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FREE LOCALLY INVERSE *-SEMIGROUPS

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1. INTRODUCTION

An *involution* * of a semigroup S is a unary operation $x \mapsto x^*$ satisfying

(1) $(xy)^* = y^*x^*$,

(2) $(x^*)^* = x$.

The algebra $(S, \cdot, ^*)$ is a *semigroup with involution* or an *involutorial semigroup*. If in addition

(3) $xx^*x = x$

holds then * is a *regular* involution and the algebra is called a *regular $*$-semigroup*. The study of such algebras was suggested by Nordahl and Scheiblich [13] and then conducted by several authors, for instance by Adair [1], Auinger [2, 3], Gerhard and Petrich [5, 6], Nambooripad and Pastijn [12], Petrich [16], Polák [18], Pondělíček [19], Scheiblich [21] and Szendrei [23, 24].

For a class $\mathscr{C}$ of regular semigroups let $\mathscr{C}^*$ denote the class of all regular $*$-semigroups $(S, \cdot, ^*)$ whose underlying semigroups $(S, .)$ are contained in $\mathscr{C}$. The members of $\mathscr{C}^*$ will be termed $\mathscr{C}$-$*$-semigroups (such as *completely simple* $*$-semigroups, *orthodox* $*$-semigroups etc.). If $\mathscr{C}$ is an e-variety (see Hall [7, 8]), that is, if $\mathscr{C}$ is closed under taking direct products, regular subsemigroups and homomorphic images then $\mathscr{C}^*$ forms a variety of algebras of type $\langle 2, 1\rangle$. The relatively free objects have been described for several varieties of regular $*$-semigroups (see [2, 3, 5, 6, 18, 20, 21, 23, 24]).

A regular semigroup S is *locally inverse* if for each idempotent e in S, the local submonoid eSe is an inverse semigroup. Locally inverse semigroups have been studied by several authors (see, for instance Pastijn [14, 15] and Nambooripad [11]). It is well known that this class—denoted by $\mathscr{L}\mathscr{I}$—is closed under taking direct products, regular subsemigroups and homomorphic images. Hence the class of all locally inverse

$*$-semigroups $\mathscr{L}\mathscr{I}^*$ is a variety. The purpose of the present paper is to describe the free objects in $\mathscr{L}\mathscr{I}^*$. The main result will be an analogue of Scheiblich's well-known description of the free inverse semigroup [20]. Roughly speaking, the completely simple $*$-semigroups (that is, "$*$-local groups") play the role for the free locally inverse $*$-semigroups that groups play for the free inverse semigroups. In fact we shall obtain canonical forms for the elements of the free object $F\mathscr{L}\mathscr{I}^*(X)$, similar to Schein's ones for the inverse case (see [22]). Furthermore, we shall describe $F\mathscr{L}\mathscr{I}^*(X)$ as a certain subsemigroup of a semidirect product of a semilattice by a free completely simple $*$-semigroup. This will be done in section 4. In section 2 we shall present some preliminaries, in section 3 some information on free completely simple $*$-semigroups will be given. Finally we shall obtain some further properties of the relatively free objects $F\mathscr{L}\mathscr{I}^*(X)$ in section 5.

2. Preliminaries

For definitions and results concerning semigroups the reader is referred to the books of Howie [9] and Petrich [17] (inverse semigroups).

Let X be any non-empty set, $X^* = \{x^* \mid x \in X\}$ be a disjoint copy of X such that $x \mapsto x^*$ is a bijection between X and X^*. Throughout the paper the set $X \cup X^*$ will be denoted by I. The mapping $^*: I \to I$ then denotes the bijection $x \mapsto x^*$, $x^* \mapsto x$, $x \in X$. Let $F^*(X)$ be the free semigroup on I which is equipped with the unary operation

$$^*: x_1 \dots x_n \mapsto x_n^* \dots x_1^*.$$

We obtain an involutorial semigroup. In fact, $F^*(X)$ is the *free involutorial* semigroup on X. By $F^*(X)^1$ we denote the free involutorial monoid, its identity—the empty word—will be denoted by 1. Now assume that $X = \{z < z' < \dots\}$ is well ordered with the least element z. Let I be ordered by $z < z^* < z' < (z')^* < \dots$. Then I is also well ordered. For each pair $(i, j) \in I \times I$ with $z < i < j$ let p_{ij} be an element not contained in I and such that $p_{ij} \neq p_{kl}$ whenever $(i, j) \neq (k, l)$. Let P denote the set of all these elements. As above, let $P^* = \{p^* \mid p \in P\}$ be a disjoint copy of P, $p \mapsto p^*$ being a bijection and $^*: P \cup P^* \to P \cup P^*$ being extended as above. Throughout the paper let $M = P \cup P^*$. Also assume that $M \cap I = \emptyset$. Finally we make the convention that for $i > j > z$, $p_{ij} = p_{ji}^*$ and $p_{zi} = p_{iz} = p_{ii} = 1$ (denoting the empty word) for all $i \in I$.

In the following we shall introduce three different manipulations of words in $F^*(X)$ respectively $F^*(X \cup P)^1$, two kinds of reductions and one expansion. These operations will be used essentially throughout the paper. First we need some terminology. Let $x_1 \dots x_n \in F^*(Z)$ for any non-empty set Z and $k \in \mathbb{N}$. Then

$\varrho_k x_1 \ldots x_n = x_1 \ldots x_{\min\{n,k\}}$, $x_1 \ldots x_n \lambda_k = x_{n-\min\{n,k\}+1} \ldots x_n$, $\varrho = \varrho_1$, $\lambda = \lambda_1$. Each $\varrho_k w$ is an *initial segment* whereas each $w\lambda_k$ is a *terminal segment*. The *length* n of the word $w = x_1 \ldots x_n$ will be denoted by $|w|$.

The two mentioned reductions are the following; the first one is the usual reduction of words in the free group.

Definition 1. The mapping $\mathbf{r} : F^*(X \cup P)^1 \to F^*(X \cup P)^1$ is defined by $\mathbf{r}\,1 = 1$, $\mathbf{r}\,y = y$ for all $y \in I \cup M$ where 1 denotes the empty word. Let $n > 1$ and suppose that $\mathbf{r}\,x_1 \ldots x_n = y_1 \ldots y_k$ $(k \geqslant 0)$; then

$$\mathbf{r}\,x_1 \ldots x_n x_{n+1} = \begin{cases} y_1 \ldots y_{k-1} & \text{if } x_{n+1} = y_k^*, \\ y_1 \ldots y_k x_{n+1} & \text{if } x_{n+1} \neq y_k^*. \end{cases}$$

Here $y_1 \ldots y_0$ stands for the empty word.

Definition 2. The mapping $\mathbf{s} : F^*(X) \to F^*(X)$ is defined by $\mathbf{s}\,x = x$, $\mathbf{s}\,xy = xy$ for all $x, y \in I$. Let $n > 2$ and $\mathbf{s}\,x_1 \ldots x_n = y_1 \ldots y_k$. Then

$$\mathbf{s}\,x_1 \ldots x_n x_{n+1} = \begin{cases} y_1 \ldots y_{k-1} & \text{if } y_{k-1} = y_k^* = x_{n+1}, \\ y_1 \ldots y_{k-2} x_{n+1} & \text{if } y_{k-1} = y_k^*, y_{k-2} = x_{n+1}^*, y_{k-3} \neq y_{k-2}^*, \\ y_1 \ldots y_{k-3} & \text{if } y_{k-1} = y_k^*, y_{k-2} = x_{n+1}^*, y_{k-3} = y_{k-2}^*, \\ y_1 \ldots y_k x_{n+1} & \text{otherwise.} \end{cases}$$

Roughly speaking, $\mathbf{r}\,w$ is obtained by deleting successively each occurrence of some x^*x whereas $\mathbf{s}\,w$ is obtained by successively replacing each occurrence of some xx^*x by x and xyy^*x^* by xx^* $(x, y \in I)$. We call $\mathbf{r}\,w$ the *reduced* form of w and $\mathbf{s}\,w$ the *weakly reduced* form of w. The operations $xx^* \to 1$ applied for obtaining $\mathbf{r}\,w$ are *reductions* whereas the operations $xx^*x \to x$, $xyy^*x^* \to xx^*$ are *weak reductions*. Further, applying $\mathbf{r}$ respectively $\mathbf{s}$ to subsets A of $F^*(X \cup P)^1$ or $F^*(X)$ means that $\mathbf{r}$ respectively $\mathbf{s}$ will be applied to each element of A. It is well-known that the reduced words $\mathbf{r}\,F^*(X \cup P)^1$ are canonical forms for the free group on $X \cup P$. If we consider elements of the free group on $X \cup P$, inversion sometimes will be denoted by $^{-1}$ rather than by * and then the words are assumed to be in reduced form. As we shall see in the next section, the weakly reduced words $\mathbf{s}\,w$ play the role for the free completely simple $*$-semigroups that reduced words $\mathbf{r}\,w$ play for the free groups. The third operation on words is the following expansion.

Definition 3. The mapping $\mathbf{e}_z : F^*(X) \to F^*(X \cup P)^1$ is defined by $\mathbf{e}_z\,x = x$ for all $x \in I$ and $\mathbf{e}_z\,x_1 \ldots x_n = x_1 p_{x_1^* x_2} x_2 \ldots p_{x_{n-1}^* x_n} x_n$ for all $x_1 \ldots x_n \in F^*(X)$, $n > 1$.

Here the index z indicates "normalisation with respect to z", that is, $p_{zi} = p_{iz} = p_{ii} = 1$ for all $i \in I$. As the reductions $\mathbf{s}$ and $\mathbf{r}$, $\mathbf{e}_z$ will be applied to a set A by applying it to each element of A.

The following lemma will be used several times without making special mention of. It can be proved easily by induction.

Lemma 2.1. *Let $w = x_1 \dots x_n \in F^*(X)$ and b be an initial segment of $\mathbf{s}\,w$. Then there is an initial segment $\varrho_k w$ of w such that $\mathbf{s}\,\varrho_k w = b$.*

The semigroups in this paper will be regular $*$-semigroups (except specially indicated). Hence also "subsemigroups", "homomorphisms", "congruences" etc. are considered to respect multiplication *and* involution without further making mention of. Similarly, all varieties under study are varieties of algebras of type $\langle 2, 1\rangle$. Given such a variety $\mathscr{V}$, the free object in $\mathscr{V}$ on the set X will be denoted by $F\mathscr{V}(X)$.

3. Completely simple $*$-semigroups

In this section we provide some information on the free completely simple $*$-semigroup $F\mathscr{C}\mathscr{S}^*(X)$. The first lemma has been proved by Petrich [16, Theorem 3.4].

Lemma 3.1. *Let $J \neq \emptyset$, G be a group and $Q = (q_{ij})$ be a $J \times J$-matrix with entries in G such that $q_{ij}^{-1} = q_{ji}$ and $q_{ii} = 1$ for all $i, j \in J$. Then the Rees matrix semigroup $S = \mathscr{M}(J, G, J; Q)$, endowed with the usual multiplication and with the involution*

$$(i, g, j)^* = (j, g^{-1}, i)$$

is a completely simple $*$-semigroup. Conversely, every completely simple $*$-semigroup can be so constructed.

