Free profinite $\mathcal{R}$-trivial, locally idempotent and locally commutative semigroups

José Carlos Costa*

May, 1999

Abstract

This paper is concerned with the structure of implicit operations on $\mathbf{R} \cap \mathbf{LJ_1}$, the pseudovariety of all $\mathcal{R}$-trivial, locally idempotent and locally commutative semigroups. We give a unique factorization statement, in terms of component projections and idempotent elements, for the implicit operations on $\mathbf{R} \cap \mathbf{LJ_1}$. As an application we give a combinatorial description of the languages that are both $\mathcal{R}$-trivial and locally testable. A similar study is conducted for the pseudovariety $\mathbf{DA} \cap \mathbf{LJ_1}$ of locally idempotent and locally commutative semigroups in which each regular $\mathcal{D}$-class is a rectangular band.

1 Introduction

Since the publication of Reiterman's paper [14],— where he showed that pseudovarieties are defined by pseudoidentities, i.e., by formal equalities of implicit operations,— the theory of implicit operations has received a great deal of attention, particularly in the work of authors like Almeida, Azevedo, Selmi, Weil and Zeitoun [1, 3, 5, 8, 15, 19]. In fact, the description of the structure and properties of the semigroups of implicit operations (also known as free profinite semigroups) on a pseudovariety proved to be a useful tool in the study of that pseudovariety and on the variety of recognizable languages associated with it (via Eilenberg's Theorem on varieties [10]).

In a remarkable work, Almeida [2] gives a description of the structure of the free profinite $\mathcal{J}$-trivial semigroups and solves the word problem for them. He shows that each implicit operation on the pseudovariety $\mathbf{J}$ of $\mathcal{J}$-trivial semigroups admits a canonical factorization in terms of component projections and regular elements. Azevedo [7, 8] showed that this result can be partially extended to any subpseudovariety $\mathbf{V}$ of $\mathbf{DS}$, the pseudovariety of all finite semigroups in which all regular elements lie in groups. In fact, he showed that each implicit operation on $\mathbf{V}$ can be factorized as a product of component projections and regular elements. However, relatively few forms of such factorizations are known to be canonical.

Let $\mathbf{LJ_1}$ be the pseudovariety of all locally idempotent and locally commutative semigroups, that is, the pseudovariety of all finite semigroups S such that eSe is a semilattice for each idempotent e of S. Brzozowski and Simon [9] and McNaughton [11]

* Research supported by the PRC-GdR AMI, by Centro de Matemática da Universidade do Minho and by INVOTAN, grant 4/A/94/PO. The author is member of the projects *SAL* contract PBIC/C/CEN/1021/92 and *AGC* contract Praxis/2/2.1/MAT/63/94.

proved that the variety of recognizable languages associated with $\mathbf{LJ_1}$ is the important class of locally testable languages. Recall that a language L is locally testable if one can decide the membership of a given word u in L by considering the factors of a fixed length k of u and its prefix and suffix of length $k-1$.

This paper is a contribution to the study of implicit operations on subpseudovarieties of $\mathbf{DS}$ and $\mathbf{LJ_1}$. We give unique factorization statements for the implicit operations on $\mathbf{R} \cap \mathbf{LJ_1}$, $\mathbf{L} \cap \mathbf{LJ_1}$, $\mathbf{J} \cap \mathbf{LJ_1}$ (for which we give a new proof of Selmi's results [15]) and $\mathbf{DA} \cap \mathbf{LJ_1}$, where $\mathbf{R}$ (resp. $\mathbf{L}$) is the pseudovariety of all finite $\mathcal{R}$-trivial (resp. $\mathcal{L}$-trivial) semigroups, and $\mathbf{DA}$ is the pseudovariety of finite semigroups in which all regular elements are idempotents. As a consequence of this work we are able to give combinatorial descriptions of the classes of languages recognized by each of these pseudovarieties. More precisely, for each finite alphabet A, we describe a set of generators for the Boolean algebra of the recognizable languages of A^+ that are both $\mathcal{R}$-trivial (resp. $\mathcal{L}$-trivial, $\mathcal{J}$-trivial, $\mathbf{DA}$-recognizable) and locally testable. These generators are all of the form $u_0A_1^*u_1A_2^* \cdots A_n^*u_n$ where $n \geq 0$, the u_i are words over A, the A_i are pairwise disjoint subsets of A, and where the extreme letters of the u_i satisfy some conditions depending on the pseudovariety involved. Note that several varieties of languages have been described as Boolean combinations of languages of the form $u_0A_1^*u_1A_2^* \cdots A_n^*u_n$ imposing various conditions on the words u_i and on the subsets A_i of A (e.g. piecewise testable languages (Simon [16]), $\mathcal{R}$-trivial languages (Eilenberg [10]), level 2 languages in the Straubing hierarchy (Pin and Straubing [13]), etc).

As a consequence of our results we compute the join $(\mathbf{R} \cap \mathbf{LJ_1}) \vee (\mathbf{L} \cap \mathbf{LJ_1})$ which we prove is equal to $\mathbf{DA} \cap \mathbf{LJ_1}$. We then deduce that $(\mathbf{R} \vee \mathbf{L}) \cap \mathbf{LJ_1} = \mathbf{DA} \cap \mathbf{LJ_1}$. This is an interesting and somewhat unexpected equality since $\mathbf{R} \vee \mathbf{L}$ is "far from being equal" to $\mathbf{DA}$.

This paper is organized as follows. In section 2 we recall the main definitions and properties concerning pseudovarieties, implicit operations and languages. In section 3 we describe the structure of the semigroups of implicit operations on $\mathbf{R} \cap \mathbf{LJ_1}$, $\mathbf{J} \cap \mathbf{LJ_1}$ and $\mathbf{DA} \cap \mathbf{LJ_1}$, and show that $(\mathbf{R} \cap \mathbf{LJ_1}) \vee (\mathbf{L} \cap \mathbf{LJ_1}) = \mathbf{DA} \cap \mathbf{LJ_1}$. Section 4 is devoted to the characterization of the varieties of languages associated with the pseudovarieties considered in section 3. Finally, in section 5 we show that the $(\mathbf{R} \cap \mathbf{LJ_1})$-recognizable languages can be described by certain congruences.

2 Preliminaries

We begin by presenting basic definitions and notation concerning words. Next we recall the notion of pseudovariety of semigroups and define the pseudovarieties mentioned in this paper. We then review some definitions and facts concerning implicit operations and pseudoidentities. Next we present the main definitions about recognizable languages and their relations with pseudovarieties. We conclude by summarizing some properties of the implicit operations on subpseudovarieties of $\mathbf{DS}$. For omitted proofs and missing definitions, the reader is referred to the books of Almeida [3], Eilenberg [10] and Pin [12], and to the surveys [5, 17].

Words Let A be a finite non empty set, or *alphabet*. The elements of A are called *letters* and those of A^*, the free monoid on A, *words*. The identity of A^* is called the *empty word* and is denoted by 1. If $u = a_1 \cdots a_n \, (a_i \in A)$ is a word of A^+, the free semigroup on A, the number n is called the *length* of u and is denoted by $|u|$. The length of the empty word is 0.

We denote by $A^{\mathbb{N}}$ (resp. ${}^{\mathbb{N}}A$) the set of all words over A that are "infinite to the right" (resp. "infinite to the left"), that is, the set of sequences of letters of A indexed by $\mathbb{N}$ (resp. $-\mathbb{N}$). The set of all letters appearing in a word (finite or infinite) u is denoted by $c(u)$ and is called the *content* of u.

A word $u \in A^*$ is a *prefix* (resp. *suffix*, *factor*) of a word x (finite or infinite) if there exist words y and z such that $x = uy$ (resp. $x = yu$, $x = yuz$). For each integer k we denote by $p_k(x)$ (resp. $s_k(x)$, $F_k(x)$) the prefix (resp. suffix, set of factors) of x of length k, if it exists.

Pseudovarieties A *pseudovariety* of semigroups is a class of finite semigroups closed under taking subsemigroups, homomorphic images and finite direct products. The pseudovariety of all finite semigroups is denoted by $\mathbf{S}$, and $\mathbf{I}$ denotes the trivial pseudovariety, consisting only of the 1-element semigroup. The pseudovarieties $\mathbf{R}$, $\mathbf{L}$, $\mathbf{J}$ and $\mathbf{J_1}$ are respectively the classes of all $\mathcal{R}$-trivial, $\mathcal{L}$-trivial, $\mathcal{J}$-trivial (where $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{J}$ are the Green relations) and idempotent and commutative semigroups (or semilattices). We denote by $\mathbf{DS}$ (resp. $\mathbf{DA}$) the pseudovariety of all semigroups S in which each regular $\mathcal{D}$-class is a subsemigroup of S (resp. which is idempotent).

For any pseudovariety of semigroups $\mathbf{V}$, the class $\mathbf{LV}$ of all finite semigroups S such that $eSe \in \mathbf{V}$ for each idempotent e of S, is a pseudovariety of semigroups. Particularly important in this paper is the pseudovariety $\mathbf{LJ_1}$ whose elements are called locally idempotent and locally commutative semigroups. The pseudovariety $\mathbf{LI}$ of locally trivial semigroups is one of its subpseudovarieties. We will also encounter $\mathbf{K}$ (resp. $\mathbf{D}$) which is the subpseudovariety of $\mathbf{LI}$ consisting of all finite semigroups S such that $eS = e$ (resp. $Se = e$) for each idempotent e of S. The pseudovariety of nilpotent semigroups is $\mathbf{N} = \mathbf{K} \cap \mathbf{D}$.

