3.20}$$

for all $W \in \mathcal{I}_B(n+2)$. Furthermore, since $X \in \operatorname{proj}_x \mathcal{D}_{(A,B)}$, using Theorem 2.1 (3) shows $L_{W^*(A^{\oplus n+2})W}(X) \succeq 0$ for all $W \in \mathcal{I}_B(n+2)$. Combining this with equation (3.20) and another application of the Schur compliment shows that

$$L_{W^*\left(A^{\oplus d(n+2)}\right)W}\begin{pmatrix} X & \sqrt{1 \pm c\tilde{\alpha}}\hat{\beta} \\ \sqrt{1 \pm c\tilde{\alpha}}\hat{\beta}^* & \hat{\gamma} \pm \tilde{\alpha}\sigma \end{pmatrix} \succeq 0 \qquad \text{for all} \qquad W \in \mathcal{I}_B(n+2). \tag{3.21}$$

Note that we can naturally identify $I_B(n+1)$ with a subset of $I_B(n+2)$, since if $W \in \mathcal{I}_B(n+1, r)$, then $[W^* \ \ 0_{r \times nd}]^* \in \mathcal{I}_B(n+2)$. It follows that

$$L_{W^*\left(A^{\oplus d(n+1)}\right)W}\begin{pmatrix} X & \sqrt{1 \pm c\tilde{\alpha}}\hat{\beta} \\ \sqrt{1 \pm c\tilde{\alpha}}\hat{\beta}^* & \hat{\gamma} \pm \alpha\sigma \end{pmatrix} \succeq 0 \qquad \text{for all} \qquad W \in \mathcal{I}_B(n+1). \tag{3.22}$$

It follows from Theorem 2.1 that

$$\begin{pmatrix} X & \sqrt{1 \pm c\tilde{\alpha}}\hat{\beta} \\ \sqrt{1 \pm c\tilde{\alpha}}\hat{\beta}^* & \hat{\gamma} \pm \alpha\sigma \end{pmatrix} \in \operatorname{proj}_x \mathcal{D}_{(A,B)}(n+1).$$

Recalling that $\hat{Y}$ was chosen to be a maximal 1-dilation of $X$, we must have

$$\sqrt{1 \pm c\tilde{\alpha}} \leq 1$$

It follows that $c\tilde{\alpha} = 0$. Moreover, since $\tilde{\alpha} > 0$, we must have $c = 0$. From this we find

$$\hat{\gamma} \pm \tilde{\alpha}\sigma \in \Gamma_{X,\hat{\beta}}(\operatorname{proj}_x \mathcal{D}_{(A,B)}).$$

From our construction we have that if $c = 0$, then $\sigma \neq 0$, so since $\tilde{\alpha} > 0$, the above implies that $\hat{\gamma}$ is not an extreme point of the closed bounded convex set $\Gamma_{X,\hat{\beta}}(\operatorname{proj}_x \mathcal{D}_{(A,B)})$. However, this is a contradiction to $\hat{Y}$ being a maximal 1-dilation of $X$, since, from the definition of a maximal 1-dilation, $\hat{\gamma}$ is an extreme point of $\Gamma_{X,\hat{\beta}}(\operatorname{proj}_x \mathcal{D}_{(A,B)})$. Thus, the assumption that

$$\dim \mathfrak{K}_{\hat{Y}}(\operatorname{proj}_x \mathcal{D}_{(A,B)}) \geq \dim \mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)}).$$

leads to a contradiction. We conclude that

$$\dim \mathfrak{K}_{\hat{Y}}(\operatorname{proj}_x \mathcal{D}_{(A,B)}) < \dim \mathfrak{K}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)}).$$

Using Lemma 3.1 (3) then shows that

$$\dim \mathfrak{D}_{\hat{Y}}(\operatorname{proj}_x \mathcal{D}_{(A,B)}) < \dim \mathfrak{D}_X(\operatorname{proj}_x \mathcal{D}_{(A,B)}).$$

That is, maximal 1-dilations reduce the dimension of the dilation subspace, as claimed. ∎

## 4. Arveson extreme points and generalized free spectrahedra

We now handle the case of generalized free spectrahedra. In this section, $\mathcal{H}$ is a real (infinite dimensional) Hilbert space and $A \in SA(\mathcal{H})^g$ is a tuple of compact self-adjoint operators on $\mathcal{H}$. A simple observation is that with these assumptions, if $X \in \mathcal{D}_A$ then $L_A(X)$ has a smallest nonzero eigenvalue. Using this fact allows for us to define maximal 1-dilations for generalized free spectrahedra and to show that these maximal 1-dilations in generalized free spectrahedra reduce the dimension of the dilation subspace. Though the argument in this setting is again similar to arguments used in [23] (and in this article in Section 3), details are given for the sake of completeness.