The following result is from the same paper ([16, Theorem 4.1]).

Lemma 3.2. *A regular $*$-semigroup S is completely simple if and only if S satisfies the identity $xx^* = xyy^*x^*$.*

The free completely simple $*$-semigroup has been studied by Gerhard and Petrich [6] who obtained a Rees matrix representation of $F\mathscr{C}\mathscr{S}^*(X)$ similar to the model of the free completely simple semigroup due to Clifford and Rasin (see [4]). Recently, L. Polák provided a model of $F\mathscr{C}\mathscr{S}^*(X)$ by means of canonical forms. In the following, $\varrho_{\mathscr{C}\mathscr{S}^*}$ denotes the fully invariant congruence on $F^*(X)$ corresponding to the variety

$\mathscr{CS}^*$ of all completely simple $*$-semigroups. The result of Gerhard and Petrich [6, Theorem 7.3] states the following.

Theorem 3.3. *Let X, $I = X \cup X^*$, P be as in section 2 and let G denote the free group on $X \cup P$. Then the Rees matrix semigroup $S = \mathscr{M}(I, G, I; P)$, endowed with the usual multiplication and with the involution of Lemma 3.1 is the free completely simple $*$-semigroup, freely generated by the set $\{(x, x, x^*) \mid x \in X\}$.*

Theorem 3.3 can be also interpreted in the following way (see also [6, section 8]). Let $G = \mathbf{r}\, F^*(X \cup P)^1$ be the set of all reduced words in $F^*(X \cup P)^1$, endowed the involution of $F^*(X \cup P)^1$ and the multiplication $w \odot v = \mathbf{r}(wv)$ (in fact, $\mathbf{r}\, F^*(X \cup P)^1$ is the free group on $X \cup P$). Then the mapping $\varphi\colon F^*(X) \to \mathscr{M}(I, G, I; P)$, defined by $w\varphi = (\varrho w, \mathbf{r}\,\mathbf{e}_z\, w, (w\lambda)^*)$ is the canonical homomorphism of $F^*(X)$ onto $\mathscr{M}(I, G, I; P) \cong F\mathscr{CS}^*(X)$ which induces the fully invariant congruence $\varrho_{\mathscr{CS}^*}$.

On the other hand, L. Polák [18] showed that weak reduction as it is defined in section 2 provides canonical forms for the elements of $F\mathscr{CS}^*(X)$ (this result has been announced at the Conference on Semigroups in Oberwolfach, July 1991). It can be formulated as follows.

Theorem 3.4. *Let $\mathrm{s}\, F^*(X) = \{\mathrm{s}\, w \mid w \in F^*(X)\}$ be the set of all weakly reduced words endowed with the multiplication $w \otimes v = \mathrm{s}(wv)$ and with the involution of $F^*(X)$. Then the mapping $\mathrm{s} : F^*(X) \to \mathrm{s}\, F^*(X)$, $w \mapsto \mathrm{s}\, w$ is an epimorphism which induces the fully invariant congruence $\varrho_{\mathscr{CS}^*}$. In particular, weak reduction provides canonical forms of the elements of $F\mathscr{CS}^*(X)$.*

Let σ denote the equivalence relation on $F^*(X)$ defined by $u \;\sigma\; v \Leftrightarrow \mathrm{s}\, u = \mathrm{s}\, v$. By Lemma 3.2 and Theorem 3.3 it follows immediately that $\sigma \subseteq \varrho_{\mathscr{CS}^*}$. The result of L. Polák states that in fact $\sigma = \varrho_{\mathscr{CS}^*}$. For completeness we shall give an independent proof of this result in the following. Denote by $\varphi\colon F^*(X) \to \mathscr{M}(I, G, I; P)$ the canonical homomorphism $w\varphi = (\varrho w, \mathbf{r}\,\mathbf{e}_z\, w, (w\lambda)^*)$.

Lemma 3.5. *Let $w = x_1 \dots x_n \in F^*(X)$ be a word such that $p_{x_k^* x_{k+1}} = 1$ for all k, $1 \leqslant k < n$ and $w\varphi = (x_1, 1, x_1)$; then $\mathrm{s}\, w = x_1 x_1^*$.*

P r o o f. Let w be as above. Immediately we have $x_n = x_1^*$ since $x_1 = (w\lambda)^*$. We show the following. If $w \neq x_1 x_1^*$ then w is not weakly reduced. We may assume that w does not contain a subword of the form $xx^*x = x$, $x \in I$. Suppose first that $x_1 \notin \{z, z^*\}$. The assumptions on w imply that it is a word of the following form

$$w = x_1[x_1^*]zw_0z^*u_1[u_1^*]zw_1z^* \dots u_k[u_k^*]zw_kz^*[x_1]x_1^*$$

where each w_i is a word in the variables z and/or z^* or the empty word, $u_i \notin \{z, z^*\}$ and the brackets [ ] indicate that the respective element may or may not occur. If for some i, u_i^* in the brackets [ ] actually occurs then w can be weakly reduced. Also, if some w_i contains z as well as z^* then zw_iz^* and thus also w can be weakly reduced. Hence we may assume that w is of the following form

$$w = x_1[x_1^*]zz_0z^*u_1zz_1z^* \dots u_kzz_kz^*[x_1]x_1^*$$

where each z_i is a power of either z or z^* or is the empty word. We know that

$$1 = \mathbf{r}\,\mathbf{e}_z\, w = \mathbf{r}\, w = \mathbf{r}(x_1[x_1^*]z_0u_1z_1 \dots u_kz_k[x_1]x_1^*)$$

and thus also

$$(*) \qquad \mathbf{r}([x_1^*]z_0u_1z_1 \dots u_kz_k[x_1]) = 1.$$

Let $u_0 = [x_1^*]$, that is, $u_0 = x_1^*$ if x_1^* actually occurs in the brackets and $u_0 = 1$ otherwise. Similarly let $u_{k+1} = [x_1]$. By relation $(*)$ it follows that there is some i such that $z_i = 1$ and $u_i^* = u_{i+1}$. Then w contains a subword of the form $u_izz^*u_i^*$. Hence if w contains some u_i $(1 \leqslant i \leqslant k)$ then w can be weakly reduced. We therefore may assume that w has the form

$$w = x_1[x_1^*]zz_0z^*[x_1]x_1^*.$$

Again using $\mathbf{r}\,w = \mathbf{r}\,\mathbf{e}_z\,w = 1$ we obtain $z_0 = 1$ and either each or none of the elements in brackets [ ] occurs. In any case, w can be weakly reduced to $x_1x_1^*$. If $x_1 \in \{z, z^*\}$ then, as above, we may assume that w is of the form

$$w = x_1w_0z^*u_1[u_1^*] \dots u_k[u_k^*]zw_kx_1^*.$$

Now we apply the same procedure as for the previous case. □

Corollary 3.6. *Let $w = x_1 \dots x_n = \mathrm{s}(x_1 \dots x_n) \in \mathrm{s}\,F^*(X)$ be a weakly reduced word. If $p_{x_k^*x_{k+1}} = 1$ for all k, $1 \leqslant k < n$ and $w\varphi = (x_1, 1, x_1)$ then $w = x_1x_1^*$.*

Lemma 3.7. *If $w = x_1 \dots x_n \in F^*(X)$ is weakly reduced and $w\varphi = (x_1, 1, x_1)$ then $w = x_1x_1^*$.*

Proof. Again it is clear that $x_n = x_1^*$. If $p_{x_k^*x_{k+1}} = 1$ for all k then the assertion is proved by Corollary 3.6. Now suppose that there is some k such that $p_{x_k^*x_{k+1}} \neq 1$. Let $p_{x_k^*x_{k+1}} = p_k$. Since $\mathbf{r}(x_1p_1x_2 \dots x_{n-1}p_{n-1}x_n) = 1$ there are $k < l$ such that $1 \neq$

$p_k = p_l^*$, $p_{k+1} = \ldots = p_{l-1} = 1$ and $\mathbf{r}(x_{k+1}p_{k+1}x_{k+2}\ldots p_{l-1}x_l) = \mathbf{r}(x_{k+1}\ldots x_l) = 1$. Since $p_{x_k^* x_{k+1}} = p_k = p_l^* = p^*_{x_l^* x_{l+1}} = p_{x_{l+1}x_l^*}$ we observe that $x_k^* = x_{l+1}$, that is, $x_k = x_{l+1}^*$, and $x_{k+1} = x_l^*$. Since $p_{k+1} = \ldots = p_{l-1} = 1$ and $\mathbf{r}(x_{k+1}\ldots x_l) = 1$ we have $(x_{k+1}\ldots x_l)\varphi = (x_{k+1}, 1, x_l^*) = (x_{k+1}, 1, x_{k+1})$. Since w is weakly reduced, the subword $x_{k+1}\ldots x_l$ is also weakly reduced so that by Corollary 3.6, $x_{k+1}\ldots x_l = x_{k+1}x_{k+1}^*$ (and thus $l = k+2$). But then w contains a subword $x_k x_{k+1} x_{k+1}^* x_k^*$ which contradicts the assumption that w is weakly reduced. Therefore, $p_{x_k^* x_{k+1}} \neq 1$ cannot be true for any k and thus the assertion follows by Corollary 3.6. □

Now we are able to obtain the following result.

Corollary 3.8. *If $u = \mathrm{s}\,u = x_1\ldots x_n$ and $v = \mathrm{s}\,v = y_1\ldots y_m \in \mathrm{s}\,F^*(X)$ are weakly reduced words such that $u\varphi = v\varphi$ then $u = v$.*

Proof. First, $u\varphi = (\varrho u, \mathbf{r}\,\mathbf{e}_z\,u, (u\lambda)^*)$ and $v\varphi = (\varrho v, \mathbf{r}\,\mathbf{e}_z\,v, (v\lambda)^*)$ so that $x_1 = y_1$, $x_n^* = y_m^*$ and $\mathbf{r}\,\mathbf{e}_z\,u = \mathbf{r}(x_1 p_{x_1^* x_2} x_2 \ldots x_n) = \mathbf{r}(y_1 p_{y_1^* y_2} y_2 \ldots y_m) = \mathbf{r}\,\mathbf{e}_z\,v$. Put $w = uv^* = x_1 \ldots x_n y_m^* \ldots y_1^*$. Notice that $p_{x_n^* y_m^*} = 1$ since $x_n^* = y_m^*$. Hence $(\mathbf{e}_z\,u)(\mathbf{e}_z\,v^*) = \mathbf{e}_z\,w$. Using $\mathbf{e}_z\,v^* = (\mathbf{e}_z\,v)^*$,

$$\mathbf{r}\,\mathbf{e}_z\,w = \mathbf{r}[(\mathbf{e}_z\,u)(\mathbf{e}_z\,v^*)] = \mathbf{r}[(\mathbf{e}_z\,u)(\mathbf{e}_z\,v)^*] = \mathbf{r}[(\mathbf{r}\,\mathbf{e}_z\,u)(\mathbf{r}\,\mathbf{e}_z\,v)^*] = 1$$

and thus $w\varphi = (x_1, 1, y_1) = (x_1, 1, x_1)$. By Lemma 3.7 it follows that $\mathrm{s}\,w = \mathrm{s}(uv^*) = x_1 x_1^*$. The weak reduction of uv^* to $x_1x_1^*$ necessarily starts with a subword containing a terminal segment of u and an initial segment of v^*. The first step of weak reduction therefore is one of the following possibilities:

(1) $x_{n-2}x_{n-1}x_n y_m^* \to x_{n-2}y_m^*$ where $x_{n-1} = x_n^*$ and $x_{n-2} = y_m$,
(2) $x_n y_m^* y_{m-1}^* y_{m-2}^* \to x_n y_{m-2}^*$ where $y_{m-1}^* = y_m$ and $x_n = y_{m-2}$,
(3) $x_{n-1}x_n y_m^* y_{m-1}^* \to x_{n-1}y_{m-1}^*$ where $x_{n-1} = y_{m-1}$ (and $x_n = y_m$),
(4) $x_{n-1}x_n y_m^* \to y_m^* = x_{n-1}$ $(= x_n^*)$,
(5) $x_n y_m^* y_{m-1}^* \to x_n = y_{m-1}^*$ $(= y_m)$.

