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FREE FIBONACCI ALGEBRAS

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Dedicated to Professor B.H. Neumann on his eightieth birthday

Fibonacci varieties were introduced by one of us in 1978 and a natural generalisation was studied shortly afterwards. We carry this investigation one stage further by giving a description of the free objects in these varieties. This is done in terms of the n-abelian groups of Levi.

0. INTRODUCTION

We are concerned with the variety $\mathcal{V}(m)$ of universal algebras G of the following kind : G is a group equipped with a unary operation ϕ that is an automorphism of G and satisfies the one-variable law

$$xx\phi\ldots x\phi^{m-1} = x\phi^m, \tag{1}$$

where m is a positive integer and any occurrence of ϕ or one of its powers is understood to apply only to the symbol immediately preceding it. It seems natural to call such objects *Fibonacci algebras*, or *ϕ-algebras* for short, in contradistinction to Fibonacci groups [**2**]. Note that we omit the condition in [**4**] that ϕ have some specified finite order, and hope to extend our results to this "modular" case in a future article. Our chief aim is to prove the following

THEOREM. *The free object of rank d in $\mathcal{V}(m)$ is given by*

$$V_d(m) = Z^{\times(m-1)d} \times F/\big(F' \cap F^m \cap F^{m-1}\big)\big(F' \cap F^{m-1} \cap F^{m-2}\big),$$

where the first factor is the free abelian group of rank $(m-1)d$ and, in the second, $F = F_d$ is the (absolutely) free group of rank d, F' is its derived group, and F^m the subgroup generated by all mth powers.

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1. Preliminaries

We begin with a brief review of the main results of [4].

By evaluating the product $xx\phi\dots x\phi^{m-1}x\phi^m$ in two different ways, we obtain the one-variable law

$$xx\phi^{m+1} = x\phi^m x\phi^m. \tag{2}$$

Next, apply (1) to xy:

$$\begin{aligned} x\phi^m y\phi^m = (xy)\phi^m &= xy(xy)\phi\dots(xy)\phi^{m-1} \\ &= xyx\phi y\phi\dots x\phi^{m-1}y\phi^{m-1} \\ &= x(yx\phi)(yx\phi)\phi\dots(yx\phi)\phi^{m-1}x^{-1}\phi^m \\ &= x(yx\phi)\phi^m x^{-1}\phi^m, \text{ applying (1) to } yx\phi, \\ &= xy\phi^m x\phi^{m+1}x^{-1}\phi^m \\ &= xy\phi^m x^{-1}x\phi^m, \text{ by (2).} \end{aligned}$$

Thus, $x^{-1}x\phi^m$ commutes with $y\phi^m$ and so is central. But

$$x^{-1}x\phi^m = x\phi^{m+1}x^{-1}\phi^m = \left(x\phi x^{-1}\right)\phi^m$$

by (2), and we have the following central result.

Proposition 1. *For all $x \in G$, $x^{-1}x\phi$ is central.*

It follows easily that the set

$$H = \{x^{-1}x\phi \mid x \in G\}$$

is closed under multiplication, inverses, and the action of ϕ, so that $G/H \in \mathcal{V}(m)$ and admits the trivial ϕ-action. It is thus the largest ϕ-trivial factor-algebra G^ϕ of G. If we let G_ϕ denote the largest ϕ-trivial subalgebra of G, then Proposition 1 can be restated as follows.

Corollary 1. *The mapping $\theta : G \to G$, $x \mapsto x^{-1}x\phi$, is a homomorphism into $Z(G)$ with $\operatorname{Ker}\theta = G_\phi$ and $\operatorname{Coker}\theta = G^\phi$.*

Since ϕ-trivial groups have exponent dividing $m-1$, it follows that $G^{m-1} \leqslant \operatorname{Im}\theta \leqslant Z(G)$, and we have

Corollary 2. *The law $[x^{m-1}, y] = 1$ holds in G.*

This is (1.4) in Theorem 3 of [4], and it follows that the algebras in $\mathcal{V}(2)$ are all commutative, which is the Lemma in [3].

