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FREE INVOLUTIONS ON NON-PRIME 3-MANIFOLDS

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1. The purpose of this paper is to study free (i.e., fixed point free) involutions on non-prime closed 3–manifolds and to suggest the general problem of characterizing the relationship between the connected sum decomposition of a closed 3–manifold and that of its closed covering spaces. Recall from [5] the following definitions. A closed connected 3–manifold is *prime* if there is no decomposition $M \approx M_1 \# M_2$, where M_1 and M_2 are non-trivial closed 3–manifolds (that is, different from the 3–sphere S^3). A closed 3–manifold is *irreducible* if every tame 2–sphere in M bounds a 3–cell. We say that M is *non-irreducible* if M contains a tame 2–sphere that does not bound a 3–cell. All manifolds are connected in this paper.

2. If $T: M \rightarrow M$ is a free involution we will denote the orbit space M/T by M^*. If $M \approx A \# B$ and T induces a free involution $T': A \rightarrow A$, we denote A/T' by A^* without specifically referring to T'. We remark that a free involution on a closed 3–manifold is always simplicial with respect to some triangulation.

Lemma 1. *A non-irreducible closed 3–manifold M admitting a free involution $T: M \rightarrow M$ contains a tame 2–sphere S not bounding a 3–cell in M such that either $TS=S$ or $TS \cap S=\phi$.*

Proof. Using Brouwer's fixed point theorem it is easy to show that a tame 2–sphere S in M such that $TS=S$ does not bound a cell. So it is only necessary to consider the case when there are no such 2–spheres in M. Suppose then that $TS \neq S$ for every tame 2–sphere S in M. It will follow that there is one such that $TS \cap S=\phi$.

Take any tame 2–sphere in M that does not bound a cell. One exists since M is not irreducible. By performing (if necessary) a series of small *pl* isotopies we can obtain a tame 2–sphere $S_0 \subset M$ not bounding a cell and such that $TS_0 \cap S_0 = \{c_1 \cdots c_n\}$, where each c_i is a simple closed curve disjoint from c_j if $i \neq j$. Let $n(TS_0 \cap S_0)$ denote the number of components of $TS_0 \cap S_0$.

From the class (non-empty) of tame 2–sphere Σ in M such that Σ does not bound a cell and $T\Sigma \cap \Sigma$ is a finite collection of pairwise disjoint simple closed curves, we select a 2–sphere S such that $n(TS \cap S)$ is minimal. We show that $n(TS \cap S)=0$.

Suppose that $n(TS \cap S)>0$. Let c be an innermost curve on TS. Then c bounds a closed disk $E \subset TS$ such that Int $E \cap S=\phi$. c separates S into two closed disks, E_1 and E_2. By proper choice of notation we have $TE \subset E_1$. Consider the tame 2–spheres $S_1=E \cup E_1$ and $S_2=E \cup E_2$. $Tc=c$ if and only if $TS_1=S_1$. Hence $Tc \neq c$ and TE is properly contained in E_1.

Both S_1 and S_2 cannot bound cells in M, so suppose S_i does not. Let c' be a simple closed curve on E_i close to c such that $c \cup c'$ bounds a closed annulus $A \subset E$, with $A \cap TS=c$. Span a closed disk E' on c' close to E so that $E' \cap TE'=\phi$, $E' \cap TS=\phi$, $E' \cap S=c'$, and the 2–sphere $S'=(E_i-A) \cup E'$ does not bound a cell. We have constructed a tame 2–sphere S' such that $n(TS' \cap S')<n(TS \cap S)$, contradicting our choice of S. Therefore, $n(TS \cap S)=0$.

Let N denote the non-orientable 2–sphere bundle over the circle. P^n denotes real projective n-space, $n=2$, 3.

Corollary (Tao [7]). *The orbit space of a free involution on $S^1 \times S^2$ is homeomorphic to $S^1 \times S^2$, N, $P^2 \times S^1$, or $P^3 \# P^3$.*

Corollary. *The orbit space of a free involution on N is homeomorphic to $P^2 \times S^1$.*

Proof. According to Lemma 1 there are two cases.

Case 1. There is a tame 2–sphere $S \subset N$ such that $TS=S$. Since S does not bound a cell, $N-S$ is connected. Cutting N by S we get a space homeomorphic to $S^2 \times I$. T induces a free involution T': $S^2 \times I \rightarrow S^2 \times I$ such that $T'(S^2 \times i)=S^2 \times i$, $i=1$, 2. By [2], T' is equivalent to $A \times e$, where A: $S^2 \rightarrow S^2$ is the antipodal map and e the identity on I. So the orbit space is homeomorphic to $P^2 \times S^1$.