Cases (1) and (2) cannot occur since $x_n = y_m$ would imply $w\lambda_3 = x_n x_n^* x_n$ or $v\lambda_3 = (\varrho_3 v^*)^* = y_m y_m^* y_m$. In case (3) we immediately observe that $x_{n-1} = y_{m-1}$. In case (4), after the first weak reduction, we get the word $x_1 \ldots x_{n-1} y_{m-1}^* \cdots y_1^*$. The next weak reduction is of the form either $[x_{n-3}]x_{n-2}x_{n-1}y_{m-1}^* \to [x_{n-3}]y_{m-1}^*$ or $x_{n-1}y_{m-1}^* y_{m-2}^*[y_{m-3}^*] \to x_{n-1}[y_{m-3}^*]$ or $x_{n-2}x_{n-1}y_{m-1}^* y_{m-2}^* \to x_{n-2}y_{m-2}^*$. (Brackets [ ] indicate that the respective element may or may not be involved.) In the first case, $x_{n-2} = x_{n-1}^* = x_n$ which is impossible since $x_1 \ldots x_n$ is weakly reduced. In the second case, either $x_{n-1}y_{m-1}^* y_{m-2}^* = x_{n-1}x_{n-1}^* x_{n-1}$ or $y_{m-2} = y_{m-1}^*$ and $x_{n-1} = y_{m-3}$. In any case, $y_m^* = x_{n-1} = x_n^*$ implies that $y_m^* y_{m-1}^* \cdots y_1^*$ is not weakly reduced, a contradiction. Therefore only the third case is possible and we infer that $x_{n-1} = y_{m-1}$. In case (5), we get $x_{n-1} = y_{m-1}$ in an analogous way. The assertion now follows by induction on $\min\{|u|, |v|\}$. □

Remark. In the definition of the weakly reduced word s w we started the weak reductions on the left hand side of the word w and moved successively to the right in order to avoid ambiguity. Corollary 3.8 now in particular implies that the weak reductions $xx^*x \to x$, $xyy^*x^* \to xx^*$ may be executed in any order to obtain s w. We shall use this fact in the sequel without making mention of.

4. Free locally inverse $*$-semigroups

In this section we first obtain two identities each of which defines the variety $\mathscr{L}\mathscr{I}^*$ of all locally inverse $*$-semigroups (within the variety of all regular $*$-semigroups). Then we show that each element of $F\mathscr{L}\mathscr{I}^*(X)$ can be written as a product of certain commuting idempotents and a weakly reduced word. Furthermore, we shall show that this rewriting process provides canonical forms for the elements of $F\mathscr{L}\mathscr{I}^*(X)$. This will be done by showing that $F\mathscr{L}\mathscr{I}^*(X)$ can be realized as a subsemigroup of a certain semidirect product of a semilattice by the free completely simple $*$-semigroup $F\mathscr{C}\mathscr{S}^*(X)$.

Theorem 4.1. *Let S be a regular $*$-semigroup. Then S is locally inverse if and only if S satisfies either*

(4) $xyy^*x^*xzz^*x^* = xzz^*x^*xyy^*x^*$

or

(4′) $(xyx^*)(xyx^*)^*(xyx^*)^*(xyx^*) = (xyx^*)^*(xyx^*)(xyx^*)(xyx^*)^*$.

Proof. Let $e \in E(S)$; then $e \mathscr{D} ee^*$ and therefore eSe and $ee^*See^* = eSe^*$ are isomorphic *as semigroups* (via the mapping $x \mapsto xe^*$). The semigroup eSe^* is invariant under the involution so that eSe^* is a regular $*$-semigroup. The identity (4) implies that eSe^* satisfies the identity $yy^*xx^* = xx^*yy^*$ whereas (4′) implies that eSe^* satisfies the identity $xx^*x^*x = x^*xxx^*$. In any case, eSe^* is an inverse semigroup (see [17, Chap. XII]). Consequently eSe is an inverse semigroup. Conversely, let S be a locally inverse $*$-semigroup. Then $xSx^* = xx^*Sxx^*$ is a regular $*$-semigroup which in addition is an inverse semigroup. Hence on xx^*Sxx^*, $u \mapsto u^*$ is the unique inverse operation. The elements $xyy^*x^* = (xy)(xy)^*$ and $xzz^*x^* = (xz)(xz)^*$ are idempotents in xx^*Sxx^* and therefore commute. In particular, the identity (4) holds in S. Similarly, the elements $(xyx^*)(xyx^*)^*$ and $(xyx^*)^*(xyx^*)$ are idempotents in xx^*Sxx^* and thus commute. This implies the identity (4′). □

Recall that $\mathrm{s}(x_1 \dots x_n)$ denotes the weakly reduced word of $x_1 \dots x_n$. For $w, v \in F^*(X)$ the identity $w = v$ holds in $S \in \mathscr{L}\mathscr{I}^*$ if and only if $wf = vf$ for each homomorphism $f\colon F^*(X) \to S$. The identity $w = v$ holds in $\mathscr{L}\mathscr{I}^*$ if it holds in each member of $\mathscr{L}\mathscr{I}^*$. Similarly as for the inverse case (see [17, Chap. VIII]) we have the following rewriting process for locally inverse $*$-semigroups. (The proof is a natural analogue of the corresponding proof in [17, p. 360]). Here equality $=$ stands for equality in a locally inverse $*$-semigroup S, that is, equality in $\mathscr{L}\mathscr{I}^*$.

Theorem 4.2. *Let S be a locally inverse $*$-semigroup and $x_1, \dots, x_n \in S$. Then*

$$x_1 \dots x_n = \prod_{i=1}^{n-1} [\mathrm{s}(x_1 \dots x_i)\, \mathrm{s}(x_1 \dots x_i)^*]\, \mathrm{s}(x_1 \dots x_n).$$

P r o o f. Notice that all idempotents $\mathrm{s}(x_1 \dots x_k)\, \mathrm{s}(x_1 \dots x_k)^*$ commute since they belong to the local inverse submonoid $x_1 S x_1^*$ of S. The argument is by induction on n. For $n = 1$ the assertion is trivial. Let $v = x_1 \dots x_{n-1}$ and $\mathrm{s}\, v = y_1 \dots y_k$. If $\mathrm{s}(v) x_n \neq \mathrm{s}(v x_n)$ then either $\mathrm{s}\, v = y_1 \dots y_{k-3} x_n^* y_k^* y_k$ (that is, $y_{k-1} = y_k^*$ and $y_{k-2} = x_n^*$) or $\mathrm{s}\, v = y_1 \dots y_{k-2} x_n x_n^*$ (that is, $y_{k-1} = x_n$ and $y_k = x_n^*$). For the former case we have

$$\begin{aligned}
\mathrm{s}(v)\, \mathrm{s}(v)^*\, \mathrm{s}(v x_n) &= (y_1 \dots y_k)(y_1 \dots y_k)^* y_1 \dots y_{k-3} [x_n^* x_n] \\
&= (y_1 \dots y_k)(y_1 \dots y_k)^* y_1 \dots y_{k-3} x_n^* x_n \\
&= (y_1 \dots y_{k-3} x_n^* y_k^* y_k)(y_k^* y_k x_n y_{k-3}^* \dots y_1^*)(y_1 \dots y_{k-3} x_n^* x_n) \\
&= (y_1 \dots y_{k-3})(x_n^* y_k^* y_k x_n)(x_n^* x_n y_{k-3}^* \dots y_1^* y_1 \dots y_{k-3} x_n^* x_n) \\
&= (y_1 \dots y_{k-3})(x_n^* x_n y_{k-3}^* \dots y_1^* y_1 \dots y_{k-3} x_n^* x_n)(x_n^* y_k^* y_k x_n) \\
&= (y_1 \dots y_{k-3} x_n^*)(x_n y_{k-3}^* \dots y_1^*)(y_1 \dots y_{k-3} x_n^*)(y_k^* y_k x_n) \\
&= (y_1 \dots y_{k-3} x_n^*) y_k^* y_k x_n = (\mathrm{s}\, v) x_n.
\end{aligned}$$

The notation $[x_n^* x_n]$ means that $x_n^* x_n$ actually occurs if $y_{k-3} \neq x_n$ and is omitted if $y_{k-3} = x_n$. For the latter case we have

$$\begin{aligned}
(\mathrm{s}\, v)(\mathrm{s}\, v)^*\, \mathrm{s}(v x_n) &= \mathrm{s}(v)\, \mathrm{s}(v)^* y_1 \dots y_{k-2} x_n \\
&= \mathrm{s}(v)\, \mathrm{s}(v)^* y_1 \dots y_{k-2} x_n x_n^* x_n \\
&= \mathrm{s}(v)\, \mathrm{s}(v)^*\, \mathrm{s}(v) x_n = \mathrm{s}(v) x_n.
\end{aligned}$$

Finally, if $\mathrm{s}(v) x_n = \mathrm{s}(v x_n)$ then trivially $\mathrm{s}(v)\, \mathrm{s}(v)^*\, \mathrm{s}(v x_n) = \mathrm{s}(v) x_n$. Now let $n > 1$ and suppose that the assertion of the Theorem be true for all $n' < n$. That is,

$$(*) \qquad x_1 \dots x_{n-1} = \prod_{i=1}^{n-2} [\mathrm{s}(x_1 \dots x_i)\, \mathrm{s}(x_1 \dots x_i)^*]\, \mathrm{s}(x_1 \dots x_{n-1}).$$

By the above argument,

$$\mathrm{s}(x_1\dots x_{n-1})x_n = \mathrm{s}(x_1\dots x_{n-1})\,\mathrm{s}(x_1\dots x_{n-1})^*\,\mathrm{s}(x_1\dots x_{n-1}x_n).$$

Multiplying $(*)$ by x_n on the right then implies the assertion. □

Theorem 4.2 provides strong candidates for canonical forms of the elements of $F\mathscr{L}\mathscr{I}^*(X)$. One could expect that for two given words $x_1\dots x_n$, $y_1\dots y_m \in F^*(X)$, the identity $x_1\dots x_n = y_1\dots y_m$ holds in $F\mathscr{L}\mathscr{I}^*(X)$ if and only if

(1) $\{\mathrm{s}(x_1\dots x_i) \mid 1 \leqslant i \leqslant n\} = \{\mathrm{s}(y_1\dots y_j) \mid 1 \leqslant j \leqslant m\}$,
(2) $\mathrm{s}(x_1\dots x_n) = \mathrm{s}(y_1\dots y_m)$.