By applying Proposition 1 to $x, x\phi, x\phi^2, \dots$ in turn, it follows that $x^{-1}x\phi^k$ is central for all $k \in \mathbb{N}$, whence the images of x under all integral powers of ϕ commute in pairs, which yields Theorem 1 of [4]:

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Corollary 3. *The monogenic free algebra in $\mathcal{V}(m)$ is commutative.*

Next, apply (1) to xy as above to obtain:

$$xx\phi\dots x\phi^{m-1}yy\phi\dots y\phi^{m-1} = xyx\phi y\phi\dots x\phi^{m-1}y\phi^{m-1}.$$

Since the terms x, $x\phi$, ..., $x\phi^{m-1}$ each appear once on both sides and are congruent to x modulo $Z(G)$, they can all be replaced by x. Similarly, each of y, $y\phi$, ..., $y\phi^{m-1}$ can be replaced by y.

Corollary 4. *The law $x^m y^m = (xy)^m$ holds in G.*

This law defines the variety $\mathcal{A}(m)$ of m-abelian groups introduced in [**5**] and classified in [**1**]. In view of the fact that any two of the laws

$$(3) \qquad [x^{m-1}, y] = 1, \quad x^m y^m = (xy)^m, \quad x^{m-1}y^{m-1} = (xy)^{m-1}$$

imply the third (easy exercise), Corollaries 2 and 4 can be combined into

Corollary 5. $\mathcal{V}(m) \leqslant \mathcal{A}(m) \cap \mathcal{A}(m-1)$.

It follows that the algebras of $\mathcal{V}(3)$ are commutative, which is Theorem 2 of [**4**].

2. A criterion

For every x in a ϕ-algebra G, the ϕ-subalgebra generated by x is abelian (Corollary 1.3); we write it additively for the moment and work with the homomorphisms

$$(4) \qquad \theta : x \mapsto -x + x\phi, \quad \mu : (m-1)x,$$

noting that $\operatorname{Im}\mu \leqslant \operatorname{Im}\theta \leqslant Z(G)$. For each $x \in G$ and every $k = 0, 1, 2, \dots$, we have

$$x\phi^k = x(1+\theta)^k,$$

and (1) can be rewritten in the form

$$mx + \sum_{k=0}^{m-1} x\{(1+\theta)^k - 1\} = x + x\{(1+\theta)^m - 1\},$$

that is,

$$(5) \qquad x\mu = xf_m(\theta), \qquad \text{where } f_m(t) \in \mathbb{Z}[t]$$

is a monic polynomial of degree m with zero constant term. Since θ is a homomorphism with $\operatorname{Im}\theta \subseteq Z(G)$, the same is true of $f_m(\theta)$, and we have another proof of Corollary 1.2. We also have the following characterisation of ϕ-algebras.

PROPOSITION 2. *Let G be any group, X a set of generators for G, and ϕ an automorphism of G such that :*

(i) (1) *holds for all $x \in X$,*

(ii) $x^{-1}x\phi \in Z(G)$, $\forall x \in X$,

(iii) *the map $\mu : g \mapsto g^{m-1}$ is a homomorphism.*

Then (1) *holds for all $x \in G$, that is, G is a ϕ-algebra.*

PROOF: Since ϕ is a homomorphism, it follows from (ii) that $g^{-1}g\phi \in Z(G)$ for all $g \in G$, and that $\theta : g \mapsto g^{-1}g\phi$ is a homomorphism. Thus, using (i), (5) holds for all $x \in X$, and so for all $x \in G$, by (iii). Since every $x \in G$ commutes with all the $x\phi^k$, $k \in \mathbb{N}$, (1) now follows from (5). ∎

COROLLARY 1. *If G is an abelian group and $\phi \in \operatorname{Aut} G$ satisfies* (1) *on a set of generators of G, then* (1) *is a law in G.*

COROLLARY 2. *Let G be a group and $\theta : G \to Z(G)$ a homomorphism such that*

$$(m-1)g = gf_m(\theta), \qquad \forall g \in G,$$

where $f_m(t) \in \mathbb{Z}[t]$ is given by

$$f_m(t) = (1+t)^m - \frac{(1+t)^m - 1}{t} + (m-1).$$

Then G is a ϕ-algebra with respect to $\phi : g \mapsto g(1+\theta)$.