Case 2. $TS \neq S$ for every tame 2–sphere S in N. An analysis similar to that of [6] in the proof of the previous corollary reveals that this case does not occur.

Lemma 2. *If a closed 3–manifold M admits a free involution T: $M \rightarrow M$ such that the orbit space M^* is irreducible and contains no 2–sided projective planes, then M is also irreducible.*

Proof. Suppose that M is not irreducible. According to Lemma 1 there is a tame 2–sphere $S \subset M$ that does not bound a cell and such that either $TS \cap S=\phi$ or $TS=S$. Let p: $M \rightarrow M^*$ be the projection.

Case 1. Suppose there is a tame 2–sphere $S \subset M$ not bounding a cell such that $TS \cap S=\phi$. Then $p(S)$ is a tame 2–sphere in M^*. But M^* is irreducible, so $p(S)$ must bound a cell in M^* and hence S also bounds a cell in M. This is in contradiction to our choice of S.

Case 2. Suppose there is no tame 2-sphere $S \subset M$ not bounding a cell such that $TS \cap S = \phi$, i.e. suppose Case 1 does not occur. Then there is a tame 2-sphere S such that $TS = S$. S must separate M, otherwise $p(S)$ would be a two-sided projective plane in M^*. Let $M = A' \cup B'$, where A', B' are the closures of the components of $M - S$. Since the Euler characteristic of P^2 is odd, P^2 cannot bound a manifold and hence $TA' = B'$. Let A be the non-trivial closed 3-manifold obtained by capping the 2-sphere boundary of A'. It follows that $M^* \approx A \# P^3$. But this contradicts M^* being irreducible.

Therefore we must have M irreducible.

3. We adopt the following notational conventions. Let H_1 denote the collection of all non-trivial prime closed 3-manifolds and let C_1 denote the collection of all non-trivial irreducible closed 3-manifolds. Then $H_1 = C_1 \cup \{S^1 \times S^2, N\}$. For $n \geq 2$ we let $H_n(C_n)$ denote the collection of closed 3-manifolds which are homeomorphic to the connected sum of exactly n members of $H_1(C_1)$.

Lemma 3. *Let $M \in C_m(H_m)$ and suppose $T\colon M \to M$ is a free involution. Then $M^* \in C_n(H_n)$, where $n \leq m (n \leq m+1)$.*

Proof. A proof for the case when $M \in C_m$ may be found in [8]. A similar argument establishes the case when $M \in H_m$, noting the corollaries to Lemma 1.

Theorem. *Let $T\colon M \to M$ be a free involution on $M \in C_m$, $m > 1$. Then there exist closed 3-manifolds A and B, with B irreducible (possibly trivial) such that $M \approx A \# B \# A$ and $M^* \approx A \# B^*$.*

Proof. The proof follows by a straight forward induction. We present an argument for the case when $m = 2$ which indicates the general technique. It follows from Lemmas 2 and 3 that $M^* \in C_2$ when $m = 2$. Write $M^* \approx A \# B = A' \cup B'$, where A, $B \in C_1$ and A', B' are obtained from A, B respectively, by deleting tame open 3-cells so that $A' \cap B' = S$ is a 2-sphere. Let $p\colon M \to M^*$ be the projection. $p^{-1}(S) = S_1 \cup S_2$, a pair of disjoint 2-spheres each separating M. Let U_1, U_2, V be the three components of M-$p(S)$, labeled so that $Bd(ClU_i) = S_i$ and $Bd(ClV) = S_1 \cup S_2$. Capping the 2-sphere boundary components of ClU_1, ClU_2, ClV with 3-cells we obtain the closed 3-manifolds Q_1, Q_2, R respectively. Then $M \approx Q_1 \# R \# Q_2$.

But $TU_1 = U_2$ and $TV = V$. Since Q_1 and Q_2 both cover either A or B, say A, exactly once and $M \in C_2$, it follows that $A \approx Q_1 \approx Q_2$ and $R \approx S^3$. Since $B \approx R^*$, Livesay's result [4] gives us $B \approx P^3$. Therefore $M \approx A \# S^3 \# A$ and $M^* \approx A \# P^3$.

Corollary. *A 3-manifold M belonging to C_2 admits a free involution if and* *only if $M \approx A \# A$ for some $A \in C_1$, in which case $M^* \approx A \# P^3$.*

We remark that Kwun [3] first observed that $P^3 \# P^3$ is the only non-prime orientable closed 3–manifold to double-cover itself.

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