By Theorem 4.2, (1) and (2) are sufficient in order that $x_1\dots x_n = y_1\dots y_m$ holds in $\mathscr{L}\mathscr{I}^*$. However, the converse is not true.

Example. The identity $xy = xyy^*y$ holds in $\mathscr{L}\mathscr{I}^*$. Also $\mathrm{s}(xy) = \mathrm{s}(xyy^*y)$. But $\{\mathrm{s}(x), \mathrm{s}(xy)\} = \{x, xy\} \neq \{x, xy, xyy^*\} = \{\mathrm{s}(x), \mathrm{s}(xy), \mathrm{s}(xyy^*), \mathrm{s}(xyy^*y)\}$.

In the set of weakly reduced initial segments in (1) one has to take into account the element $\mathrm{s}(x_1\dots x_i)$ as well as $\mathrm{s}(x_1\dots x_i x_i^*)$ for each i. For $w = x_1\dots x_n$ let $\hat{\mathrm{s}}\,w = \{\mathrm{s}(x_1\dots x_i), \mathrm{s}(x_1\dots x_i x_i^*) \mid 1 \leqslant i \leqslant n\}$. In the following we shall prove that the identity $x_1\dots x_n = y_1\dots y_m$ holds in $\mathscr{L}\mathscr{I}^*$ if and only if

(1) $\hat{\mathrm{s}}(x_1\dots x_n) = \hat{\mathrm{s}}(y_1\dots y_m)$
(2) $\mathrm{s}(x_1\dots x_n) = \mathrm{s}(y_1\dots y_m)$.

Notice that the product in Theorem 4.2 will not be influenced if the first part is multiplied by all elements of the form $\mathrm{s}(x_1\dots x_i x_i^*)\,\mathrm{s}(x_1\dots x_i x_i^*)^*$ since

$$\mathrm{s}(x_1\dots x_i)\,\mathrm{s}(x_1\dots x_i)^* = \mathrm{s}(x_1\dots x_i x_i^*)\,\mathrm{s}(x_1\dots x_i x_i^*)^*$$

(= denoting equality in $\mathscr{L}\mathscr{I}^*$) and all such idempotents commute. Next we obtain some auxiliary definitions and results. The purpose is to reconstruct a weakly reduced element $x_1\dots x_n \in \mathrm{s}\,F^*(X)$ from $\mathbf{re}_z(x_1\dots x_n)$. By Corollary 3.8 this will not be completely possible since $x_1\dots x_n = \mathrm{s}(x_1\dots x_n)$ is determined by x_1, x_n and $\mathbf{re}_z(x_1\dots x_n)$. However, we shall try to obtain as much information as possible. The idea is the following. Let $\mathbf{re}_z(x_1\dots x_n) = q_1\dots q_k$ where $q_i \in I \cup M$. If $q_i \in M$, that is, $q_i = p_{u^*v}$ for some $u, v \in I$ then q_i will be replaced by u^*uvv^*. If $q_i = x \in I$ then x will be left unchanged. However, if xy occurs in $\mathbf{re}_z(x_1\dots x_n)$ and $x^* \neq z \neq y$, $x^* \neq y$ then xy has to be replaced by xzz^*y rather than by xy since there is no element from M between x and y. Formally we proceed as follows.

Definition 4. Let $q_1 \ldots q_k \in \mathbf{r}\, F^*(X \cup P)^1$ and $x \in I$. For $l = 0, 1, \ldots, k$ let $w_l = w_l(x, q_1 \ldots q_k) \in F^*(X)$ be defined by induction. First, $w_0 = xx^*$. Suppose that $w_l \in F^*(X)$ is already defined for some $l \geqslant 0$. Let

$$w_{l+1} = \begin{cases} w_l q_{l+1} & \text{if } (q_{l+1} \in I \text{ and}) \ w_l\lambda = q_{l+1}^*, \\ w_l zz^* q_{l+1} & \text{if } q_{l+1} \in I \text{ and } w_l\lambda \neq q_{l+1}^*, \\ w_l u^* uvv^* & \text{if } q_{l+1} = p_{u^*v} \text{ and } w_l\lambda = u, \\ w_l zz^* u^* uvv^* & \text{if } q_{l+1} = p_{u^*v} \text{ and } w_l\lambda \neq u. \end{cases}$$

Notice that for the latter two cases, $u^* \neq v$ and $u^* \neq z \neq v$ since $p_{u^*v} \neq 1$. In the following statements let $w_l = w_l(x, q_1 \ldots q_k)$.

Lemma 4.3. *If $w_l\lambda_3 = z^*yy^*$ for some $y \in I$ then $w_l = w_1$ and $x = y^* = z^*$.*

Proof. If w_l contains more than two letters then $l > 0$. If $w_l\lambda_3 = z^*yy^*$ then the first case in the definition of w_l applies: $w_l = w_{l-1}q_l$ where $q_l \neq 1$ and $q_l^* = w_{l-1}\lambda$. If $l = 1$ then this necessarily implies $w_0 = q_l q_l^*$. Thus $x = q_l$. Then $q_l q_l^* q_l = z^*yy^*$ implies $y^* = x = z^*$. If $l > 1$ then, since $q_{l-1}^* \neq q_l$, $w_{l-1} = w_{l-2}u^*uvv^*$ or $w_{l-1} = w_{l-2}zz^*u^*uvv^*$ and then $w_l = w_{l-2}u^*uvv^*q_l$ or $w_l = w_{l-2}zz^*u^*uvv^*q_l$. Both alternatives are in contradiction to the assumption on $w_l\lambda_3$ so that $l > 1$ is impossible. □

Lemma 4.4. *If $w_l\lambda_2 = zz^*$ then either $l = 0$ and $z = x$ or $l = 1$ and $z^* = x$. Further, $w_l\lambda_4 \neq tt^*tt^*$ for any $t \in I$.*

Proof. If $l = 0$ then $w_0\lambda_2 = zz^*$ if and only if $z = x$. Suppose that $w_1\lambda_2 = zz^*$. Then $w_1 = w_0z^*$ and $w_0 = z^*z$. Hence $z^* = x$. Let $l > 1$ and suppose that $w_l\lambda_2 = zz^*$. Then $w_l = w_{l-1}z^*$, $q_l = z^*$ and $w_{l-1}\lambda = z$. But then $q_{l-1} = z$ which is in contradiction to $q_l = z^*$ since $q_1 \ldots q_l$ is reduced. Hence $w_l\lambda_2 \neq zz^*$ whenever $l > 1$. The assertion on $w_l\lambda_4$ is easy to see. □

Lemma 4.5. *The word w_l does not contain a subword of the form st^*ts^* for $s \neq t$ nor a subword of the form $stt^*tt^*s^*$ for any $s, t \in I$.*

Proof. We consider the case st^*ts^*, $s \neq t$, first. For $l = 0, 1$ the assertion can be checked easily. Let $l > 1$ and assume that the assertion be true for all $l' < l$. If $w_l = w_{l-1}q_l$ then $w_{l-1}\lambda = q_l^*$ and so the induction hypothesis on w_{l-1} implies the assertion. If $w_l = w_{l-1}zz^*q_l$ then the assertion follows by $w_{l-1}\lambda \neq q_l^*$, $w_{l-1}\lambda_3 \neq z^*ss^*$ (Lemma 4.3) and the induction hypothesis on w_{l-1}. Similarly the assertion follows if $w_l = w_{l-1}zz^*u^*uvv^*$. If $w_l = w_{l-1}u^*uvv^*$ then $w_{l-1}\lambda = u \neq v^*$ and the assertion follows in this case, too.

Now consider the word $stt^*tt^*s^*$ for some $s, t \in I$. Again the assertion can be checked directly if $l = 0, 1$. Let $l > 1$ and assume that the assertion be true for all $l' < l$. If $w_l = w_{l-1}q_l$, then $w_l\lambda_6 \neq stt^*tt^*s^*$ by Lemma 4.4 and thus by the hypothesis of induction on w_{l-1} the assertion follows. If $w_l = w_{l-1}zz^*q_l$ then $(w_{l-1}zz^*q_l)\lambda_6 \neq stt^*tt^*s^*$ since $w_{l-1}\lambda_2 \neq zz^*$ by Lemma 4.4 and $(w_{l-1}z)\lambda_6 \neq stt^*tt^*s^*$ since $w_{l-1}\lambda_4 \neq tt^*tt^*$. Also, $(w_{l-1}zz^*)\lambda_6 = stt^*tt^*s^*$ implies $w_{l-1}\lambda_4 = zz^*zz^*$ which is impossible. Again by hypothesis on w_{l-1} the assertion follows. Now consider the case $w_l = w_{l-1}u^*uvv^*$, that is, $q_l = p_{u^*v}$ and $w_{l-1}\lambda = u$. Similarly as above, $(w_{l-1}u^*)\lambda_6, (w_{l-1}u^*u)\lambda_6, (w_{l-1}u^*uvv^*)\lambda_6 \neq stt^*tt^*s^*$. If $(w_{l-1}u^*uv)\lambda_6 = stt^*tt^*s^*$ then $w_{l-1}\lambda_3 = v^*u^*u$. It is impossible that $w_{l-1} = w_{l-2}u$, that is, $w_{l-2}\lambda_2 = v^*u^*$ and $q_{l-1} = u$ for then $q_{l-2} = u^*$, a contradiction. Hence $w_{l-1} = w_{l-2}vv^*u^*u$ or $w_{l-1} = w_{l-2}zz^*vv^*u^*u$, that is, $q_{l-1} = p_{vu^*}$. But this implies $q_{l-1} = q_l^*$ which is also impossible. Again the assertion follows by hypothesis on w_{l-1}. The case $w_l = w_{l-1}zz^*u^*uvv^*$ can be treated in an analogous way. □

R e m a r k . Lemma 4.5 in fact assures that $\mathrm{s}\, w_l$ can be obtained by using solely weak reductions of the form $xx^*x \to x$, $x \in I$.

