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FREE-INERTIAL AND DAMPED-INERTIAL NAVIGATION MECHANIZATION AND ERROR EQUATIONS

Warren G. Heller

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The equations that describe both the navigation mechanization and the propagation of errors in an unaided inertial system are detailed. Extensions of these equations which apply to continuous speed and altitude damping are also given. A general vector-matrix notation is employed, thereby eliminating the need to specify a particular navigation mechanization before setting down the error equations.

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Specific application of the general equations to the local-level, wander-azimuth mechanization is outlined. The detailed form of the error equations is given for both the free-inertial case and various choices of continuous damping.

FOREWORD

This document contains the first subset of equations to be furnished to the Defense Mapping Agency Aerospace Center (DMAAC) as a part of Task II of contract DMA700-74-C-0075, "Aided Inertial Navigation System Error Analysis." A total of three such equation subsets will be issued:

1. Free-Inertial and Damped-Inertial Navigation Mechanization and Error Equations (the present report)

2. Covariance Propagation Equations for Optimal and Suboptimal Kalman-Filter-Integrated Multi-Sensor Inertial Systems

3. Models for Aided Inertial Navigation Instrument Errors

The intent of these equation subsets is to provide DMAAC with complete, self-contained mathematical "modules" suitable for studying modern multi-sensor inertial navigation systems such as those of the B-1 and F-15 aircraft. These subsets of equations are rendered in a form sufficiently general as to be applicable to the inertial systems in all terrestrial vehicles. Included in this category are missile, aircraft and marine navigation. Also detailed is the particular form of the equations most suited to performance studies of these two aircraft.

ABSTRACT

The equations that describe both the navigation mechanization and the propagation of errors in an unaided inertial system are detailed. Extensions of these equations which apply to continuous speed and altitude damping are also given. A general vector-matrix notation is employed, thereby eliminating the need to specify a particular navigator mechanization before setting down the error equations.

Specific application of the general equations to the local-level, wander-azimuth mechanization is outlined. The detailed form of the error equations is given for both the free-inertial case and various choices of continuous damping.

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1. INTRODUCTION

The errors of an unaided inertial navigation system (INS) typically grow with time in an unbounded manner. As a consequence, the cruise inertial navigation systems used in modern aircraft and submarines are normally aided or "damped" with data from external aids, such as:

- altimeter or depth gauge
- speed reference (doppler radar or EM-log)
- position reference (LORAN, NAVSAT, ...)

The errors of an aided INS are typically bounded, although the errors grow in-between position fixes.

The techniques that are used to combine external reference data with INS outputs fall into two categories: "conventional continuous-feedback damping" and "Kalman-filter damping". Conventional continuous-feedback techniques are usually used with altitude and speed reference devices, whereas Kalman-filter techniques are frequently used with position reference devices. However, the trend in recent years has been toward more extensive use of Kalman-filter techniques. An overall conceptual diagram of a multi-sensor aided INS is shown in Fig. 1-1.

The purpose of this report is to present the mechanization and linearized error propagation equations for the following cases:

- Unaided INS
- Speed-damped INS
  - First-order damping
  - Second-order damping

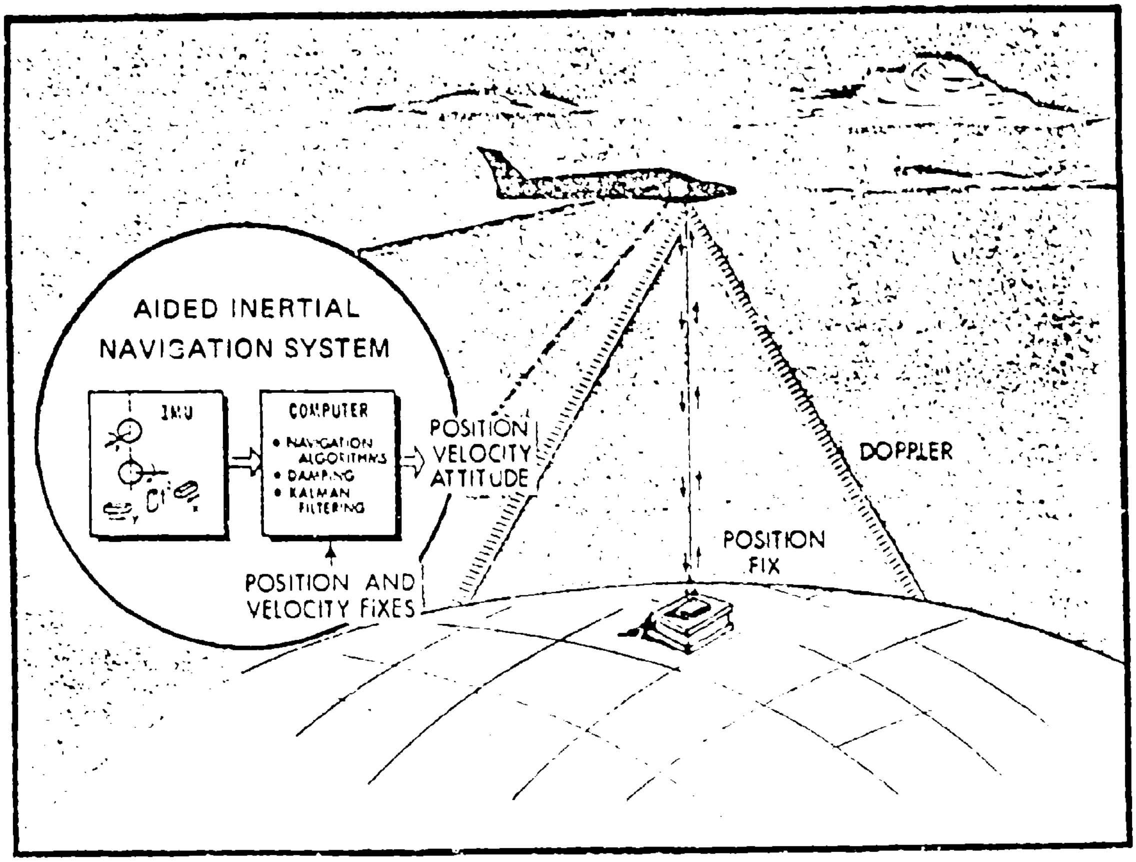

Figure 1-1   Multi-Sensor Aided Inertial Navigation System Operation

- Altitude-damped INS
  - First-order damping
  - Second-order damping

In this report, it is assumed that speed and altitude damping are accomplished with conventional continuous-feedback loops. A forthcoming report will treat the case of Kalman-filter damping with speed,altitude, and position references.

The mechanization and error propagation equations given in this report are applicable to the inertial navigation system in any terrestrial vehicle--for example, a jeep, aircraft, submarine, or missile. However, it should be kept in mind that missile inertial navigation systems are typically not damped with external aids. In such a case, only the free-inertial equations are applicable.

Error Sources - The error sources which are considered in this equation subset are those inherent to the INS itself, to velocity damping and to altitude stabilization. Detailed error models and treatment of errors in externally derived measurements will be deferred until the third equation subset. The error quantities of interest in this subset are:

accelerometer errors	platform misalignments
altitude reference errors	velocity reference errors
gyro drift rates	vertical deflections
position and velocity errors	gravity anomalies

Coordinate Frames - Several coordinate systems whose origins are common with the center of the earth are used in the equations. They are:

- An inertial frame which has its axes fixed with respect to the "fixed" stars,
- A local level frame which has two axes tangent to and a third axis normal to a reference ellipsoid at the locality under consideration,
- A true frame corresponding to the ideal (i.e., error free) orientation of the inertial platform (mechanization dependent) at the vehicle's actual position,
- An earth fixed frame with axes embedded in and non-rotating with respect to the earth,
- The platform frame with axes parallel to the nominal accelerometer input axes,
- The frame in which the navigation equation mechanization actually occurs. Because of errors, this frame will not be the same as that in which the equations are nominally mechanized (true frame). This reference frame, which is specified by the navigation system outputs of position and velocity is the so-called computer frame.

Usually, the true frame, platform frame, and computer frame are coincident; inertial navigation errors lead to small-angle misalignments among these frames.

In this report the INS mechanization and error equations are derived and presented for a variety of applications. No attempt is made to actually solve the equations, although in the course of deriving them qualitative observations about the form of the solutions are given. Where background material is of a lengthly nature, particularly with regard to equation derivations, appropriate references are cited.

2. VECTOR FORMULATION OF MECHANIZATION AND ERROR EQUATIONS FOR AN UNDAMPED INERTIAL SYSTEM

2.1 UNIFIED EQUATION SUBSET

A vector formulation of the navigation mechanization and error equations is presented in this Chapter. This formulation is completely general--that is, valid for any dynamically-exact* INS mechanization (local level, space stable, tangent plane, strapdown, etc.). Such a unified approach allows one equation "module" to serve all cases and provides notational consistency. It is not surprising that a single set of equations can properly describe all inertial systems. Any such system is a mechanization of Newton's second law which itself is invariant. The inertial equations are but a more detailed formulation of the force-momentum relationship specialized to the navigation process. Expansion of the general vector equations in a form specific to a local-level INS mechanization is demonstrated in Chapter 3.

Although the error propagation is frame-independent, the instrument error models are not, hence the sensor error equations must be tailored to each mechanization. For example, a drifting gyro in a locally-level mechanized INS does not induce the same dynamical position, velocity and tilt errors as the same drifiting gyro would cause in a space-stable INS (Ref. 1). In this document consideration of error sources (such as gyro drift rate) will be limited to their treatment as driving terms in the equations. Detailed models of error sources will be presented in the third equation subset.

*"Dynamically exact" implies that if there were no instrument measuring errors or initial alignment errors, the INS outputs would be error-free. This is sometimes referred to as "no errors due to true vehicle motion."