Finally $\mathbf{V} \vee \mathbf{W}$ denotes the least pseudovariety containing both the pseudovarieties $\mathbf{V}$ and $\mathbf{W}$.

Implicit operations and pseudoidentities We first review some definitions and facts concerning free profinite semigroups. For details and proofs, the reader is referred to Almeida's book [3] and to the survey [5].

Let $\mathbf{V}$ be a pseudovariety. A *profinite* (resp. *pro-*$\mathbf{V}$) semigroup is a projective limit of finite semigroups (resp. in $\mathbf{V}$). A topological semigroup is profinite (resp. pro-$\mathbf{V}$) if and only if it is compact and 0-dimensional (resp. and all its finite continuous homomorphic images are in $\mathbf{V}$). If A is an alphabet, we say that a profinite semigroup S is A-*generated* if there exists a mapping $\mu : A \to S$ such that the subsemigroup generated by $\mu(A)$ is dense in S. We denote by $\hat{F}_A(\mathbf{V})$ the projective limit of the A-generated elements of $\mathbf{V}$. The elements of $\hat{F}_A(\mathbf{V})$ are usually called (*$|A|$-ary*) *implicit operations* (*on* $\mathbf{V}$).

If $\mathbf{V}$ admits a finite free object $F_A(\mathbf{V})$ over the alphabet A, then $\hat{F}_A(\mathbf{V}) = F_A(\mathbf{V})$. This is the case, for instance, of $\mathbf{J_1}$: $\hat{F}_A(\mathbf{J_1})$ is the semigroup $\mathcal{P}(A)$ of non empty subsets of A under union.

The following important properties of $\hat{F}_A(\mathbf{V})$, will be used freely in this paper.

Proposition 2.1 *Let A be an alphabet and let $\mathbf{V}$ be a non trivial pseudovariety.*

(1) *There exists a natural injective mapping $\iota : A \to \hat{F}_A(\mathbf{V})$ such that $\iota(A)$ generates a dense subsemigroup of $\hat{F}_A(\mathbf{V})$.*

(2) $\hat{F}_A(\mathbf{V})$ *is the free pro-*$\mathbf{V}$ *semigroup over* A*: if* μ *is a mapping from* A *into a pro-*$\mathbf{V}$ *semigroup* S*, then* μ *admits a unique continuous extension* $\hat{\mu} : \hat{F}_A(\mathbf{V}) \to S$ *such that* $\hat{\mu} \circ \iota = \mu$*.* ∎

Usually we will ignore the mapping $\iota : A \to \hat{F}_A(\mathbf{V})$, and consider A as a subset of $\hat{F}_A(\mathbf{V})$. Observe that, if $\mathbf{W}$ is a subpseudovariety of $\mathbf{V}$, then every pro-$\mathbf{W}$ semigroup is also pro-$\mathbf{V}$. So, in particular, we have the following important application of Proposition 2.1: the identity of A induces a continuous onto homomorhism $\pi : \hat{F}_A(\mathbf{V}) \to \hat{F}_A(\mathbf{W})$, called the *canonical projection of* $\hat{F}_A(\mathbf{V})$ *onto* $\hat{F}_A(\mathbf{W})$. The image $\pi(x)$ of an element $x \in \hat{F}_A(\mathbf{V})$ is called the *restriction of* x *to* $\mathbf{W}$. In particular, when $\mathbf{V}$ is a pseudovariety containing $\mathbf{J_1}$, the canonical projection $c : \hat{F}_A(\mathbf{V}) \to \hat{F}_A(\mathbf{J_1}) = \mathcal{P}(A)$ is called the *content homomorphism* on $\mathbf{V}$. As one can easily show, c extends to the elements of $\hat{F}_A(\mathbf{V})$ the notion of content for words of A^+.

For each $x \in \hat{F}_A(\mathbf{V})$, the sequence $(x^{n!})_n$ converges in $\hat{F}_A(\mathbf{V})$. Its limit, denoted by x^ω, is the only idempotent in the topological closure of the subsemigroup generated by x.

Let $\mathbf{V}$ be a pseudovariety and let A be an alphabet. A *pseudoidentity* on $\mathbf{V}$ on the alphabet A (or, in $|A|$ variables) is a pair (u, v) of elements of $\hat{F}_A(\mathbf{V})$, and is usually denoted $u = v$. It is said to be non trivial if the elements u and v are distinct. We say that $u = v$ is an *identity* if u and v are words, i.e., finite products of elements of A, or elements of $\iota(A^+)$. We say that a pro-$\mathbf{V}$ semigroup S *satisfies* a pseudoidentity $u = v$ on $\mathbf{V}$, and we write $S \models u = v$, if, for any continuous morphism $\mu : \hat{F}_A(\mathbf{V}) \to S$, we have $\mu(u) = \mu(v)$. We say that a class $\mathbf{W}$ of pro-$\mathbf{V}$ semigroups satisfies a set Σ of pseudoidentities on $\mathbf{V}$, and we write $\mathbf{W} \models \Sigma$, if each element of $\mathbf{W}$ satisfies each element of Σ. The class of all finite semigroups which satisfy Σ is said to be *defined by* Σ and is denoted $[\![\Sigma]\!]_\mathbf{V}$ ($[\![\Sigma]\!]$ if $\mathbf{V} = \mathbf{S}$). For instance, we have the following equalities:

- $\mathbf{J_1} = [\![xy = yx, x^2 = x]\!]$;
- $\mathbf{R} = [\![(xy)^\omega x = (xy)^\omega]\!]$; $\mathbf{L} = [\![y(xy)^\omega = (xy)^\omega]\!]$;
- $\mathbf{K} = [\![x^\omega y = x^\omega]\!]$; $\mathbf{D} = [\![yx^\omega = x^\omega]\!]$;
- $\mathbf{LI} = [\![x^\omega y x^\omega = x^\omega]\!]$;
- $\mathbf{LJ_1} = [\![x^\omega y x^\omega y x^\omega = x^\omega y x^\omega, x^\omega y x^\omega z x^\omega = x^\omega z x^\omega y x^\omega]\!]$;
- $\mathbf{DA} = [\![(xy)^\omega (yx)^\omega (xy)^\omega = (xy)^\omega, x^\omega x = x^\omega]\!]$;
- $\mathbf{DS} = [\![((xy)^\omega (yx)^\omega (xy)^\omega)^\omega = (xy)^\omega]\!]$.

The following remark will be useful.

Proposition 2.2 *Let* $\mathbf{W} \subseteq \mathbf{V}$ *be pseudovarieties and let* A *be an alphabet. Let* $\pi : \hat{F}_A(\mathbf{V}) \to \hat{F}_A(\mathbf{W})$ *be the canonical projection and let* $x, y \in \hat{F}_A(\mathbf{V})$*. Then,* $\mathbf{W} \models x = y$ *if and only if* $\pi(x) = \pi(y)$*.* ∎

The following fundamental theorem is due to Reiterman [14].

Theorem 2.3 *Let* $\mathbf{V}$ *be a pseudovariety and let* $\mathbf{W}$ *be a class of semigroups in* $\mathbf{V}$*. Then* $\mathbf{W}$ *is a pseudovariety if and only if there exists a set* Σ *of pseudoidentities on* $\mathbf{V}$ *such that* $\mathbf{W} = [\![\Sigma]\!]_\mathbf{V}$*.* ∎

Languages recognized by a pseudovariety V Let A be an alphabet and let $\mathbf{V}$ be a pseudovariety. A subset L of A^+ is called a *language*. It is said to be *recognizable* (resp. $\mathbf{V}$-*recognizable*) if there exists a finite semigroup S (resp. in $\mathbf{V}$) and a morphism $\mu : A^+ \to S$ such that $L = \mu^{-1}(\mu(L))$. In that case, we say that S *recognizes* L. The *syntactic congruence* of a language L is the congruence $\sim_L$ over A^+ given by

$$u \sim_L v \qquad \text{if and only if} \qquad xuy \in L \Leftrightarrow xvy \in L \ \text{ for all } x, y \in A^*.$$

The *syntactic semigroup* of L, denoted by $S(L)$, is the quotient of A^+ by $\sim_L$. We know that L is recognizable (resp. $\mathbf{V}$-recognizable) if and only if $S(L)$ is finite (resp. $S(L) \in \mathbf{V}$). Furthermore, a semigroup S recognizes a language L if and only if $S(L)$ *divides* S (that is, if $S(L)$ is a homomorphic image of a subsemigroup of S). For more details on recognizable languages, the reader is referred to [10, 12].

A *class of (recognizable) languages* is a correspondence $\mathcal{C}$ associating with each alphabet A a set $A^+\mathcal{C}$ of (recognizable) languages of A^+. A *variety of languages* is a class $\mathcal{V}$ of recognizable languages such that

(1) for every alphabet A, $A^+\mathcal{V}$ is closed under finite union, finite intersection and complement;

(2) for every morphism $\varphi : A^+ \to B^+$, $L \in B^+\mathcal{V}$ implies $\varphi^{-1}(L) \in A^+\mathcal{V}$;

(3) if $L \in A^+\mathcal{V}$ and $a \in A$, then $a^{-1}L = \{u \in A^+ : au \in L\}$ and $La^{-1} = \{u \in A^+ : ua \in L\}$ are in $A^+\mathcal{V}$.