Corollary 4.6. *$(\mathrm{s}\, w_l)\lambda_3 \neq z^*yy^*$ for any $y \in I$.*

P r o o f . This is trivial if $l = 0$ and can be checked directly if $l = 1$. Let $l > 1$. If $w_l = w_{l-1}q_l$ then $w_{l-1} = w_{l-2}u^*uvv^*$ or $w_{l-1} = w_{l-2}zz^*u^*uvv^*$ and $q_l = v$. The last three letters in the word obtained by the weak reduction $w_l\lambda_3 = vv^*v \to v$ are u^*uv. By Lemma 4.5 (and the above remark), the element u in u^*uv cannot be eliminated by further weak reductions. Since $u^* \neq v$ the assertion follows. If $w_l = w_{l-1}zz^*q_l$ we consider two cases. Case (i) $q_l = z$. Then $w_l\lambda_4 = szz^*z$ for some $s \neq z^*$. Now $w_l\lambda_3 = zz^*z$ will be weakly reduced to z, but using Lemma 4.5 again, the element s cannot be removed by any further weak reduction. Case (ii) $q_l \neq z$. Then $w_l\lambda_2 = z^*q_l$ and again the letter z^* cannot be eliminated by further weak reduction. Finally, if $w_l = w_{l-1}u^*uvv^*$ or $w_l = w_{l-1}zz^*u^*uvv^*$ then in both cases $z^* \neq u$ and since $u \neq v^*$, again by Lemma 4.5 and the remark thereafter, u cannot be removed by weak reduction. □

Corollary 4.7. *If $(\mathrm{s}\, w_l)\lambda_2 = zz^*$ then $l = 0$ and $x = z$.*

P r o o f . By Lemma 4.5 (and the remark thereafter), $(\mathrm{s}\, w_l)\lambda_2 = zz^*$ implies $w_l\lambda_2 = zz^*$. Hence by Lemma 4.4, $l = 0$ and $x = z$ or $l = 1$. But in the latter case $w_1 = z^*zz^*$ and then $\mathrm{s}\, w_l = z^*$. □

Lemma 4.8. *$\mathbf{re}_z\, w_l(x, q_1 \ldots q_k) = q_1 \ldots q_l$ and $\mathbf{re}_z\, w_0 = 1$.*

Proof. The argument is by induction on l. If $l = 0$ then this is trivial. Let $l > 0$ and suppose the assertion be true for all $l' < l$. For the respective cases of Definition 4 we have

$$w_l = \begin{cases} w_{l-1}q_l, \\ w_{l-1}zz^*q_l, \\ w_{l-1}u^*uvv^*, \\ w_{l-1}zz^*u^*uvv^* \end{cases} \quad \text{and } \mathrm{e}_z\, w_l = \begin{cases} (\mathrm{e}_z\, w_{l-1})q_l, \\ (\mathrm{e}_z\, w_{l-1})zz^*q_l, \\ (\mathrm{e}_z\, w_{l-1})u^*up_{u^*v}vv^*, \\ (\mathrm{e}_z\, w_{l-1})zz^*u^*up_{u^*v}vv^*. \end{cases}$$

From this it follows easily that $\mathbf{r}\,\mathrm{e}_z\, w_l = q_1 \dots q_l$ if $\mathbf{r}\,\mathrm{e}_z\, w_{l-1} = q_1 \dots q_{l-1}$. □

Definition 5. Let $x_1 \dots x_n \in F^*(X)$ be a word, let $q_1 \dots q_k = \mathbf{r}\,\mathrm{e}_z(x_1 \dots x_n) = \mathbf{r}(x_1 p_{x_1^* x_2} x_2 \dots x_n)$ and let w_k be as in Definition 4. Put

$$w(x_1, \mathbf{r}\,\mathrm{e}_z(x_1 \dots x_n)) = \mathrm{s}(w_k(x_1, q_1 \dots q_k)).$$

Corollary 4.9. *Let $x_1 \dots x_n \in F^*(X)$; then*

$$\mathbf{r}\,\mathrm{e}_z\, w(x_1, \mathbf{r}\,\mathrm{e}_z(x_1 \dots x_n)) = \mathbf{r}\,\mathrm{e}_z(x_1 \dots x_n).$$

Proof. If for $a, b \in F^*(X)$, $\mathrm{s}\,a = \mathrm{s}\,b$ then by Theorem 3.3, $\mathbf{r}\,\mathrm{e}_z\, a = \mathbf{r}\,\mathrm{e}_z\, b$. Let $\mathbf{r}\,\mathrm{e}_z(x_1 \dots x_n) = q_1 \dots q_k$. Using Lemma 4.8, we obtain

$$\begin{aligned} \mathbf{r}\,\mathrm{e}_z\, w(x_1, \mathbf{r}\,\mathrm{e}_z(x_1 \dots x_n)) &= \mathbf{r}\,\mathrm{e}_z(\mathrm{s}\, w_k(x_1, \mathbf{r}\,\mathrm{e}_z(x_1 \dots x_n))) \\ &= \mathbf{r}\,\mathrm{e}_z\, w_k(x_1, \mathbf{r}\,\mathrm{e}_z(x_1 \dots x_n)) \\ &= q_1 \dots q_k = \mathbf{r}\,\mathrm{e}_z(x_1 \dots x_n). \end{aligned}$$

□

We are able to formulate the following important result.

Theorem 4.10. *Let $x_1 \dots x_n = \mathrm{s}(x_1 \dots x_n) \in \mathrm{s}\,F^*(X)$ be a weakly reduced word. Let $w = w(x_1, \mathbf{r}\,\mathrm{e}_z(x_1 \dots x_n))$. Then*

$$x_1 \dots x_n = \begin{cases} w & \text{iff } w\lambda = x_n, \\ wx_n^*x_n & \text{iff } w\lambda = z^* \neq x_n, \\ wzz^* & \text{iff } w\lambda \neq z^* = x_n, \\ wzz^*x_n^*x_n & \text{iff } w\lambda \neq z^* \neq x_n, w\lambda \neq x_n. \end{cases}$$

Proof. Denote these four different cases by (1)–(4). Notice that (1)–(4) are pairwise disjoint and each possible case is covered by one of these. In case (1), w is clearly weakly reduced. If in case (2) $wx_n^*x_n$ could be weakly reduced then $|w| \geqslant 3$ and $w\lambda_3 = x_n zz^*$ which is a contradiction to Corollary 4.7. Hence $wx_n^*x_n$ is weakly reduced. The respective elements of cases (3) and (4) are weakly reduced by Corollary 4.6. Now consider the canonical mapping $\varphi\colon F^*(X) \to F\mathscr{C}\mathscr{S}^*(X) = \mathscr{M}(I, G, I; P)$ given by $a \mapsto a\varphi = (\varrho a, \mathbf{r}\,\mathbf{e}_z\, a, (a\lambda)^*)$. Letting $a \in \{w, wx_n^*x_n, wzz^*, wzz^*x_n^*x_n\}$ denote any one of the respective cases (1)–(4) then $(x_1 \ldots x_n)\varphi = a\varphi$. Since $x_1 \ldots x_n$ as well as a is weakly reduced, by Corollary 3.8 we have $x_1 \ldots x_n = a$. $\square$

Immediately we have the following result.

Corollary 4.11. *Let $x_1 \ldots x_n = \mathrm{s}(x_1 \ldots x_n) \in \mathrm{s}\,F^*(X)$ be a weakly reduced word and let $w = w(x_1, \mathbf{r}\,\mathbf{e}_z(x_1 \ldots x_n))$. If $x_{n-1} \neq x_n^*$ then $w = x_1 \ldots x_n$.*

By Theorem 4.2 we know that for $a, b \in F^*(X)$, $a\ \varrho_{\mathscr{L}\mathscr{I}^*}\ b$ if $(\hat{\mathrm{s}}\,a, \mathrm{s}\,a) = (\hat{\mathrm{s}}\,b, \mathrm{s}\,b)$. In the following we shall prove the converse. For this purpose we construct a locally inverse $*$-semigroup in which the identity $a = b$ holds if and only if $(\hat{\mathrm{s}}\,a, \mathrm{s}\,a) = (\hat{\mathrm{s}}\,b, \mathrm{s}\,b)$. As in section 2 let $G = F\mathscr{G}(X \cup P)$ be the free group on $X \cup P$. In the following, inverses in this group will be indicated by ${}^{-1}$ rather than by *. In particular, $p_{x^*y} = p_{yx^*}^{-1}$ for any $x, y \in I$ and we assume that multiplication automatically results in reduced words. Let $Y = F\mathscr{S}(G)$ be the free semilattice generated by G. That is, Y consists of all finite non-empty subsets of G, endowed with the binary operation of set theoretical union. For $A \in Y$, $g \in G$ let $gA = \{ga \mid a \in A\}$. According to this definition, the group G acts on the semilattice Y as a group of automorphisms. Now let $S = I \times Y \times G \times I$, endowed with the multiplication

$$(i, A, g, j)(k, B, h, l) = (i, A \cup gp_{jk}B, gp_{jk}h, l)$$

and involution

$$(i, A, g, j)^* = (j, g^{-1}A, g^{-1}, i).$$

By [12, Example 1.7], S is a locally inverse $*$-semigroup. In fact, S is a perfect rectangular band of E-unitary inverse semigroups (see [14]). Let $\chi\colon F^*(X) \to S$ be the unique extension of the mapping $x \mapsto (x, \{1, x\}, x, x^*)$, $x \in X$, to a homomorphism. Let $x_1 \ldots x_n \in F^*(X)$. Using induction, it can be easily seen that

$$(x_1 \ldots x_n)\chi = (x_1, \{1, x_1, x_1 p_{x_1^* x_2}, \ldots, x_1 p_{x_1^* x_2} x_2 \ldots x_n\}, x_1 p_{x_1^* x_2} x_2 \ldots x_n, x_n^*).$$

Since the elements $x_1 p_{x_1^* x_2} x_2 \ldots$ are in the group G and

$$x_1 p_{x_1^* x_2} x_2 \ldots p_{x_{k-1}^* x_k} = x_1 p_{x_1^* x_2} x_2 \ldots p_{x_{k-1}^* x_k} x_k x_k^{-1},$$

the homomorphism χ provides the following information on a given word $a = x_1 \dots x_n \in F^*(x)$:

(1) x_1,
(2) $\{\mathbf{r}\,\mathbf{e}_z(x_1 \dots x_i x_i^*), \mathbf{r}\,\mathbf{e}_z(x_1 \dots x_i) \mid 1 \leqslant i \leqslant n\}$,
(3) $\mathbf{r}\,\mathbf{e}_z(x_1 \dots x_n)$,
(4) x_n.

