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Free Products of Compact Quantum Groups

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Abstract. We construct and study compact quantum groups from free products of C^*-algebras. In this connection, we discover two mysterious classes of natural compact quantum groups, $A_u(m)$ and $A_o(m)$. The $A_u(m)$'s (respectively $A_o(m)$'s) are non-isomorphic to each other for different m's, and are not obtainable by the ordinary quantization method. We also clarify some basic concepts in the theory of compact quantum groups.

1. Introduction

In this paper, we give general constructions of quantum groups from free products of C^*-algebras. Surprisingly, two mysterious classes of compact matrix quantum groups, $A_u(m)$ and $A_o(m)$, naturally arise in this connection. The quantum groups constructed in this paper are of very different nature from the ones constructed by Woronowicz [50, 52], and the ones studied in [41, 38, 35, 18, 19] and [16] based respectively on the construction of Drinfeld and Jimbo [5, 6, 10] and the construction of Manin [22, 21] – the quantum groups in this paper are not obtainable by the quantization method.

The origin of quantum groups goes back at least to the early sixties. They were called ring groups by Kac when he used them to generalize the Pontryagin duality to locally compact groups [11]. The ring groups are also called Kac algebras. These are certain Hopf von Neumann algebras with a very beautiful theory (see [14, 7, 8]). But there are very few non-trivial known examples of Kac algebras that are not groups (see [12, 13, 20]), their constructions are highly technical. The first nontrivial example of these were constructed about thirty years ago by quantization of the Heisenberg Lie group [12]. This example was studied more recently at the C^*-algebra level independently by Rieffel [30] and Van Daele [42].

Recently, motivated by the work of the Faddeev school on the quantum inverse scattering method (QISM), Drinfeld and Jimbo [5, 6, 10] constructed a remarkable class of Hopf algebras from the universal enveloping algebras of semisimple Lie algebras using the method of quantization, thus realizing the ideas proposed by Kac and Palyutkin [12]. These examples have since been intensively studied and have surprising applications in low dimension topology and physics. The approach of Drinfeld and Jimbo can be termed the approach of quantum infinitesimal groups. This was soon followed by the approach of Faddeev and Reshetikhin and Takhtajan [9], in which the authors introduced the Hopf algebras of "algebraic functions on simple quantum Lie groups" by using the main RTT algebraic relations in QISM. Manin developed an approach to quantum groups based on the general study of quadratic algebras [22, 21], in which the quantum groups are described as symmetry objects of quantum spaces. The two latter approaches are dual to that of Drinfeld and Jimbo in a certain sense, and may be termed as the approach of quantum algebraic (Lie) groups. A defect with these three purely algebraic approaches is that there is no *axiomatic* definition of a quantum group.

On the other hand, motivated by the theory of non-commutative differential geometry of Connes [4], Woronowicz at the same time (in fact, he started the program in 1979 [49]) independently constructed the first examples of non-commutative differential geometric groups [50] and laid down the foundation for compact matrix quantum groups [51] via an axiomatic approach, using the C^*-algebra language, which is more appropriate than the von Neumann algebra language used in Kac algebras. The quantum groups constructed by Woronowicz are known to be essentially the same as those constructed by Drinfeld and Jimbo [34, 35, 41]. The representation theory of compact quantum groups gives satisfactory interpretation of the long mystery of the q-analogues of certain special functions [41, 17, 23]. But there are also a few defects with Woronowicz's approach as developed in [51]. One of these is that it was not clear how to define morphisms from one compact matrix quantum group to another ([51] p. 617) so that the compact matrix quantum groups would form a category such that the category of compact matrix groups sits therein as a full subcategory. Another defect with [51] is that there was no definition of compact quantum groups such that the compact groups, in addition to the compact matrix quantum groups, are examples of them. (But see [54].)

More recently, generalizing the Kac–Takesaki unitary operators for locally compact groups, Baaj and Skandalis [2, 37] axiomatized the notion of multiplicative unitary operators and developed a very beautiful and deep theory for them: the points of view of Kac and Woronowicz are unified in their approach. They introduced the so-called Woronowicz C^*-algebras and proved that Woronowicz C^*-algebras are essentially the same as the compact-type multiplicative unitaries. As pointed out in the introduction of their paper, a multiplicative unitary should be viewed as a quantum group. But the morphisms between multiplicative unitaries are not defined in [2, 37], and so the multiplicative unitaries are not made into a category either. Therefore Theorem 2.2 of their paper [2] does not give a true Gelfand–Naimark type duality, due to this defect and the multiplicity occurring in the theorem.

The purposes of this paper are to clarify some of the basic concepts in the theory of compact quantum groups mentioned above and to give general constructions of compact quantum groups of a different nature from ones constructed so far. We now summarize the main results of this paper.

Section 2 of this paper is devoted to clarification of the following concepts in the theory of compact quantum groups: compact quantum groups as opposed to compact matrix quantum groups [51] (see also [54] for this), morphisms between compact quantum groups, quantum subgroups, normal quantum subgroups and quotient quantum groups. The main justifications for our definitions of these concepts are that they *coincide* with the corresponding ones in group theory when they are applied to compact groups, and that there is a nontrivial beautiful theory for the more general case of quantum groups.

The main result in Sect. 3 is that the free product of compact quantum groups (i.e. Woronowicz C^*-algebras) is again a compact quantum group. This construction of compact quantum groups is the fundamental one for the other constructions considered in this paper and elsewhere (see [45, 46]). It is stated more precisely as follows (see Theorems 3.4, 3.8, 3.10).

1.1. Theorem. *Let A and B be Woronowicz C^*-algebras. Then there is a unique structure of a Woronowicz C^*-algebra on the free product $A * B$ such that A and B sit therein as Woronowicz C^*-subalgebras, where the free product C^*-algebra $A * B$ is amalgamated over the complex numbers.*

Let h_A and h_B be the normalised Haar measures on the quantum groups of A and B respectively. Denote by $\{u^\gamma\}$ and $\{v^\beta\}$ the complete sets of irreducible representations of the quantum groups of A and B respectively. Then the Haar measure for the quantum group of $A * B$ is the free product $h_A * h_B$, and a complete set of mutually inequivalent irreducible representations of the quantum group of $A * B$ is given by the set consisting of the trivial representation together with the collection of interior tensor product representations of the form

$$w^{\gamma_1} \otimes_{in} w^{\gamma_2} \otimes_{in} \cdots \otimes_{in} w^{\gamma_n} ,$$

where w^{γ_i} is a nontrivial representation belonging to either $\{u^\gamma\}$ or $\{v^\beta\}$, and w^{γ_i} and $w^{\gamma_{i+1}}$ are in different sets.

The universal noncommutative unitary algebras $U_{nc}(m)$ of Brown [3] are defined in connection with the study of free products of C^*-algebras. There is a natural coproduct on them. However, this coproduct does not define the structure of a compact quantum group (see Sect. 4.1). Surprisingly, we obtain two infinite classes of nontrivial compact matrix quantum groups from these algebras if we impose the appropriate additional commutation relations (see Sect. 4.2 and 4.5):

1.2. Theorem. *Let m be a natural number greater than one. Let $A_u(m)$ be the universal C^*-algebra generated by m^2 elements a_{ij} subject to the relations*

$$\sum_{k=1}^{m} a_{ik}a_{jk}^* = \delta_{ij}, \quad \sum_{k=1}^{m} a_{ki}^* a_{kj} = \delta_{ij},$$

$$\sum_{k=1}^{m} a_{ki}a_{kj}^* = \delta_{ij}, \quad \sum_{k=1}^{m} a_{ik}^* a_{jk} = \delta_{ij}$$

for $i,j = 1,\ldots,m$. Then $A_u(m)$ is a compact matrix quantum group. For $m \neq n$, $A_u(m)$ and $A_u(n)$ are non-isomorphic to each other as C^*-algebras, and therefore non-isomorphic to each other as quantum groups.

Similarly, let $A_o(m)$ be the universal C^*-algebra generated by m^2 elements a_{ij} subject to the relations

$$\sum_{k=1}^{m} a_{ik}a_{jk} = \delta_{ij}, \quad \sum_{k=1}^{m} a_{ki}a_{kj} = \delta_{ij},$$

$$a_{ij}^* = a_{ij}$$

for $i,j = 1,\ldots,m$. Then $A_o(m)$ is a compact matrix quantum group. For $m \neq n$, $A_o(m)$ and $A_o(n)$ are non-isomorphic to each other as C^*-algebras, and therefore *non-isomorphic to each other as quantum groups. Furthermore, the quantum group* $A_o(m)$ *is a quantum subgroup of* $A_u(m)$.