Let $\mathbf{V}$ be a pseudovariety and let $\mathcal{V}$ be the class of recognizable languages which associates with each alphabet A the set $A^+\mathcal{V}$ of $\mathbf{V}$-recognizable languages of A^+. One can show that $\mathcal{V}$ is a variety of languages. Moreover, Eilenberg [10] proved the following fundamental result.

Theorem 2.4 *The correspondence $\mathbf{V} \mapsto \mathcal{V}$ defines a bijective correspondence between pseudovarieties of semigroups and varieties of languages.* ■

We summarize in the next theorem the well-known characterizations of the varieties of languages associated with **LI**, $\mathbf{LJ_1}$, **R**, **L** and **DA** (see Pin [12]).

Theorem 2.5 *For each alphabet A, the following hold.*

(1) *The Boolean algebra of all* **LI**-*recognizable languages of A^+ is generated by the languages of the form wA^* and A^*w where $w \in A^+$.*

(2) *The Boolean algebra of all* $\mathbf{LJ_1}$-*recognizable (or, equivalently, locally testable) languages of A^+ is generated by the languages of the form wA^*, A^*w and A^*wA^* where $w \in A^+$.*

(3) *The Boolean algebra of all* **R**-*(resp.* **L**-*)recognizable languages of A^* is generated by the languages of the form*

$$A_0^*a_1A_1^* \cdots a_nA_n^*$$

where $A_i \subseteq A\,(i = 0, \ldots, n)$ and $a_i \in A \setminus A_{i-1}$ (resp. $a_i \in A \setminus A_i$) $(i = 1, \ldots, n)$.

(4) *The Boolean algebra of all* **DA***-recognizable languages of* A^+ *is generated by the languages of the form*

$$A_0^* a_1 A_1^* \cdots a_n A_n^* \tag{1}$$

with $a_1, \ldots, a_n \in A$, $A_0, \ldots, A_n \subseteq A$ *and the product* (1) *is unambiguous in the sense that each of its elements* w *has a unique factorization* $w = u_0 a_1 u_1 \cdots a_n u_n$ *with* $u_i \in A_i^*$ $(i = 0, \ldots, n)$. ■

We say that a family $\mathcal{X}$ of subsets of $\hat{F}_A(\mathbf{V})$ *separates* the points of $\hat{F}_A(\mathbf{V})$ if, for each pair of distinct elements x and y in $\hat{F}_A(\mathbf{V})$, there exists an element X of $\mathcal{X}$ such that either $x \in X$ and $y \notin X$, or $x \notin X$ and $y \in X$. The next result, due to Almeida [3, 5], will be very useful.

Proposition 2.6 *Let* A *be an alphabet, let* $\mathbf{V}$ *be a pseudovariety satisfying no non trivial identity and let* $\mathcal{V}$ *be the corresponding variety of languages. Let* $\mathcal{L}$ *be a subset of* $A^+\mathcal{V}$ *and let* $\overline{\mathcal{L}}$ *be the set of the topological closures in* $\hat{F}_A(\mathbf{V})$ *of the elements of* $\mathcal{L}$.

The Boolean algebra $A^+\mathcal{V}$ *is generated by* $\mathcal{L}$ *if and only if the points of* $\hat{F}_A(\mathbf{V})$ *are separated by* $\overline{\mathcal{L}}$. ■

Subpseudovarieties of DS Almeida and Azevedo [4] gave a number of factorization and regularity results for the implicit operations on subpseudovarieties of **DS**, which will prove fundamental in this paper. Some of these results are summarized in the following proposition.

Proposition 2.7 *Let* $\mathbf{V}$ *be a subpseudovariety of* **DS** *containing* $\mathbf{J_1}$ *and let* $x, y \in \hat{F}_A(\mathbf{V})$.

(1) *x can be written as a product of the form*

$$x = u_0 x_1 u_1 \cdots x_n u_n$$

where the u_i *are words and the* x_i *are regular implicit operations on* $\mathbf{V}$.

(2) *If* x *and* y *are regular, then* $x \mathcal{J} y$ *if and only if* $c(x) = c(y)$.

(3) *If* $w \in \hat{F}_A(\mathbf{V})$, $c(w) \subseteq c(y)$, $x = wy$ *(resp.* $x = yw$*) and* y *is regular, then* x *is regular and* $x \mathcal{L} y$ *(resp.* $x \mathcal{R} y$*).* ■

We now consider the pseudovarieties **N**, **K**, **D** and **LI**. It is well known that **N** satisfies no non trivial identity. This means that the natural morphism $\iota : A^+ \to \hat{F}_A(\mathbf{N})$ is injective for each alphabet A. In particular, we may identify the free semigroup A^+ with a subsemigroup of $\hat{F}_A(\mathbf{N})$. Since **N** is contained in **K**, **D** and **LI** the same is true for each of these pseudovarieties.

Furthermore, we have (see [3]) that:

- $\hat{F}_A(\mathbf{N}) = A^+ \cup \{0\}$ and the product in $\hat{F}_A(\mathbf{N})$ is extended from the product in A^+ by letting $0w = w0 = 0$ if $w \in A^+ \cup \{0\}$;
- $\hat{F}_A(\mathbf{K}) = A^+ \cup A^{\mathbb{N}}$ and the product in $\hat{F}_A(\mathbf{K})$ is extended from the product in A^+ by letting $ww' = w$ if $w \in A^{\mathbb{N}}$;
- $\hat{F}_A(\mathbf{D}) = A^+ \cup {}^{\mathbb{N}}A$ and the product in $\hat{F}_A(\mathbf{D})$ is extended from the product in A^+ by letting $w'w = w$ if $w \in {}^{\mathbb{N}}A$;

- $\hat{F}_A(\mathbf{LI}) = A^+ \cup (A^{\mathbb{N}} \times {}^{\mathbb{N}}A)$ and the product in $\hat{F}_A(\mathbf{LI})$ is given, for all $u, u' \in A^+$ and $(v,w),(v',w') \in A^{\mathbb{N}} \times {}^{\mathbb{N}}A$, by:

$$\begin{aligned} u \cdot u' &= uu' \\ u \cdot (v,w) &= (uv, w) \\ (v,w) \cdot u &= (v, wu) \\ (v,w) \cdot (v',w') &= (v, w'). \end{aligned}$$

Note that if $x = (u,v)$ is an element of $\hat{F}_A(\mathbf{LI}) \setminus A^+$, then u (resp. v) is the restriction of x to $\mathbf{K}$ (resp. $\mathbf{D}$). In particular, $\mathbf{LI}$ satisfies a pseudoidentity $x = y$ if and only if $\mathbf{K}$ and $\mathbf{D}$ satisfy $x = y$. This is another way of stating the well known equality $\mathbf{LI} = \mathbf{K} \vee \mathbf{D}$.

3 Implicit Operations on $\mathbf{R} \cap \mathbf{LJ_1}$, $\mathbf{DA} \cap \mathbf{LJ_1}$ and $\mathbf{J} \cap \mathbf{LJ_1}$

This section is concerned with the structure of semigroups of implicit operations on $\mathbf{R} \cap \mathbf{LJ_1}$, $\mathbf{DA} \cap \mathbf{LJ_1}$ and $\mathbf{J} \cap \mathbf{LJ_1}$. We prove that every element of each of these semigroups can be written in a canonical form as a product of words and idempotents. We apply these results to the computation of the pseudovariety $(\mathbf{R} \cap \mathbf{LJ_1}) \vee (\mathbf{L} \cap \mathbf{LJ_1})$ which we prove is equal to $\mathbf{DA} \cap \mathbf{LJ_1}$.

Implicit operations on $\mathbf{R} \cap \mathbf{LJ_1}$ We begin by proving a crucial result about the implicit operations on $\mathbf{LJ_1}$ (see [3]). Since $\mathbf{LJ_1}$ satisfies no non trivial identity (say because $\mathbf{N} \subseteq \mathbf{LJ_1}$) the free semigroup A^+ can be seen as a subsemigroup of $\hat{F}_A(\mathbf{LJ_1})$. Let $\hat{F}_A(\mathbf{LJ_1})^1$ denote the monoid $\hat{F}_A(\mathbf{LJ_1}) \cup \{1\}$ and let $x \in \hat{F}_A(\mathbf{LJ_1})$. We denote by $\mathrm{Fact}(x)$ the set of all words $u \in A^+$ such that u is a factor of x, i.e., such that $x = yuz$ for some $y, z \in \hat{F}_A(\mathbf{LJ_1})^1$.

Proposition 3.1 *Let A be an alphabet and let $x, y \in \hat{F}_A(\mathbf{LJ_1})$. Then, $x = y$ if and only if $\mathrm{Fact}(x) = \mathrm{Fact}(y)$ and $\mathbf{LI} \models x = y$.*

Proof. Since $\mathbf{LI}$ is contained in $\mathbf{LJ_1}$ the necessary condition is immediate. Suppose now that $\mathrm{Fact}(x) = \mathrm{Fact}(y)$ and that $\mathbf{LI} \models x = y$. As we recalled in Theorem 2.5, the Boolean algebra of all $\mathbf{LJ_1}$-recognizable languages of A^+ is generated by the set $\mathcal{L} = \{wA^*, A^*w, A^*wA^* : w \in A^+\}$. The set of the topological closures in $\hat{F}_A(\mathbf{LJ_1})$ of the elements of $\mathcal{L}$ is

$$\overline{\mathcal{L}} = \{w\hat{F}_A(\mathbf{LJ_1})^1, \hat{F}_A(\mathbf{LJ_1})^1 w, \hat{F}_A(\mathbf{LJ_1})^1 w \hat{F}_A(\mathbf{LJ_1})^1 : w \in A^+\}.$$

Indeed we have $wA^* \subseteq w\hat{F}_A(\mathbf{LJ_1})^1 \subseteq \overline{wA^*}$ since A^+ is dense in $\hat{F}_A(\mathbf{LJ_1})$. Now, since $w\hat{F}_A(\mathbf{LJ_1})^1$ is closed, we deduce that $w\hat{F}_A(\mathbf{LJ_1})^1 = \overline{wA^*}$. The computation of the closure of A^*w or A^*wA^* is similar.