By Theorem 3.3 and Corollary 3.8 it follows that $\mathrm{s}\,a$ is uniquely determined by the triple $(x_1, \mathbf{r}\,\mathbf{e}_z\, a, x_n^*)$. Theorem 4.10 shows how $\mathrm{s}\,a$ can be reconstructed from the data $x_1, \mathbf{r}\,\mathbf{e}_z\, a$ and x_n. Further, by Theorem 3.3, the canonical homomorphism $\varphi\colon F^*(X) \to F\mathscr{C}\mathscr{S}^*(X) = \mathscr{M}(I, G, I; P)$ is given by $a \mapsto (\varrho a, \mathbf{r}\,\mathbf{e}_z\, a, (a\lambda)^*)$. In particular, $\mathbf{r}\,\mathbf{e}_z\, a = \mathbf{r}\,\mathbf{e}_z\, a'$ whenever $a \;\varrho_{\mathscr{C}\mathscr{S}^*}\; a'$. Consequently, (2) in fact is the following set

$$\{\mathbf{r}\,\mathbf{e}_z\, \mathrm{s}(x_1 \dots x_i x_i^*), \mathbf{r}\,\mathbf{e}_z\, \mathrm{s}(x_1 \dots x_i) \mid 1 \leqslant i \leqslant n\}.$$

For each i let $s_i = \mathrm{s}(x_1 \dots x_i)$ and $t_i = \mathrm{s}(x_1 \dots x_i x_i^*)$. Then either $s_i\lambda_2 \neq (s_i\lambda)^*(s_i\lambda)$ or $t_i\lambda_2 \neq (t_i\lambda)^*(t_i\lambda)$. Let $u_i = w(x_1, \mathbf{r}\,\mathbf{e}_z\, s_i)$ and $v_i = w(x_1, \mathbf{r}\,\mathbf{e}_z\, t_i)$ according to Definition 5. Hence by Corollary 4.11, either $u_i = s_i$ or $v_i = t_i$. Also, since $\mathrm{s}(\mathrm{s}(a_1 \dots a_k)a_{k+1}) = \mathrm{s}(a_1 \dots a_k a_{k+1})$, $a_j \in I$, we have

$$t_i = \mathrm{s}(s_i(t_i\lambda)) = \mathrm{s}(s_i(s_i\lambda)^*)$$

and

$$s_i = \mathrm{s}(t_i(s_i\lambda)) = \mathrm{s}(t_i(t_i\lambda)^*).$$

In particular,

$$\{s_i, t_i\} \subseteq \{u_i, \mathrm{s}(u_i(u_i\lambda)^*), v_i, \mathrm{s}(v_i(v_i\lambda)^*)\}$$

for each i and thus

$$\hat{\mathrm{s}}\, a \subseteq \{u_i, v_i, \mathrm{s}(u_i(u_i\lambda)^*), \mathrm{s}(v_i(v_i\lambda)^*) \mid 1 \leqslant i \leqslant n\}.$$

Now take any $\mathrm{s}(x_1 \dots x_i) = x_1 y_2 \dots y_k x_i$ and consider the word $w = w(x_1, \mathbf{r}\,\mathbf{e}_z\, s_i) = w(x_1, \mathbf{r}\,\mathbf{e}_z\, x_1 y_2 \dots y_k x_i)$. We apply Theorem 4.10. If $w\lambda = x_i$ then $w = s_i = \mathrm{s}(x_1 \dots x_i)$ and $\mathrm{s}(w(w\lambda)^*) = \mathrm{s}(\mathrm{s}(x_1 \dots x_i)x_i^*) = \mathrm{s}(x_1 \dots x_i x_i^*)$. If $w\lambda \neq x_i$ then either $\mathrm{s}(x_1 \dots x_i) = w x_i^* x_i$ or $\mathrm{s}(x_1 \dots x_i) = w z z^* x_i^* x_i$. In any case, by Lemma 2.1, $w = \mathrm{s}(x_1 \dots x_l)$ for some suitable $l < i$ and thus $w \in \hat{\mathrm{s}}\, a$. Further, $\mathrm{s}(w(w\lambda)^*) = \mathrm{s}(w x_l^*) = \mathrm{s}(x_1 \dots x_l x_l^*) \in \hat{\mathrm{s}}\, a$. Similarly it can be shown that $w, \mathrm{s}(w(w\lambda)^*) \in \hat{\mathrm{s}}\, a$ for $w = w(x_1, \mathbf{r}\,\mathbf{e}_z\, t_i)$ for each i. Consequently,

$$\hat{\mathrm{s}}\, a = \{u_i, v_i, \mathrm{s}(u_i(u_i\lambda)^*), \mathrm{s}(v_i(v_i\lambda)^*) \mid 1 \leqslant i \leqslant n\}.$$

In fact, the set $\hat{\mathrm{s}}\,a$ is uniquely determined by the element x_1 and the set

$$\mathbf{r}\,\mathrm{e}_z\,\hat{\mathrm{s}}\,a = \{\mathbf{r}\,\mathrm{e}_z\,t_i, \mathbf{r}\,\mathrm{e}_z\,s_i \mid 1 \leqslant i \leqslant n\}.$$

Summarizing the results we have the following. Let $\varrho_{\mathscr{L}\mathscr{I}^*}$ denote the fully invariant congruence on $F^*(X)$ corresponding to $\mathscr{L}\mathscr{I}^*$ and let $\bar{\varrho}_{\mathscr{I}}$ be the least inverse congruence on $F^*(X \cup P)^1$, that is, the fully invariant congruence corresponding to the variety $\mathscr{I}$ of all inverse semigroups.

Theorem 4.12. *Let $a = x_1 \ldots x_n$, $b = y_1 \ldots y_m \in F^*(X)$ be two words. Then the following assertions are equivalent:*

(1) $a\ \varrho_{\mathscr{L}\mathscr{I}^*}\ b$,

(2) $(\hat{\mathrm{s}}\,a, \mathrm{s}\,a) = (\hat{\mathrm{s}}\,b, \mathrm{s}\,b)$,

(3) $\varrho a = \varrho b$, $\mathrm{e}_z\,a\ \bar{\varrho}_{\mathscr{I}}\ \mathrm{e}_z\,b$, $a\lambda = b\lambda$,

(4) $(\varrho a, \mathbf{r}\,\mathrm{e}_z\,\hat{\mathrm{s}}\,a, \mathbf{r}\,\mathrm{e}_z\,a, (a\lambda)^*) = (\varrho b, \mathbf{r}\,\mathrm{e}_z\,\hat{\mathrm{s}}\,b, \mathbf{r}\,\mathrm{e}_z\,b, (b\lambda)^*)$.

Furthermore, the mapping $\chi\colon F^*(X) \to S = I \times Y \times G \times I$, defined by

$$a\chi = (\varrho a, \mathbf{r}\,\mathrm{e}_z\,\hat{\mathrm{s}}\,a, \mathbf{r}\,\mathrm{e}_z\,a, (a\lambda)^*)$$

is a homomorphism which induces $\varrho_{\mathscr{L}\mathscr{I}^*}$. In particular, the ($*$-)subsemigroup of S which is generated by the set $\{(x, \{1, x\}, x, x^*) \mid x \in X\}$ is a model of the free locally inverse $*$-semigroup on X. The mapping $\psi\colon (\varrho a, \mathbf{r}\,\mathrm{e}_z\,\hat{\mathrm{s}}\,a, \mathbf{r}\,\mathrm{e}_z\,a, (a\lambda)^*) \mapsto (\varrho a, \mathbf{r}\,\mathrm{e}_z\,a, (a\lambda)^*)$ is the canonical homomorphism of $F\mathscr{L}\mathscr{I}^*(X)$ onto $F\mathscr{C}\mathscr{S}^*(X)$.

Proof. By Theorem 4.2 we have (2) $\Rightarrow$ (1). Since $\chi\colon F^*(X) \to S$ is a homomorphism and $S \in \mathscr{L}\mathscr{I}^*$ we have (1) $\Rightarrow$ (4). Since, as shown above, $(\hat{\mathrm{s}}\,a, \mathrm{s}\,a)$ can be uniquely reconstructed from $(\varrho a, \mathbf{r}\,\mathrm{e}_z\,\hat{\mathrm{s}}\,a, \mathbf{r}\,\mathrm{e}_z\,a, (a\lambda)^*)$ the implication (4) $\Rightarrow$ (2) follows. Using the fact that $\mathbf{r}\,\mathrm{e}_z = \mathbf{r}\,\mathrm{e}_z\,\mathrm{s}$ it follows from the well-known description of $\bar{\varrho}_{\mathscr{I}}$ (see [17, Chap. VIII] and [20]) that $\mathrm{e}_z\,a\ \bar{\varrho}_{\mathscr{I}}\ \mathrm{e}_z\,b \Leftrightarrow (\mathbf{r}\,\mathrm{e}_z\,\hat{\mathrm{s}}\,a, \mathbf{r}\,\mathrm{e}_z\,a) = (\mathbf{r}\,\mathrm{e}_z\,\hat{\mathrm{s}}\,b, \mathbf{r}\,\mathrm{e}_z\,b)$, showing the equivalence of (4) and (3). Since (1) $\Leftrightarrow$ (4), the homomorphism $\chi\colon F^*(X) \to S$ induces the congruence $\varrho_{\mathscr{L}\mathscr{I}^*}$ on $F^*(X)$. Consequently $F\mathscr{L}\mathscr{I}^*(X) \cong F^*(X)\chi \subseteq S = I \times Y \times G \times I$ and $F^*(X)\chi$ is precisely the ($*$-)subsemigroup of S which is generated by the set $\{(x, \{1, x\}, x, x^*) \mid x \in X\}$. Finally, $\varphi = \chi\psi$ where $\varphi\colon F^*(X) \to F\mathscr{C}\mathscr{S}(X) = \mathscr{M}(I, G, I; P)$ is the canonical homomrophism. This implies the assertion on ψ. □

Since $\mathrm{s}(x_1 \ldots x_i)\,\mathrm{s}(x_1 \ldots x_i)^*\ \varrho_{\mathscr{L}\mathscr{I}^*}\ \mathrm{s}(x_1 \ldots x_i x_i^*)\,\mathrm{s}(x_1 \ldots x_i x_i^*)^*$ by Theorem 4.2 we are motivated to define a canonical form of $x_1 \ldots x_n \varrho_{\mathscr{L}\mathscr{I}^*}$ as follows. For $i = 1, \ldots, n-1$ put

$$r_i = \begin{cases} s_i = \mathrm{s}(x_1 \ldots x_i) & \text{if } \mathrm{s}(x_1 \ldots x_i)\lambda_2 \neq x_i^* x_i, \\ t_i = \mathrm{s}(x_1 \ldots x_i x_i^*) & \text{if } \mathrm{s}(x_1 \ldots x_i)\lambda_2 = x_i^* x_i. \end{cases}$$

Then $r_i r_i^* \varrho_{\mathscr{L}\mathscr{I}^*} s_i s_i^* \varrho_{\mathscr{L}\mathscr{I}^*} t_i t_i^*$ and $(\prod r_i r_i^*)\,\mathrm{s}(x_1 \ldots x_n) \varrho_{\mathscr{L}\mathscr{I}^*} x_1 \ldots x_n$ so that the element $(\prod r_i r_i^*)\,\mathrm{s}(x_1 \ldots x_n)$ can be interpreted as a canonical form of $x_1 \ldots x_n$ in $F\mathscr{L}\mathscr{I}^*(X)$. The product will be taken over the set $\{r_i \mid 1 \leqslant i \leqslant n\}$ rather than $\{i \mid 1 \leqslant i \leqslant n\}$ since several of the elements r_i may coincide. All idempotents $r_i r_i^*$ commute.

5. Some properties of the relatively free object $F\mathscr{L}\mathscr{I}^*(X)$

Concerning the description of $F\mathscr{L}\mathscr{I}^*(X)$ as the subsemigroup of the semidirect product $I \times Y \times G \times I$ which is generated by the set $\{(x, \{1, x\}, x, x^*) \mid x \in X\}$ (Theorem 4.12), the following question arises. Given $(i, A, g, j) \in I \times Y \times G \times I$; is (i, A, g, j) contained in $F^*(X)\chi = F\mathscr{L}\mathscr{I}^*(X)$ or not? According to Theorem 3.3 and 4.12, for each $(i, g, j) \in I \times G \times I$ there is some $A \in Y$ such that $(i, A, g, j) \in F\mathscr{L}\mathscr{I}^*(X)$. Hence the question may be formulated as follows. Given $A \in Y$, $(i, g, j) \in I \times G \times I$; is it true or not that $(i, A, g, j) \in F\mathscr{L}\mathscr{I}^*(X)$? For given $i \in I$, $g = q_1 \ldots q_k \in G$ (in reduced form) let $w(i, g) = \mathrm{s}(w_k(i, g))$ where $w_k(i, g)$ is as in Definition 4.