**Lemma 4.1.** *Let $A$ be a $g$-tuple of bounded self-adjoint compact operators on $\mathcal{H}$ and let $X \in SM_n(\mathbb{R})^g$ be a $g$-tuple of self-adjoint $n \times n$ matrices. Then*

$$\Lambda_A(X) := A_1 \otimes X_1 + \cdots + A_g \otimes X_g$$

*is a compact operator self-adjoint on $\mathcal{H} \otimes \mathbb{R}^n$*

*Proof.* Straightforward. ∎

**Lemma 4.2.** *Let $Q \in B(\mathcal{H})$ be a compact self-adjoint operator. Then $I - Q$ has a smallest nonzero eigenvalue.*

*Proof.* Since $Q$ is compact and self-adjoint, $Q$ is diagonalizable and only can have zero as a limit point of its spectrum. It follows that

$$L_A(X) := I - \Lambda_A(X)$$

is diagonalizable and can only have one as a limit point of its spectrum. Therefore, $I - Q$ has a smallest nonzero eigenvalue. ∎

**Corollary 4.3.** *Let $A$ be a $g$-tuple of compact self-adjoint operators on $\mathcal{H}$ such that $\mathcal{D}_A$ is a generalized free spectrahedron and let $X \in SM_n(\mathbb{R})^g$ be any $g$-tuple of self-adjoint $n \times n$ matrices. Then $L_A(X)$ has a smallest nonzero eigenvalue.*

*As an immediate consequence $L_A(X)^\dagger$ is a bounded self-adjoint operator on $\mathcal{H} \otimes \mathbb{R}^n$.*

### 4.1. The dilation subspace of generalized free spectrahedra.

Similar to as was done for free spectrahedrops, we now describe an alternative characterization of the dilation subspace for generalized free spectrahedra. To this end, define the subspace $\mathfrak{K}_X(\mathcal{D}_A)$ by

$$\mathfrak{K}_X(\mathcal{D}_A) = \{\beta \in M_{n\times 1}(\mathbb{R})^g | \ \ker L_A(X) \subseteq \ker \Lambda_A(\beta^*)\}.$$

**Lemma 4.4.** *Let $\mathcal{D}_A$ be a generalized free spectrahedron and let $X \in \mathcal{D}_A(n)$.*

*(1) If $\beta \in M_{n\times 1}(\mathbb{R})^g$ and*

$$Y = \begin{pmatrix} X & \beta \\ \beta^* & \gamma \end{pmatrix} \in \mathcal{D}_A(n+1)$$

*is a 1-dilation of $X$, then $\beta \in \mathfrak{K}_X(\mathcal{D}_A)$.*

*(2) Let $\beta \in M_{n\times 1}(\mathbb{R})^g$. Then $\beta \in \mathfrak{K}_X(\mathcal{D}_A)$ if and only if there is a tuple $\gamma \in \mathcal{D}_A(1)$ real number $c_\gamma > 0$ such that*

$$\begin{pmatrix} X & c_\gamma \beta \\ c_\gamma \beta^* & \gamma \end{pmatrix} \in \mathcal{D}_A(n+1).$$

*In particular, one may take $\gamma = 0 \in \mathbb{R}^g$.*

*(3) One has*

$$\mathfrak{D}_X(\mathcal{D}_A) = \mathfrak{K}_X(\mathcal{D}_A).$$

*As a consequence, $X$ is an Arveson extreme point of $\mathcal{D}_A$ if and only if*

$$\dim \mathfrak{K}_X(\mathcal{D}_A) = 0.$$

*Proof.* Corollary 4.3 shows that $L_A(\gamma)^\dagger$ is a well-defined bounded operator on $\mathcal{H}$. From this point, the proof if Item (1) is essentially identical to the proof of Lemma 3.1 (1).

We now prove Item (2). Note that $L_A(0) = I_{\mathcal{H}}$, so using the Schur complement shows

$$Y_0 = \begin{pmatrix} X & c\beta \\ c\beta^* & 0 \end{pmatrix} \in \mathcal{D}_A(n+1)$$

if and only if

$$L_A(X) - c^2 \Lambda_A(\beta)\Lambda_A(\beta^*) \succeq 0. \tag{4.1}$$

Using Corollary 4.3 shows that $L_A(X)$ has a smallest nonzero eigenvalue, so we may pick $c$ small enough so that $\|c^2\Lambda_A(\beta)\Lambda_A(\beta^*)\|_2$ is less than the smallest nonzero eigenvalue of $L_A(X)$. Furthermore, $\beta \in \mathfrak{K}_X(\mathcal{D}_A)$ implies $\ker L_A(X) \subseteq \ker \Lambda_A(\beta)\Lambda_A(\beta^*)$. Thus this choice of $c$ then guarantees that inequality (4.1) holds, hence $Y_0 \in \mathcal{D}_A(n+1)$. The reverse direction is a consequence of Item (1).

Item (3) follows from Items (1) and (2). ■

4.2. **Maximal 1-dilations for generalized free spectrahedra.** We now present our notion of maximal 1-dilations for generalized free spectrahedra.