The proof of the above theorem uses the free products of C^*-algebras. The quantum groups $A_u(m)$ and $A_o(m)$ are the universal analogues of the unitary and orthogonal groups respectively, and seem to be the first infinite quantum groups not obtainable by the ordinary quantization method.

Further applications of the free product construction of quantum groups are considered in our paper [46], in which we study compact quantum groups from the maximal and minimal tensor products and crossed products of C^*-algebras.

We remark that the obvious purely algebraic versions of Theorems 1.1. *and* 1.2 *are still valid, which to the best of our knowledge have not appeared in the algebra literature.*

This paper is culled from the first chapter and Sect. 2 of the second chapter of the author's thesis [45] submitted to the University of California at Berkeley in partial fulfillment for the Ph.D. degree. It is a pleasure of the author to record here his thanks to E. Effros, for catching a blunder of the author concerning Brown's universal non-commutative unitary algebras in an early version of this paper that lead the author to a more careful investigation of the related matters; N.C Phillips, for some fruitful conversations and communications; S. L. Woronowicz, for communicating to him a unitary matrix that is used in this paper to prove the non-existence of Woronowicz C^*-algebra structure on the universal non-commutative unitary algebras, whose fundamental papers on the theory of quantum groups have inspired the author tremendously. Most of all, the author would like to express his deep gratitude to his supervisor Professor Marc Rieffel, for suggesting that he work on the subject of quantum groups, and for constant mathematical and moral support throughout the preparation of this work.

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2. The Category of Compact Quantum Groups

This section serves to clarify some basic concepts in the theory of compact quantum groups and to fix some notation. The reader is cautioned that some of the corresponding concepts defined in the purely algebraic context (e.g. [25]) are inappropriate for compact quantum groups. We omit detailed proofs of the results in this section (except Theorem 2.11), for which the reader is referred to the author's dissertation [45].

For every natural number d, and any $*$-algebra A, $M_d(A)$ denotes the $*$-algebra $M_d \otimes A$ of $d \times d$ matrices with entries in A. We use spatial a C^*-tensor product for all C^*-algebras considered in this paper unless otherwise specified.

2.1. Definition. cf. [51, 2]) *A* **Woronowicz** C^***-algebra** *is a unital* C^**-algebra* A *together with a dense* $*$*-subalgebra* $\mathscr{A}$ *generated by* u_{ij}^{α} *(where* $\alpha \in N$ *and* $i,j \in \{1,\dots,d_\alpha\}$*, and* N *is an index set), a* C^**-homomorphism* $\Phi : A \longrightarrow A \otimes A$*, and a linear algebra-antihomomorphism* $\kappa : \mathscr{A} \longrightarrow \mathscr{A}$*, such that,*

(1) *The matrix* $u^\alpha = (u_{ij}^\alpha)$ *is a unitary element of* $M_{d_\alpha} \otimes A$*, for all* $\alpha \in N$;

(2) *For* $\alpha \in N$*, and* $i,j \in \{1,\dots,d_\alpha\}$, $\Phi(u_{ij}^\alpha) = \sum_{k=1}^{d_\alpha} u_{ik}^\alpha \otimes u_{kj}^\alpha$;

(3) *For* $\alpha \in \mathscr{A}, \kappa(\kappa(a^*)^*) = a$*, and for* $\alpha \in N, (id \otimes \kappa)(u^\alpha) = (u^\alpha)^{-1}$.

We denote the above Woronowicz C^*-algebra by $(A, \mathscr{A}, \Phi, \kappa)$ or (A, Φ), or simply A. A Woronowicz C^*-algebra A is called **commutative** if its underlying C^*-algebra is commutative; it is called **cocommutative** if $\sigma\Phi = \Phi$, where σ is the flip map on $A \otimes A$ sending $x \otimes y$ to $y \otimes x$ for $x, y \in A$.

Modifying the definition in [53], we have the following equivalent Definition 2.1′.

2.1′. Definition. *A* **Woronowicz C^*-algebra** *is a unital C^*-algebra A together with a dense *-subalgebra $\mathscr{A}$ generated by u_{ij}^{a} (where $\alpha \in N$ and $i, j \in \{1, \ldots, d_\alpha\}$, and N is an index set), a C^*-homomorphism $\Phi : A \longrightarrow A \otimes A$, such that,*

(1′) *The matrix $u^\alpha = (u_{ij}^\alpha)$ is a unitary element of $M_{d_\alpha} \otimes A$, for all $\alpha \in N$;*

(2′) *For $\alpha \in N$, and $i, j \in \{1, \ldots, d_\alpha\}, \Phi(u_{ij}^\alpha) = \sum_{k=1}^{d_\alpha} u_{ik}^\alpha \otimes u_{kj}^\alpha$;*

(3′) *For $\alpha \in N$, the transpose $(u^\alpha)^T$ is invertible.*

Recently, we received a preprint of Woronowicz [54], in which it is shown that Definition 2.1 is also equivalent to the following Definition 2.1″.

2.1″. Definition. *A* **Woronowicz C^*-algebra** *is a unital C^*-algebra A together with a C^*-homomorphism $\Phi : A \longrightarrow A \otimes A$, such that,*

(1″) *The map Φ is coassociative, namely, $(id \otimes \Phi)\Phi = (\Phi \otimes id)\Phi$;*

(2″) *The linear subsets*

$$\{\textstyle\sum(b_i \otimes I)\Phi(c_i) | b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n \in A\}$$

and

$$\{\textstyle\sum(I \otimes b_i)\Phi(c_i) | b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n \in A\}$$

are both dense in $A \otimes A$.

2.2. Remarks.(1) The terminology "Woronowicz C^*-algebra" is introduced in [2] to honor Woronowicz for *his* fundamental contributions to the theory of compact quantum groups [50-52]. For a Woronowicz C^*-algebra A defined above, it can be proved that the canonical dense $*$-subalgebra $\mathscr{A}$ is a Hopf $*$-algebra, that there is a unique Haar state (or measure) [54, 43], and that Woronowicz's Peter–Weyl theory (see Sect. 4 and 5 of [51]) is still valid, as in the case of a compact matrix quantum group [51]. We will use these results in the present paper without explanation.

(2) Note that Woronowicz [54] assumes the C^*-algebra A in Definition 2.1″ to be separable. The separability condition is removed in an ingenious recent paper of Van Daele [43].

(3) If we change the requirement that (u_{ij}^α) be unitary matrices to the requirement that (u_{ij}^α) be invertible matrices in Definition 2.1, we obtain an equivalent definition. (One may see this from [51].)

2.3. Definition. *A* **morphism** *from a Woronowicz C^*-algebra A_1 to another A_2 is a unital C^*-morphism $\pi : A_1 \longrightarrow A_2$ such that*

$$(\pi \otimes \pi)\Phi_1 = \Phi_2\pi \, .$$

The Woronowicz C^*-algebras form a category under these morphisms.

The category of **compact quantum groups** is defined to be the dual category of the category of Woronowicz C^*-algebras. For each Woronowicz C^*-algebra. A, the corresponding compact quantum group will be called **the compact quantum group of** A, and will be denoted by G_A (we will also call A a compact quantum group, referring to the object G_A). Conversely, if G is a compact quantum group, the corresponding Woronowicz C^*-algebra will be called **the Woronowicz C^*-algebra of** G, and will be denoted by A_G.

A **compact matrix quantum group** [51] is defined to be the dual object of a Woronowicz C^*-algebra $(A, \mathscr{A}, \Phi, \kappa)$ such that the set N (see Definition 2.1) is a singleton. A compact quantum group will also be denoted (A, u), as in [51]. The **morphisms between compact matrix quantum groups** are defined to be the morphisms coming from the underlying Woronowicz C^*-algebras. It is clear that compact matrix quantum groups form a category under these morphisms.

A compact quantum group G_A is said to be **connected** if the center of the C^*-algebra A has no projections other than zero and the identity; it is called **extremely connected** if the C^*-algebra A is noncommutative and has no projections other than zero and the identity; it is called **finite** if A is finite dimensional. Note that a finite quantum group G_A whose algebra A is commutative is precisely a finite group. (Use Theorem 2.6 to see these!)

2.4. Definition *Let A be a Woronowicz C^*-algebra, H a Hilbert space (not necessarily finite-dimensional) and $\mathscr{K} = \mathscr{K}(H)$ the C^*-algebra of compact operators on H. An invertible element υ of the multiplier C^*-algebra $M(\mathscr{K} \otimes A)$ is called a* **representation** *of G_A (or a* **corepresentation** *of A) if*

$$(id \otimes \Phi)\upsilon = \upsilon_{12}\upsilon_{13} ,$$

where υ_{12} and υ_{13} are the leg numbering notation of [27] *(see* p.385 *there). A representation υ is called* **unitary** *if υ is an unitary element of $M(\mathscr{K} \otimes A)$. One may also define the notions of subrepresentations, direct sum of representations, interior tensor product representations, and irreducible representations.*

Part (2) of the following proposition means that C^*-algebra morphisms between Woronowicz C^*-algebras preserving the coproducts automatically preserve the counits and antipodes (coinverses), which is not true for general Hopf algebras. Geometrically, it says that homomorphisms of (compact) quantum groups preserve the identities and inverses.