3. FREE OBJECTS IN $\mathcal{V}(m)_{ab}$

Because of Corollary 2.2, the direct product of two ϕ-algebras is a ϕ-algebra in the natural way. Moreover, when they are abelian, it is their free product in $\mathcal{V}(m)_{ab}$. Since the free object of rank d in any variety is just the dth free power of the monogenic free object, it suffices to describe the latter. Consider the group

(6) $$F(m) = \langle x | Y_m, C \rangle,$$

where $$X = \{x_i \mid i \in \mathbb{Z}\},$$

(7) $$Y_m = \{x_i x_{i+1} \dots x_{i+m-1} x_{i+m}^{-1} \mid i \in \mathbb{Z}\},$$

$$C = \{[x_i, x_j] \mid i, j \in \mathbb{Z}\},$$

which is clearly free abelian of rank m. Now the map $\phi : x_i \mapsto x_{i+1}$ clearly extends to an automorphism of $F(m)$ satisfying (1) for all $x \in X$. $F(m)$ is thus a ϕ-algebra by Corollary 2.1, and generated as such by x_0. Given any element y in any G in $\mathcal{V}(m)_{ab}$, it is clear how to define a ϕ-homomorphism from $F(m)$ to G sending x_0 to y. Thus $F(m)$ is the monogenic free object in $\mathcal{V}(m)_{ab}$.

PROPOSITION 3. *The free object of rank d in $\mathcal{V}(m)_{ab}$ is the ϕ-algebra $F(m)^{\times d}$, where $F(m)$ is defined by* (6) *and* (7).

COROLLARY 1. *$F(m)$ is the monogenic free object in $\mathcal{V}(m)$.*

COROLLARY 2. *$F(m)^{\times d}$ is the free object of rank d in $\mathcal{V}(m)$ when $m = 2$ or 3.*

These are consequences of Corollary 1.3 and the fact that ϕ-algebras are abelian when $m = 2$ or 3, respectively.

For later use, we replace the basis $\{x_i \mid 0 \leqslant i \leqslant m-1\}$ of $F(m)$ by

$$x = x_0, \quad y_i = x_{i-1}^{-1} x_i, \quad 1 \leqslant i \leqslant m-1; \tag{8}$$

then $F(m)\theta$ has basis x^{m-1}, y_i, $1 \leqslant i \leqslant m-1$.

4. FREE PRODUCTS IN $\mathcal{V}(m)$

Given ϕ-algebras G and H, out strategy is to factor out from their ordinary free product $F := G * H$ "just enough" extra relators to yield a ϕ-algebra $G *_\phi H$. Specifically, let G and H be presented as groups by

$$G = \langle X | R \rangle, \quad H = \langle Y | S \rangle,$$

so that

$$F = \langle X, Y | R, S \rangle.$$

Now ϕ is defined on the generators $X \cup Y$; let $\nu : F \to K$ be any group homomorphism into a ϕ-algebra K that commutes with ϕ on $X \cup Y$. Then ν annihilates all $[z\theta, z']$, z, $z' \in X \cup Y$, θ as above. Moreover, if $w = w(X \cup Y)$ is any word in $\langle X, Y | \ \rangle$, then $w\mu = w^{m-1}$ and $w((X \cup Y)\mu)$ have the same image under ν. (The latter word is that obtained from w by replacing each letter by its $(m-1)$th power.) Now hopefully put

$$G *_\phi H = \langle X, Y | R, S, [X\theta, Y], [X, Y\theta], W \rangle, \tag{9}$$

where θ is defined on X and Y as in (4), and

$$W = \{ w^{m-1} w((X \cup Y)\mu)^{-1} \mid w \in \langle X, Y | \ \rangle \}. \tag{10}$$

From what has been said, any ν of the above type factors through $G *_\phi H$. Furthermore, the group presented in (9) clearly satisfies all the conditions of Proposition 2. It is thus the biggest ϕ-homomorphic image of F, and as such is the free product of G and H in $\mathcal{V}(m)$.

PROPOSITION 4. *The free product of groups* G, H *in* $\mathcal{V}(m)$ *is given by* (9).

COROLLARY 1. *The free object of rank* d *in* $\mathcal{V}(m)$ *is* $F(m)^{*_\phi d}$.