Now, we claim that $\overline{\mathcal{L}}$ does not separate x and y. Let us suppose first that $x \in \hat{F}_A(\mathbf{LJ_1})^1 w \hat{F}_A(\mathbf{LJ_1})^1$. This means that w is a factor of x and, by hypothesis, w is also a factor of y. Then, $y \in \hat{F}_A(\mathbf{LJ_1})^1 w \hat{F}_A(\mathbf{LJ_1})^1$. Suppose now that $x \in w\hat{F}_A(\mathbf{LJ_1})^1$. This means that w is a prefix of the restriction of x to $\mathbf{K}$. Since $\mathbf{LI}$ satisfies $x = y$, we have that $\mathbf{K}$ satisfies $x = y$. Hence w is also a prefix of the restriction of y to $\mathbf{K}$, meaning that $y \in w\hat{F}_A(\mathbf{LJ_1})^1$. Finally, if we suppose $x \in \hat{F}_A(\mathbf{LJ_1})^1 w$ we can show analogously that $y \in \hat{F}_A(\mathbf{LJ_1})^1 w$, proving the claim. So, by Proposition 2.6, we have that $x = y$. ∎

Now we characterize the idempotents of $\hat{F}_A(\mathbf{R} \cap \mathbf{LJ_1})$.

Proposition 3.2 *Let A be an alphabet and let x and y be idempotents in $\hat{F}_A(\mathbf{R} \cap \mathbf{LJ_1})$. Then, $x = y$ if and only if $c(x) = c(y)$ and $\mathbf{K} \models x = y$.*

Proof. Suppose first that $x = y$. Since $\mathbf{J_1}$ and $\mathbf{K}$ are contained in $\mathbf{R} \cap \mathbf{LJ_1}$, we conclude immediately that $c(x) = c(y)$ and that $\mathbf{K}$ satisfies $x = y$.

Suppose now that $c(x) = c(y)$ and that $\mathbf{K}$ satisfies $x = y$. Since $\mathbf{R} \cap \mathbf{LJ_1}$ is a subpseudovariety of $\mathbf{DS}$ containing $\mathbf{J_1}$, we have $x \mathcal{J} y$ by Proposition 2.7. But $\hat{F}_A(\mathbf{R} \cap \mathbf{LJ_1})$ is $\mathcal{R}$-trivial, and so $x \mathcal{L} y$. This implies that $xy = x$ and $yx = y$.

Let $\pi : \hat{F}_A(\mathbf{LJ_1}) \to \hat{F}_A(\mathbf{R} \cap \mathbf{LJ_1})$ be the canonical projection, let $x', y' \in \hat{F}_A(\mathbf{LJ_1})$ be such that $\pi(x') = x$ and $\pi(y') = y$ and let $z = (x'y')^\omega x'$ and $w = (y'x')^\omega$. Then, $\mathrm{Fact}(z) = \mathrm{Fact}(w)$ and $\mathbf{D}$ satisfies $z = w$. Moreover, $\mathbf{K}$ satisfies $x' = y'$ because it satisfies $x = y$. So, $\mathbf{K}$ satisfies $z = w$. It follows that $\mathbf{LI}$ satisfies $z = w$. Hence, by Proposition 3.1, $z = w$, and therefore $\pi(z) = \pi(w)$. On the other hand we have

$$\begin{aligned} \pi(z) &= (\pi(x')\pi(y'))^\omega \pi(x') \\ &= (xy)^\omega x \\ &= x^\omega x \quad \text{since } xy = x \\ &= x \quad \text{since } x \text{ is idempotent.} \end{aligned}$$

Analogously we have $\pi(w) = y$ which shows that $x = y$. ■

We represent by $[w, B]$ ($\emptyset \neq B \subseteq A, w \in B^{\mathbb{N}}$) the only idempotent in $\hat{F}_A(\mathbf{R} \cap \mathbf{LJ_1})$ of content B and restriction w to $\mathbf{K}$.

The following result presents some important properties of the idempotent implicit operations on $\mathbf{R} \cap \mathbf{LJ_1}$.

Proposition 3.3 *Let $[w, B]$ and $[w', B']$ be idempotents in $\hat{F}_A(\mathbf{R} \cap \mathbf{LJ_1})$ such that $B \cap B' \neq \emptyset$, let $x \in \hat{F}_A(\mathbf{R} \cap \mathbf{LJ_1})^1$ and let $a \in B$. Then*

(1) $[w, B]a = [w, B]$;

(2) $a[w, B] = [aw, B]$;

(3) $[w, B]x[w', B'] = [w, B \cup c(x) \cup B']$.

Proof. Let us first prove that $[w, B]a$, $a[w, B]$ and $[w, B]x[w', B']$ are regular. By Proposition 2.7, this is immediate for $[w, B]a$ and $a[w, B]$. Let now $y = [w, B]x[w', B']$ and let $b \in B \cap B'$. Again by Proposition 2.7, we have $b^\omega[w, B] \mathcal{L} [w, B]$ and $[w', B']b^\omega = [w', B']$ (since $\hat{F}_A(\mathbf{R} \cap \mathbf{LJ_1})$ is $\mathcal{R}$-trivial). This implies that $b^\omega y b^\omega \mathcal{L} y$. Since $\hat{F}_A(\mathbf{R} \cap \mathbf{LJ_1})$ is locally idempotent, the product $b^\omega y b^\omega$ is idempotent. Consequently y is regular.

As $\mathbf{R} \cap \mathbf{LJ_1}$ is a contained in $\mathbf{DA}$, every regular element in $\hat{F}_A(\mathbf{R} \cap \mathbf{LJ_1})$ is idempotent. So, it follows that $[w, B]a$, $a[w, B]$ and $[w, B]x[w', B']$ are idempotents. Now, to conclude the proof, it suffices to apply Proposition 3.2. ■

Let $x \in \hat{F}_A(\mathbf{R} \cap \mathbf{LJ_1})$. We say that a factorization of x of the form

$$x = u_0[x_1, A_1]u_1 \cdots u_{n-1}[x_n, A_n]u_n$$

is *normal* if

- $u_i \in A^*$ for all $0 \leq i \leq n$, $u_0 \neq 1$ if $x = u_0$;
- $\emptyset \neq A_i \subseteq A$ and $x_i \in A_i^{\mathbb{N}}$ for all $1 \leq i \leq n$;
- $A_i \cap A_j = \emptyset$ for $i \neq j$;
- for each $1 \leq i \leq n$ such that u_i (resp. u_{i-1}) is not the empty word, the first (resp. last) letter of u_i (resp. u_{i-1}) does not lie in A_i.

As a consequence of Proposition 3.3 we have the following result.

Proposition 3.4 *Every element x of $\hat{F}_A(\mathbf{R} \cap \mathbf{LJ_1})$ admits a normal factorization.*

Proof. Let $x \in \hat{F}_A(\mathbf{R} \cap \mathbf{LJ_1})$ and let $x = u_0[x_1, A_1]u_1 \cdots u_{n-1}[x_n, A_n]u_n$ be a factorization of x as a product of words u_i and idempotents $[x_i, A_i]$. The existence of such a factorization is ensured by Proposition 2.7 since in $\hat{F}_A(\mathbf{R} \cap \mathbf{LJ_1})$ every regular element is idempotent.

On the factorization of x we can apply the following three rules.

r.1) Suppose that the first letter of u_i, say a, lies in A_i, for some $1 \leq i \leq n$. Then $u_i = au'_i$ and $[x_i, A_i]a = [x_i, A_i]$ by Proposition 3.3. In this case, we modify the factorization of x by replacing u_i by u'_i.

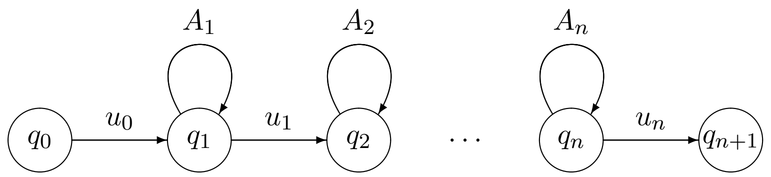
r.2) Suppose that the last letter of u_{i-1}, say a, lies in A_i, for some $1 \leq i \leq n$. Then $u_{i-1} = u'_{i-1}a$ and $a[x_i, A_i] = [ax_i, A_i]$, by Proposition 3.3. In this case, we replace u_{i-1} by u'_{i-1} and $[x_i, A_i]$ by $[ax_i, A_i]$.

r.3) If $A_i \cap A_j \neq \emptyset$ for some $1 \leq i < j \leq n$, then $y = [x_i, A_i]u_i \cdots u_{j-1}[x_j, A_j]$ is equal to $y' = [x_i, A_i \cup A_{i+1} \cup \ldots \cup A_j \cup c(u_i u_{i+1} \cdots u_{j-1})]$, again by Proposition 3.3. We again modify the factorization of x by replacing y by y'.