2.5. Proposition. *Let $\pi : A \longrightarrow B$ be a morphism of Woronowicz C^*-algebras. Then we have*

(1) *For every unitary representation (u, H) of G_A, $(id \otimes \pi)u$ is a unitary representation of G_B.*

(2) *π preserves the Hopf $*$-algebra structures. Namely, we have*

$$\pi(\mathscr{A}) \subseteq \mathscr{B},\ \pi\kappa_A = \kappa_B\pi,\ e_B\pi = e_A ,$$

where for instance, e_A means the counit on $\mathscr{A}$.

The next theorem is the analogue of Gelfand–Naimark duality for commutative Woronowicz C^*-algebras.

2.6. Theorem. *Let G be a compact group. Let $A = C(G)$ be the algebra of continuous functions on G, and let $\mathscr{A} = \mathscr{A}(G)$ be the algebra of representative functions on G generated by the coefficients of a complete set $\hat{G} = \{(u_{ij}^{\alpha})\}$ of irreducible unitary representations. Define Φ by*

$$\Phi(a)(s,t) = a(st)$$

for all $a \in C(G)$. *This defines a Woronowicz* C^**-algebra structure on* A, *denoted by* $W(G)$.

Conversely, let A *be a commutative Woronowicz* C^**-algebra endowed with a coproduct* Φ. *Denoted by* $H(A)$ *the Gelfand spectrum of* A *with the following product structure:*

$$\chi\chi' = (\chi \otimes \chi')\Phi, \ \chi, \chi' \in H(A) .$$

Then $H(A)$ *is a compact group.*

Furthermore, W *and* H *defined above are duality functors inverse to each other.*

The following is the quantum analogue of the solution of the Hilbert 5^{th} problem for compact Lie groups [24].

2.7. Theorem. *Let* $G = G_A$ *be a compact quantum group. Then the following are equivalent:*

(1) *The quantum group* G *is isomorphic to a compact matrix quantum group;*

(2) *The algebra* $\mathscr{A}$ *is finitely generated;*

(3) *The quantum group* G *has a faithful finite dimensional representation. Therefore, the category of compact matrix quantum groups is antiequivalent to the category of finitely generated Woronowicz* C^**-algebras.*

In view of Theorems 2.6 and 2.7, we have the following result.

2.8. Theorem. *The category of compact Lie groups is antiequivalent to the category of finitely generated commutative Woronowicz* C^**-algebras. Therefore the category of compact matrix quantum groups includes the category of compact Lie groups as a full subcategory.*

2.9. Definition. *A* **Woronowicz** C^***-ideal** *of a Woronowicz* C^**-algebra* A *is a* C^**-ideal* J *of* A *such that* $\Phi(J) \subseteq ker(\pi \otimes \pi)$, *where* π *is the quotient map from* A *to* A/J.

Note that for any closed ideal J of a C^*-algebra A, We have

$$J \otimes A + A \otimes J \subseteq ker(\pi \otimes \pi) .$$

A **Woronowicz** C^***-subalgebra of** $(A, \mathscr{A}, \Phi, \kappa)$ is defined to be a Woronowicz Hopf C^*-algebra $(B, \mathscr{B}, \Phi', \kappa')$ together with an injective morphism $B \to A$ of Woronowicz C^*-algebras. Note that A and the scalars are both Woronowicz C^*-subalgebras of A, which will be called the **trivial** Woronowicz C^*-subalgebras of A.

2.10. Remarks. Recall that a Hopf algebra ideal I of a Hopf algebra $(A, m, \Delta, i, \varepsilon, S)$ is an ideal of the underlying algebra A that satisfies the additional conditions

$$\Delta(I) \subseteq I \otimes A + A \otimes I, \quad S(I) \subseteq I, \quad \varepsilon(I) = 0 .$$

However as a bonus of the axioms of Woronowicz C^*-algebras, we have automatically $\kappa(J) \subseteq J$ and $e(J) = 0$ for a Woronowicz C^*-ideal J of any finite dimensional Woronowicz C^*-algebra A, where e and κ are respectively the counit and antipode of the Woronowicz C^*-algebra A.

2.11. Theorem. (*The Fundamental Isomorphism Theorem*)

(1) *The quotient of a Woronowicz* C^**-algebra by a Woronowicz* C^**-ideal has a unique Woronowicz* C^**-algebra structure such that the quotient map is a morphism of Woronowicz* C^**-algebras.*

(2) *For every morphism $\theta : A \longrightarrow B$ of Woronowicz C^*-algebras, the kernel of θ is a Woronowicz C^*-ideal. The image of θ is a Woronowicz C^*-algebra isomorphic to the quotient Woronowicz C^*-algebra $A/ker(\theta)$ as defined in* (1). *Furthermore, this image is a Woronowicz C^*-subalgebra of B.*

(3) *Let θ be the morphism in* (2) *above. If $J \subset ker(\theta)$, then there is a unique morphism of Woronowicz C^*-algebras $\tilde{\theta} : A/J \to B$ such that $\tilde{\theta}\pi = \theta$, where π is the quotient map from A to A/J.*

Proof. (1) Let $\tilde{A}$ be the quotient A/J. Define $\Phi_{\tilde{A}}$ by

$$\Phi_{\tilde{A}}(\tilde{a}) = (\pi \otimes \pi)\Phi(a) ,$$

where $\tilde{a} = \pi(a)$ for $a \in A$. It can be checked that $\Phi_{\tilde{A}}$ is well-defined. Let u^α be as in Definition 2.1$'$. Put $\tilde{u}^\alpha_{ij} = \pi(u^\alpha_{ij})$. One may check the conditions (1$'$)–(3$'$) of Definition 2.1$'$ for $\tilde{A}$. Thus $\tilde{A}$ is a Woronowicz C^*-algebra. The fact that π is a morphism of Woronowicz C^*-algebras is immediate from the definition of the Woronowicz C^*-algebra structure on $\tilde{A}$.

The uniqueness part is clear.

(2) Let $J = ker(\theta)$. Define an isomorphism of C^*-algebras ρ from A/J to $\theta(A)$ by

$$\rho(\tilde{a}) = \theta(a)$$

for $\tilde{a} \in \tilde{A}$. This is the same thing as $\rho\pi = \theta$. Under this isomorphism, the quotient map $\pi : A \longrightarrow \tilde{A}$ is identified with the map $\theta : A \longrightarrow \theta(A)$. At this point we still do not know yet whether J is a Woronowicz C^*-ideal or not. But the identification ρ helps us to see this. Since θ is a morphism of Woronowicz C^*-algebras, we have

$$(\theta \otimes \theta)\Phi_A(a) = \Phi_B\theta(a) .$$

Thus we have

$$(\pi \otimes \pi)\Phi_A(a) = (\rho^{-1} \otimes \rho^{-1})\Phi_B\theta(a) .$$

In particular, if $c \in J$, then $\Phi_A(c) \in ker(\pi \otimes \pi)$. From these, together with (1), we see that the first two statements of (2) are true and the isomorphism ρ is an isomorphism of Woronowicz C^*-algebras from $\tilde{A}$ onto $\theta(A)$. The above also shows that the coproduct on $\tilde{A}$ is given by $\Phi_{\tilde{A}} = (\rho^{-1} \otimes \rho^{-1})\Phi_B\rho$.

We show the last statement of (2). The above shows that the image $\theta(A)$ is a Woronowicz C^*-algebra if we restrict the coproduct Φ_B to it. Let $\imath$ be the natural injection from $\theta(A)$ to B. Then we see that $\theta(A)$ is indeed a Woronowicz C^*-subalgebra of B.

(3) Straightforward. Q.E.D.

The following result is immediate from 2.11 and 2.6.

2.12. Proposition. *Let G be a compact group. Then we have the following natural one to one correspondences:*

(1) *The closed subgroups of G correspond to the quotients of $W(G)$ by its Woronowicz C^*-ideals;*

(2) *The quotient groups of G by its closed normal subgroups correspond to the Woronowicz C^*-subalgebras of $W(G)$.*

The following definition is natural in view of the theorem above.