Now consider the result $(G *_\phi H)^\phi$ of factoring $G *_\phi H$ out by its central subgroup $\langle X\theta, Y\theta \rangle$. The relators $[X\theta, Y]$, $[X, Y\theta]$ become redundant and W reduces to

$$P = \{w^{m-1} \mid w \in \langle X, Y| \ \rangle\},$$

as $X\mu \subseteq G\theta$, $Y\mu \subseteq H\theta$. It follows that $(G *_\phi H)^\phi$ has the presentation

$$\langle X, Y | R, S, X\theta, Y\theta, P \rangle = G^\phi *_B H^\phi,$$

where the right-hand side is the free product in the Burnside variety $\mathcal{B}(m-1)$ of groups of exponent $m-1$. Because of the natural ϕ-homomorphism from $G *_\phi H$ onto $G \times H$, it is clear that

$$\langle X\theta, Y\theta \rangle = (G *_\phi H)\theta = G\theta \times H\theta.$$

COROLLARY 2. *There is a central extension*

$$1 \to G\theta \times H\theta \to G *_\phi H \to G^\phi *_B H^\phi \to 1.$$

COROLLARY 3. $G *_\phi H$ *is the result of factoring out* $G * H$ *by the intersection of the kernels of the natural maps onto* G, H *and* $G^\phi *_B H^\phi$.

COROLLARY 4. *The free object* $V_d(m)$ *of rank* d *in* $\mathcal{V}(m)$ *is a central extension of the corresponding object in* $\mathcal{B}(m-1)$ *by a free abelian group of rank* md.

Corollary 3 follows at once from Corollary 2, and Corollary 4 by induction on d from Corollaries 1 and 2.

5. FREE OBJECTS IN $\mathcal{V}(m)$

We conclude by describing the presentation of $V_d(m) = F(m)^{*_\phi d}$ arrived at using (9), where each ϕ-free factor is generated by a set of the form (8). We refer to these m generators of the ith factor as i-generators, $1 \leqslant i \leqslant d$, and to the d generators x and $(m-1)d$ generators y_i, $1 \leqslant i \leqslant m-1$, as x-generators and y-generators respectively. Letting Z denote the set of all such generators, the defining relators for $F(m)^{*_\phi d}$ are now of three types (corresponding respectively to R and S, $[X\theta, Y]$ and $[X, Y\theta]$, W respectively).

(1) the m i-generators commute along themselves, $1 \leqslant i \leqslant d$.
(2) the y-generators are all central, and so are the $(m-1)$th powers of the x-generators.
(3) $w^{m-1} = w(Z\mu)$, for all $w \in \langle Z| \ \rangle$.

Now the centrality of the y-generators asserted in (2) ensures that:

(i) the relators (1) are superfluous, and

(ii) only words w in the x-generators are needed in (3).

It follows that $V_d(m)$ is the direct product of the subgroups C and A generated by the x-generators and y-generators, respectively. Moreover, it follows from Corollary 4.2 that A is free abelian of rank $(m-1)d$, and that C is the free object $C_d(m-1)$ of rank d in the variety $\mathcal{C}(m-1) = \mathcal{A}(m) \cap \mathcal{A}(m-1)$ defined by the laws (3).

PROPOSITION 5. $V_d(m) \cong C_d(m-1) \times Z^{\times(m-1)d}$.

The theorem is an immediate consequence of this, by a result of Alperin [1] which asserts that the free object of rank d in $\mathcal{A}(m)$ is given by

$$A_d(m) = \frac{F}{F' \cap F^m \cap F^{m-1}},$$

where $F = F_d$ is the (absolutely) free group of rank d.

Examples. Since $F^0 = \{1\}$, $F^1 = F$, and $F^2 \supseteq F'$, we read off

$$C_d(0) = F, \quad C_d(1) = C_d(2) = F/F', \quad C_d(3) = F/F' \cap F^3.$$

Putting $d = 2$ in the last case, it can be shown using Corollary 4.4 that

$$F/F' \cap F^3$$
$$= \left\langle x, y \mid [x^3, y] = [x, y^3] = 1, (xy)^3 = x^3y^3, (xy^{-1})^3 = x^3y^{-3} \right\rangle.$$

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