2.2 NOTATION AND DEFINITIONS FOR THE MECHANIZATION EQUATIONS

$\frac{d}{dt}(\ldots)_S$ = Time rate of change of (...) as seen by an observer in a general S frame

$\bar{R}$ = Vehicle position vector

$\bar{V} = \frac{d}{dt}(\bar{R})_E$ = Vehicle Velocity in an earth fixed frame - "ground speed"

$\bar{A}$ = Specific force, i.e., vehicle acceleration due to all forces acting except gravity (ideal accelerometer output)

$\bar{g}(\bar{R})$ = Plumb bob gravity acceleration on vehicle

$\bar{\Omega}$ = Angular rate of earth fixed axes with respect to inertial space

$\bar{\omega}_{RS}$ = Angular rate of a general S frame with respect to another general R frame

$\bar{\omega}_{IC}$ = Angular rate of computer frame with respect to inertial space

$\bar{\omega}_{EC}$ = Angular rate of computer frame with respect to an earth fixed frame

$\bar{A}\cdot\bar{B}$ = Scalar product of vectors $\bar{A}$ and $\bar{B}$

$\bar{A}\times\bar{B}$ = Vector cross product of vectors $\bar{A}$ and $\bar{B}$

2.3 MECHANIZATION EQUATIONS

The mechanization equations for an unaided INS are (Ref. 2):

$$\frac{d}{dt}(\bar{V})_C = \bar{A} + \bar{g}(\bar{R}) - (\bar{\omega}_{IC} + \bar{\Omega}) \times \bar{V} \tag{2-1}$$

$$\frac{d}{dt}(\bar{R})_C = \bar{V} - \bar{\omega}_{EC} \times \bar{R} \tag{2-2}$$

Note that the vector quantities (overbar notation) in these equations require no subscripts to designate a coordinate system since the magnitudes and directions they specify are independent of the reference frame in which they are expressed. The vectors $\bar{R}$, $\bar{V}$, $\bar{A}$, and $\bar{g}$ in Eqs. (2-1) and (2-2) are the "true" position, velocity, specific force, and gravity acceleration. However, since the navigation computer deals only with "computed" quantities ($\bar{R}_C$, $\bar{V}_C$, $\bar{A}_C$, $\bar{g}_C$), the equations actually mechanized are:

$$\frac{d}{dt}(\bar{V}_C)_C = \bar{A}_C + \bar{g}_C(\bar{R}_C) - (\bar{\omega}_{IC} + \bar{\Omega}) \times \bar{V}_C \qquad (2\text{-}3)$$

$$\frac{d}{dt}(\bar{R}_C)_C = \bar{V}_C - \bar{\omega}_{EC} \times \bar{R}_C \qquad (2\text{-}4)$$

where

$\bar{A}_C$ = accelerometer outputs

$\bar{V}_C$ = INS outputs of vehicle groundspeed (a 3-vector)

$\bar{R}_C$ = INS outputs of vehicle position (a 3-vector)

The term $\bar{g}_C(\bar{R}_C)$ in Eq. (2-3) must include earth oblateness terms in order to provide an accurate gravity computation (Ref. 10). Neglecting the earth's ellipticity can cause position errors on the order of ten nautical miles (Ref. 3). Although the non-spherical character of the earth is explicitly expressed in the mechanization equations, usual practice is to neglect it in the error equations. Further discussion of Eqs. (2-3) and (2-4) is given in Appendix A.

2.4 NOTATION AND DEFINITIONS FOR THE ERROR EQUATIONS

$$\delta\overline{V} = \overline{V}_C - \overline{V} \tag{2-5a}$$

$$\delta\overline{R} = \overline{R}_C - \overline{R} \tag{2-5b}$$

$$\delta\overline{g} = \overline{g}(\overline{R}_C) - \overline{g}(R) \tag{2-5c}$$

$\overline{\psi}$ = small angle misalignment between platform and computer frames (defined positive for rotation from computer frame to platform frame)

$\overline{\mu}$ = total accelerometer error (difference between ideal and actual accelerometer output)

$\overline{\Delta g}$ = gravity anomaly and vertical deflections (gravity disturbance vector)

g = nominal value of gravity (scalar)

$\overline{\epsilon}$ = total gyro drift rate error due to all gyro error sources

R_0 = nominal radius of the earth (scalar magnitude)

R = magnitude of $\overline{R}$

2.5 ERROR EQUATIONS

The free (unaided) inertial system error equations are derived in Appendix A. They are:

$$\frac{d}{dt}(\overline{\delta V})_S = \overline{\mu} - \overline{\psi} \times \overline{A} + \overline{\delta g} - (\overline{\omega}_{IS} + \overline{\Omega}) \times \overline{\delta V} \tag{2-6}$$

$$\frac{d}{dt}(\overline{\delta R})_S = \overline{\delta V} - \overline{\omega}_{ES} \times \overline{\delta R} \tag{2-7}$$

$$\frac{d}{dt}(\overline{\psi})_S = -\overline{\omega}_{IS} \times \overline{\psi} + \overline{\epsilon} \tag{2-8}$$

where it is noted that they are coordinatized in a general S frame.

The gravity error $\overline{\delta g}$ is given by (Ref. 3):

$$\overline{\delta g} = \overline{\Delta g} - \frac{gR_o^2}{R^3}\,\overline{\delta R} + \frac{3\,gR_o^2}{R^5}\,(\overline{\delta R}\cdot\overline{R})\,\overline{R} \qquad (2\text{-}9)$$

Note that Eq. (2-9) contains no explicit treatment of earth ellipticity errors. Such errors are of second order size and, as such, may be neglected.

For a slowly-moving vehicle, equations (2-6) to (2-9) contain undamped oscillatory dynamics at both the Schuler (84 minute period) and earth (24 hour period) rates. This dynamical behavior is encountered in all terrestrial inertial navigation systems, regardless of the nominal platform orientation (local level, space stable, strapdown, etc.) and may be seen by writing out Eqs. (2-6) to (2-8) in a particular reference frame and taking eigenvalues. That such is the case is not surprising when it is recalled that Eqs. (2-6) through (2-9) express vector quantities in a generalized coordinate frame. The dynamics they describe must be invariant with regard to the reference system in which they are cast. Later in this report these equations will be written out term by term.

Note that the solution of Eq. (2-8) for the computer to platform misalignments is independent of the position and velocity error equations (Eqs. (2-6), (2-7)). The navigation error equations may thus be considered as being driven by the $\overline{\psi}$ errors. This partial decoupling of the equations is depicted in Fig. 2-1 and the attendant simplified form of the error equations is a major rationale for expressing them in terms of the $\overline{\psi}$ angles.

It is appropriate to point out the difference in the character of Eqs. (2-6) through (2-8) depending upon whether the vehicle travels at "high" or "low" speeds. In general, for no limitations on rates of motion the error

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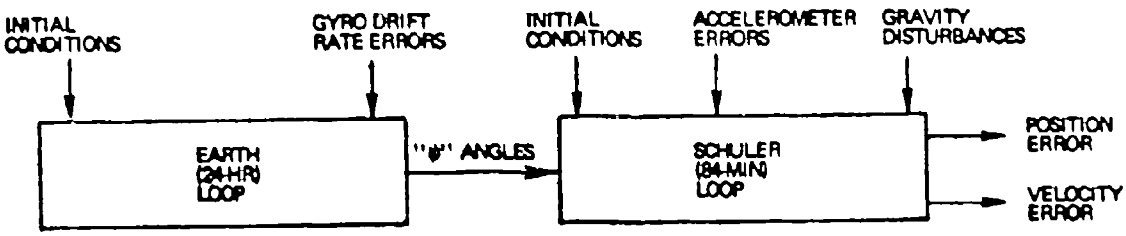

Figure 2-1 Inertial Navigation Error Dynamics

equations possess time-varying, trajectory-dependent coefficients. However, if the maximum vehicle accelerations and velocities are sufficiently small so that:

1. $\overline{A} << \overline{g}$ in Eq. (2-6)
2. $\overline{\omega}_{IS} \approx \overline{\Omega}$ in Eqs. (2-6), (2-8)(earth rate >> vehicle rate)
3. $\overline{\omega}_{ES} \times \overline{\delta R} << \overline{\delta V}$ in Eq. (2-7)

the error equations reduce to the essentially constant coefficient relations:

$$\frac{d}{dt}(\overline{\delta V})_S = \overline{\mu} - \overline{\psi} \times \overline{g} + \overline{\delta g} - 2\overline{\Omega} \times \overline{\delta V} \quad (2\text{-}10)$$

$$\frac{d}{dt}(\overline{\delta R})_S = \overline{\delta V} \quad (2\text{-}11)$$

$$\frac{d}{dt}(\overline{\psi})_S = -\overline{\Omega} \times \overline{\psi} + \overline{\epsilon} \quad (2\text{-}12)$$

The "small" velocity and acceleration assumptions required for the validity of Eqs. (2-10) through (2-12) are generally met for marine vehicles.

2.6 PLATFORM MISALIGNMENT EQUATIONS

The angular misalignment between the true (S) frame and the computer (C) frame is defined as the small angle $\bar{\theta}$ where positive $\bar{\theta}$ is taken from the true frame to the computer frame. This angle is a function of position error only.

$$\bar{\theta} = \bar{\theta}(\overline{\delta R}) \tag{2-13}$$

The small angular misalignment, $\bar{\phi}$, between the platform frame and the true frame (platform "tilt") is given by:

$$\bar{\phi} = \bar{\theta} + \bar{\psi} \tag{2-14}$$

where positive $\bar{\phi}$ corresponds to rotation from the true frame to the platform frame. Note that the solution of Eq. (2-14) for the vector angle $\bar{\phi}$ provides the azimuth error although the required combination of $\bar{\phi}$ components needed is reference frame dependent.