2.13. Definition. *A compact quantum G' is called a* **quantum subgroup** *of another compact quantum group G_A if there is a Woronowicz C^*-ideal J of A such that*

G' is the quantum group of A/J. If there is a surjective morphism of Woronowicz C^*-algebras $A \longrightarrow B$, then we say G_B is an **embedded quantum subgroup** *of G_A.*

An embedded quantum subgroup of a quantum group G is isomorphic to a quantum subgroup of G. We use the terminology quantum subgroup to mean that it is a "subset" of the (quantum) space G. If no confusions arise, we do not distinguish these two concepts.

Recently Podles [26] also formulates a notion of quantum subgroups of compact *matrix* quantum groups (see his Definition 1.3). It is easy to see that his definition coincides with our notion of **embedded quantum subgroups** if we specialize to compact matrix quantum groups, even though he imposes condition on the "sizes" of the compact matrix quantum groups in the definition.

Let N be a quantum subgroup of some compact quantum group $G = G_A$, with $\theta : A \longrightarrow A_N$ the quotient map. We say that N is a **normal quantum subgroup** of G if for every irreducible representation υ of G, the multiplicity of the trivial representation of N in the representation $(id \otimes \theta)\upsilon$ is either zero or the dimension of υ. Let N be normal in G_A. Let $N\backslash G$ be the right quotient space [26] defined via the C^*-subalgebra

$$C(N\backslash G) = \{x \in A : (\theta \otimes id)\Phi_A(x) = I \otimes x\} .$$

Then by 2.1′ and the remarks preceding Theorem 1.7 in [26] we see that $N\backslash G$ is also a compact quantum group (namely, $C(N\backslash G)$ is a Woronowicz C^*-subalgebra of A), which will be called the **right quotient (or factor) quantum group** of G by N. Using Podles' left quotient spaces, we can also define the notion of **left quotient quantum group**. In general, a right quotient quantum group is different from the corresponding left quotient quantum group.

Note that for every compact quantum group $G = G_A$, the group G is a normal quantum subgroup of G, and the one element group is also one such if (and only if) the counit of A is continuous. They will be called the **trivial normal quantum subgroups**. A compact quantum group is called **simple** if it has no non-trivial normal quantum subgroups.

2.14. Theorem and Definition. *Let A be a Woronowicz C^*-algebra such that the space $X(A)$ of nonzero $*$-homomorphisms from A to the algebra $\mathbf{C}$ of complex numbers is nonempty. Then $X(A)$ is a compact subgroup of the quantum group G_A with the property that every compact subgroup H of G_A is a subgroup of $X(A)$. If the Woronowicz C^*-algebra A is finited generated, then this subgroup is a compact Lie group. The group $X(A)$ will be called the* **maximal compact subgroup of** *G_A.*

For the computation of the maximal compact subgroups of $SU_q(2)$, see Podles [26]; for the computations of maximal compact subgroups of other nontrivial compact quantum groups, see Sect. 4 below.

3. The Free Products

In this section, we shall make use of the results on compact matrix quantum groups in [51] *as* **modified**(!) *to the more general case of compact quantum groups without further explanation* (*see the remarks in* 2.2.(1)).

We first construct the Woronowicz C^*-algebra structure on the free product of two Woronowicz C^*-algebras.

General Constructions

For later use, we prove a few simple results on the inductive limits of Woronowicz C^*-algebras. For generalities on inductive limits (also called direct limits) of C^*-algebras, see the books of Kadison and Ringrose [15] and Sakai [36].

3.1. Proposition. *Let A_λ be an inductive family of Woronowicz C^*-algebras, where the connecting morphisms $\pi_{\lambda'\lambda}$ from A_λ to $A_{\lambda'}(\lambda < \lambda')$ are injective morphisms of Woronowicz C^*-algebras and the set of the λ's is countable. Then the inductive limit $A = \lim_\lambda A_\lambda$ has a unique Woronowicz C^*-algebra structure with the following property: For every Woronowicz C^*-algebra C and any family of morphisms*

$$\phi_\lambda : A_\lambda \longrightarrow C$$

of Woronowicz C^*-algebras such that $\phi_{\lambda'}\pi_{\lambda'\lambda} = \phi_\lambda$, the uniquely defined morphism $\lim_\lambda \phi_\lambda$ in the category of unital C^*-algebras is a morphism in the category of Woronowicz C^*-algebras.

Proof. Let π_λ be the canonical injection from A_λ into A such that $\pi_\lambda = \pi_{\lambda'}\pi_{\lambda'\lambda}$, for $\lambda < \lambda'$. Define

$$\Phi : \pi_\lambda(A_\lambda) \longrightarrow A \otimes A$$

by $\Phi = (\pi_\lambda \otimes \pi_\lambda)\Phi_\lambda \pi_\lambda^{-1}$. Using the coherences of π_λ and $\pi_{\lambda'\lambda}$, it is straightforward to check that Φ is a well-defined bounded $*$-morphism from the dense $*$-subalgebra $\cup \pi_\lambda(A_\lambda)$ (where the union is taken over the set of the indices λ's) of A to $A \otimes A$. Therefore Φ extends to a bounded C^*-morphism from A to $A \otimes A$. It is also straightforward to check the axioms of Definition 2.1$'$.

For the uniqueness part of the theorem, it suffices to observe that (A_λ, π_λ) is a Woronowicz C^*-subalgebra of the Woronowicz C^*-algebra $\lim_\lambda A_\lambda$ defined above. This completes the proof. Q.E.D.

Recall that the underlying C^*-algebras of finite dimensional Woronowicz C^*-algebras (i.e. finite quantum groups) are direct sums of matrix algebras. We call the Woronowicz C^*-algebras obtained by taking inductive limits of finite dimensional ones **Woronowicz AF-algebras.** By taking the trivial inductive systems, we see that Woronowicz AF-algebras contains all finite dimensional Woronowicz C^*-algebras and therefore the function algebras and the group C^*-algebras of all finite groups. The AF-algebras are completely classified by their K-theory. But this is no longer true for the Woronowicz AF-algebras. Thus the classification of the Woronowicz AF-algebras is not only a C^*-algebra theoretic problem, but also a (quantum) group theoretic problem.

We now determine the Haar state on the inductive limit of A_λ. For this we need a lemma. Recall that for continuous functionals ϕ and ψ on a Woronowicz C^*-algebra A with coproduct Φ (or functionals ϕ and ψ on the dense subalgebra $\mathscr{A}$ of A), the convolution $\phi * \psi$ is defined by

$$\phi * \psi = (\phi \otimes \psi)\Phi .$$

see formula (1.50) of [51].

3.2. Lemma. *Let $i : A \longrightarrow B$ be an injective morphism of Woronowicz C^*-algebras. Then the Haar state on A is the Restriction of that B to A.*

Proof. Use Theorem 4.2.2 of [51] and the Hahn-Banach Theorem. Q.E.D.

Note that the above lemma contains the following well-known result: Let π be a surjective homomorphism from a compact group G to another G'. Then the normalized Haar measure on G' is equal to the push-forward of the Haar measure on G under π_*.

We go back to the question of the Haar state on A considered in 3.1. Let h_λ be the Haar state on A_λ. Since $\pi_{\lambda'\lambda}$, is injective for each pair $\lambda < \lambda'$, by the above lemma we have $h_{\lambda'}\pi_{\lambda'\lambda} = h_\lambda$. Thus the inductive limit $\lim h_\lambda$ exists (see Sakai [36]).

3.3. Proposition. *Under the hypotheses of* 3.1, *the Haar state on* A *is equal to the inductive limit* $\lim_\lambda h_\lambda$.

Proof. Apply 3.2 and the uniqueness of inductive limit of states Q.E.D.

Now we turn to the free products. For generalities on free products see A vitzour [1], Brown [3] and Voiculescu [44]. For unital C^*-algebras A and B, the notation $A * B$ will mean the C^*-algebra free product of A and B amalgamated over the scalars, and $A *_D B$ will mean the C^*-algebra free product of A and B amalgamated over D if D is a C^*-subalgebra of both A and B.