**Definition 4.5.** *Given a bounded generalized free spectrahedron $\mathcal{D}_A$ and a tuple $X \in \mathcal{D}_A(n)$, say the dilation*

$$\hat{Y} = \begin{pmatrix} X & \hat{\beta} \\ \hat{\beta}^* & \hat{\gamma} \end{pmatrix} \in \mathcal{D}_A(n+1)$$

*is a* **maximal** **1-dilation** *of $X$ if $\hat{\beta} \in M_{n\times 1}(\mathbb{R})^g$ is nonzero and the following two conditions hold:*

*(1) The real number 1 satisfies*

$$\begin{array}{rl} 1 = & \max_{\alpha\in\mathbb{R},\gamma\in\mathbb{R}^g} \quad \alpha \\ \text{s.t.} & L_A\begin{pmatrix} X & \alpha\hat{\beta} \\ \alpha\hat{\beta}^* & \gamma \end{pmatrix} \succeq 0 \end{array}$$

*(2) $\hat{\gamma}$ is an extreme point of the closed bounded convex set $\Gamma_{A,\hat{\beta}}(\mathcal{D}_A)$ where $\Gamma_{A,\hat{\beta}}(\mathcal{D}_A)$ is as defined in equation (3.8).*

We now show that maximal 1-dilations in generalized free spectrahedra reduce the dimension of the dilation subspace.

**Theorem 4.6.** *Let $A \in B(\mathcal{H})^g$ be a $g$-tuple of compact self-adjoint operators on $\mathcal{H}$ such that $\mathcal{D}_A$ is a bounded real free spectrahedron and let $X \in \mathcal{D}_A(n)$. Assume $X$ is not an Arveson extreme point of $\mathcal{D}_A$. Then there exists a nontrivial maximal 1-dilation $\hat{Y} \in \mathcal{D}_A(n+1)$ of $X$. Furthermore, any such $\hat{Y}$ satisfies*

$$\dim \mathfrak{D}_{\hat{Y}}(\mathcal{D}_A) < \dim \mathfrak{D}_X(\mathcal{D}_A).$$

*Proof.* The existence of maximal 1-dilations in a bounded generalized free spectrahedron follows from a routine compactness argument together with Lemma 3.4.

Now, let $\hat{Y}$ be a maximal 1-dilation of $X$. Using Lemma 4.4 (3), it is sufficient to show

$$\dim \mathfrak{K}_{\hat{Y}}(\mathcal{D}_A) < \dim \mathfrak{K}_X(\mathcal{D}_A).$$

Assume towards a contradiction that

$$\dim \mathfrak{K}_{\hat{Y}}(\mathcal{D}_A) \geq \dim \mathfrak{K}_X(\mathcal{D}_A).$$

Following the same argument as was given at the beginning of the proof of Theorem 3.7 shows that there exists a real number $c$ and a tuple $\sigma \in \mathbb{R}^g$ so that

$$L_A\begin{pmatrix} X & \hat{\beta} & c\hat{\beta} \\ \hat{\beta}^* & \hat{\gamma} & \sigma \\ c\hat{\beta}^* & \sigma & 0 \end{pmatrix} \succeq 0. \tag{4.2}$$

and such that either $c \neq 0$ or $\sigma \neq 0$.

Now, applying the NC LDL*-decomposition (up to canonical shuffles) shows that inequality (4.2) holds if and only if $L_A(X) \succeq 0$ and the Schur complements

$$I_{\mathcal{H}} - c^2 Q \succeq 0 \tag{4.3}$$

and

$$L_A(\hat{\gamma}) - Q - (\Lambda_A(\sigma) - cQ)^* \left(I_{\mathcal{H}} - c^2 Q\right)^\dagger (\Lambda_A(\sigma) - cQ) \succeq 0 \tag{4.4}$$

where

$$Q := \Lambda_A(\hat{\beta}^*) L_A(X)^\dagger \Lambda_A(\hat{\beta}). \tag{4.5}$$

It follows that

$$L_A(\hat{\gamma}) - Q \succeq 0 \tag{4.6}$$

and

$$\ker[L_A(\hat{\gamma}) - Q] \subseteq \ker[\Lambda_A(\sigma) - cQ]. \tag{4.7}$$

Recall from Corollary 4.3 and Lemma 4.1 that $L_A(X)^\dagger$ is a bounded self-adjoint operator and $\Lambda_A(\hat{\beta})$ and $\Lambda_A(\hat{\beta}^*)$ are compact operators. It follows that $Q$ is a compact self-adjoint operator. Therefore

$$L_A(\hat{\gamma}) - Q = I_{\mathcal{H}} - (\Lambda_A(\hat{\gamma}) + Q)$$

is the identity minus a compact self-adjoint operator and by Lemma 4.2 has a smallest nonzero eigenvalue. Therefore, picking $\tilde{\alpha} > 0$ so that $\tilde{\alpha}\|\Lambda_A(\sigma) - cQ\|$ is smaller than the smallest nonzero eigenvalue of $L_A(\hat{\gamma}) - Q$ and using inequalities (4.6) and (4.7) guarantees

$$L_A(\hat{\gamma}) - Q \;\pm\; \tilde{\alpha}\,(\Lambda_A(\sigma) - cQ) \succeq 0.$$