It is not necessary to solve Eqs. (2-6) through (2-9) prior to finding $\bar{\phi}$ in Eq. (2-14). Instead, the expressions (2-13), (2-14) can be substituted into Eqs. (2-6) to (2-8) and the error equation set expressed in terms of the platform misalignment angles, $\bar{\phi}$, instead of the computer misalignment, $\bar{\psi}$. However this has not been done here for several reasons:

1. The decoupling associated with the $\bar{\psi}$ angle representation (see discussion in previous Section) does not occur when $\bar{\phi}$ is the misalignment variable in the equations.
2. The additional relations between $\bar{\theta}$ and $\overline{\delta R}$ (Eq. (2-13)) must be included necessitating the specification of a reference frame in which the error equations are to be written. A resultant loss of generality occurs.
3. The $\bar{\psi}$ form is directly suitable for use with externally supplied stellar observation information. ($\bar{\psi}$ is the angle which is measured by a star sensor.)

In the interest of completeness the $\bar{\phi}$ formulation of the error equations is given in Appendix B.

Equations (2-6) through (2-9), Eqs. (2-13) and (2-14) comprise the complete description of unaided inertial system error propagation. This equation subset, summarized in Table 2-1, properly describes the error behavior for any navigation mechanization. An example of one such mechanization is given in the next Chapter.

TABLE 2-1

GENERAL FREE-INERTIAL NAVIGATION ERROR EQUATIONS

$$\frac{d}{dt}(\overline{\delta V})_S = \bar{\mu} - \bar{\psi} \times \bar{A} + \overline{\delta g} - (\bar{\omega}_{IS} + \bar{\Omega}) \times \overline{\delta V}$$

$$\frac{d}{dt}(\overline{\delta R})_S = \overline{\delta V} - \bar{\omega}_{ES} \times \overline{\delta R}$$

$$\frac{d}{dt}(\bar{\psi})_S = -\bar{\omega}_{IS} \times \bar{\psi} + \bar{\epsilon}$$

$$\overline{\delta g} = \overline{\Delta g} - \frac{gR_o^2}{R^3}\overline{\delta R} + \frac{3gR_o^2}{R^5}(\overline{\delta R} \cdot \bar{R})\bar{R}$$

$$\bar{\theta} = \bar{\theta}(\overline{\delta R}) \quad \text{(reference frame dependent)}$$

$$\bar{\phi} = \bar{\theta} + \bar{\psi}$$

3. SPECIALIZATION OF THE ERROR EQUATIONS TO A WANDER-AZIMUTH LOCAL-LEVEL INERTIAL SYSTEM MECHANIZATION

Some inertial systems are designed to mechanize the navigation equations in a local-level, wander-azimuth configuration. Accordingly, in order to be directly applicable to analysis involving such systems, the generalized error propagation equations listed in the previous Chapter will now be detailed in this mechanization.

3.1 DESCRIPTION OF WANDER-AZIMUTH, LOCAL-LEVEL MECHANIZATION

A wander-azimuth, local level mechanization involves aligning the INS platform to be perpendicular to the local geodetic vertical. The gyro which senses rotation about the vertical is untorqued* and, as a result, the platform will not maintain a particular terrestrial heading reference. Instead, orientation with respect to north will vary with time and vehicle position by the so-called "wander angle." The geometry is depicted in Fig. 3-1.

* The vertical gyro is untorqued only insofar as navigation variables are concerned. Torques applied to cancel bias error or to align the platform are left unaffected by this disucssion.

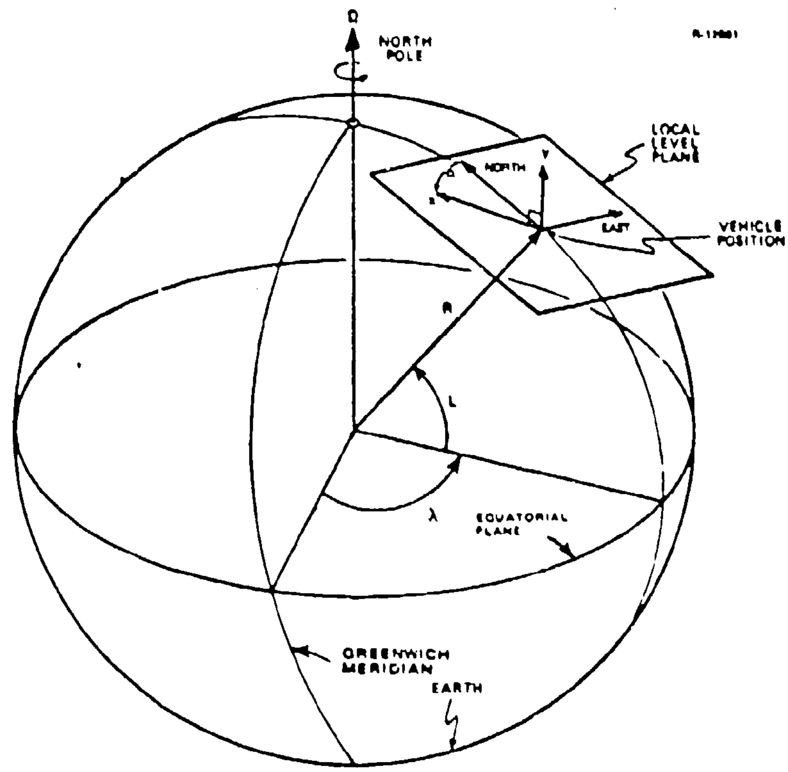

Figure 3-1 Local-Level, Wander-Azimuth Geometry

<u>Notation and Definitions</u>

$\bar{1}_N$ = unit vector pointing in the north direction

$\bar{1}_E$ = unit vector pointing in the east direction

$\bar{1}_Z$ = unit vector pointing down along the position radius vector

α = counterclockwise rotation of the platform X axis from north alignment (viewed from above); wander angle

A_N, A_E, A_Z = north, east and down components of specific force

A_X, A_Y = X and Y components of specific force (X, Y are platform coordinates)

h = altitude above the reference ellipsoid

L = latitude

λ = east longitude

η = east vertical deflection

ξ = north vertical deflection

Δg = gravity anomaly

(A more detailed description of gravity disturbances is given in Ref. 5.)

V_N = north component of groundspeed

V_E = east component of groundspeed

V_Z = rate of change of altitude above the reference ellipsoid

$\epsilon_N, \epsilon_E, \epsilon_Z$ = north, east and vertical gyro drift rate components of $\bar{\epsilon}$

δ = error in quantity which follows; e.g. $\delta V_N = V_N$ (measured) - V_N (correct value)

$$\bar{R} = R\bar{i}_Z = -(R_o + h)\bar{i}_Z \tag{3-1}$$

$$\delta\bar{R} = -\delta h\,\bar{i}_Z \tag{3-2}$$

$$\frac{d}{dt}(\delta\bar{R})_S = -\delta\dot{h}\,\bar{i}_Z \tag{3-3}$$

3.2 COMPONENTS OF ANGULAR RATE VECTORS IN N, E, Z COORDINATES

In north, east, down (N, E, Z) coordinates, the earth's angular rate $\bar{\Omega}$ is:

$$\bar{\Omega} = \Omega\cos L\,\bar{i}_N - \Omega\sin L\,\bar{i}_Z \tag{3-4}$$

The angular rate of the platform with respect to the earth (platform rate) is:

$$\bar{\omega}_{ES} = \frac{V_E}{R}\bar{i}_N - \frac{V_N}{R}\bar{i}_E - \left(\dot{\alpha} + \frac{V_E}{R}\tan L\right)\bar{i}_Z \tag{3-5}$$

The platform rate plus the earth's rate is:

$$\bar{\omega}_{IS} = \left(\Omega \cos L + \frac{V_E}{R}\right) \bar{1}_N - \frac{V_N}{R} \bar{1}_E - \left(\dot{\alpha} + \Omega \sin L + \frac{V_E}{R} \tan L\right) \bar{1}_Z \tag{3-6}$$

Because the Z gyro is untorqued, the $\bar{1}_Z$ component of $\bar{\omega}_{IS}$ is zero. Consequently Eq. (3-6) becomes:

$$\bar{\omega}_{IS} = \left(\Omega \cos L + \frac{V_E}{R}\right) \bar{1}_N - \frac{V_N}{R} \bar{1}_E \tag{3-7}$$

and the rate $\dot{\alpha}$ is given by:

$$\dot{\alpha} = - \Omega \sin L - \frac{V_E}{R} \tan L \tag{3-8}$$

Equation (3-8) is the mechanization equation for the wander angle, α. Note that α is not an error quantity and must be calculated from the assumed trajectory via this relation prior to the solution of the error equations.

3.3 ERROR EQUATIONS IN COMPONENT FORM

There are two useful choices of coordinates in which to solve the error equations, namely the north, east, down(N, E, Z) or platform (X, Y, Z) frames.

1. For the N, E, Z frame solution it is necessary to project the sensor errors, $\bar{\mu}$ and $\bar{\epsilon}$ which are given in X, Y coordinates into components along the N, E axes. The solutions to the equations are the North, East and Down referenced velocity, position and alignment errors. These are the errors of interest in analysis.

2. A solution of the error equations in X, Y, Z coordinates allows the use of the gyro and accelerometer errors without transformation but does require transformation of the gravity errors which are conventionally specified in terms of N, E, Z coordinates. In addition, the solutions to the equations (velocity, position and alignment) are usually required in N, E, Z coordinates.

Each of the representations above requires a transformation to be applied to driving errors (gyro, accelerometer vs. gravity), but since the N, E, Z choice requires no output transformation, this frame was selected for solution of the equations. Note that this choice effects computational savings as the necessary transformations apply only to two components (X, Y) of each of two three-vectors ($\bar{\mu}$, $\bar{\epsilon}$). Choice two above would have required transformation of two components (N, E) of one three-vector ($\overline{\delta g}$) and six components of one nine-vector ($\delta\bar{V}$, $\delta\bar{R}$, $\bar{\psi}$), a somewhat greater computational load.