As these rules effectively reduce the length of the factorization of x, we may apply them only a finite number of times, and so we obtain a factorization of x with the announced properties. ∎

In order to prove uniqueness of the normal factorization of the implicit operations on $\mathbf{R} \cap \mathbf{LJ_1}$, we need some test semigroups to separate distinct factorizations. We will use the transition semigroups of suitable automata.

For $n \geq 0$, let $u_0, \ldots, u_n \in A^*$ and $\emptyset \neq A_1, \ldots, A_n \subseteq A$ be such that $u_i \neq 1$ $(0 < i < n)$, $u_0 \cdots u_n \neq 1$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and the first letter (if it exists) of each u_i $(0 < i \leq n)$ does not lie in A_i. Let $\mathcal{A} = \mathcal{A}(u_0, A_1, u_1, \ldots, A_n, u_n)$ be the following automaton



The condition on the first letter of the words u_i guarantees that $\mathcal{A}$ is a deterministic automaton, meaning that each letter (and, consequently, each word) defines a partial transformation of the set Q of states.

Lemma 3.5 *Let $\mu : A^+ \to S_{\mathcal{A}}$ be the transition homomorphism of the automaton $\mathcal{A}$. The transition semigroup $S_{\mathcal{A}}$ lies in $\mathbf{R} \cap \mathbf{LJ_1}$.*

Moreover, if $w \in A^+, k > |u_0 \cdots u_n| + n$ and $\mu(w^k)$ is not the empty tranformation, then there is some unique $i \in \{1, \ldots, n\}$ such that $w \in A_i^+$, and such that $\mu(w^k)$ has range $\{q_i\}$ and contains q_i in its domain.

Proof. Let us first assume that $\mu(w^k)$ is not the empty tranformation. By the assumption on k it is clear that $q_i \cdot w = q_i$ for some $i \in \{1, \ldots, n\}$. Now, we have clearly $w \in A_i^+$ and the claim that i is unique follows from the assumption that $A_i \cap A_j = \emptyset$ for $i \neq j$. So, the domain of $\mu(w^k)$ contains q_i. The assumption that the first letter of u_i does not belong to A_i implies that the range of $\mu(w^k)$ is $\{q_i\}$.

Let $x, y, z \in A^+$. To prove that $S_{\mathcal{A}}$ belongs to $\mathbf{R} \cap \mathbf{LJ_1}$, it suffices to prove that $\mu((xy)^k x) = \mu((xy)^k)$, $\mu(x^k y x^k y x^k) = \mu(x^k y x^k)$ and $\mu(x^k y x^k z x^k) = \mu(x^k z x^k y x^k)$.

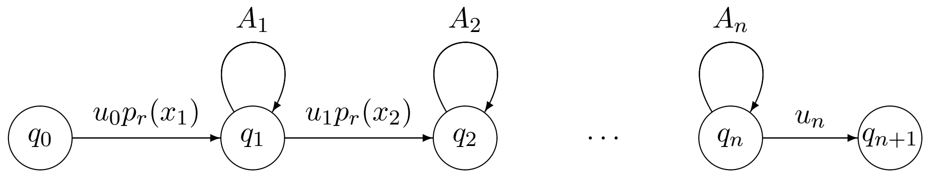
As we proved above, $\mu((xy)^k)$ is not the empty transformation if and only if $xy \in A_i^+$ for some unique $i \in \{1, \ldots, n\}$. In that case, the range of $\mu((xy)^k)$ is $\{q_i\}$ and, as one can easily verify $\mu((xy)^k x)$ has the same domain and range as $\mu((xy)^k)$. Since the range is a singleton set we deduce that $\mu((xy)^k) = \mu((xy)^k x)$.

Similarly, one can show that $\mu(x^k y x^k y x^k)$ and $\mu(x^k y x^k)$ are the same transformation, by simple analysis of their domain (which is not empty if and only if $x, y \in A_i^+$ for some unique $i \in \{1, \ldots, n\}$) and range. The third equality can be proved by an entirely analogous process. ∎

Now we are able to prove that each implicit operation x on $\mathbf{R} \cap \mathbf{LJ_1}$ admits only one normal factorization (which we will call, from now on, the *canonical* factorization of x).

Theorem 3.6 *Let $x, y \in \hat{F}_A(\mathbf{R} \cap \mathbf{LJ_1})$ and let $x = u_0[x_1, A_1]u_1 \cdots [x_n, A_n]u_n$ and $y = v_0[y_1, B_1]v_1 \cdots [y_m, B_m]v_m$ be factorizations in normal form. Then $x = y$ if and only if $n = m,\ u_i = v_i,\ x_i = y_i$ and $A_i = B_i$ for all i.*

Proof. Let us fix $r > |v_i|$ $(1 \leq i \leq m)$, let $\mathcal{A}$ be the following automaton



and let $\mu : A^+ \to S$ be its transition homomorphism. By Lemma 3.5, $S \in \mathbf{R} \cap \mathbf{LJ_1}$. So, let $\hat{\mu} : \hat{F}_A(\mathbf{R} \cap \mathbf{LJ_1}) \to S$ be the unique continuous homomorphic extension of μ and let $k > max\{|u_0 \ldots u_n| + (r+1)n, |v_0 \ldots v_m| + (r+1)m\}$.

For each $1 \leq i \leq n$, let $x'_i \in A_i^{\mathbb{N}}$ be such that $x_i = p_r(x_i)x'_i$ so that $[x_i, A_i] = p_r(x_i)[x'_i, A_i]$. Since $[x'_i, A_i]$ is idempotent, its image in S, $\hat{\mu}([x'_i, A_i])$ is idempotent. By density of A^+ in $\hat{F}_A(\mathbf{R} \cap \mathbf{LJ_1})$, there is a word w_i such that $c(w_i) = A_i$ and $\hat{\mu}([x'_i, A_i]) = \mu(w_i^k)$. Now, it is immediate that $\mu(w_i^k)$ is a partial function whose domain includes q_i and whose range is $\{q_i\}$, by Lemma 3.5. Then, we have clearly $q_0 \cdot u_0 p_r(x_1) w_1^k u_1 p_r(x_2) \cdots w_n^k u_n = q_{n+1}$ and $\hat{\mu}(x) = \mu(u_0 p_r(x_1) w_1^k u_1 p_r(x_2) \cdots w_n^k u_n)$.

Consider now words $y'_i \in B_i^{\mathbb{N}}$ $(1 \leq i \leq m)$ such that $y_i = p_r(y_i)y'_i$ so that $[y_i, B_i] = p_r(y_i)[y'_i, B_i]$. Consider also words w'_i such that $c(w'_i) = B_i$ and $\hat{\mu}([y'_i, B_i]) = \mu({w'_i}^k)$. Again by Lemma 3.5, $\mu({w'_i}^k)$ is a partial function with range either the empty set, or the set $\{q_j\}$ if there is some $j \in \{1, \ldots, n\}$ (unique since $A_l \cap A_j = \emptyset$ for $l \neq j$) such that $B_i \subseteq A_j$. Note that, in this case we have $B_i \cap A_l = \emptyset$ for $l \neq j$.

As $x = y$, $\hat{\mu}(x) = \hat{\mu}(y)$ and so $q_0 \cdot v_0 p_r(y_1) {w'_1}^k v_1 p_r(y_2) \cdots {w'_m}^k v_m = q_{n+1}$. Therefore, $q_0 \cdot v_0 p_r(y_1) {w'_1}^k = q_j$ for some unique $j \in \{1, \ldots, n\}$. This implies that $u_0 p_r(x_1)$ is a prefix of $v_0 p_r(y_1) {w'_1}^k$. Hence, since $r > |v_0|$, $p_r(x_1)$ and $p_r(y_1) {w'_1}^k$ must have some common factor, which means that $A_1 \cap B_1 \neq \emptyset$. It follows that $j = 1$. Therefore $B_1 \subseteq A_1$ and, by symmetry, we have $A_1 = B_1$. Now, since the last letter (if it exists) of u_0 does not belong to A_1, we have that u_0 is a prefix of v_0. Again by symmetry, it follows that $u_0 = v_0$ and consequently $p_r(x_1) = p_r(y_1)$. Hence, since this holds for any r arbitrarily large we conclude that $x_1 = y_1$.

Iterating the above argument, we deduce that $n = m$, $u_i = v_i$, $x_i = y_i$ and $A_i = B_i$ for all i, which concludes the proof. ∎

By the last proof we may deduce that the transition semigroups of the automata $\mathcal{A}$ suffice to separate two distinct implicit operations on $\mathbf{R} \cap \mathbf{LJ_1}$. So, as a consequence of Reiterman's Theorem we have the following.