Definition 6. Let $i, j \in I$, $g \in G$. The element $w(i, g, j)$ will be defined by

$$w(i,g,j) = \begin{cases} w(i,g) & \text{if } w(i,g)\lambda = j^*, \\ w(i,g)jj^* & \text{if } w(i,g)\lambda \neq j^* \text{ and either } z = j \text{ or } w(i,g)\lambda = z^*, \\ w(i,g)zz^*jj^* & \text{if } w(i,g)\lambda \neq j^* \neq z^* \text{ and } w(i,g)\lambda \neq z^*. \end{cases}$$

By Theorem 4.10, $w(i, g, j)$ is the uniquely determined (weakly reduced) word $w \in \mathrm{s}\,F^*(X)$ such that $w\varphi = (\varrho w, \mathbf{r}\,\mathrm{e}_z\, w, (w\lambda)^*) = (i, g, j)$. Recall that for a given word $a = x_1 \ldots x_n \in F^*(X)$, $\hat{\mathrm{s}}\, a = \{\mathrm{s}(x_1 \ldots x_i), \mathrm{s}(x_1 \ldots x_i x_i^*) \mid 1 \leqslant i \leqslant n\}$. By Theorem 4.12 we have that $(i, A, g, j) \in F^*(X)\chi = F\mathscr{L}\mathscr{I}^*(X)$ if and only if there is some $a = x_1 \ldots x_n \in F^*(X)$ such that

(1) $\varrho a = i$,
(2) $\mathbf{r}\,\mathrm{e}_z\, \hat{\mathrm{s}}\, a = A$,
(3) $\mathbf{r}\,\mathrm{e}_z\, a = g$,
(4) $(a\lambda)^* = j$.

We formulate the following criterion.

Theorem 5.1. *Let $(i, A, g, j) \in I \times Y \times G \times I$. Then $(i, A, g, j) \in F^*(X)\chi$ if and only if*

(1) $\mathbf{r}\,\mathrm{e}_z\, \hat{\mathrm{s}}\, w(i, g, j) \subseteq A$,
(2) $\mathbf{r}\,\mathrm{e}_z\, \hat{\mathrm{s}}\, w(i, h) \subseteq A$ *for all* $h \in A$.

Proof. Suppose that $(i, A, g, j) \in F^*(X)\chi$. Then there is $a = x_1 \dots x_n \in F^*(X)$ such that

$$(i, A, g, j) = a\chi = (\varrho a, \mathbf{r}\,\mathbf{e}_z\, \hat{\mathrm{s}}\, a, \mathbf{r}\,\mathbf{e}_z\, a, (a\lambda)^*).$$

Since $a\varphi = (\varrho a, \mathbf{r}\,\mathbf{e}_z\, a, (a\lambda)^*) = (i, g, j)$ we have $\mathrm{s}\, a = w(i, g, j)$. Let b be an initial segment of $w(i, g, j)$. Then there is an initial segment a' of a such that $\mathrm{s}\, a' = b = \mathrm{s}\, b$ (Lemma 2.1). Then also

$$\mathrm{s}(a'(a'\lambda)^*) = \mathrm{s}((\mathrm{s}\, a')((\mathrm{s}\, a')\lambda)^*) = \mathrm{s}(b(b\lambda)^*) = \mathrm{s}((\mathrm{s}\, b)((\mathrm{s}\, b)\lambda^*)).$$

Consequently, $\hat{\mathrm{s}}\, w(i, g, j) \subseteq \hat{\mathrm{s}}\, a$ and thus $\mathbf{r}\,\mathbf{e}_z\, \hat{\mathrm{s}}\, w(i, g, j) \subseteq \mathbf{r}\,\mathbf{e}_z\, \hat{\mathrm{s}}\, a = A$ showing (1). (2) will be shown by a similar argument. Let $h \in A$. Then $h \in \mathbf{r}\,\mathbf{e}_z\, \hat{\mathrm{s}}\, a$. That is, $h = \mathbf{r}\,\mathbf{e}_z\, \mathrm{s}(x_1 \dots x_l)$ or $h = \mathbf{r}\,\mathbf{e}_z\, \mathrm{s}(x_1 \dots x_l x_l^*)$ for some $l \leqslant n$. Suppose that $h = \mathbf{r}\,\mathbf{e}_z\, \mathrm{s}(x_1 \dots x_l)$. By Theorem 4.10, $w(i, h)$ is an initial segment of $\mathrm{s}(x_1 \dots x_l)$ (or coincides with $\mathrm{s}(x_1 \dots x_l)$). Each initial segment $b = \mathrm{s}\, b$ of $w(i, h)$ is of the form $b = \mathrm{s}(x_1 \dots x_{l'}) \in \hat{\mathrm{s}}\, a$ for some $l' \leqslant l$ (Lemma 2.1). Furthermore, $\mathrm{s}(b(b\lambda)^*) = \mathrm{s}(x_1 \dots x_{l'} x_{l'}^*) \in \hat{\mathrm{s}}\, a$. In particular, $\hat{\mathrm{s}}\, w(i, h) \subseteq \hat{\mathrm{s}}\, a$. If $h = \mathbf{r}\,\mathbf{e}_z\, \mathrm{s}(x_1 \dots x_l x_l^*)$ then a similar argument applies. In any case we have thus shown the direct part. To prove the converse suppose that (1) and (2) hold for a given $(i, A, g, j) \in I \times Y \times G \times I$. Consider the element

$$a = \prod_{h \in A} [w(i, h) w(i, h)^*] w(i, g, j).$$

Notice that all idempotents $w(i, h)w(i, h)^*$ commute. It is clear that $\varrho a = \varrho w(i, h) = i$ for each $h \in A$, $a\lambda = w(i, g, j) = j^*$ and $\mathbf{r}\,\mathbf{e}_z\, a = \mathbf{r}\,\mathbf{e}_z\, \mathrm{s}\, a = \mathbf{r}\,\mathbf{e}_z\, w(i, g, j) = g$. Let the elements of A be indexed in some way: $A = \{h_1, \dots, h_q\}$. Taking into account that $\varrho w(i, h_l) = i = \varrho w(i, g, j)$ for all $h_l \in A$ we have the following

$$a = (i a_{11} \dots a_{1m_1} a_{1m_1}^* \dots a_{11}^* i^*) \dots (i a_{l1} \dots a_{lm_l} a_{lm_l}^* \dots a_{l1}^* i^*) \dots i a_1 \dots a_k$$

where $w(i, h_l) = i a_{l1} \dots a_{lm_l}$ and $w(i, g, j) = i a_1 \dots a_k$. Consider any initial segment b of a. Then $\mathrm{s}\, b$ is one of the following:

$$\mathrm{s}\, b \in \{i a_{l1} \dots a_{lk_l}, i a_{l1} \dots a_{lk_l} a_{lk_l}^*, i a_1 \dots a_l\}$$

where $0 \leqslant k_l \leqslant m_l$ and $0 \leqslant l \leqslant k$ (here $k_l = 0$ means $\mathrm{s}\, b = i$ or $\mathrm{s}\, b = ii^*$ and $l = 0$ means $\mathrm{s}\, b = i$). Consequently, $\mathrm{s}(b(b\lambda^*))$ is one of the following:

$$\mathrm{s}(b(b\lambda)^*) \in \{i a_{l1} \dots a_{lk_l} a_{lk_l}^*, i a_{l1} \dots a_{lk_l}, i a_1 \dots a_l a_l^*\}$$

(provided the same convention on k_l and l). In any case we have $\mathbf{s}\,b$, $\mathbf{s}(b(b\lambda)^*) \in \hat{\mathbf{s}}\,w(i,h_l)$ for some $h_l \in S$ or $\mathbf{s}\,b$, $\mathbf{s}(b(b\lambda)^*) \in \hat{\mathbf{s}}\,w(i,g,j)$. By conditions (1) and (2) it follows that $\mathbf{re}_z\,\hat{\mathbf{s}}\,a \subseteq A$. On the other hand, for $h_l \in A$ we have

$$\mathbf{s}\,ia_{11}\dots a_{1m_1}a^*_{1m_1}\dots i^*\dots ia_{l1}\dots a_{l1}\dots a_{lm_l} = ia_{l1}\dots a_{lm_l} = w(i,h_l) \in \hat{\mathbf{s}}\,a.$$

By Lemma 4.8 also $\mathbf{re}_z\,w(i,h_l) = h_l$ and thus $h_l = \mathbf{re}_z\,w(i,h_l) \in \mathbf{re}_z\,\hat{\mathbf{s}}\,a$. The element $h_l \in A$ is arbitrarily chosen so that $A \subseteq \mathbf{re}_z\,\hat{\mathbf{s}}\,a$ and thus $A = \mathbf{re}_z\,\hat{\mathbf{s}}\,a$. Summarizing the converse part we have shown that

$$(i,A,g,j) = (\varrho a, \mathbf{re}_z\,\hat{\mathbf{s}}\,a, \mathbf{re}_z\,a, (a\lambda)^*) \in F^*(X)\chi = F\mathscr{L}\mathscr{I}^*(X).$$

□

The next results concern idempotents and the natural partial order in $F^*(X)\chi = F\mathscr{L}\mathscr{I}^*(X)$.

Lemma 5.2. *Let $(i,A,g,j) \in I \times Y \times G \times I$. Then $(i,A,g,j)^2 = (i,A,g,j)$ if and only if $g = p_{ij}$.*

Proof. We have $(i,A,g,j)(i,A,g,j) = (i, A \cup gp_{ji}A, gp_{ji}g, j)$. Hence (i,A,g,j) is idempotent if and only if $g = gp_{ji}g$ and $A \cup gp_{ji}A = A$. The first condition is equivalent to $g = p_{ji}^{-1}$ and thus $g = p_{ij}$. Conversely, if $g = p_{ij}$ then immediately $(i,A,g,j) \in E(I \times Y \times G \times I)$. □

Corollary 5.3. *Let $w = x_1\dots x_n \in F^*(X)$. Then $w\ \varrho_{\mathscr{L}\mathscr{I}^*}\ w^2$ (that is, w is an idempotent in $F\mathscr{L}\mathscr{I}^*(X)$) if and only if either $\mathbf{s}\,w = x_1x_1^*x_n^*x_n$ or $\mathbf{s}\,w = x_1x_1^* = x_n^*x_n$.*

Proof. We have $w\ \varrho_{\mathscr{L}\mathscr{I}^*}\ w^2$ if and only if $w\chi$ is an idempotent in $F^*(X)\chi$. That is, $w\chi = (i,A,p_{ij},j)$ by Lemma 5.3. The element $\mathbf{s}\,w$ is uniquely determined by the parameters i, p_{ij}, j, namely $\mathbf{s}\,w = w(i,p_{ij},j)$. By Definition 6,

$$w(i,p_{ij},j) = \begin{cases} ii^* = jj^* & \text{if } i = j, \\ ii^*jj^* & \text{if } i \neq j. \end{cases}$$

Since $i = \varrho(\mathbf{s}\,w) = \varrho w = x_1$ and $j = ((\mathbf{s}\,w)\lambda)^* = (w\lambda)^*$ we observe that $\mathbf{s}\,w = x_1x_1^*x_n^*x_n$ or $\mathbf{s}\,w = x_1x_1^* = x_n^*x_n$. Conversely, if $\mathbf{s}\,w = x_1x_1^*x_n^*x_n$ then $w\chi = (x_1, A, p_{x_1x_n^*}, x_n^*)$ and $w\chi$ is idempotent. Similarly, if $\mathbf{s}\,w = x_1x_1^* = x_n^*x_n$ then $w\chi = (x_1, A, 1, x_1) = (x_n^*, A, 1, x_n)$ which is idempotent. □

The natural partial order on a regular semigroup has been introduced by Nambooripad [11]. A list of equivalent definitions is given by Mitsch [10].