In expanded (component) form, the velocity error Eqs. (2-6) become:

$$\delta\dot{V}_N = \mu_X \cos\alpha + \mu_Y \sin\alpha - A_Z\psi_E + A_E\psi_Z - \frac{g}{R}\,\delta R_N + g\xi - \left(2\,\Omega \sin L + \frac{V_E}{R}\tan L\right)\delta V_E + \frac{V_N}{R}\,\delta V_Z \tag{3-9}$$

$$\delta\dot{V}_E = \mu_Y \cos\alpha - \mu_X \sin\alpha + A_Z\psi_N - A_N\psi_Z - \frac{g}{R}\,\delta R_E + g\eta + \left(2\,\Omega \sin L + \frac{V_E}{R}\tan L\right)\delta V_N + \left(2\,\Omega \cos L + \frac{V_E}{R}\right)\delta V_Z \tag{3-10}$$

$$\delta\dot{V}_Z = \mu_Z + A_N\psi_E - A_E\psi_N - \frac{2g}{R}\delta h + \Delta g - \frac{V_N}{R}\delta V_N - \left(2\Omega\cos L + \frac{V_E}{R}\right)\delta V_E \quad (3\text{-}11)$$

The position error equations are:

$$\delta\dot{R}_N = \delta V_N - \frac{V_E}{R}\tan L\,\delta R_E - \frac{V_N}{R}\,\delta h \quad (3\text{-}12)$$

$$\delta\dot{R}_E = \delta V_E + \frac{V_E}{R}\tan L\,\delta R_N - \frac{V_E}{R}\,\delta h \quad (3\text{-}13)$$

$$\delta\dot{h} = -\delta V_Z + \frac{V_N}{R}\,\delta R_N + \frac{V_E}{R}\,\delta R_E \quad (3\text{-}14)$$

The computer-to-platform misalignment equations are:

$$\dot{\psi}_N = \frac{V_N}{R}\psi_Z - \left(\Omega\sin L + \frac{V_E}{R}\tan L\right)\psi_E + \epsilon_X\cos\alpha + \epsilon_Y\sin\alpha \quad (3\text{-}15)$$

$$\dot{\psi}_E = \left(\Omega\cos L + \frac{V_E}{R}\right)\psi_Z + \left(\Omega\sin L + \frac{V_E}{R}\tan L\right)\psi_N - \epsilon_X\sin\alpha + \epsilon_Y\cos\alpha \quad (3\text{-}16)$$

$$\dot{\psi}_Z = -\frac{V_N}{R}\psi_N - \left(\Omega\cos L + \frac{V_E}{R}\right)\psi_E + \epsilon_Z \quad (3\text{-}17)$$

3.4 RECOVERY OF AZIMUTH POINTING ERROR

Azimuth error is contained implicitly in Eq. (2-14). The angle $\overline{\theta}$ results from errors in computed position with components given by:

$$\theta_N = \frac{\delta R_E}{R} \tag{3-18}$$

$$\theta_E = - \frac{\delta R_N}{R} \tag{3-19}$$

$$\theta_Z = - \frac{\delta R_E}{R} \tan L \tag{3-20}$$

The components of the platform misalignment angle, $\bar{\phi}$, are:

$$\phi_N = \psi_N + \frac{\delta R_E}{R} \tag{3-21}$$

$$\phi_E = \psi_E - \frac{\delta R_N}{R} \tag{3-22}$$

$$\phi_Z = \psi_Z - \frac{\delta R_E}{R} \tan L \tag{3-23}$$

The last equation specifies azimuth pointing error ϕ_Z. Note that this angle is defined in the opposite sense of the quantity commonly referred to as "heading error." These relations are depicted in Fig. 3-2.

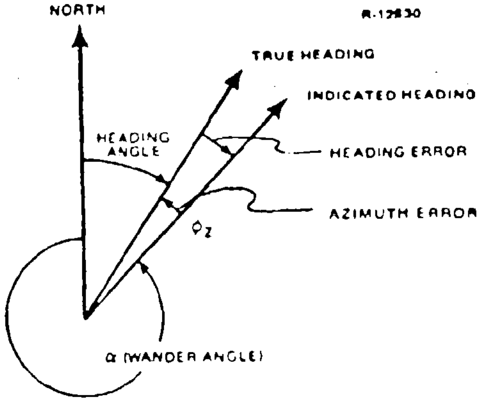

Figure 3-2 Azimuth and Heading Error Sign Conventions

Equations (3-9) through (3-20) and Eq. (3-23) comprise the subset which describes error propagation in an unaided local-level, free-azimuth inertial navigator. Equations (3-21, 22) are auxilliary relations which may be used to find platform tilt errors.

4. INERTIAL SYSTEM MECHANIZATION AND ERROR EQUATIONS WITH STABILIZED ALTITUDE CHANNEL, ALTITUDE DAMPING AND VELOCITY DAMPING

4.1 RATIONALE FOR DAMPING

It is well known that the altitude channel of an unaided inertial system is unstable (Ref. 2). This characteristic is commonly corrected by employing an externally measured altitude reference (e.g., barometric or radar altimeter) to stabilize the INS vertical axis. A detailed discussion of altitude damping is given in Ref. 6.

The horizontal channels of an unaided INS, while not unstable, are undamped and hence only marginally stable. A small amount of spectral energy from error sources which drive the Schuler-loop (accelerometer noise, vertical deflections, earth loop errors ($\underline{\psi}$) etc.) is sufficient to cause unbounded growth of the navigation errors. These errors are "damped" by making use of external velocity measurements such as are furnished by doppler radar.

In the discussion above, a distinction between "vertical" and "horizontal' channels has been made largely to take advantage of intuitive notions which are typically directed toward local-level mechanization schemes. In general, as is illustrated in the equations below, externally supplied altitude and velocity information may be used to stabilize and damp an INS regardless of its mechanization (local level, space stable, strapdown, etc.).

4.2 NOTATION AND DEFINITIONS FOR MECHANIZATION AND ERROR EQUATIONS

h_r = Vehicle altitude supplied by external altitude reference

$\bar{V}_r$ = Vehicle groundspeed supplied by external velocity reference

$\bar{V}_d$ = Velocity damping state variables (a 3-vector)

K_1 = First order damping constant matrix

K_2 = Second order damping constant matrix

τ = Gyro torquing feedback gain matrix (earth rate gyrocompassing constant)

$\bar{g}(\bar{R}_c, h_r)$ = Plumb bob gravity acceleration calculated from both externally and inertially indicated altitude

κ = Weighting constant (scalar) governing the proportion of externally and inertially measured altitude used to calculate gravity

$\delta\bar{V}_r$ = External velocity reference error

$\delta\bar{V}_d$ = Velocity damping state error

4.3 GENERAL MECHANIZATION

4.3.1 Schuler Loop Damping Mechanization

The specific force equation (2-3), modified to include continuous second-order external-velocity damping and altitude stabilization is given below

$$\frac{d}{dt}(\bar{V}_C)_C = \bar{A}_C + \bar{g}(\bar{R}_C, h_r) - (\bar{\omega}_{IC} + \bar{\Omega}) \times \bar{V}_C - K_1(\bar{V}_C - \bar{V}_r) + K_2\bar{V}_d \quad (4\text{-}1)$$

with the auxiliary relation:

$$\frac{d}{dt}(\bar{V}_d)_C = K_1(\bar{V}_C - \bar{V}_r) - K_2\bar{V}_d \quad (4\text{-}2)$$

Note that for an error-free system, the reference velocity, $\bar{V}_r$, is equal to the computed velocity, $\bar{V}_C$, and the initial condition of the second order damping variable, $\overline{V}_d$, is zero. Also, in such an ideal system $\overline{R}_C$ and h_r provide the same value of gravity. Under these conditions Eq. (4-1) reduces to Eq. (2-3) hence demonstrating the dynamic exactness of the aided system. Equations (4-1) and (4-2) describe a second-order velocity-damped INS mechanization. This consists of proportional and integral feedback of the difference between externally and inertialilly measured velocity to the acceleration summing node. First order velocity damping which involves only proportional feedback may also be described by Eq. (4-1) by the setting of the gain constant, K_2, to zero. The damping state variable, $\overline{V}_d$ and Eq. (4-2) become redundant in this case.

4.3.2 Earth Loop Damping Mechanization

While a complete discussion of the various platform alignment procedures is beyond the scope of this equation subset, error equation generality requires consideration of platform misalignment angle recovery from the externally supplied velocity data. For this purpose it is assumed that the gyros are torqued with a signal which is proportional to the difference between inertially-derived and externally-measured velocity, i.e.,

$$\text{Total Command Gyro Rate} = \text{Mechanization Torqued Rate} + \underline{\tau\,(V_C - V_r)}$$

Note that the first term on the right hand side of the expression above is the commanded gyro rate corresponding to the particular mechanization involved; hence it is generally different for each mechanization. The underlined term is the additional gyro rate which incorporates the velocity feedback. As will be seen in the next section, this term mechanizes the earth loop damping.
In considering this procedure, it is necessary to distinguish between the INS "navigate mode" and "alignment mode." The latter case in which navigation

information is not being extracted from the INS but in fact is being supplied to it for purposes of system alignment is not considered here. Attention is confined the navigate mode during which the system continuously supplies the user with current values of his velocity, position and heading. One may look at this velocity-to-gyro feedback in two ways, either as an alignment procedure in which the navigate mode is maintained or as simply another use of external velocity data to improve knowledge of an INS variable, in this case $\bar{\psi}$. Mechanization considerations pertinent to the use of an external velocity signal to drive the gyros are:

- The external velocity reference must be very accurate in order to prevent significant velocity errors from being introduced into the earth rate ($\bar{\psi}$) loop.
- Even a "good" external velocity reference will contain enough error to require small values for the feedback gain constants ("light" damping).