3.4. Theorem. *Let* A *and* B *be Woronowicz* C^**-algebras. Let* D *be a Woronowicz* C^**-subalgebra of both* A *and* B *with embeddings* j_A *and* j_B *respectively. Then the free product* C^**-algebra* $A *_D B$ *has a unique Woronowicz* C^**-algebra structure with the following properties: The canonical injections*

$$i_A : A \longrightarrow A *_D B \quad \text{and} \quad i_B : B \longrightarrow A *_D B$$

are morphisms of Woronowicz C^**-algebras; and for every Woronowicz* C^**-algebra* C *and any morphisms*

$$\pi_A : A \longrightarrow C \quad \text{and} \quad \pi_B : B \longrightarrow C$$

of Woronowicz C^**-algebras such that* $\pi_A j_A = \pi_B j_B$, *the uniquely defined morphism* $\pi_A *_D \pi_B$ *in the category of unital* C^**-algebras is a morphism in the category of Woronowicz* C^**-algebras.*

Proof. The main difficulty is to define the coproduct on $A *_D B$. Note that we cannot naively take it to be $\Phi_A *_D \Phi_B$ because the target of $\Phi_A *_D \Phi_B$ is $(A \otimes A) *_{D\otimes D} (B \otimes B)$ instead of the desired algebra $(A *_D B) \otimes (A *_D B)$. To fix this, let

$$i_A : A \longrightarrow A *_D B \quad \text{and} \quad i_B : B \longrightarrow A *_D B$$

be the canonical injections. Then $i_A j_A = i_B j_B$ from the definition of $A *_D B$. Put

$$\rho_A = (i_A \otimes i_A)\Phi_A \quad \text{and} \quad \rho_B = (i_B \otimes i_B)\Phi_B ,$$

where

$$i_A \otimes i_A : A \otimes A \longrightarrow (A *_D B) \otimes (A *_D B)$$

and

$$i_B \otimes i_B : B \otimes B \longrightarrow (A *_D B) \otimes (A *_D B)$$

are defined as in Proposition IV 4.22 of Takesaki [40]. Then we have $\rho_A j_A = \rho_B j_B$ according to the assumption that j_A and j_B are morphisms of Woronowicz C^*-algebras. Thus by the universal property of $A *_D B$, there is a well-defined map

$$\Phi : A *_D B \to (A *_D B) \otimes (A *_D B)$$

such that $\Phi i_A = \rho_A$ and $\Phi i_B = \rho_B$.

Put $E = A *_D B$. Assume that $\{u_{ij}^{\alpha}\}$ are the generators of $\mathscr{A}$ and $\{v_{rs}^{\beta}\}$ are the generators of $\mathscr{B}$. Then the *-subalgebra of E generated by the $i_A(u_{ij}^{\alpha})$'s and $i_B(v_{kl}^{\beta})$'s is clearly dense in E. The conditions (1′)–(3′) of Definition 2.1′ are also easily checked. Thus E together with the above Φ is a Woronowicz C^*-algebra, and the canonical injections i_A, i_B are morphisms of Woronowicz C^*-algebras.

Let C be any other Woronowicz C^*-algebra, and

$$\pi_A : A \to C \quad \text{and} \quad \pi_B : B \to C$$

be morphisms of Woronowicz C^*-algebras such that $\pi_A j_A = \pi_B j_B$. Then the uniquely defined morphism $\pi_A *_D \pi_B$ in the category of unital C^*-algebras is a morphism in the category of Woronowicz C^*-algebras when restricted to $i_A(A)$ and $i_B(B)$. Since $i_A(A)$ and $i_B(B)$ generate $A *_D B$ as a C^*-algebra, $\pi_A *_D \pi_B$ is a morphism of Woronowicz C^*-algebra.

To show the uniqueness part of the theorem, let (E, Φ') be another Woronowicz C^*-algebra structure on E with the property specified in the theorem, where E denotes the algebra $A *_D B$. Consider the canonical embeddings i_A and i_B of A and B into (E, Φ). Since they are morphisms of Woronowicz C^*-algebras, the uniquely defined C^*-algebra morphism $i_A *_D i_B$ from E to E is a morphism from the Woronowicz C^*-algebra (E, Φ') to the Woronowicz C^*-algebra (E, Φ). But it is clear that $i_A *_D i_B$ is the identity map at the C^*-algebra level. Therefore $\Phi' = \Phi$. Q.E.D.

3.5. Corollary. *Assume the conditions in* 3.4. *Let* $\langle D \rangle$ *be the closed ideal of* $A * B$ *generated by* $i'_A j_A(d) - i'_B j_B(d)$ *with* d *in* D, *where* i'_A *and* i'_B *are the canonical injections from* A *and* B *into* $A * B$ *respectively. Then* $\langle D \rangle$ *is a Woronowicz* C^**-ideal, and the Woronowicz* C^**-algebra* $(A * B)/\langle D \rangle$ *is isomorphic to the Woronowicz* C^**-algebra* $A *_D B$.

Proof. Let $C = A *_D B$. Consider the canonical injections

$$i_A : A \longrightarrow C \quad \text{and} \quad i_B : B \longrightarrow C\,.$$

From the definition of the Woronowicz C^*-algebra structure on $A *_D B$, both i_A and i_B are morphisms of Woronowicz C^*-algebras. Then by Theorem 3.4, the uniquely defined morphism $i_A * i_B$ from $A * B$ to C is a morphism of Woronowicz C^*-algebras. It is clear from the definition of the free product of C^*-algebras that the kernel of this map is equal to $\langle D \rangle$. Now the corollary follows from Theorem 2.11. Q.E.D.

One can also prove the above corollary directly using the method employed in the maximal tensor product and the crossed product (compare with sections 2 and 3 of [46]). We omit the details.

The following theorem provides an interesting method of constructing nontrivial examples of compact matrix quantum groups.

3.6. Corollary. *Let* $G = (A, u)$ *and* $G' = (A', u')$ *be two compact matrix quantum groups. Then the compact quantum group of* $A * A'$ *has a structure of compact matrix quantum group of the form* $(A * A', u \oplus u')$, *where*

$$u \oplus u' = \begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix},$$

and the coproduct of $(A * A', u \oplus u')$ *is defined as in Theorem* 3.4.

Proof. It is clear that both v and v^T are invertible, where $v = u \oplus u'$. In view of Definition 2.1′ we only need to observe that the coefficients of the matrix $v = u \oplus u'$ generate $\mathscr{A} * \mathscr{A}'$ as a $*$-algebra. Q.E.D.

3.7. Corollary. *The free product of an arbitrary sequence of Woronowicz C^*-algebras has a natural Woronowicz C^*-algebra structure.*

Proof. Apply 3.4 and 3.1. Q.E.D.

Haar Measure and Irreducible Representations

3.8. Theorem. *Let A, B be Woronowicz C^*-algebras. Then the Haar state on the free product Woronowicz C^*-algebra $C = A * B$ is given by the free product $h = h_A * h_B$.*

More generally, if A_λ is an arbitrary family of Woronowicz C^*-algebras with Haar states h_λ respectively, then the Haar state of $*_\lambda A_\lambda$ is given by $h = *_\lambda h_\lambda$.

Proof. Let us first recall the definition of free product of states (see Avitzour [1] or Voiculescu [44]). Let ϕ, ψ be states on arbitrary C^*-algebras A, B respectively.

The free product state $\phi * \psi$ on $A * B$ is defined to be the unique state with the property that it restricts to the states ϕ and ψ and that

$$\begin{aligned}(\phi * \psi)(c_1 \cdots c_n) = \sum_{1 \leqq i \leqq n} (\phi * \psi)(c_i)(\phi * \psi)(c_1 \cdots \hat{c}_i \cdots c_n) \\ - \sum_{1 \leqq i < j \leqq n} (\phi * \psi)(c_i)(\phi * \psi)(c_j)(\phi * \psi)(c_1 \cdots \hat{c}_i \cdots \hat{c}_j \cdots c_n) \\ + \cdots + (-1)^{n-1}(\phi * \psi)(c_1) \cdots (\phi * \psi)(c_n),\end{aligned}$$

where the adjacent c_i's are elements of different algebras taken from A and B. In particular,

$$(\phi * \psi)(c_1 c_2) = (\phi * \psi)(c_1)(\phi * \psi)(c_2)\,.$$

The reader should be careful about the difference between the meaning of the product $\phi * \psi$ and that of the "convolution" in (1.50) of [51] (see also the end of 3.1)! Although we use the same symbol $*$ for these two operations, it should not cause confusion if we keep the context in mind.