Definition 7. Let S be a regular semigroup, $a, b \in S$. Then $a \leqslant b$ if and only if there are idempotents $e, f \in E(S)$ such that $a = eb = bf$.

Lemma 5.4. *Let S be a regular $*$-semigroup. Then $^*: x \mapsto x^*$ is an order automorphism of $(S, \leqslant)$.*

Proof. Let $a \leqslant b$, that is, $a = eb = bf$ for some $e, f \in E(S)$. Then $a^* = b^* e^* = f^* b^*$. Since $e^*, f^* \in E(S)$, $a^* \leqslant b^*$. Since * is self-inverse the assertion follows. □

For locally inverse $*$-semigroups we give a further characterization of $\leqslant$ which is a natural anologon of the well known definition of $\leqslant$ for the inverse case. In [11] Nambooripad has shown that a regular semigroup is locally inverse if and only if $\leqslant$ is compatible with the multiplication.

Proposition 5.5. *Let S be a locally inverse $*$-semigroup. Then $a \leqslant b$ if and only if $a = aa^*b = ba^*a$.*

Proof. If $a \leqslant b$ then $a^* \leqslant b^*$ by Lemma 5.4. Compatibility of $\leqslant$ implies $a \leqslant aa^*b$, $a \leqslant ba^*a$, $a^* \leqslant a^*ab^*$, $a^* \leqslant b^*aa^*, a^*a \leqslant b^*b$ and $aa^* \leqslant bb^*$. Now $a^* \leqslant a^*ab^*$ implies $a^*b \leqslant a^*ab^*b = a^*a$. Hence $aa^*b \leqslant aa^*a$ so that $a = aa^*b$. Similarly, $a^* \leqslant b^*aa^*$ implies $ba^* \leqslant bb^*aa^* = aa^*$. Hence $ba^*a \leqslant a$ so that $a = ba^*a$. The converse is obvious. □

Remark. In the same fashion as for the inverse case several equivalent characterizations of $\leqslant$ in a locally inverse $*$-semigroup can be obtained.

Corollary 5.6. *Let $(i, A, g, j), (k, B, h, l) \in I \times Y \times G \times I$. Then $(i, A, g, j) \leqslant (k, B, h, l)$ if and only if $(i, g, j) = (k, h, l)$ and $B \subseteq A$.*

Proof. A straightforward calculation shows

$$(i, A, g, j)(i, A, g, j)^*(k, B, h, l) = (i, A \cup p_{ik}B, p_{ik}h, l)$$

and

$$(k, B, h, l)(i, A, g, j)^*(i, A, g, j) = (k, B \cup hp_{lj}g^{-1}A, hp_{lj}, j).$$

If $(i, g, j) = (k, h, l)$ and $B \subseteq A$ then immediately from Proposition 5.5. $(i, A, g, j) \leqslant (k, B, h, l)$. Conversely suppose $(i, A, g, j) \leqslant (k, B, h, l)$. By Proposition 5.5, $l = j, k = i, g = p_{ik}h = h$ and $A = A \cup p_{ik}B = A \cup B$ so that $B \subseteq A$. □

Corollary 5.7. *Let $u, v \in F^*(X)$. Then $u\varrho_{\mathcal{L}\mathcal{J}^*} \leqslant v\varrho_{\mathcal{L}\mathcal{J}^*}$ if and only if*

(1) $\mathbf{s}\,u = \mathbf{s}\,v$,

(2) $\hat{\mathbf{s}}\,u \supseteq \hat{\mathbf{s}}\,v$.

Proof. The inequality $u \leqslant v$ holds in $F\mathscr{L}\mathscr{I}^*(X)$ if and only if $u\chi \leqslant v\chi$ in $F^*(X)\chi$. Now

$$u\chi \leqslant v\chi \Leftrightarrow (\varrho u, \mathbf{r}\,\mathbf{e}_z\, u, (u\lambda)^*) = (\varrho v, \mathbf{r}\,\mathbf{e}_z\, v, (v\lambda)^*) \text{ and } \mathbf{r}\,\mathbf{e}_z\, \hat{\mathbf{s}}\, v \subseteq \mathbf{r}\,\mathbf{e}_z\, \hat{\mathbf{s}}\, u.$$

Immediately we thus have that (1) and (2) imply $u\varrho_{\mathscr{L}\mathscr{I}^*} \leqslant v\varrho_{\mathscr{L}\mathscr{I}^*}$. Suppose conversely that $(\varrho u, \mathbf{r}\,\mathbf{e}_z\, u, (u\lambda)^*) = (\varrho v, \mathbf{r}\,\mathbf{e}_z\, v, (v\lambda)^*)$ and $\mathbf{r}\,\mathbf{e}_z\, \hat{\mathbf{s}}\, v \subseteq \mathbf{r}\,\mathbf{e}_z\, \hat{\mathbf{s}}\, u$. First we have $\mathbf{s}\, u = w(\varrho u, \mathbf{r}\,\mathbf{e}_z\, u, (u\lambda)^*) = w(\varrho v, \mathbf{r}\,\mathbf{e}_z\, v, (v\lambda)^*) = \mathbf{s}\, v$. By the process which reconstructs $\hat{\mathbf{s}}\, u$ from ϱu and $\mathbf{r}\,\mathbf{e}_z\, \hat{\mathbf{s}}\, u$ and $\hat{\mathbf{s}}\, v$ from ϱv and $\mathbf{r}\,\mathbf{e}_z\, \hat{\mathbf{s}}\, v$ (see end of section 4) it follows that $\hat{\mathbf{s}}\, v \subseteq \hat{\mathbf{s}}\, u$. □

Definition 8. Let $A \subseteq S$ be a subset of a regular semigroup. Then $A\omega = \{x \in S \mid a \leqslant x \text{ for some } a \in A\}$.

It is well known that the free inverse semigroup $F\mathscr{I}(X)$ is E-unitary (see [17]). This is not true for locally inverse $*$-semigroups as an E-unitary regular semigroup must be orthodox. However, for inverse semigroups S the property of being E-unitary is equivalent to the property that the idempotents form a closed subset of S under the natural order, that is $E\omega = E$. This seems to be the appropriate analogue for the locally inverse case.

Corollary 5.8. *For the free locally inverse $*$-semigroup $F\mathscr{L}\mathscr{I}^*(X)$, $E\omega = E$.*

Proof. Let (i, A, p_{ij}, j), $(k, B, h, l) \in F^*(X)\chi$ such that $(i, A, p_{ij}, j) \leqslant (k, B, h, l)$. By Corollary 5.6, $(i, p_{ij}, j) = (k, h, l)$. Hence by Lemma 5.2, $(k, B, h, l) = (i, B, p_{ij}, j)$ is an idempotent. □

Finally we mention some more properties of the relatively free object $F\mathscr{L}\mathscr{I}^*(X)$. By Nordahl and Scheiblich [13], Green's relations $\mathscr{R}$ and $\mathscr{L}$ on a regular $*$-semigroup admit the following description.

Lemma 5.9. *Let S be a regular $*$-semigroup and $a, b \in S$. Then*

(1) $a \,\mathscr{R}\, b \Leftrightarrow aa^* = bb^*$,
(2) $a \,\mathscr{L}\, b \Leftrightarrow a^*a = b^*b$.

For two elements of the semidirect product $I \times Y \times G \times I$ this yields the following characterization:

(1) $(i, A, g, j) \,\mathscr{R}\, (k, B, h, l) \Leftrightarrow i = k$ and $A = B$,
(2) $(i, A, g, j) \,\mathscr{L}\, (k, B, h, l) \Leftrightarrow j = l$ and $g^{-1}A = h^{-1}B$.

Since for each $w \in F^*(X)$, $(\varrho w, \mathbf{r}\,\mathbf{e}_z\, \hat{\mathbf{s}}\, w)$ is uniquely determined by $\hat{\mathbf{s}}\, w$ and conversely, this leads to the following characterization of Green's relations in $F\mathscr{L}\mathscr{I}^*(X)$.

Propostion 5.10. *Let $v, w \in F^*(X)$. Then*

(1) $v\varrho_{\mathscr{L}\mathscr{I}^*} \,\mathscr{R}\, w\varrho_{\mathscr{L}\mathscr{I}^*} \Leftrightarrow \hat{\mathbf{s}}\, v = \hat{\mathbf{s}}\, w$,
(2) $v\varrho_{\mathscr{L}\mathscr{I}^*} \,\mathscr{L}\, w\varrho_{\mathscr{L}\mathscr{I}^*} \Leftrightarrow \hat{\mathbf{s}}\, v^* = \hat{\mathbf{s}}\, w^*$.

The description of the relation $\mathscr{L}$ also could be formulated directly in terms of v and w. However, for this purpose the dual of the operator $\hat{\mathrm{s}}$ is needed. Using a similar idea as in [17, VIII.1.14] the description of $\mathscr{L}$ respectively $\mathscr{R}$ in $I \times Y \times G \times I$ can be used to show that this semidirect product is combinatorial.

Corollary 5.11. *$F\mathscr{L}\mathscr{I}^*(X)$ is combinatorial.*

Corollary 5.12. *$F\mathscr{L}\mathscr{I}^*(X)$ has finite $\mathscr{R}$- and $\mathscr{L}$-classes. In particular, $F\mathscr{L}\mathscr{I}^*(X)$ is completely semisimple with finite $\mathscr{D}$-classes and is finite-$\mathscr{R}(\mathscr{L}, \mathscr{D})$-above.*

Proof. Let $v \in F^*(X)$. Then $v\varrho_{\mathscr{L}\mathscr{I}^*}\mathscr{R}$ is determined by $\hat{\mathrm{s}}\, v$. But $v\varrho_{\mathscr{L}\mathscr{I}^*}$ is determined by $(\mathrm{s}\, v, \hat{\mathrm{s}}\, v)$ and $\mathrm{s}\, v \in \hat{\mathrm{s}}\, v$. Since $\hat{\mathrm{s}}\, v$ is finite, the $\mathscr{R}$-class of $v\varrho_{\mathscr{L}\mathscr{I}^*}$ is finite for any v. The mapping $x \mapsto x^*$ induces a bijection between R_x and L_{x^*}. Hence each $\mathscr{L}$-class of $F\mathscr{L}\mathscr{I}^*(X)$ is finite. But then each $\mathscr{D}$-class is finite and $F\mathscr{L}\mathscr{I}^*(X)$ is completely semisimple. A similar argument proves the ascending chain condition for $F\mathscr{L}\mathscr{I}^*(X)/\mathscr{R}$ respectively $F\mathscr{L}\mathscr{I}^*(X)/\mathscr{L}$. □

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