4.4 GENERALIZED ERROR EQUATIONS

4.4.1 Damped Schuler Loop Error Propagation

Perturbation of Eqs. (4-1) and (4-2) leads to the following error equations (Ref. 7).

$$\frac{d}{dt}(\delta\bar{V})_S = \delta\bar{A} - \bar{\psi}\times\bar{A} + \delta\bar{g}(\delta h, \delta h_r) - (\bar{\omega}_{IS} + \bar{\Omega})\times\delta\bar{V} - K_1(\delta\bar{V} + \bar{\psi}\times\bar{V} - \delta\bar{V}_r) + K_2\delta\bar{V}_d \qquad (4\text{-}3)$$

$$\frac{d}{dt}(\delta\bar{V}_d)_S = K_1(\delta\bar{V} + \bar{\psi}\times\bar{V} - \delta\bar{V}_r) - K_2\delta\bar{V}_d \qquad (4\text{-}4)$$

In the derivation it is assumed that the externally-measured velocity is available in platform-frame coordinates. (This is the usual case for aircraft doppler or shipboard velocity log measurements.) The fact that the platform and computer frames are misaligned by the angle $\overline{\Psi}$ gives rise to the $\overline{\Psi} \times \overline{V}$ terms in Eqs. (4-3) and (4-4).

4.4.2 Damped Earth Loop Error Propagation

As a result of the applied gyro torques which are proportional to the measured velocity error, the $\overline{\Psi}$ error equation (24 hour dynamics) is damped. Equation (2-8) then becomes:

$$\frac{d}{dt}(\overline{\Psi})_S = -\overline{\omega}_{IS} \times \overline{\Psi} + \tau\,(\delta\overline{V} + \overline{\Psi} \times \overline{V} - \delta\overline{V}_r) + \overline{\epsilon} \qquad (4\text{-}5)$$

The τ term in Eq. (4-5) results in extraction of alignment information ($\overline{\Psi}$) from the reference velocity. Because this is the same process as is used in gyrocompassing (when the reference velocity is accurately known to be zero) it is referred to (in the aircraft case) as "doppler inertial gyrocompassing" (Ref. 8). In this instance the velocity reference is doppler radar. Additional discussion of doppler-inertial gyrocompassing while in the navigate mode is given in Ref. 9. Treatment of this topic as part of the alignment proceedure may be found in Refs. 2 and 10 within the sections describing earth-rate gyrocompassing.

For analysis of inertial systems which do not employ the earth loop damping feature, the terms involving τ in Eq. (4-5) and the sequel may be omitted. The presence of externally supplied velocity and altitude does not affect error Eqs. (2-7), (2-9), (2-13) or (2-14). However when these external measurements are used, these equations in conjunction with Eqs. (4-3) through (4-5) above describe error propagation in any velocity-damped, altitude-stabilized inertial system regardless of mechanization. Note that this set of equations which are summarized in Table 4-1, reduce to the unaided navigation error equations given in Table 2-1 when K_1, K_2, and τ are set to zero.

TABLE 4-1

VELOCITY AND ALTITUDE-AIDED INERTIAL NAVIGATION SYSTEM ERROR EQUATIONS

$$\frac{d}{dt}(\delta\bar{V})_S = \delta\bar{A} - \bar{\psi}\times\bar{A} + \overline{\delta g}(\delta h, \delta h_r) - (\bar{\omega}_{IS} + \bar{\Omega})\times\delta\bar{V}$$
$$- K_1(\delta\bar{V} + \bar{\psi}\times\bar{V} - \delta\bar{V}_r) + K_2\delta\bar{V}_d$$

$$\frac{d}{dt}(\delta\bar{V}_d)_S = K_1(\delta\bar{V} + \bar{\psi}\times\bar{V} - \delta\bar{V}_r) - K_2\delta\bar{V}_d$$

$$\frac{d}{dt}(\delta\bar{R})_S = \delta\bar{V} - \bar{\omega}_{ES}\times\delta\bar{R}$$

$$\frac{d}{dt}(\bar{\psi})_S = -\bar{\omega}_{IS}\times\bar{\psi} + \tau(\delta\bar{V} + \bar{\psi}\times\bar{V} - \delta\bar{V}_r) + \bar{\epsilon}$$

$$\bar{\theta} = \bar{\theta}(\overline{\delta R}) \quad \text{(reference frame dependent)}$$

$$\bar{\phi} = \bar{\theta} + \bar{\psi}$$

4.5 ALTITUDE STABILIZED AND VELOCITY DAMPED INS ERROR EQUATIONS FOR A LOCAL LEVEL MECHANIZATION

In a local-level system the gravity errors given by Ref. 11 can be written as:

$$\delta\overline{g} = g\xi\overline{I}_N + g\eta\overline{I}_E + \left[\frac{(\kappa-2)}{R}g\delta h - \frac{\kappa g}{R}\delta h_r + \Delta g\right]\overline{I}_Z \qquad (4\text{-}6)$$

where R_0 is the earth's radius. The constant, κ as defined in Eq. (4-6) determines the relative weighting of the inertially-calculated altitude and that measured by the altimeter, with the altimeter being weighted more heavily for larger κ. Note that κ may be any positive number but must be greater than 2 if the vertical channel is to be stable.

The complete set of error equations which correspond to a velocity- and altitude-aided local-level, wander-azimuth mechanized INS is given below. These equations have been derived directly from the general error equation set given in Table 4-1. Notation which has not previously been defined is:

$\delta V_{rN,\, rE,\, rZ}$ = north, east and down components of $\overline{\delta V}_r$

$\delta V_{dN,\, dE,\, dZ}$ = north, east and down components of $\overline{\delta V}_d$.

It is common practice in velocity-damped systems to set all of the off-diagonal elements of K_1 and K_2 to zero. This simplifies the form of the equations and still provides all of the feedback which will significantly improve the navigation errors. The remaining elements of K_1, K_2 and the nonzero elements of τ are defined as follows:

$$\text{Diag.}\ [K_1] = (k_1\ k_1\ 0) \tag{4-7}$$

$$\text{Diag.}\ [K_2] = (k_2\ k_2\ 0) \tag{4-8}$$

$$\tau = \begin{bmatrix} 0 & \tau_L & 0 \\ \tau_L & 0 & 0 \\ 0 & \tau_V & 0 \end{bmatrix} \tag{4-9}$$

Note from Eqs. (4-7) and (4-8) that the altitude channel is not velocity damped. While such damping could be implemented, it is usual instead to damp the vertical channel with the external altitude signal. This is discussed in greater detail in the next section.

The aided inertial navigation equations for a local-level, free-azimuth mechanized system which incorporates the gains specified by Eqs. (4-7) through (4-9) are given below:

$$\begin{aligned} \delta\dot{V}_N = {} & \mu_X \cos\alpha + \mu_Y \sin\alpha - A_Z\psi_E + A_E\psi_Z - \frac{g}{R}\,\delta R_N + g\xi \\ & - \left(2\Omega \sin L + \frac{V_E}{R}\tan L\right)\delta V_E + \frac{V_N}{R}\,\delta V_Z - k_1\delta V_N - k_1 V_E\psi_Z \\ & + k_1 V_Z\psi_E + k_1\delta V_{rN} + k_2\delta V_{dN} \end{aligned} \tag{4-10}$$

$$\begin{aligned} \delta\dot{V}_E = {} & \mu_Y \cos\alpha - \mu_X \sin\alpha + A_Z\psi_N - A_N\psi_Z - \frac{g}{R}\,\delta R_E + g\eta \\ & + \left(2\Omega \sin L + \frac{V_E}{R}\tan L\right)\delta V_N + \left(2\Omega \cos L + \frac{V_E}{R}\right)\delta V_Z - k_1\delta V_E - k_1 V_Z\psi_N \\ & + k_1 V_N\psi_Z + k_1\delta V_{rE} + k_2\delta V_{dE} \end{aligned} \tag{4-11}$$

$$\delta\dot{V}_Z = \mu_Z + A_N\psi_E - A_E\psi_N - \frac{2g}{R_o}\delta h + \frac{\kappa g}{R}(\delta h - \delta h_r) + \Delta g$$

$$- \frac{V_N}{R}\delta V_N - \left(2\Omega\cos L + \frac{V_E}{R}\right)\delta V_E - k_1 V_N\psi_E + k_1 V_E\psi_N \qquad (4\text{-}12)$$

$$\delta\dot{V}_{dN} = -k_2\delta V_{dN} + k_1(\delta V_N + V_E\psi_Z - V_Z\psi_E - \delta V_{rN}) \qquad (4\text{-}13)$$

$$\delta\dot{V}_{dE} = -k_2\delta V_{dE} + k_1(\delta V_E + V_Z\psi_N - V_N\psi_z - \delta V_{rE}) \qquad (4\text{-}14)$$

$$\delta\dot{R}_N = \delta V_N - \frac{V_E}{R}\tan L\,\delta R_E - \frac{V_N}{R}\delta h \qquad (4\text{-}15)$$

$$\delta\dot{R}_E = \delta V_E + \frac{V_E}{R}\tan L\,\delta R_N - \frac{V_E}{R}\delta h \qquad (4\text{-}16)$$

$$\delta\dot{h} = -\delta V_Z + \frac{V_N}{R}\delta R_N + \frac{V_E}{R}\delta R_E \qquad (4\text{-}17)$$

$$\dot{\psi}_N = \frac{V_N}{R}\psi_Z - \left(\Omega\sin L + \frac{V_E}{R}\tan L\right)\psi_E + \tau_L\delta V_E + \tau_L V_N\psi_Z - \tau_L V_Z\psi_N - \tau_L\delta V_{rE}$$

$$+ \epsilon_X\cos\alpha + \epsilon_Y\sin\alpha \qquad (4\text{-}18)$$

$$\dot{\psi}_E = \left(\Omega\cos L + \frac{V_E}{R}\right)\psi_Z + \left(\Omega\sin L + \frac{V_E}{R}\tan L\right)\psi_N - \tau_L\delta V_N - \tau_L V_Z\psi_E$$

$$+ \tau_L V_E\psi_Z + \tau_L\delta V_{rN} - \epsilon_X\sin\alpha + \epsilon_Y\cos\alpha \qquad (4\text{-}19)$$

$$\dot{\psi}_Z = -\frac{V_N}{R}\psi_N - \left(\Omega\cos L + \frac{V_E}{R}\right)\psi_E + \tau_V\delta V_N$$

$$+ \tau_V V_N\psi_Z - \tau_V V_Z\psi_N - \tau_V\delta V_{rN} + \epsilon_Z \qquad (4\text{-}20)$$