Corollary 3.7 *The pseudovariety $\mathbf{R} \cap \mathbf{LJ_1}$ is generated by the transition semigroups of the automata $\mathcal{A} = \mathcal{A}(u_0, A_1, u_1, \ldots, A_n, u_n)$ where $u_0, \ldots, u_n \in A^*$ and $\emptyset \neq A_1, \ldots, A_n \subseteq A$ are such that $u_i \neq 1$ $(0 < i < n)$, $u_0 \cdots u_n \neq 1$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and the first letter (if it exists) of each u_i $(0 < i \leq n)$ does not lie in A_i.* ∎

Naturally all that we have done for $\mathbf{R} \cap \mathbf{LJ_1}$ can be done for $\mathbf{L} \cap \mathbf{LJ_1}$. In particular each idempotent x of $\hat{F}_A(\mathbf{L} \cap \mathbf{LJ_1})$ is determined by its content and by its restriction to $\mathbf{D}$. We denote $x = [B, w]$ where $B = c(x)$ and $w \in {}^{\mathbb{N}}B$ is the restriction of x to $\mathbf{D}$.

Theorem 3.8 *Each element x of $\hat{F}_A(\mathbf{L} \cap \mathbf{LJ_1})$ can be written as a product*

$$x = u_0[A_1, x_1]u_1 \cdots u_{n-1}[A_n, x_n]u_n$$

where

- *$u_i \in A^*$ for all $0 \leq i \leq n$, $u_0 \neq 1$ if $x = u_0$;*
- *$\emptyset \neq A_i \subseteq A$ and $x_i \in {}^{\mathbb{N}}A_i$ for all $1 \leq i \leq n$;*
- *$A_i \cap A_j = \emptyset$ for $i \neq j$;*
- *for each $1 \leq i \leq n$ such that u_i (resp. u_{i-1}) is not the empty word, the first (resp. last) letter of u_i (resp. u_{i-1}) does not lie in A_i.*

Moreover this factorization is canonical, i.e., if $x = u_0[A_1, x_1]u_1 \cdots u_{n-1}[A_n, x_n]u_n$ and $y = v_0[B_1, y_1]v_1 \cdots v_{m-1}[B_m, y_m]v_m$ are factorizations of this type, then $x = y$ if and only if $n = m$, $u_i = v_i$, $x_i = y_i$ and $A_i = B_i$ for all i. ∎

Implicit operations on $\mathbf{DA} \cap \mathbf{LJ_1}$ We now turn to the implicit operations on $\mathbf{DA} \cap \mathbf{LJ_1}$. By application of the results of the preceding section we describe canonical factorizations for them. We start by characterizing the idempotents.

Proposition 3.9 *Let A be an alphabet and let x and y be idempotents in $\hat{F}_A(\mathbf{DA} \cap \mathbf{LJ_1})$. Then, $x = y$ if and only if $c(x) = c(y)$ and $\mathbf{LI} \models x = y$.*

Proof. Suppose that $x = y$. Since $\mathbf{J_1}$ and $\mathbf{LI}$ are contained in $\mathbf{DA} \cap \mathbf{LJ_1}$, we conclude immediately that $c(x) = c(y)$ and that $\mathbf{LI}$ satisfies $x = y$.

Suppose now that $c(x) = c(y)$ and that $\mathbf{LI}$ satisfies $x = y$. Since $\mathbf{DA} \cap \mathbf{LJ_1}$ is a subpseudovariety of $\mathbf{DS}$ containing $\mathbf{J_1}$, $x \mathcal{J} y$ by Proposition 2.7. Then, xy and yx are idempotent since, in $\mathbf{DA}$, regular $\mathcal{D}$-classes are idempotent subsemigroups.

Let $\pi : \hat{F}_A(\mathbf{LJ_1}) \to \hat{F}_A(\mathbf{DA} \cap \mathbf{LJ_1})$ be the canonical projection, let $x', y' \in \hat{F}_A(\mathbf{LJ_1})$ be such that $\pi(x') = x$ and $\pi(y') = y$ and let $z = (x'y')^\omega x'$ and $w = (y'x')^\omega y'$. Then, $\mathrm{Fact}(z) = \mathrm{Fact}(w)$ and, since $\mathbf{LI}$ satisfies $x' = y'$ (because it satisfies $x = y$), $\mathbf{LI}$ satisfies $z = w$. Hence, by Proposition 3.1, $z = w$, and therefore $\pi(z) = \pi(w)$. On the other hand we have

$$\begin{aligned} \pi(z) &= (\pi(x')\pi(y'))^\omega \pi(x') \\ &= (xy)^\omega x \\ &= xyx \quad \text{since } xy \text{ is idempotent} \\ &= x \quad \text{since } xyx \,\mathcal{H}\, x \text{ and } \hat{F}_A(\mathbf{DA} \cap \mathbf{LJ_1}) \text{ is } \mathcal{H}\text{-trivial.} \end{aligned}$$

Analogously we have $\pi(w) = y$ which shows that $x = y$. ■

If x is an idempotent of $\hat{F}_A(\mathbf{DA} \cap \mathbf{LJ_1})$ we denote it by $x = [w, B, z]$ where $B = c(x)$ and $(w, z) \in B^{\mathbb{N}} \times {}^{\mathbb{N}}B$ is the restriction of x to $\mathbf{LI}$. The following properties of the idempotents of $\hat{F}_A(\mathbf{DA} \cap \mathbf{LJ_1})$, are proved as in Proposition 3.3.

Proposition 3.10 *Let $[w, B, z]$ and $[w', B', z']$ be idempotent elements of $\hat{F}_A(\mathbf{DA} \cap \mathbf{LJ_1})$ such that $B \cap B' \neq \emptyset$, let $x \in \hat{F}_A(\mathbf{DA} \cap \mathbf{LJ_1})^1$ and let $a \in B$. Then*

(1) $[w, B, z]a = [w, B, za]$;

(2) $a[w, B, z] = [aw, B, z]$;

(3) $[w, B, z]x[w', B', z'] = [w, B \cup c(x) \cup B', z']$. ■

Now we can prove the main result of this section.

Theorem 3.11 *Each element x of $\hat{F}_A(\mathbf{DA} \cap \mathbf{LJ_1})$ can be written as a product $x = u_0[x_1, A_1, x'_1]u_1 \cdots u_{n-1}[x_n, A_n, x'_n]u_n$ where*

- $u_i \in A^*$ *for all* $0 \leq i \leq n$, $u_0 \neq 1$ *if* $x = u_0$;
- $\emptyset \neq A_i \subseteq A$, $x_i \in A_i^{\mathbb{N}}$ *and* $x'_i \in {}^{\mathbb{N}}A_i$ *for all* $1 \leq i \leq n$;
- $A_i \cap A_j = \emptyset$ *for* $i \neq j$;
- *for each* $1 \leq i \leq n$ *such that* u_i *(resp.* u_{i-1}*) is not the empty word, the first (resp. last) letter of* u_i *(resp.* u_{i-1}*) does not lie in* A_i.

Moreover this factorization is canonical, i.e., if $x = u_0[x_1, A_1, x'_1] \cdots [x_n, A_n, x'_n]u_n$ and $y = v_0[y_1, B_1, y'_1] \cdots [y_m, B_m, y'_m]v_m$ are factorizations of this type, then $x = y$ if and only if $n = m$, $u_i = v_i$, $x_i = y_i$, $A_i = B_i$ and $x'_i = y'_i$ for all i.

Proof. We can prove as in Proposition 3.4 that a factorization of the required form exists. For the proof of uniqueness let us assume that

$$x = u_0[x_1, A_1, x'_1]u_1 \cdots u_{n-1}[x_n, A_n, x'_n]u_n$$

and

$$y = v_0[y_1, B_1, y'_1]v_1 \cdots v_{m-1}[y_m, B_m, y'_m]v_m$$

are factorizations of this form and that $x = y$. Let

$$\rho : \hat{F}_A(\mathbf{DA} \cap \mathbf{LJ_1}) \to \hat{F}_A(\mathbf{R} \cap \mathbf{LJ_1})$$

and

$$\lambda : \hat{F}_A(\mathbf{DA} \cap \mathbf{LJ_1}) \to \hat{F}_A(\mathbf{L} \cap \mathbf{LJ_1})$$

be the canonical projections. Since the image by ρ (resp. λ) of an idempotent is an idempotent, it is immediate that, if $e = [w, B, z]$ is an idempotent in $\hat{F}_A(\mathbf{DA} \cap \mathbf{LJ_1})$, then $\rho(e) = [w, B]$ and $\lambda(e) = [B, z]$. Therefore we have

$$\rho(x) = u_0[x_1, A_1]u_1 \cdots [x_n, A_n]u_n$$

and

$$\rho(y) = v_0[y_1, B_1]v_1 \cdots [y_m, B_m]v_m$$

and these factorizations are in canonical form. Thus, as $\rho(x) = \rho(y)$, it follows from Theorem 3.6 that $n = m$, $u_i = v_i$, $x_i = y_i$ and $A_i = B_i$ for all i. Furthermore,

$$\lambda(x) = u_0[A_1, x'_1]u_1 \cdots [A_n, x'_n]u_n$$

and

$$\lambda(y) = v_0[B_1, y'_1]v_1 \cdots [B_m, y'_m]v_m$$

are canonical factorizations by Theorem 3.8. So, since $\lambda(x) = \lambda(y)$, we conclude that $x'_i = y'_i$ for all i, which concludes the proof. ■

Corollary 3.12 $(\mathbf{R} \cap \mathbf{LJ_1}) \vee (\mathbf{L} \cap \mathbf{LJ_1}) = (\mathbf{R} \vee \mathbf{L}) \cap \mathbf{LJ_1} = \mathbf{DA} \cap \mathbf{LJ_1}$.