It follows from the above that

$$\begin{aligned} & L_A(\hat{\gamma} \pm \tilde{\alpha}\sigma) - (1 \pm c\tilde{\alpha})Q \\ = \;& L_A(\hat{\gamma} \pm \alpha\sigma) - \left(\Lambda_A(\sqrt{1 \pm c\tilde{\alpha}}\hat{\beta}^*) L_A(X)^\dagger \Lambda_A(\sqrt{1 \pm c\tilde{\alpha}}\hat{\beta})\right) \;\succeq\; 0. \end{aligned} \tag{4.8}$$

Since $L_A(X) \succeq 0$, equation (4.8) implies

$$L_A \begin{pmatrix} X & \sqrt{1 \pm c\tilde{\alpha}}\hat{\beta} \\ \sqrt{1 \pm c\tilde{\alpha}}\hat{\beta}^* & \hat{\gamma} \pm \tilde{\alpha}\sigma \end{pmatrix} \succeq 0. \tag{4.9}$$

Recalling that $\hat{Y}$ is a maximal 1-dilation of $X$, we must have

$$\sqrt{1 \pm c\tilde{\alpha}} \leq 1.$$

It follows that $c\tilde{\alpha} = 0$. Since $\tilde{\alpha} > 0$, it follows that $c = 0$. From our construction, this in turn implies that $\sigma \neq 0$. But then equation (4.9) implies that

$$\hat{\gamma} \pm \tilde{\alpha}\sigma \in \Gamma_{X,\hat{\beta}}(\mathcal{D}_A),$$

which contradicts the fact that $\hat{\gamma}$ is an extreme point of the convex set $\Gamma_{X,\hat{\beta}}(\mathcal{D}_A)$. We conclude that

$$\dim \mathfrak{K}_{\hat{Y}}(\mathcal{D}_A) < \dim \mathfrak{K}_X(\mathcal{D}_A),$$

from which we can use Lemma 4.4 (3) to conclude that

$$\mathfrak{D}_{\hat{Y}}(\mathcal{D}_A) < \mathfrak{D}_X(\mathcal{D}_A),$$

as claimed. ■

## 5. Free spectrahedrops versus generalized free spectrahedra

We now discuss a class of generalized free spectrahedra that are not free spectrahedrops. The class arises by considering the polar duals of the matrix convex sets introduced in [21] which have no free extreme points. Following the notation in [21], let $A \in SA(\mathcal{H})^g$ and for each $n \in \mathbb{N}$ define the set $K_A(n) \subseteq SM_n(\mathbb{R})^g$ by

$$K_A(n) = \{Y \in SM_n(\mathbb{R})^g | Y = V^*(I_\mathcal{H} \otimes X)V \text{ for some isometry } V : \mathbb{R}^n \to \oplus_1^\infty \mathcal{H}\}. \tag{5.1}$$

We then define $K_A \subseteq SM(\mathbb{R})^g$ by

$$K_A = \cup_n K_A(n). \tag{5.2}$$

We call $K_A$ the **noncommutative convex hull** of X. The set $K_A$ is closely related to the matrix range of $A$, see [45] for further discussion. For a $g$-tuple $A \in SA(\mathcal{H})^g$, say 0 **is in the finite interior of** $K_A$ if there exist an integer $d \in \mathbb{N}$ and a nonzero vector $v \in (\mathcal{H})^d$ such that.

$$v^*(I_d \otimes A)v = 0 \in \mathbb{R}^g.$$

**Theorem 5.1.** *Let $A \in SA(\mathcal{H})^g$ be a tuple of compact operators such that $\mathcal{D}_A$ is a bounded generalized free spectrahedron. Additionally assume that $A$ has no finite dimensional reducing subspaces and that 0 is in the finite interior of $K_A$. Then $\mathcal{D}_A$ is not a free spectrahedrop.*

*Proof.* Using [21] shows that $K_A$ is a closed bounded matrix convex set that has no free extreme points. It then follows from Theorem 2.3 that $K_A$ cannot be a free spectrahedrop. On the other hand, combining the discussion in [45, Example 4.6] with [12, Proposition 3.1] shows that $K_A^\circ = \mathcal{D}_A$. Furthermore, using [34, Lemma 4.2] (also see [12, Lemma 3.2]) shows that since $0 \in K_A$, we have that $(K_A^\circ)^\circ = K_A$, hence $\mathcal{D}_A^\circ = K_A$. To complete the proof note that if $\mathcal{D}_A$ was a closed bounded free spectrahedrop, then [34, Theorem 4.11] would imply that $\mathcal{D}_A^\circ = K_A$ is also a closed bounded free spectrahedrop. We have already shown that $K_A$ is not a free spectrahedrop, so it follows that $\mathcal{D}_A$ cannot be a free spectraherop. ■