To see that $h = h_A * h_B$ is indeed the Haar state for C, we only need to check $c * h = h * c = h(c)I$ for all $c \in \mathscr{C}$ (see Theorem 4.2 of [51]). Since $\mathscr{C}$ is generated by $\mathscr{A}$ and $\mathscr{B}$, and every element of $\mathscr{A}$ (resp. of $\mathscr{B}$) is a linear combination of a finite number of elements u_{ij}^{α} (resp. v_{kl}^{β}), we only need to check these equalities for the products of elements u_{ij}^{α} and v_{kl}^{β}. To simplify computation, let us use w_{ij}^{γ} to denote u_{ij}^{α} or v_{kl}^{β}, and agree that the adjacent elements of the product $w_{i_1 j_1}^{\gamma_1} \cdots w_{i_n j_n}^{\gamma_n}$ are taken from different algebras A, B. Without loss of generality, we can assume that none of the $w_{i_k j_k}^{\gamma_k}$'s is the identity. Then using

$$w_{i_k j_k}^{\gamma_k} * h = \sum_{r_k} h(w_{i_k r_k}^{\gamma_k}) w_{r_k j_k}^{\gamma_k} = h(w_{i_k j_k}^{\gamma_k})I = 0\,,$$

which follows from Theorem 5.7.4 of [51], we obtain

$$\begin{aligned} h(w^{\gamma_1}_{i_1j_1}\cdots w^{\gamma_n}_{i_nj_n}) &= \sum_{1\leq k\leq n} h(w^{\gamma_k}_{i_kj_k})h(w^{\gamma_1}_{i_1j_1}\cdots \hat{w}^{\gamma_k}_{i_kj_k}\cdots w^{\gamma_n}_{i_nj_n}) \\ &\quad - \sum_{1\leq k<l\leq n} h(w^{\gamma_k}_{i_kj_k})h(w^{\gamma_l}_{i_lj_l})h(w^{\gamma_1}_{i_1j_1}\cdots \hat{w}^{\gamma_k}_{i_kj_k}\cdots \hat{w}^{\gamma_l}_{i_lj_l}\cdots w^{\gamma_n}_{i_nj_n}) \\ &\quad + \cdots + (-1)^{n-1}h(w^{\gamma_1}_{i_1j_1})\cdots h(w^{\gamma_n}_{i_nj_n}) \\ &= 0. \end{aligned}$$

Thus we have

$$\begin{aligned} (w^{\gamma_1}_{i_1j_1}\cdots w^{\gamma_n}_{i_nj_n}) * h &= (h\otimes id)\Big(\sum_{r_1\cdots r_n} w^{\gamma_1}_{i_1r_1}\cdots w^{\gamma_n}_{i_nr_n}\otimes w^{\gamma_1}_{r_1j_1}\cdots w^{\gamma_n}_{r_nj_n}\Big) \\ &= \sum_{r_1\cdots r_n} h(w^{\gamma_1}_{i_1r_1}\cdots w^{\gamma_n}_{i_nr_n})w^{\gamma_1}_{r_1j_1}\cdots w^{\gamma_n}_{r_nj_n} \\ &= 0. \end{aligned}$$

So we have shown that

$$(w^{\gamma_1}_{i_1j_1}\cdots w^{\gamma_n}_{i_nj_n}) * h = h(w^{\gamma_1}_{i_1j_1}\cdots w^{\gamma_n}_{i_nj_n})I\ .$$

Exactly the same method gives

$$h * (w^{\gamma_1}_{i_1j_1}\cdots w^{\gamma_n}_{i_nj_n}) = h(w^{\gamma_1}_{i_1j_1}\cdots w^{\gamma_n}_{i_nj_n})I\ .$$

This proves the case of the free product of two Woronowicz C^*-algebras.

Now the general case follows from this special case and Proposition 3.3. Q.E.D.

The next example shows the relation between the free product of discrete groups and that of Woronowicz C^*-algebras.

3.9. Example. (1) Let Γ be a discrete abelian group. Then the natural isomorphism $C^*(\Gamma)\cong C(\hat{\Gamma})$ is an isomorphism of Woronowicz C^*-algebras.

(2) Let Γ_i be discrete groups (i = 1,2). Then the natural isomorphism

$$C^*(\Gamma_1) * C^*(\Gamma_2) \cong C^*(\Gamma_1 * \Gamma_2)$$

is an isomorphism of Woronowicz C^*-algebras.

Proof. Routine. Q.E.D.

This example serves to find the irreducible representations of the quantum group of $A * B$? For general Woronowicz C^*-algebras A and B, we can still do the following heuristic computation: Let $A_i = C(G_i)$ be the Woronowicz C^*-algebra of a compact quantum group $G_i (i = 1,2)$ and $\hat{G}_i$ the complete set of irreducible representations of G_i. Then we have the "formula":

$$A_1 * A_2 \cong C(G_1) * C(G_2) \cong C^*(\hat{G}_1) * C^*(\hat{G}_2) \cong C^*(\hat{G}_1 * \hat{G}_2)$$

as Woronowicz C^*-algebras, which is strictly correct for compact abelian groups by the above example. For a general Woronowicz C^*-algebra $A = C(G)$ of a compact quantum group G, the formula $C(G) = C^*(\hat{G})$ is part of the Tannaka-Krein duality theorem for compact quantum groups (see [47] and [52]). The following theorem answers the question of how we make sense out of the above "formula." If w_1

and w_2 are representations of a compact quantum group, we will use $w_1 \otimes_{in} w_2$ to denote the interior tensor product representation of w_1 by w_2 (see p.632 of [51] as well as [45, 46]).

3.10. Theorem. *Let A and B be Woronowicz C^*-algebras and $\{u^\alpha\}$ and $\{v^\beta\}$ the complete sets of irreducible representations for the corresponding quantum groups respectively. Then a complete set of mutually irreducible representations of the quantum group for $A * B$ consists of the trivial representation together with the collection of representations of the form*

$$w^{\gamma_1} \otimes_{in} w^{\gamma_2} \otimes_{in} \cdots \otimes_{in} w^{\gamma_n} ,$$

where w^{γ_i} is a nontrivial representation belonging to either $\{u^\alpha\}$ or $\{v^\beta\}$ and w^{γ_i} and $w^{\gamma_{i+1}}$ are in different sets.

Proof. First we note that as a consequence of the definition of the Woronowicz C^*-algebra $A * B$, each of u^α and v^β can be viewed as a representation of the quantum group of $A * B$. Thus each

$$w^{\gamma_1} \otimes_{in} w^{\gamma_2} \otimes_{in} \cdots \otimes_{in} w^{\gamma_n}$$

is a representation of this quantum group. Let w denote this representation. To show that it is irreducible, we make use of Theorem 5.8 of [51].

Let h be the Haar state on $A * B$, and let χ_i be the character of the representation w^{γ_i} of the quantum group of $A * B$. Then the character of w is $\chi_1\chi_2 \cdots \chi_n$ (see [51]).

We assert that

$$h(\chi_1\chi_2 \cdots \chi_n\chi_{n+1}\chi_{n+1}^*\chi_n^* \cdots \chi_2^*\chi_1^*) = h(\chi_1\chi_2 \cdots \chi_n\chi_n^* \cdots \chi_2^*\chi_1^*)$$

for all $n \geqq 1$. Since $h(\chi_1\chi_1^*) = 1$, this will imply

$$h(\chi_1\chi_2 \cdots \chi_n\chi_n^* \cdots \chi_2^*\chi_1^*) = 1$$

for all $n \geqq 1$ and therefore w is irreducible by Theorem 5.8 of [51]. Without loss of generality, we can assume that $\chi_i \neq 1$ for all i. Since $h(\chi_i) = h(\chi_i^*) = 0$ by the Peter-Weyl orthogonality relations (see Theorem 5.7.4 of [51]), the truthfulness of the equality

$$h(\chi_1\chi_2 \cdots \chi_n\chi_{n+1}\chi_{n+1}^*\chi_n^* \cdots \chi_2^*\chi_1^*) = h(\chi_1\chi_2 \cdots \chi_n\chi_n^* \cdots \chi_2^*\chi_1^*)$$

is immediately seen from the definition of h given at the beginning of the proof of Theorem 3.8.

Next we show that

$$w^{\gamma_1} \otimes_{in} w^{\gamma_2} \otimes_{in} \cdots \otimes_{in} w^{\gamma_n} \text{ and } w^{\gamma'_1} \otimes_{in} w^{\gamma'_2} \otimes_{in} \cdots \otimes_{in} w^{\gamma'_{n'}}$$

are inequivalent if $(\gamma_1, \gamma_2, \cdots, \gamma_n) \neq (\gamma'_1, \gamma'_2, \cdots, \gamma'_{n'})$. Let k be the least number no greater than either of n and n' and such that $\gamma_{n-k} \neq \gamma'_{n'-k}$. Denoting by χ'_i the characters of $w^{\gamma'_i}$, then again by the Peter-Weyl orthogonality relations and the definition of h given at the beginning of the proof of Theorem 3.8 we have

$$h(\chi'_1\chi'_2 \cdots \chi'_{n'}\chi_n^* \cdots \chi_2^*\chi_1^*) = h(\chi'_1\chi'_2 \cdots \chi'_{n'-k}\chi_{n-k}^* \cdots \chi_2^*\chi_1^*) .$$