Proof. Let $x, y \in \hat{F}_A(\mathbf{DA} \cap \mathbf{LJ_1})$. It is clear, by the proof of the preceding proposition, that $x = y$ if and only if $\rho(x) = \rho(y)$ and $\lambda(x) = \lambda(y)$. Thus, by Reiterman's Theorem, $\mathbf{DA} \cap \mathbf{LJ_1} = (\mathbf{R} \cap \mathbf{LJ_1}) \vee (\mathbf{L} \cap \mathbf{LJ_1})$. Moreover, we have

$$\begin{aligned} (\mathbf{R} \cap \mathbf{LJ_1}) \vee (\mathbf{L} \cap \mathbf{LJ_1}) &\subseteq (\mathbf{R} \vee \mathbf{L}) \cap \mathbf{LJ_1} \\ &\subseteq \mathbf{DA} \cap \mathbf{LJ_1} \end{aligned}$$

which implies that $(\mathbf{R} \vee \mathbf{L}) \cap \mathbf{LJ_1} = \mathbf{DA} \cap \mathbf{LJ_1}$. ■

Implicit operations on $\mathbf{J} \cap \mathbf{LJ_1}$ In this section we describe the semigroups of implicit operations on $\mathbf{J} \cap \mathbf{LJ_1}$ using the same method applied for $\mathbf{R} \cap \mathbf{LJ_1}$. We remark that this result was obtained earlier by Selmi [15] using other techniques.

Proposition 3.13 *Let A be an alphabet and let x and y be idempotents in $\hat{F}_A(\mathbf{J} \cap \mathbf{LJ_1})$. Then, $x = y$ if and only if $c(x) = c(y)$.*

Proof. Naturally, we only need to prove the sufficient condition. So, suppose that $c(x) = c(y)$. By Proposition 2.7, $x \mathcal{J} y$. Hence, since $\hat{F}_A(\mathbf{J} \cap \mathbf{LJ_1})$ is $\mathcal{J}$-trivial, we conclude that $x = y$. ■

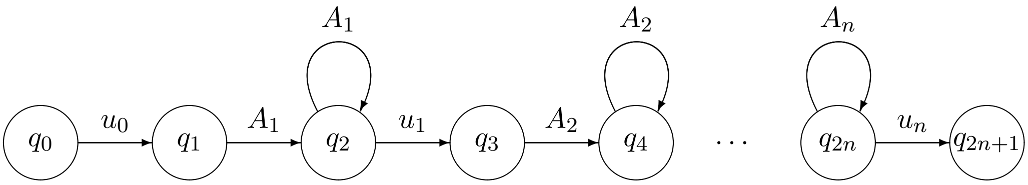
If x is an idempotent of $\hat{F}_A(\mathbf{J} \cap \mathbf{LJ_1})$ we denote it by $x = [B]$ where $B = c(x)$. Analogously to Proposition 3.3 we have the following properties of the idempotents of $\hat{F}_A(\mathbf{J} \cap \mathbf{LJ_1})$.

Proposition 3.14 *Let $[B]$ and $[B']$ be idempotents in $\hat{F}_A(\mathbf{J} \cap \mathbf{LJ_1})$ such that $B \cap B' \neq \emptyset$, let $x \in \hat{F}_A(\mathbf{J} \cap \mathbf{LJ_1})^1$ and let $a \in B$. Then*

(1) $[B]a = [B] = a[B]$;

(2) $[B]x[B'] = [B \cup c(x) \cup B']$. ■

For $n \geq 0$, let $u_0, \ldots, u_n \in A^*$ and $\emptyset \neq A_1, \ldots, A_n \subseteq A$ be such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and, for each $1 \leq i \leq n$ such that u_i (resp. u_{i-1}) is not the empty word, the first (resp. last) letter of u_i (resp. u_{i-1}) does not lie in A_i. Let $\mathcal{B}$ be the following automaton



Selmi [15] proved the following lemma.

Lemma 3.15 *Let $\mu : A^+ \to S$ be the transition homomorphism of the automaton $\mathcal{B}$. The transition semigroup S lies in $\mathbf{J} \cap \mathbf{LJ_1}$.*

Moreover, if $w \in A^+, k > |u_0 \cdots u_n| + n$ and $\mu(w^k)$ is not the empty tranformation, then there is some unique $i \in \{1, \ldots, n\}$ such that $w \in A_i^+$, $\mu(w^k)$ has range $\{q_{2i}\}$ and its domain is $\{q_{2i-1}, q_{2i}\}$. ■

Now we are able to give a new proof of the following result.

Theorem 3.16 ((Selmi)) *Each element x of $\hat{F}_A(\mathbf{J} \cap \mathbf{LJ_1})$ can be written as a product $x = u_0[A_1]u_1 \cdots [A_n]u_n$ where*

- *$u_i \in A^*$ for all $0 \leq i \leq n$, $u_0 \neq 1$ if $x = u_0$;*
- *$\emptyset \neq A_i \subseteq A$ for all $1 \leq i \leq n$;*
- *$A_i \cap A_j = \emptyset$ for $i \neq j$;*
- *for each $1 \leq i \leq n$ such that u_i (resp. u_{i-1}) is not the empty word, the first (resp. last) letter of u_i (resp. u_{i-1}) does not lie in A_i.*

Moreover this factorization is canonical, that is, if $x = u_0[A_1]u_1 \cdots [A_n]u_n$ and $y = v_0[B_1]v_1 \cdots [B_m]v_m$ are factorizations of this type, then $x = y$ if and only if $n = m$, $u_i = v_i$ and $A_i = B_i$ for all i. ■

Proof. That a factorization of the required form exists can be proved as in Proposition 3.4. Let now $x = u_0[A_1]u_1 \cdots [A_n]u_n$ and $y = v_0[B_1]v_1 \cdots [B_m]v_m$ be factorizations of this type and suppose that $x = y$. Consider the automaton $\mathcal{B}$ above. Let $\mu : A^+ \to S$ be its transition homomorphism. By Lemma 3.15, $S \in \mathbf{J} \cap \mathbf{LJ_1}$. Let $\hat{\mu} : \hat{F}_A(\mathbf{J} \cap \mathbf{LJ_1}) \to S$ be the unique continuous homomorphic extension of μ.

Let $k > max\{|u_0 \ldots u_n| + n, |v_0 \ldots v_m| + m\}$ and for each $1 \leq i \leq n$, let w_i be a word such that $c(w_i) = A_i$ and $\hat{\mu}([A_i]) = \mu(w_i^k)$. By Lemma 3.15, $\mu(w_i^k)$ is an idempotent partial function whose range is $\{q_{2i}\}$ and whose domain is $\{q_{2i-1}, q_{2i}\}$. Then, we have $q_0 \cdot u_0 w_1^k u_1 \cdots w_n^k u_n = q_{2n+1}$ and $\hat{\mu}(x) = \mu(u_0 w_1^k u_1 \cdots w_n^k u_n)$.

Consider now words w'_i $(1 \leq i \leq m)$ such that $c(w'_i) = B_i$ and $\hat{\mu}([B_i]) = \mu({w'_i}^k)$. Again by Lemma 3.15, $\mu({w'_i}^k)$ is a partial function whose domain and range are, if not empty, $\{q_{2j-1}, q_{2j}\}$ and$\{q_{2j}\}$, respectively, where $j \in \{1, \ldots, n\}$ is the unique index such that $B_i \subseteq A_j$.

As $x = y$, we have $\hat{\mu}(x) = \hat{\mu}(y)$ and so $q_0 \cdot v_0 {w'_1}^k v_1 \cdots {w'_m}^k v_m = q_{2n+1}$. Therefore, for each $1 \leq i \leq m$ there is some unique $j_i \in \{1, \ldots, n\}$ such that $q_0 \cdot v_0 {w'_1}^k v_1 \cdots {w'_i}^k = q_{2j_i}$, whence $B_i \subseteq A_{j_i}$. By symmetry, we also have $A_{j_i} \subseteq B_{l_i}$ for some $l_i \in \{1, \ldots, m\}$. Since, by hypothesis, the B_i's are pairwise disjoint, this implies $l_i = i$. Consequently, we have $B_i = A_{j_i}$ and we deduce that $1 \leq j_1 < j_2 < \cdots < j_m \leq n$ which shows that $m \leq n$. By symmetry it follows that $n = m$ and, consequently, that $j_i = i$ for all i. In particular we have $A_i = B_i$. Now, for each $0 \leq i \leq n$, the condition on the extreme letters of the words v_i allows us to deduce that $q_0 \cdot v_0 {w'_1}^k v_1 \cdots {w'_i}^k v_i = q_{2i+1}$ and that $u_i = v_i$, which concludes the proof. ■

4 The corresponding varieties of languages

In this section we give combinatorial descriptions of the varieties of languages associated with the pseudovarieties $\mathbf{R} \cap \mathbf{LJ_1}$, $\mathbf{J} \cap \mathbf{LJ_1}$ and $\mathbf{DA} \cap \mathbf{LJ_1}$. For each of these varieties we describe a set of generators. This is done by simple translation, via Eilenberg's correspondence, of the results of the preceding section.

$\mathcal{R}$-trivial and locally testable languages Let A be a finite alphabet. Denote by $\mathcal{L}_A(\mathbf{R} \cap \mathbf{LJ_1})$ the class of all languages of the form $u_0 A_1^* u_1 \cdots A_n^* u_n$ where $n \geq 0$, $u_i \in A^*$, $\emptyset \neq A_i \subseteq A$ and:

- $u_i \neq 1$ for $0 < i < n$, $u_0 \cdots u_n \neq 1$;
- $A_i \cap A_j = \emptyset$ for $i \neq j$;
- the first letter (if it exists) of each u_i $(0 < i \leq n)$ does not lie in A_i.