As it turns out, the precise examples of tuples $A$ such that $A$ has no finite dimensional reducing subspaces and $K_A$ contains 0 in the finite interior considered in [21, Proposition 4.1] do not have the property that $\mathcal{D}_A$ is bounded. It is possible however to lightly modify these example to make them appropriate for our current use. Similar to [21, Proposition 4.1], set

$\mathcal{H} = \ell^2(\mathcal{N})$ and set $\mathcal{H}_2 = \ell^2(\{1,2\})$. Additionally, given a weight vector $w = (w_1, w_2, \dots) \in \mathbb{R}^\infty$, define the weighted forward shift $S_w : \mathcal{H} \to \mathcal{H}$ by

$$S_w v = (0, w_1 v_1, w_2, v_2, \dots)$$

for all $v \in \mathcal{H}$.

**Proposition 5.2.** *Let $A_1 = \operatorname{diag}(\lambda_1, \lambda_2, \dots)$ where the $\lambda_i$ are nonzero real numbers converging to $0$ with distinct norms and where that $\lambda_1 > 0$ and that $\lambda_2 < 0$. Also let $S_w$ be a weighted shift where the weight vector $w \in \mathbb{R}^\infty$ is chosen so $w_i \neq 0$ for all $i$ and such that $S_w$ is compact. Set*

$$A_2 = S_w + S_w^*.$$

*Then $(A_1, A_2)$ is a tuple of compact operators on $\mathcal{H}$ which has no finite dimensional reducing subspaces such that $K_A$ contains $0$ in its finite interior and such that $\mathcal{D}_A$ is bounded.*

*Proof.* The proof that $A$ has no finite dimensional reducing subspaces is identical to proof appearing in [21, Proposition 5.1]. We next show that $\mathcal{D}_A$ is bounded. To this end, consider the inclusion map $\iota : \mathcal{H}_2 \to \mathcal{H}$. By identifying $\mathcal{H}_2$ with $\mathbb{R}^2$ we have

$$\iota^* A \iota = \left( \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} 0 & w_1 \\ w_1 & 0 \end{pmatrix} \right)$$

Since $\lambda_1 > 0$ and $\lambda_2 < 0$ and $w_1 \neq 0$, no linear combination of $\iota^* A_1 \iota$ and $\iota^* A_2 \iota$ can be positive semidefinite, from which it follows that $\mathcal{D}_{\iota^* A \iota}$ is bounded. It is straightforward to check that

$$\mathcal{D}_A \subseteq \mathcal{D}_{\iota^* A \iota},$$

so we conclude that $\mathcal{D}_A$ is bounded.

It remains to show that $0$ is in the finite interior of $K_A$. Set

$$v_1 = \begin{pmatrix} \sqrt{-\lambda_2} \\ 0 \end{pmatrix} \qquad \text{and} \qquad v_2 = \begin{pmatrix} 0 \\ \sqrt{\lambda_1} \end{pmatrix}$$

Then one can verify that

$$v_1^*(\iota^* A_1 \iota) v_1 + v_2^*(\iota^* A_1 \iota) v_2 = v_1^*(\iota^* A_2 \iota) v_1 + v_2^*(\iota^* A_2 \iota) v_2 = 0.$$

It follows that if one sets

$$w = \begin{pmatrix} \iota v_1 \\ \iota v_2 \end{pmatrix} \in (\mathcal{H})^2,$$

then $w^*(I_2 \otimes A) w = 0 \in \mathbb{R}^2$. That is $0$ is in the finite interior of $K_A$.

■

5.1. **Thoughts on "generalized" free spectrahedrops.** To end the article, we briefly discuss the case of generalized free spectrahedrops. From our results, it is natural to wonder if it is necessary to consider finite dimensional defining tuples when working with free spectrahedrops. That is, suppose $\mathcal{H}$ is an infinite dimensional real Hilbert space and $A \in SA(\mathcal{H})^g$ and $B \in SA(\mathcal{H})^h$ are tuples of compact self-adjoint operators on $\mathcal{H}$ such that $\operatorname{proj}_x \mathcal{D}_{(A,B)}$ is a closed bounded "generalized" free spectrahedrop. One may wonder if $\operatorname{proj}_x \mathcal{D}_{(A,B)}$ is the matrix convex hull of its free extreme points.

As it stands, it is unclear how to extend our approach to this setting. A key issue is that in the proof of [18, Theorem 3.2], given an element of $X \in \operatorname{proj}_x \mathcal{D}_{(A,B)}(n)$ one considers a map

$$W \to tr(W)I_n - \sum_{i=1}^{g} \langle A_i, W \rangle X_i$$

defined on $B(\mathcal{H})$. Of course, such a map is not defined for general operators in $B(\mathcal{H})$, which leads to difficulties.