If $w^{\gamma'_{n'-k}}$ and $w^{\gamma_{n-k}}$ are representations of different quantum groups, then we have that

$$h(\chi'_1\chi'_2 \cdots \chi'_{n-k}\chi_{n-k}^* \cdots \chi_2^*\chi_1^*) = h(\chi'_1\chi'_2 \cdots (\chi'_{n'-k}\chi_{n-k}^*) \cdots \chi_2^*\chi_1^*) = 0$$

by the same reasons; if $w^{\gamma'_{n'-k}}$ and $w^{\gamma_{n-k}}$ are representations of the same quantum group, then we also have

$$\begin{aligned} &h(\chi'_1\chi'_2\cdots\chi'_{n'-k}\chi^*_{n-k}\cdots\chi^*_2\chi^*_1) \\ &= h(\chi'_{n'-k}\chi^*_{n-k})h(\chi'_1\chi'_2\cdots\chi'_{n'-k-1}\chi^*_{n-k-1}\cdots\chi^*_2\chi^*_1) = 0. \end{aligned}$$

Thus the above two representations are inequivalent by Theorem 5.8 of [51]. It is also clear that the trivial representation is inequivalent to the representation

$$w^{\gamma_1} \otimes_{in} w^{\gamma_2} \otimes_{in} \cdots \otimes_{in} w^{\gamma_n} .$$

Since the linear span of the coefficients of these representations is dense in $A * B$, we conclude that they form a complete set of irreducible representations of the quantum group of $A * B$ by virtue of Theorem 5.7.4 of [51]. Q.E.D.

4. The compact Matrix Quantum Groups $A_u(m)$ and $A_o(m)$

Comparing the notion of a compact matrix quantum group [51] with the construction of Brown's universal non-commutative unitary algebras $U_{nc}(m)$ [3], one might think the later are compact matrix quantum groups. But they are not:

4.1. Non-example. Let $U_{nc}(m)$ be Brown's [3] universal noncommutative C^*-algebra generated by the u_{ij}'s subject to the relations that make (u_{ij}) into an $m \times m$ unitary matrix, where m is some positive integer $\geqq 1$. Then it has a natural co-product Φ defined by

$$\Phi(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} ,$$

because the element $(\sum_{k=1}^m u_{ik} \otimes u_{kj})$ of $M_m(U_{nc}(m) \otimes U_{nc}(m))$ is unitary. We show that this Φ however does not define a Woronowicz C^*-algebra structure on $U_{nc}(m)$. If this defined a Woronowicz C^*-algebra structure on $U_{nc}(m)$, in other words $U_{nc}(m)$ were a compact matrix quantum group, then $u = (u_{ij})$ would be a unitary representation of the quantum group, and so u^T would be invertible by 2.1′ and Theorem 4.5 of [51]. We show that this is not the case.

Let $u = (u'_{ij})$ be Woronowicz's unitary $m \times m$ matrix (with entries in the 2×2 scalar matrix algebra M_2) defined by

$$u' = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1_{2m-4} \end{pmatrix},$$

where 1_{2m-4} is the $(2m-4) \times (2m-4)$ unit scalar matrix and

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(The author is grateful to Professor Woronowicz for communicating to him the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in answer to a question of the author on the second definition of compact matrix quantum groups given in [53].) By the universal property of the algebra $U_{nc}(m)$, there exists a unital C^*-algebra morphism π from $U_{nc}(m)$ into M_2 that sends u_{ij} to u'_{ij}. If $v = (v_{ij})$ is the inverse of u^T, then a direct computation shows that $(\pi(v_{ij}))$ is the inverse of $(u')^T$. But a direct computation shows that $(u')^T$ is not invertible. Q.E.D.

Thus the $U_{nc}(m)$'s are compact quantum semigroups but not quantum groups, and they are the universal ones in the sense that every compact matrix quantum group is a quantum subsemigroup of one them (see the equivalent definition of a compact matrix quantum group given in [53], see also Definition 2.1′). However, we obtain two mysterious infinite classes of compact matrix quantum groups if we impose correct additional commutation relations (see 4.2 and 4.5 below).

4.2. Example. Let m be a natural number greater than 1. Let $A_u(m)$ be the universal C^*-algebra generated by m^2 elements a_{ij} subject to the relations

$$\sum_{k=1}^m a_{ik}a_{jk}^* = \delta_{ij}, \quad \sum_{k=1}^m a_{ki}^* a_{kj} = \delta_{ij},$$
$$\sum_{k=1}^m a_{ki}a_{kj}^* = \delta_{ij}, \quad \sum_{k=1}^m a_{ik}^* a_{jk} = \delta_{ij}$$

for $i, j = 1, \ldots, m$. Then $A_u(m)$ is a non-commutative non-cocommutative Woronowicz C^*-algebra, and so it is a compact matrix quantum group. For $m \neq n, A_u(m)$ and $A_u(n)$ are non-isomorphic to each other as C^*-algebras, and therefore non-isomorphic to each other as Woronowicz C^*-algebras.

Proof. First we remark that representations of the above relations are bounded by 1. Thus the C^*-algebra $A_u(m)$ is well-defined.

By the universal property of $A_u(m)$, there exists a uniquely determined C^*-algebra homomorphism Φ from $A_u(m)$ to $A_u(m) \otimes A_u(m)$ such that

$$\Phi(a_{ij}) = \sum_{k=1}^m a_{ik} \otimes a_{kj}$$

for $i, j = 1, \ldots, m$. From the commutation relations, it is straightforward to check that both matrices (a_{ij}) and $(a_{ij})^T$ are invertible. As a matter of fact, the inverse of (a_{ij}) is $(a_{ij})^*$, where the * denotes the involution of the C^*-algebra $M_m(A_u(m))$, and the inverse of the transpose $(a_{ij})^T$ is $((a_{ij})^T)^*$. Therefore by Definition 2.1′, $A_u(m)$ is a Woronwicz C^*-algebra.

We determine the coinverse κ of $A_u(m)$. Let $c_{ij} = a_{ji}^*$, and let $A_u(m)^\circ$ denote the opposite C^*-algebra of $A_u(m)$. Denote the product on $A_u(m)^\circ$ by $\circ$. Then we have

$$\sum_{k=1}^m c_{ik} \circ c_{jk}^* = \delta_{ij}, \quad \sum_{k=1}^m c_{ki}^* \circ c_{kj} = \delta_{ij},$$
$$\sum_{k=1}^m c_{ki} \circ c_{kj}^* = \delta_{ij}, \quad \sum_{k=1}^m c_{ik}^* \circ c_{jk} = \delta_{ij}$$

for $i, j = 1, \ldots, m$. Thus by the universal property of $A_u(m)$, there exists a uniquely determined unital C^*-algebra homomorphism κ from $A_u(m)$ to $A_u(m)^\circ$ such that

$$\kappa(a_{ij}) = c_{ij} = a_{ji}^*$$

for $i, j = 1, \cdots, m$. We note that the coidentity is also an everywhere defined character.

We show that the Woronowicz C^*-algebra $A_u(m)$ is non-commutative and non-cocommutative. Since the entries of the $m \times m$ matrix $u' := diag(z_1, \cdots, z_m)$ also satisfy the communication relations for the a_{ij}'s, where T is the ordinary 1×1 unitary group and z_i is the generator of the C^*-algebra $A_i := C(T)$, there is a natural surjective morphism from $A_u(m)$ to the free product $A_1 * \cdots * A_m$ by the universal property of $A_u(m)$. Thus $A_u(m)$ is non-commutative. The noncocommutativity of $A_u(m)$ follows immediately from the following lemma because $C(U(m))$ is noncocommutative, whose proof is straightforward and is omitted.

4.3. Lemma. *The image of a cocommutative Woronowicz C^*-algebra under a morphism of Woronowicz C^*-algebras is cocommutative.*

Finally, we show that $A_u(m)$ and $A_u(n)$ are non-isomorphic to each other as C^*-algebras for $m \neq n$.

We determine the space $X(A_u(m))$ of nonzero $*$-homomorphisms from the algebra $A_u(m)$ to the algebra $\mathbf{C}$ of complex numbers, which is the maximal compact Lie subgroup of $G_{A_*}(m)$ (see 2.14). For every $u = (u_{ij}) \in U(m)$, by the universal property of the C^*-algebra $A_u(m)$, there exists a unique C^*-algebra morphism χ_u from $A_u(m)$ onto the algebra $\mathbf{C}$ of complex numbers such that $\chi_u(a_{ij}) = u_{ij}$. Conversely for every χ in $X(A_u(m))$, let $u = (\chi(a_{ij}))$; then $u \in U(m)$ and it is easy to see that $\chi = \chi_u$, where χ_u is defined as above. This shows that the space $X(A_u(m))$ of nonzero homomorphisms from $A_u(m)$ to $\mathbf{C}$ is equal to $U(m)$. It is easy to see that under the weak*-topology of the Banach dual $A^*, X(A_u(m))$ and $U(m)$ are homeomorphic. The manifolds $U(m)$ and $U(n)$ are not homeomorphic to each other if $m \neq n$ because they have different dimensions. Hence the C^*-algebras $A_u(m)$ and $A_u(n)$ are not isomorphic if $m \neq n$. This completes the proof of the statements in 4.2. Q.E.D.