4.6 EXTENSION OF THE LOCAL-LEVEL ERROR EQUATIONS TO THE ALTITUDE DAMPED CASE

The mixture of externally-measured and inertially-derived altitude which occurs in Eq. (4-12) via the gain constant κ renders the altitude channel stable for $\kappa > 2$. Note that this removes the static vertical axis instability but, in the absence of additional feedback signals to provide damping results in a marginally-stable altitude loop. This additional feedback is introduced into the mechanization Eqs. (4-1) and (4-2) in the manner shown below with the altitude damping terms expressed in north, east, down coordinates.

$$\frac{d}{dt}(\overline{V}_C)_C = \overline{A}_C + \overline{g}(\overline{R}_C, h_r) - (\overline{\omega}_{IC} + \overline{\Omega}) \times \overline{V}_C - K_1(\overline{V}_C - \overline{V}_r) + K_2\overline{V}_d + \underline{\begin{bmatrix} 0 \\ 0 \\ a_d \end{bmatrix}} \tag{4-21}$$

$$\frac{d}{dt}(\overline{R}_C) = \overline{V}_C - \omega_{EC} \times \overline{R}_C - \underline{\begin{bmatrix} 0 \\ 0 \\ C_1(h - h_r) \end{bmatrix}} \tag{4-22}$$

The equation for the additional damping state, a_d is:

$$\dot{a}_d = C_2(h - h_r) \tag{4-23}$$

The damping required to establish stability margin is supplied by the feedback constant C_1 (Ref. 12). Second-order damping which improves the low-frequency error behavior is provided by the C_2 feedback.

Changes in the local-level, wander-azimuth error equations which result from the perturbation of Eqs. (4-21) to (4-23) respectively, are given below. Equation (4-12) becomes:

$$\delta\dot{V}_Z = \mu_Z + A_N\psi_E - A_E\psi_N - \frac{2g}{R}\,\delta h + \frac{\kappa g}{R}\,(\delta h - \delta h_r) + \Delta g \qquad (4\text{-}24)$$

$$- \frac{V_N}{R}\,\delta V_N - \left(2\Omega\cos L + \frac{V_E}{R}\right)\delta V_E - k_1 V_N \psi_E + k_1 V_E \psi_N + \underline{\delta a_d}$$

The additional terms appear in Eq. (4-17) as:

$$\delta\dot{h} = -\delta V_Z + \frac{V_N}{R}\,\delta R_N + \frac{V_E}{R}\,\delta R_E \underline{- C_1\,(\delta h - \delta h_r)} \qquad (4\text{-}25)$$

where the extra terms due to the altitude damping have been underscored. The damping state variable error equation is:

$$\delta\dot{a}_d = C_2\,(\delta h - \delta h_r) \qquad (4\text{-}26)$$

These error equations are depicted in Fig. 1-1. Note that the solid lines of the figure correspond to the undamped Eqs. (4-12) and (4-17).

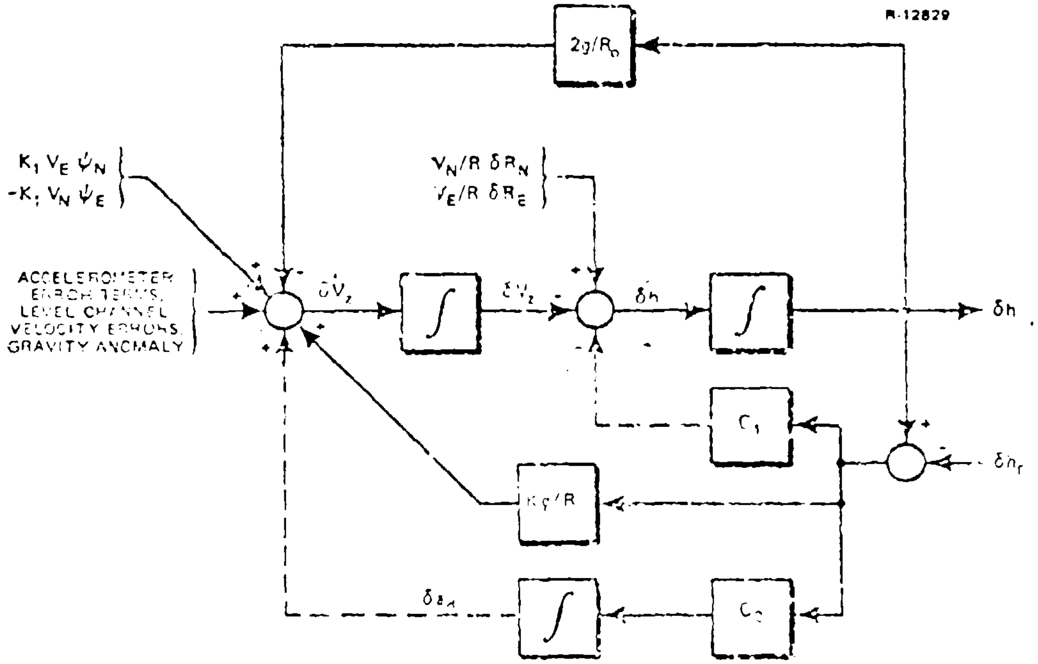

Figure 4-1 Damped Altitude Channel Error Equation Diagram

The complete set of error propagation relations for a velocity and altitude aided local-level, wander-azimuth inertial system have now been set down. They are given again in Table 4-2 for the convenience of the reader. Equation numbers identify the portion of the text in which each equation is originally stated. The following points are noted in regard to these equations.

- The equations are valid for any combination of damping configurations including zero damping; there are no restrictions on the values of κ, k_1, k_2, τ_L, τ_V, C_1 and C_2. Thus analysis of a continuously damped INS may be performed for any permutation of the following external aid schemes:

  External altitude measurements
  External velocity measurements
  First order damping (level and/or altitude channels)
  Second order damping (level and/or altitude channels)
  Navigation mode gyrocompassing

- The equations may easily be assembled into "state space form", namely

$$\dot{\underline{x}} = F\underline{x} + \underline{u} \tag{4-27}$$

  which is the formulation most suited for covariance error analysis.

- This set of equations also describes error behavior in a north-slaved, locally-level INS if α is set to zero.

- The error equations of a continuously-damped inertial system have been shown to be the same for both local-level and space-stable coordinate frame mechanizations (Refs. 1, 13). While the gyro and accelerometer error models differ for these two mechanizations, the dynamics of the error propagation are invariant if the damping is implemented in the same manner for both mechanizations. Consequently the local-level equations may be modified to be suitable for space-stable inertial platform analysis.

- The presence of the gyro torquing feedback terms in the $\underline{\psi}$ equations (4-18 through 4-20) results in additional terms which couple them with the velocity equations (4-10 and 4-11). Hence damping of the 24 hour dynamics ($\underline{\psi}$ equations) links the earth-rate (24 hour) and Schuler (84 minute) loops.

- The Foucault oscillatory mode (approximately the frequency difference between the two level-axis Schuler-loop frequencies) which is manifest as amplitude modulation of the velocity and position errors is also damped by the level channel velocity damping (k_1). For a more complete discussion of Foucault mode decay characteristics see p. 148 ff of Ref. 2. The topic is also treated in Refs. 3, 6, and 11.

- With the INS natural modes damped as described by the equations of Table 4-2, all errors except those rising from the polar component of gyro drift rate are bounded.

[illegible]

$$[illegible] \Big) \delta V_E + \frac{V}{R}\, [illegible] - k\, \delta V_N - [illegible] + [illegible] + k\, \delta V_{rN} + k\, \delta V_{dN} \qquad (4\text{-}[illegible])$$

$$[illegible] - A_Z \sin\alpha + A_E\phi_N - A_N\phi_Z - \frac{g}{R}\,\delta R_E + g\, [illegible]$$
$$+ \left(2\Omega \sin L + \frac{V_E}{R}\tan L\right)\delta V_N + \left(2\Omega\cos L + \frac{V_E}{R}\right)\delta V_Z - k_1\,\delta V_E - k_1 V_Z\phi_N \qquad (4\text{-}11)$$

$$\delta\dot{V}_Z = A_Z + A_N\phi_E - A_E\phi_N - \frac{2g}{R}\,\delta h + \frac{kg}{R}(\delta h - \delta h_r) + \Delta g$$
$$- \frac{V_N}{R}\,\delta V_N - \left(2\Omega\cos L + \frac{V_E}{R}\right)\delta V_E - k_1 V_N\phi_E + k_1 V_E\phi_N + \delta a \qquad (4\text{-}24)$$

$$\delta\dot{h}_a = C_2(\delta h - \delta h_r) \qquad (4\text{-}26)$$

$$\delta\dot{V}_{dN} = k_2\,\delta V_{dN} + k_1\,(\delta V_N + V_E\phi_Z - V_Z\phi_E - \delta V_{rN}) \qquad (4\text{-}13)$$

$$\delta\dot{V}_{dE} = -k_2\,\delta V_{dE} + k_1\,(\delta V_E + V_Z\phi_N - V_N\phi_Z - \delta V_{rE}) \qquad (4\text{-}14)$$

$$\delta\dot{R}_N = \delta V_N + -\frac{V_E}{R}\tan L\,\delta R_E - \frac{V_N}{R}\,\delta h \qquad (4\text{-}15)$$