Aside from this, we mention that other problems do appear to arise when one makes the jump from free spectrahedrops to generalized free spectrahedrops. For example, [34, Theorem 4.11] shows that if $\mathcal{H}$ is finite dimensional and $\operatorname{proj}_x \mathcal{D}_{(A,B)}$ is a closed bounded free spectrahedrop, then $\operatorname{proj}_x \mathcal{D}_{(A,B)}$ is the projection of a bounded free spectrahedron. That is, one can assume that $\mathcal{D}_{(A,B)}$ is bounded if $\operatorname{proj}_x \mathcal{D}_{(A,B)}$ is closed and bounded. The proof of this result constructs a bounded free spectrahedron $K$ such that $\operatorname{proj}_x \mathcal{D}_{(A,B)}$ is a projection of $K$. However, the number of variables in $K$ depends on the dimension of $\mathcal{H}$. Thus, if one attempts to naively extend this proof to generalized free spectrahedrops, one would encounter dimension free sets in infinitely many variables.

Working with dimension free sets in infinitely many variables causes a number of challenges. For example, this would cause the approach of successive maximal 1-dilations to fail. A critical aspect of this approach is that the dimension of the dilation subspace is bounded above by $ng$, hence is finite.

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## 6. Appendix

The appendix gives a self-contained proof of Theorem 2.1, our version of [18, Theorem 3.2].

*Proof.* We first prove Item (1). Suppose $X \in \operatorname{proj}_x \mathcal{D}_{(A,B)}$ and $W \in \mathcal{Z}_B(n)$. Then there exists some $Y \in SM_n(\mathbb{R})^g$ such that

$$I_{dn} - \Lambda_A(X) - \Lambda_B(Y) \succeq 0$$

from which it follows that

$$I_{d^2n^2} - \Lambda_{A^{\oplus nd}}(X) - \Lambda_{B^{\oplus nd}}(Y) \succeq 0.$$

Left multiplying by $W^* \otimes I_n$ and right multiplying by $W \otimes I_n$ shows that (2.1) holds.

The proof of the converse closely follows the proof of [18, Theorem 3.2]. We first note that using [34, Theorem 4.11], since $\operatorname{proj}_x \mathcal{D}_{(A,B)}$ is closed and bounded and contains 0, we can assume that $\mathcal{D}_{(A,B)}$ is bounded. Now, set

$$\mathcal{S} := \operatorname{span}\{B_1, \dots, B_h\} \qquad \text{and} \qquad \mathcal{V} = S^{\perp}.$$

Since $\mathcal{D}_{(A,B)}$ is bounded, the vector space $\mathcal{S}$ is an indefinite subspace of $SM_d(\mathbb{R})$. That is, $\mathcal{S}$ does not contain a positive semidefinite element. Furthermore, following the discussion in [18], we can without loss of generality assume that $\operatorname{tr}(B_i) = 0$ for all $i = 1, \dots, h$, that the $B_1, \dots, B_h$ are orthonormal, and that $A_i \in \mathcal{S}^{\perp}$ for each $i = 1, \dots, g$.

With this set up, we have that $I_d \in \mathcal{V}$ and that $\mathcal{V}$ is closed under taking adjoints since each element of $\mathcal{S}$ is a real symmetric matrix. Thus $\mathcal{V}$ is an operator system contained in $M_d(\mathbb{R})$. Given an element $X \in \operatorname{proj}_x \mathcal{D}_{(A,B)}(n)$, consider the following linear map $\phi : \mathcal{V} \to M_n(\mathbb{R})$ defined by

$$\phi(W) = tr(W) I_n - \sum_{i=1}^{g} \langle A_i, W \rangle X_i.$$

We will show that $\phi$ is $n$-positive on $\mathcal{V}$.

To this end, let $(W_{i,j})_{i,j} \in M_n(\mathcal{V})$ be positive semidefinite. Since $(W_{i,j})_{i,j}$ has rank at most $nd$, there is a collection of vectors $\{w_k\}_{k=1}^{nd} \subseteq \mathbb{R}^{nd}$ such that

$$(W_{i,j})_{i,j} = \sum_{k=1}^{nd} w_k w_k^*.$$

Partition each $w_k$ as

$$w_k = \begin{pmatrix} w_{1,k} \\ \vdots \\ w_{n,k} \end{pmatrix}$$

with respect to the decomposition $\mathbb{R}^{nd} \cong \mathbb{R}^n \otimes \mathbb{R}^d$. Then for each $i, j = 1, \ldots, n$ we have

$$W_{ij} = \sum_{k=1}^{nd} w_{i,k} w_{j,k}^* \in \mathcal{V} \subseteq \mathbb{R}^{d \times d}.$$