Note that we can also show directly that the derived C^*-algebras $A_u(m)/J$ is isomorphic to $C(U(m))$ (see [45]), where J is the closed ideal of $A_u(m)$ generated by the commutators $ab - ba$ for all $a, b \in A_u(m)$. From this we also see that $A_u(m)$ and $A_u(n)$ are non-isomorphic as C^*-algebras for $n \neq m$.

4.4. Remarks. There are many quantum subgroups of the quantum group $G_{A_u}(m)$. For instance, fixing any n such that $n \leqq m$, it is easy to see that the following quantum groups are all quantum subgroups of $G_{A_u(m)}$: the compact group $U(n)$ and its subgroups; the quantum group $G_{A_u}(n)$; the quantum groups of the Woronowicz C^*-algebra $C(T) * C(U(n))$ and its Woronowicz C^*-subalgebra $C^*(zu_{ij})$, where $i, j = 1, \cdots, n$, and the u_{ij}'s are the coordinate functions of the unitary group $U(n)$; and the quantum groups of $C^*(F_n)$ and $C_r^*(F_n)$, where F_n is the free group on n generators.

It is also routine to check that the quantum groups $U_{\pm 1}(m)$ studied by Manin [21] and Koelink [16] and therefore the quantum groups $SU_{\pm 1}(m)$ of Woronowicz [50, 52] ($SU_q(m)$ were studied independently in [39]) are quantum subgroups of $G_{A_u(m)}$.

Now consider $q \neq \pm 1$. Then $U_q(m)$ and $SU_q(m)$ are not quantum sub-groups of $G_{A_u(n)}$ for any n. To see this, first note that the conjugate of the matrix

$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

(see page of [50]) is not unitary, where by "conjugate" of a matrix (c_{ij}) with entries in a $*$-algebra we mean the matrix (c_{ij}^*). Since the conjugate of the matrices

$(a_{ij})_{i,j=1}^n$ for $A_u(n)$ are unitary, from this we see that $SU_q(2)$ is not a quantum subgroup of $G_{A_u(n)}$ for any n. On the other hand, $SU_q(2)$ is a quantum subgroup of the quantum groups $U_q(m)$ and $SU_q(m)$. These also show that the quantum groups $G_{A_u(m)}$ constructed here are different from the quantum groups $U_q(m)$ and $SU_q(m)$

We note further that the algebras $A_u(m)$ are non-nuclear because the non-nuclear C^*-algebras $C^*(F_m)$ are quotients of them, while for any q the C^*-algebras $C(U_q(m))$ [16], $C(SU_q(m))$ [52] and the C^*-algebras of continuous functions on the more general q-deformations of compact Lie groups studied in [35, 38] are type I and so nuclear. From this we also see that the quantum groups $A_u(m)$ are different from these q-deformations of compact Lie groups. In [18, 19], compact quantum groups with non-type I function algebras are introduced. These algebras are nuclear in view of [32]; hence our quantum groups are also different from these quantum groups.

It is interesting to note that the elements c_{ij} of the C^*-algebra $A_u(m)$ (instead of viewing them as in $A_u(m)^\circ$!) also satisfy the same relations as for the a_{ij}'s, where $c_{ij} = a_{ji}^*$. Thus there exists a unital C^*-algebra homomorphism α from $A_u(m)$ into itself satisfying

$$\alpha(a_{ij}) = c_{ij} = a_{ji}^*$$

for $i, j = 1, \cdots m$. It is easy to see that α is an automorphism of period two of the C^*-algebra $A_u(m)$ but not an automorphism of the Woronowicz C^*-algebra $A_u(m)$. For more on the automorphisms of the Woronowicz C^*-algebra $A_u(m)$, see Sect. 4 of [46]. These automorphisms motivated interesting results (see our paper [48]) concerning strict deformation quantizations of compact quantum groups in the sense of Rieffel [28, 29, 31, 33].

It would be interesting to solve the following problem.

Work out the representation theory of the quantum groups $A_u(m)$, and find the Haar state on $A_u(m)$ and the multiplicative unitary associated with $A_u(m)$.

The quantum groups in the following example are the analogues of the orthogonal groups. They are quantum subgroups of the quantum groups $G_{A_u(m)}$.

4.5. Example. Let m be a natural number greater than 1. Let $A_o(m)$ be the universal C^*-algebra generated by m^2 elements b_{ij} subject to the relations

$$\sum_{k=1}^m b_{ik} b_{jk} = \delta_{ij}, \quad \sum_{k=1}^m b_{ki} b_{kj} = \delta_{ij},$$

$$b_{ij}^* = b_{ij}$$

for $i, j = 1, \cdots, m$. Then $A_o(m)$ is a non-commutative non-cocommutative Woronowicz C^*-algebra, and so it is a compact matrix quantum group. For $m \neq n$, $A_o(m)$ and $A_o(n)$ are non-isomorphic to each other as C^*-algebras, and therefore non-isomorphic to each other as Woronowicz C^*-algebras. Furthermore, the quantum groups of the $A_o(m)$'s are quantum subgroups of those of the $A_u(m)$'s.

Proof. The proof that $A_o(m)$ is a Woronowicz C^*-algebra is the same as the proof for $A_u(m)$ (see the beginning of the proof of 4.2).

There are a surjection from $A_o(m)$ to $C^*(Z/2Z) * \cdots * C^*(Z/2Z)$ (there are m copies of $C^*(Z/2Z)$ in this free product) and $C(O(m))$ respectively, where $Z/2Z$ is the two-element group and $O(m)$ is the ordinary group of real $m \times m$ orthogonal matrices. So $A_o(m)$ is non-commutative and non-cocommutative. Similarly, $A_o(m)$

and $A_o(n)$ are non-isomorphic to each other as C^*-algebras for $m \neq n$, because of the fact that the space $X(A_o(m))$ (i.e. the maximal compact Lie subgroup of the quantum group $G_{A_o(m)}$) is homemorphic to $O(m)$ and that the spaces $O(m)$ and $O(n)$ are non-homeomorphic to each other.

Let J be the closed two sided ideal of the C^*-algebra $A_u(m)$ generated by $a_{ij} - a_{ij}^*$, where $i, j = 1, \cdots, m$. Then it is easy to see that the C^*-algebra $A_o(m)$ is isomorphic to the quotient C^*-algebra $A_u(m)/J$. It is also routine to check that the ideal J is a Woronowicz C^*-ideal of $A_u(m)$. This proves the last statement of 4.5. Q.E.D.

Similar to 4.2, we can also show directly that the derived C^*-algebra $A_o(m)/J$ is isomorphic to $C(O(m))$ (see [45]), where J is the closed ideal of $A_o(m)$ generated by the commutators $ab - ba$ for all $a, b \in A_o(m)$. From this we also see that $A_o(m)$ and $A_o(n)$ are non-isomorphic as C^*-algebras for $n \neq m$.

4.6. Remarks. Fixing any n such that $n \leqq m$, it is easy to see that the following quantum groups are all quantum subgroups of $G_{A_o(m)}$: the compact group $O(n)$ and its subgroups; the quantum group $G_{A_o(n)}$; and the quantum groups of $C^*(Z/2Z * \cdots * Z/2Z)$ and $C_r^*(Z/2Z * \cdots * Z/2Z)$.

We conclude this paper with the following important problem, whose solution should constitute a major contribution to the theory of quantum groups.

How does one construct the analogues of the classical compact Lie groups $SU(m)$, $SO(m)$ and $Sp(m)$ from $A_u(m)$ and develop the corresponding theory for them? (*These would be called* **the universal classical compact quantum Lie groups**.)

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Note added in proof. The compact matrix quantum groups $A_u(m)$ and $A_o(m)$ constructed in this paper are now put in more general context in my joint paper with Alfons Van Daele: *Universal quantum groups* (to appear in *Lett. Math. Phys.*), in which two compact matrix quantum groups $A_u(Q)$ and $A_o(Q)$ are constructed for each invertible compext scalar matrix Q, and the $A_u(Q)$'s are shown to be universal in the sense that every compact matrix quantum group is a quantum subgroup of some $A_u(Q)$. The quantum groups $A_u(m)$ and $A_o(m)$ are precisely $A_u(Q)$ and $A_o(Q)$ respectively by taking Q to be the $m \times m$ identity matrix.