(continued on next page)

$$[illegible] + \epsilon_X \cos\alpha + \epsilon_Y \sin\alpha \qquad (4\text{-}18)$$

$$\dot{\phi}_E = -\left(\Omega \cos L + \frac{V_E}{R}\right)\phi_Z + \left(\Omega \sin L + \frac{V_E}{R}\tan L\right)\phi_N - \tau_L \delta V_N - \tau_L V_N \phi_Z$$

$$+ \tau_L V_Z \phi_E + \tau_L \delta V_{sN} - \epsilon_X \sin\alpha + \epsilon_Y \cos\alpha \qquad (4\text{-}19)$$

$$\dot{\phi}_Z = -\frac{V_N}{R}\phi_N - \left(\Omega \cos L + \frac{V_E}{R}\right)\phi_E + \tau_V \delta V_N$$

$$+ \tau_V V_N \phi_Z - \tau_V V_Z \phi_N - \tau_V \delta V_{sN} + \epsilon_Z \qquad (4\text{-}20)$$

$$\phi_N = \frac{\delta R_E}{R} \qquad (3\text{-}18)$$

$$\phi_E = -\frac{\delta R_N}{R} \qquad (3\text{-}19)$$

$$\phi_Z = -\frac{\delta R_E}{R}\tan L \qquad (3\text{-}20)$$

$$\psi_N = \phi_N + \frac{\delta R_E}{R} \qquad (3\text{-}21)$$

$$\psi_E = \phi_E - \frac{\delta R_N}{R} \qquad (3\text{-}22)$$

$$\psi_Z = \phi_Z - \frac{\delta R_E}{R}\tan L \quad \text{(azimuth pointing error)} \qquad (3\text{-}23)$$

APPENDIX A
A DERIVATION OF THE NAVIGATION ERROR EQUATIONS*

A derivation of the navigation error equations which follows directly from the mechanization equations is briefly documented here. By this approach, errors in ground speed and position error of the vehicle as seen in the true frame are obtained directly. The philosophical approach is that of Ref. 2.

A.1 THE MECHANIZATION EQUATIONS

The navigation mechanization equations derived in Ref. 2 are reproduced below:

$$P_S(\overline{V}) = \overline{A} + \overline{g}(\overline{R}) - (\overline{\omega}_{IS} + \overline{\Omega}) \times \overline{V} \tag{A.1-1}$$

and

$$P_S(\overline{R}) = P_E(\overline{R}) - \overline{\omega}_{ES} \times \overline{R}$$

$$= \overline{V} - \overline{\omega}_{ES} \times \overline{R} \tag{A.1-2}$$

where

$P_S(\cdot) = \frac{d}{dt}(\cdot)_S$ = the time rate of change of $(\cdot)$ as seen by an observer in the S frame

$P_E(\cdot) = \frac{d}{dt}(\cdot)_E$ = time rate of change of $(\cdot)$ as seen in an earth fixed frame

$\overline{R}$ = vehicle position vector

$\overline{V} = P_E(\overline{R})$ = vehicle velocity in an earth fixed frame, i.e., ground speed"

*This Appendix has been extracted from Ref. 14.

Ω = angular rate of earth fixed axes with respect to inertial space

$\bar{\omega}_{IS}$ = angular rate of S frame w.r.t. inertial space

$\bar{\omega}_{ES}$ = angular rate of S frame w.r.t. an earth fixed frame

$\bar{A}$ = non gravitational force on vehicle (ideal accelerometer output)

$\bar{g}(\bar{R})$ = plumb bob gravity force on vehicle

The S (true) frame is the coordinate frame in which the solution to the navigation mechanization equations is derived. Though usually chosen with some good engineering justification, the S frame is not unique. The mechanization equations can just as well be written in some other rotating frame, in which case they would have the form:

$$P_C(\bar{V}) = \bar{A} + \bar{g} - (\bar{\omega}_{IC} + \Omega) \times \bar{V} \qquad \text{(A.1-3)}$$

and

$$P_C(\bar{R}) = \bar{V} - \bar{\omega}_{EC} \times \bar{R} \qquad \text{(A.1-4)}$$

In the sequel, "C" will denote the computer frame.

A.2 SOME COMMENTS ON ERROR ANALYSIS PHILOSOPHY

In many error analyses only two coordinate systems are considered: the S frame in which it is desired to mechanize the navigation error equations, and the frame physically instrumented by the inertial platform. Though not apparent, such an approach can lead to incorrect definition of perturbation (i.e., error) quantities. These ambiguities are easily resolved by the introduction of a third coordinate system, namely the computer frame. All errors are then defined with respect to this frame. The heuristic reasoning for this approach runs somewhat like this. At each point in time the computer may be viewed as constructing a true coordinate frame at its computed position. It then determines the angular rate of this coordinate system with respect to inertial space using both its computed position and velocity. This is the computer frame of Eq. (A.1-3) and is a valid frame in which to solve the mechanization equations. Indeed, this is exactly what happens. The computer integrates the mechanization equations in the frame defined by its output. Errors in this mechanization arise from three sources: (1) acceleration has been measured in the platform frame and not the computer frame, (2) gravity is incorrectly computed since the computer frame is not coincident with true vehicle position, (3) computed vehicle velocity does not equal true vehicle velocity. The first error results from angular misalignment of the platform and computer frames while the second and third errors are caused by relative motion between the computer and true frames.

At first glance, one may be disturbed that no error arises due to incorrect computation of the computer frame angular rate, since such a term always seems to be present when referring all errors to the true (S) frame. But one must realize that the computer frame is entirely defined by computed position and velocity. There is no error in the state of this frame. If desired it could be realized physically by constructing an S frame at the computed position and rotating at the computed angular rate. Alternately, one may say that the computer makes an error in computing the angular rate of the true (S) frame. Thus the Coriolis correction applied to $\bar{A} + \bar{g}$ is $(\bar{\omega}_{IC} + \bar{\Omega}) \times \bar{V}$ and not $(\bar{\omega}_{IS} + \bar{\Omega}) \times \bar{V}$. But this is precisely Eq. (A.1-3) and the computer does indeed generate $P_C(\bar{V})$.

A.3 THE ERROR EQUATIONS

From the foregoing discussion the computer actually mechanizes

$$P_C(\overline{V}_C) = \overline{A}_C + \overline{g}(\overline{R}_C) - (\overline{\omega}_{IC} + \overline{\Omega}) \times \overline{V}_C \qquad \text{(A.3-1)}$$

and

$$P_C(\overline{R}_C) = \overline{V}_C - \overline{\omega}_{EC} \times \overline{R}_C \qquad \text{(A.3-2)}$$

where

$\overline{V}_C$ = computed vehicle ground speed

$\overline{R}_C$ = computed vehicle position

$\overline{A}_C$ = accelerometer outputs as interpreted by the computer

Let:

$$\begin{aligned} \overline{A}_C &= \overline{A} + \overline{\delta A} \\ \overline{V}_C &= \overline{V} + \overline{\delta V} \\ \overline{g}(\overline{R}_C) &= \overline{g} + \overline{\delta g} \end{aligned} \qquad \text{(A.3-3)}$$

where the unsubscripted variables represent true quantities and the δ terms represent as set undefined errors. Putting Eqs. (A.3-3) in (A.3-1) and subtracting (A.1-3) yields

$$P_C(\overline{\delta V}) = \overline{\delta A} + \overline{\delta g} - (\overline{\omega}_{IC} + \overline{\Omega}) \times \overline{\delta V} \qquad \text{(A.3-4)}$$

Likewise putting Eq. (A.3-3) in Eq. (A.3-2) and subtracting Eq. (A.1-4)

$$P_C(\overline{\delta R}) = \overline{\delta V} - \overline{\omega}_{EC} \times \overline{\delta R} \qquad \text{(A.3-5)}$$

In an error analysis the position and velocity of the true frame are specified (this is equivalent to specifying the vehicle mission profile). Thus $\overline{\omega}_{IS}$ and $\overline{\omega}_{ES}$ are prescribed. For this reason the differential equations characterizing error propagation are best written in the "S" frame. The appropriate Coriolis conversions are:

$$P_C(\overline{\delta V}) = P_S(\overline{\delta V}) + \bar{\omega}_{CS} \times \overline{\delta V} \qquad \text{(A.3-6)}$$

and

$$P_C(\overline{\delta R}) = P_S(\overline{\delta R}) + \bar{\omega}_{CS} \times \overline{\delta R} \qquad \text{(A.3-7)}$$

Equation (A.3-6) in Eq. (A.3-4) and Eq. (A.3-7) in Eq. (A.3-9) produces the desired final relationships:

$$P_S(\overline{\delta V}) = \overline{\delta A} + \overline{\delta g} - (\bar{\omega}_{IS} + \bar{\Omega}) \times \overline{\delta V} \qquad \text{(A.3-8)}$$

and

$$P_S(\overline{\delta R}) = \overline{\delta V} - \bar{\omega}_{ES} \times \overline{\delta R} \qquad \text{(A.3-9)}$$

Equation (A.3-8) describes the time evolution of error in determining vehicle velocity with respect to an earth-fixed frame while Eq. (A.3-9) gives the time evolution of the error in determining vehicle position.