We then have that

$$\begin{aligned} \phi(W_{i,j}) &= \textstyle\sum_{k=1}^{nd} \Big( \operatorname{tr}(w_{i,k} w_{j,k}^*) I_n - \sum_{\ell=1}^{g} \langle A_\ell, w_{i,k} w_{j,k}^* \rangle X_\ell \Big) \\ &= \textstyle\sum_{k=1}^{nd} \Big( \operatorname{tr}(w_{i,k} w_{j,k}^*) I_n - \sum_{\ell=1}^{g} \operatorname{tr}(w_{j,k} w_{i,k}^* A_\ell) X_\ell \Big) \\ &= \textstyle\sum_{k=1}^{nd} \Big( \operatorname{tr}(w_{j,k}^* w_{i,k}) I_n - \sum_{\ell=1}^{g} \operatorname{tr}(w_{i,k}^* A_\ell w_{j,k}) X_\ell \\ &= \textstyle\sum_{k=1}^{nd} \Big( (w_{j,k}^* w_{i,k}) I_n - \sum_{\ell=1}^{g} (w_{i,k}^* A_\ell w_{j,k}) X_\ell \Big). \end{aligned}$$

From this we obtain

$$\begin{aligned} \phi((W_{i,j})_{i,j}) &= \textstyle\sum_{k=1}^{nd} \Big( W_k^* W_k \otimes I - \sum_{\ell=1}^{g} W_k^* A_\ell W_k \otimes X_\ell \Big) \\ &= W^* W \otimes I_n - \Lambda_{W^* \left(A^{\oplus nd}\right) W}(X) \end{aligned}$$

where $W_k$ is the matrix $d \times n$ with $w_{1,k}, \ldots, w_{n,k}$ as its columns for each $k = 1, \ldots, nd$ and where $W$ is the block matrix

$$W = \begin{pmatrix} W_1 \\ \vdots \\ W_{nd} \end{pmatrix}.$$

Now observe that that for each $i, j = 1 \ldots, n$ and $\ell = 1, \ldots, h$ the $ij$ entry of $W^* B_\ell^{\oplus nd} W = \sum_{k=1}^{nd} W_k^* B_\ell W$ is given by

$$\left( \sum_{k=1}^{nd} W_k^* B_\ell W \right)_{ij} = \sum_{k=1}^{nd} w_{i,j}^* B_\ell w_{i,j} = \operatorname{tr}\left( \sum_{k=1}^{nd} B_\ell w_{i,j} w_{i,j}^* \right) = \langle W_{ij}, B_\ell \rangle = 0$$

where the last equality follows from the fact that $W_{ij}$ is in $\mathcal{V} = (\operatorname{span}\{B_1, \ldots, B_h\})^\perp$. We conclude that $W^* B^{\oplus nd} W = 0$, i.e. that $W \in \mathcal{Z}_B(n)$. It follows from our assumptions that

$$\phi((W_{i,j})_{i,j}) = W^* W \otimes I_n - \Lambda_{W^* \left(A^{\oplus nd}\right) W}(X) \succeq 0,$$

hence $\phi$ is $n$-positive.

Having shown that $\phi$ is an $n$-positive map from $\mathcal{V} \to M_n(\mathbb{R})$ we may use [46, Theorem 6.1] to conclude that $\phi$ is completely positive. We may then use Arveson's extension theorem, see again [46], to conclude that $\phi$ has a completely positive extension $\psi : M_d(\mathbb{R}) \to M_n(\mathbb{R})$. Note that [46] works over the complexes; see [32] for a discussion of these results over the reals. Also note that it is straightforward to check that these results hold over the reals using Choi's characterization of completely positive maps [11, Theorem 2].

Now, for any matrix $U \in M_d(\mathbb{R})$ we have

$$\begin{aligned}\psi(U) =& \psi\left(U - \sum_{j=1}^{h}\langle U, B_j\rangle B_j\right) + \psi\left(\sum_{j=1}^{h}\langle U, B_j\rangle B_j\right)\\ =& \phi\left(U - \sum_{j=1}^{h}\langle U, B_j\rangle B_j\right) + \sum_{j=1}^{h}\langle U, B_j\rangle \psi(B_j)\\ =& \operatorname{tr}(U) - \sum_{\ell=1}^{g}\langle A_\ell, U\rangle X_\ell - \sum_{j=1}^{h}\langle U, B_j\rangle \psi(-B_j).\end{aligned}$$

Here the last inequality follows from the facts that $U - \sum_{\ell=1}^{h}\langle U, B_\ell\rangle B_\ell \in \mathcal{V}$ and that $\operatorname{tr}(B_j) = 0$ and $\langle A_\ell, B_j\rangle = 0$ for all $j, \ell$.

Let $E_{ij} \in M_d(\mathbb{R})$ be the matrix with 1 in the $(i,j)$-entry and zeros elsewhere. Then the Choi matrix $E = ((E_{j,k})_{j,k}) \in M_d(M_d(\mathbb{R}))$ is positive semidefinite and thus

$$\begin{aligned}0 \preceq \psi(E) =& \Big(\operatorname{tr}(E_{i,j}) - \sum_{\ell=1}^{g}\langle A_\ell, E_{i,j}\rangle X_\ell - \sum_{\ell=1}^{h}\langle E_{i,j}, B_\ell\rangle \psi(-B_\ell)\Big)_{i,j}\\ =& I - \Lambda_A(X) - \sum_{\ell=1}^{h} B_\ell \otimes \psi(-B_\ell).\end{aligned}$$

We conclude that

$$\big(X_1, \dots, X_g, \psi(-B_1), \dots, \psi(-B_h)\big) \in \mathcal{D}_{(A,B)},$$

hence $X \in \operatorname{proj}_x \mathcal{D}_{(A,B)}$. This completes the proof of Item (1).

We now prove Item (2). The forward direction of the proof uses the same argument as the forward direction of Item (1). To prove the reverse direction, we use the same strategy as in [18, Proposition 3.3]. To this end, assume that equation (2.2) holds for all $W \in \mathcal{I}_B(n)$. We will show that this implies that equation (2.1) holds for all $W \in \mathcal{Z}_B(n)$.

Let $W \in \mathcal{Z}_B(n)$ and let $r$ be the rank of $W^*W \in M_n(\mathbb{R})$. Then there is an invertible matrix $U \in M_n(\mathbb{R})$ such that

$$U^*W^*WU = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} =: P$$

Define

$$\iota := \begin{pmatrix} I_r \\ 0 \end{pmatrix} \in M_{n,r}(\mathbb{R})$$

and $V = WU\iota$. Then we have

$$V^*(B^{\oplus nd})V = \iota^* U^* (W^* B W) U \iota = 0$$

and

$$V^*V = \iota^* U^* W^* W U \iota = \iota^* P \iota = I_r.$$

That is $V \in \mathcal{I}_B(n,r)$, hence

$$L_{V^*\left(A^{\oplus nd}\right)V}(X) \succeq 0.$$

We next show that $WUP = WU$. To prove this, let $x \in \mathbb{R}^n$ and set $x_1 = Px$ and $x_2 = x - x_1$. Then we have $WUPx = WUx_1 = WUx - WUx_2$. Hence it is sufficient to show that $WUx_2 = 0$. To see this, first observe that since $P = P^2$, we have $Px = P^2x = Px_1$. Next observe that

$$\langle WUx_2, WUx_2 \rangle = \langle U^*W^*WUx_2, x_2 \rangle = \langle Px_2, x_2 \rangle = \langle Px - Px_1, x_2 \rangle = 0.$$

We conclude that $WUx_2 = 0$, hence $WUP = WU$ as claimed. Using this and the observation that $P = \iota\iota^*$ gives

$$\begin{aligned} U^*W^*WU \otimes I_n - \Lambda_{U^*W^*(A^{\oplus nd})WU}(X) &= PU^*W^*WUP \otimes I_n - \Lambda_{PU^*W^*(A^{\oplus nd})WUP}(X) \\ &= (\iota \otimes I_n) L_{V^*\left(A^{\oplus nd}\right)V}(X)(\iota \otimes I_n)^* \\ &\succeq 0. \end{aligned}$$

Moreover, since $U$ is invertible, we obtain

$$W^*W \otimes I_n - \Lambda_{W^*(A^{\oplus nd})W}(X) \succeq 0.$$

Thus we have shown that our assumptions imply that equation (2.1) holds for all $W \in \mathcal{Z}_B(n)$. Using Item (1), we conclude that $X \in \operatorname{proj}_x \mathcal{D}_{(A,B)}$ as claimed.

It remains to prove Item (3). If $m \leq n$, then the result follows immediately from Item (2) as one can naturally embed $\mathcal{I}_B(m)$ in $\mathcal{I}_B(n)$ when $m \leq n$. Now assume $m > n$. If $X \in \operatorname{proj}_x \mathcal{D}_{(A,B)}(n)$, then since $0 \in \operatorname{proj}_x \mathcal{D}_{(A,B)}$ and since $\operatorname{proj}_x \mathcal{D}_{(A,B)}$ is matrix convex, we have $X \oplus 0_{m-n} \in \operatorname{proj}_x \mathcal{D}_{(A,B)}$. Using Item (2) then shows that for any $W \in \mathcal{I}_B(m)$ we have $L_{W^*(A^{\oplus m})W}(X \oplus 0) \succeq 0$. By applying the canonical shuffle we obtain

$$L_{W^*(A^{\oplus m})W}(X \oplus 0) \cong \begin{pmatrix} L_{W^*(A^{\oplus m})W}(X) & \Lambda_{W^*(A^{\oplus m})W}(0) \\ \Lambda_{W^*(A^{\oplus m})W}(0) & L_{W^*(A^{\oplus m})W}(0) \end{pmatrix} = \begin{pmatrix} L_{W^*(A^{\oplus m})W}(X) & 0 \\ 0 & I_{m-n} \end{pmatrix} \succeq 0.$$

We conclude $L_{W^*(A^{\oplus m})W}(X) \succeq 0$, as claimed. ∎

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