Equations (A.3-6) and (A.3-9) can be obtained by an alternate and instructive route. The development for Eq. (A.3-8) is presented here. Applying Coriolis' law to $P_C(\bar{V}_C)$ one obtains:

$$P_C(\bar{V}_C) = P_S(\bar{V}_C) + \bar{\omega}_{CS} \times \bar{V}_C \qquad \text{(A.3-10)}$$

Eq. (A.3-10) in Eq. (A.3-1) yields:

$$P_S(\bar{V}_C) = \bar{A}_C + g(\bar{R}_{IS}) - (\bar{\omega}_{IS} + \bar{\Omega}) \times \bar{V}_C \qquad \text{(A.3-11)}$$

Subtracting Eq. (A.1-3) from Eq. (A.3-11) then gives Eq. (A.3-8).

The errors δA and δg must now be determined. Denoting the total accelerometer error by $\bar{\mu}$ the accelerometer output is:

$$\bar{A}_o = \bar{A} + \bar{\mu} \tag{A.3-12}$$

$\bar{\mu}$ may include random errors, scale factor errors, etc. The components of Eq. (A.3-12) which are available in platform axes are taken by the computer to be the components of $\bar{A}_o$ in the C frame. Thus the vector $\bar{A}_o$ is rotated by the computer through $-\bar{\psi}$, the small angle misalignment between computer and platform axes to become:

$$\bar{A}_C = \bar{A}_o - \bar{\psi} \times \bar{A}_o \cong \bar{A} - \bar{\psi} \times \bar{A} + \bar{\mu} \tag{A.3-13}$$

It follows from Eq. (A.3-3) that:

$$\overline{\delta A} = \bar{\mu} - \bar{\psi} \times \bar{A} \tag{A.3-14}$$

Errors in determining g include errors in computing plumb bob gravity and the computer's lack of knowledge of vertical deflections and gravity anomaly. At worst the error in computing the centripedal term in $\bar{g}(\bar{R})$ is two orders of magnitude less than that obtained in computing the mass attraction gravitational force, g_m. Denoting gravity anomaly and vertical deflections by $\overline{\Delta g}$ one may then write:

$$\overline{\delta g} = \overline{\Delta g_m} + \overline{\Delta g} \tag{A.3-15}$$

The derivation of $\overline{\Delta g_m}$ is well developed in Ref. 2 and will not be repeated here. Likewise, the differential equation governing the time evolution of the ψ angle is derived in Ref. 2. The complete set of error equations is then:

$$P_S(\overline{\delta V}) = \overline{\delta A} + \overline{\delta g} - (\bar{\omega}_{IS} + \bar{\Omega}) \times \overline{\delta V} \tag{A.3-16}$$

$$P_S(\overline{\delta R}) = \overline{\delta V} - \bar{\omega}_{ES} \times \overline{\delta R} \tag{A.3-17}$$

$$P_S(\bar{\psi}) = -\bar{\omega}_{IS} \times \bar{\psi} + \bar{\epsilon} \tag{A.3-18}$$

where $\overline{\delta A}$ and $\overline{\delta g}$ are defined by Eqs. (A.3-13) and (A.3-14) and $\overline{\epsilon}$ is the total vector gyro drift rate, including random effects, biases, and scale factor torquing errors, and g and g^2 effects.

APPENDIX B

NAVIGATION ERROR EQUATIONS EXPRESSED IN TERMS OF PLATFORM MISALIGNMENT ANGLES

As was observed in the text just prior to Eq. (2-13), the true to computer frame misalignment, $\overline{\theta}$, is a reference frame-dependent function of position error. In order to obtain the navigation error equations in terms of the $\overline{\phi}$ (platform to true) angle misalignment the relation

$$\overline{\phi} = \overline{\theta} + \overline{\psi} \tag{B-1}$$

is used. The reference frame of interest is that of a local-level, north-pointing system. Hence the north, east and down components of $\overline{\theta}$ (from Ref. 2) are:

$$\theta_N = (\cos L)\,\delta\lambda = \frac{\delta R_E}{R} \tag{B-2}$$

$$\theta_E = -\,\delta L = -\,\frac{\delta R_N}{R} \tag{B-3}$$

$$\theta_Z = -\,(\sin L)\,\delta\lambda = -\,\frac{\delta R_E}{R}\tan L \tag{B-4}$$

with derivatives:

$$\dot{\theta}_N = \frac{R\,\delta\dot{R}_E - \dot{R}\,\delta R_E}{R^2} \tag{B-5}$$

$$\dot{\theta}_E = -\,\frac{R\,\delta\dot{R}_N - \dot{R}\,\delta R_N}{R^2} \tag{B-6}$$

$$\dot{\theta}_Z = -\frac{R\delta\dot{R}_E \tan L + R\delta R_E \dot{L} \sec^2 L - \dot{R}\delta R_E \tan L}{R^2} \qquad \text{(B-7)}$$

As a consequence of Eqs. (2-7) and (3-1) of the text, the quantity $\dot{R}$ is given by:

$$\dot{R} = -V_Z \qquad \text{(B-8)}$$

Substitution of Eqs. (B-1) through (B-4) into the damped inertial acceleration error equations (4-10), (4-11) and (4-24) gives:

$$\begin{aligned}\delta\dot{V}_N = {} & \mu_X \cos\alpha + \mu_Y \sin\alpha + g\xi - (A_Z + g - k_1 V_Z)\frac{\delta R_N}{R} + (A_E - k_1 V_E)\tan L \frac{\delta R_E}{R} \\ & - k_1 \delta V_N - \left(2\Omega \sin L + \frac{V_E}{R}\tan L\right)\delta V_E + \frac{V_N}{R}\delta V_Z + k_1 \delta V_{rN} + k_2 \delta V_{dN} \\ & - (A_Z + k_1 V_Z)\phi_E + (A_E - k_1 V_E)\phi_Z \end{aligned} \qquad \text{(B-9)}$$

$$\begin{aligned}\delta\dot{V}_E = {} & \mu_Y \cos\alpha - \mu_X \sin\alpha + g\eta - (A_Z + g + k_1 V_Z + A_N \tan L - k_1 V_N \tan L)\frac{\delta R_E}{R} \\ & + \left(2\Omega \sin L + \frac{V_E}{R}\tan L\right)\delta V_N + \left(2\Omega \cos L + \frac{V_E}{R}\right)\delta V_Z \\ & - k_1 \delta V_E + k_1 \delta V_{rE} + k_2 \delta V_{dE} + (A_Z - k_1 V_Z)\phi_N \\ & + (k_1 V_N - A_N)\phi_Z \end{aligned} \qquad \text{(B-10)}$$

$$\delta\dot{V}_Z = \mu_Z + \Delta g - (A_N - k_1 V_N)\frac{\delta R_N}{R} + (A_E - k_1 V_E)\frac{\delta R_E}{R} - \frac{2g}{R}\,\delta h$$

$$+ \frac{\kappa g}{R}(\delta h - \delta h_r) - \frac{V_N}{R}\,\delta V_N - \left(2\Omega\cos L + \frac{V_E}{R}\right)\delta V_E + \delta a_d$$

$$+ (A_N - k_1 V_N)\phi_E - (A_E - k_1 V_E)\phi_N \qquad \text{(B-11)}$$

Substitution of Eqs. (B-1) through (B-8) into the ϕ equations (4-18) through (4-20) gives:

$$\dot{\phi}_N = -\tau_L V_Z \phi_N - \left(\Omega\sin L + \frac{V_E}{R}\tan L\right)\phi_E + V_N\left(\tau_L + \frac{1}{R}\right)\phi_Z - \Omega\sin L\,\frac{\delta R_N}{R}$$

$$+ \left[\left(\frac{V_N}{R} + \tau_L V_N\right)\tan L - \Omega\sin L + \left(\tau_L + \frac{1}{R}\right)V_Z\right]\frac{\delta R_E}{R}$$

$$- \frac{V_E\,\delta h}{R^2} + \left(\tau_L + \frac{1}{R}\right)\delta V_E - \tau_L\,\delta V_{rE} + \epsilon_X\cos\alpha + \epsilon_Y\sin\alpha \qquad \text{(B-12)}$$

$$\dot{\phi}_E = -\tau_L V_Z \phi_E + \left(\Omega\sin L + \frac{V_E}{R}\tan L\right)\phi_N + \left[\Omega\cos L + \left(\tau_L + \frac{1}{R}\right)V_E\right]\phi_Z$$

$$-\left(\tau_L + \frac{1}{R}\right)V_Z\,\frac{R_N}{R} + \left(\tau_L + \frac{1}{R}\right)V_E\tan L\,\frac{\delta R_E}{R} + \frac{V_N}{R^2}\,\delta h - \left(\tau_L + \frac{1}{R}\right)\delta V_N$$

$$+ \tau_L\,\delta V_{rN} - \epsilon_X\sin\alpha + \epsilon_Y\cos\alpha \qquad \text{(B-13)}$$

$$\dot{\phi}_Z = -\left(\frac{V_N}{R} + \tau_V V_Z\right)\phi_N - \left(\Omega \cos L + \frac{V_E}{R}\right)\phi_E + \tau_V V_N \phi_Z - \left(\frac{V_E}{R}\sec^2 L + \Omega \cos L\right)\frac{\delta R_N}{R}$$

$$+ \left(\tau_V - \frac{\tan L}{R}\right)(V_Z + V_N \tan L)\frac{\delta R_E}{R} + \frac{V_E}{R^2}\tan L\,\delta h - \frac{1}{R}\tan L\,\delta V_E$$

$$+ \tau_V \delta V_N - \tau_V \delta V_{rN} + \epsilon_Z \qquad \text{(B-14)}$$

where the relation for the latitude rate,

$$\dot{L} = \frac{V_N}{R} \qquad \text{(B-15)}$$

has been used.

The velocity error and damping state equations (4-13) - (4-16), (4-25) and (4-26) remain unchanged. Equations (B-9) through (B-14) in conjunction with Eqs. (4-13) - (4-16), (4-25) and (4-26) of the text provide the complete error dynamics description of a local-level, wander azimuth INS mechanization in terms of the platform tilt error angles.

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