These are precisely the languages recognized by the automata $\mathcal{A}(u_0, A_1, \ldots, A_n, u_n)$ where the initial and terminal states are, respectively, q_0 and q_{n+1}. So, as a consequence of Eilenberg's correspondence we have the following reformulation of Corollary 3.7, which gives a description of the class of $(\mathbf{R} \cap \mathbf{LJ_1})$-recognizable languages of A^+ (that is, the class of languages that are both $\mathcal{R}$-trivial and locally testable).

Theorem 4.1 *For each finite alphabet A, the class of languages in A^+ which are recognized by semigroups in $\mathbf{R} \cap \mathbf{LJ_1}$ is the Boolean algebra generated by $\mathcal{L}_A(\mathbf{R} \cap \mathbf{LJ_1})$.* ■

For the class of languages that are both $\mathcal{L}$-trivial and locally testable we have a dual description. Let $\mathcal{L}_A(\mathbf{L} \cap \mathbf{LJ_1})$ denote the class of all languages on a finite alphabet A of the form $u_0 A_1^* u_1 \cdots A_n^* u_n$ where $n \geq 0$, $u_i \in A^*$, $\emptyset \neq A_i \subseteq A$ and:

- $u_i \neq 1$ for $0 < i < n$, $u_0 \cdots u_n \neq 1$;
- $A_i \cap A_j = \emptyset$ for $i \neq j$;
- the last letter (if it exists) of each u_i $(0 \leq i < n)$ does not lie in A_{i+1}.

Then, just as above we have the following result.

Theorem 4.2 *For each finite alphabet A, the class of languages in A^+ which are recognized by semigroups in $\mathbf{L} \cap \mathbf{LJ_1}$ is the Boolean algebra generated by $\mathcal{L}_A(\mathbf{L} \cap \mathbf{LJ_1})$.* ■

Analogously for the case of the $(\mathbf{J} \cap \mathbf{LJ_1})$-recognizable languages we have the following description which is a consequence of the proof of Theorem 3.16.

Theorem 4.3 ((Selmi)) *For each finite alphabet A, the class of languages in A^+ which are recognized by semigroups in $\mathbf{J} \cap \mathbf{LJ_1}$ is the Boolean algebra generated by the languages of the form $u_0 A_1^+ u_1 \cdots A_n^+ u_n$ where $n \geq 0$, $u_i \in A^*$, $\emptyset \neq A_i \subseteq A$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and for each $1 \leq i \leq n$ such that u_i (resp. u_{i-1}) is not the empty word, the first (resp. last) letter of u_i (resp. u_{i-1}) does not lie in A_i.* ■

DA-recognizable and locally testable languages The description of the languages that are both **DA**-recognizable and locally testable is an easy consequence of previous results.

Theorem 4.4 *For each finite alphabet A, the class of languages in A^+ which are recognized by semigroups in $\mathbf{DA} \cap \mathbf{LJ_1}$ is the Boolean algebra generated by $\mathcal{L}_A(\mathbf{R} \cap \mathbf{LJ_1}) \cup \mathcal{L}_A(\mathbf{L} \cap \mathbf{LJ_1})$.*

Proof. Let us denote by $\mathbf{V}, \mathbf{V_1}$ and $\mathbf{V_2}$ the pseudovarieties $\mathbf{DA} \cap \mathbf{LJ_1}, \mathbf{R} \cap \mathbf{LJ_1}$ and $\mathbf{L} \cap \mathbf{LJ_1}$, respectively. For a set of languages $\mathcal{L}$ we will denote by $B(\mathcal{L})$ the Boolean algebra generated by $\mathcal{L}$.

We know by Corollary 3.12 that $\mathbf{V} = \mathbf{V_1} \vee \mathbf{V_2}$. So, since $A^+\mathcal{V}_1 = B(\mathcal{L}_A(\mathbf{V_1}))$ and $A^+\mathcal{V}_2 = B(\mathcal{L}_A(\mathbf{V_2}))$, we have $A^+\mathcal{V} = B(\mathcal{L}_A(\mathbf{V_1}) \cup \mathcal{L}_A(\mathbf{V_2}))$. ■

Example 4.5 *The language $L = \{a,b\}^+ac\{c,d\}^*$ on the alphabet $A = \{a,b,c,d\}$ is neither **R**- nor **L**-recognizable. But it is $(\mathbf{DA} \cap \mathbf{LJ_1})$-recognizable. Indeed*

$$\begin{aligned} L &= \{a,b\}^+a\{c,d\}^* \cap \{a,b\}^*c\{c,d\}^* \\ &= (\{a,b\}^*aa\{c,d\}^* \cup \{a,b\}^*ba\{c,d\}^*) \cap \{a,b\}^*c\{c,d\}^*, \end{aligned}$$

is clearly in the Boolean algebra generated by $\mathcal{L}_A(\mathbf{R} \cap \mathbf{LJ_1}) \cup \mathcal{L}_A(\mathbf{L} \cap \mathbf{LJ_1})$. ■

We now give another simpler description of the languages recognized by semigroups in $\mathbf{DA} \cap \mathbf{LJ_1}$. Let $\mathcal{L}_A(\mathbf{DA} \cap \mathbf{LJ_1})$ be the class of all languages of the form $u_0 A_1^* u_1 \cdots A_n^* u_n$ where $n \geq 0$, $u_i \in A^*$, $\emptyset \neq A_i \subseteq A$ and:

- $u_i \neq 1$ for $0 < i < n$, $u_0 \cdots u_n \neq 1$;
- $A_i \cap A_j = \emptyset$ for $i \neq j$.

We start by proving that every language of $\mathcal{L}_A(\mathbf{DA} \cap \mathbf{LJ_1})$ is $(\mathbf{DA} \cap \mathbf{LJ_1})$-recognizable.

Lemma 4.6 *If L is a language of $\mathcal{L}_A(\mathbf{DA} \cap \mathbf{LJ_1})$, then $S(L) \in \mathbf{DA} \cap \mathbf{LJ_1}$.*

Proof. Suppose that $L = u_0A_1^*u_1 \cdots A_n^*u_n$ and let $k > |u_0 \cdots u_n| + n$. By the assumption on k it is clear that, for each $x \in A^+$, if $rx^ks \in L$ for some $r, s \in A^*$, then $x \in A_i^+$ for some $1 \leq i \leq n$. Furthermore, this i is unique since the A_i's are pairwise disjoint.

Let $x, y, z \in A^+$. To prove that $S(L)$ belongs to $\mathbf{DA} \cap \mathbf{LJ_1}$, it suffices to prove that $(xy)^k(yx)^k(xy)^k \sim (xy)^k$, $x^kyx^kyx^k \sim x^kyx^k$ and $x^kyx^kzx^k \sim x^kzx^kyx^k$, where $\sim$ is the syntactic congruence of L.

Let $r, s \in A^*$ and suppose that $r(xy)^ks \in L$. Then xy (and so also yx) lies in A_i^+ for some unique $1 \leq i \leq n$ and $r(xy)^k(yx)^k(xy)^ks \in L$. Analogously, $r(xy)^k(yx)^k(xy)^ks \in L$ implies $r(xy)^ks \in L$. Therefore $(xy)^k(yx)^k(xy)^k \sim (xy)^k$.

Suppose now that $rx^kyx^ks \in L$. Then, $x \in A_i^+$ for some unique $1 \leq i \leq n$. The choice of k and the uniqueness of i allow us to conclude that y also lies in A_i^+, and that $rx^kyx^kyx^ks \in L$. Analogously one can prove that $rx^kyx^kyx^ks \in L$ implies $rx^kyx^ks \in L$ which shows that $x^kyx^k \sim x^kyx^kyx^k$.

The proof that $x^kyx^kzx^k \sim x^kzx^kyx^k$ is absolutely similar, so we omit it. ∎

As a consequence of the last lemma and of Theorem 4.4, and since all languages in $\mathcal{L}_A(\mathbf{R} \cap \mathbf{LJ_1}) \cup \mathcal{L}_A(\mathbf{L} \cap \mathbf{LJ_1})$ are in $\mathcal{L}_A(\mathbf{DA} \cap \mathbf{LJ_1})$, we deduce the announced result.

Theorem 4.7 *For each finite alphabet A, the class of languages in A^+ which are recognized by semigroups in $\mathbf{DA} \cap \mathbf{LJ_1}$ is the Boolean algebra generated by $\mathcal{L}_A(\mathbf{DA} \cap \mathbf{LJ_1})$.* ∎

We can now deduce, more easily, that the language $L = \{a, b\}^+ac\{c, d\}^*$ of Example 4.5 is $(\mathbf{DA} \cap \mathbf{LJ_1})$-recognizable. In fact, $L = \{a, b\}^*aac\{c, d\}^* \cup \{a, b\}^*bac\{c, d\}^*$ is the union of two languages in $\mathcal{L}_A(\mathbf{DA} \cap \mathbf{LJ_1})$.

5 Another approach to $(\mathbf{R} \cap \mathbf{LJ_1})$-recognizable languages

In this section we show that, for each alphabet A, the $\mathcal{R}$-trivial and locally testable languages of A^+ can be described by certain congruences on A^+.

First we recall the most used definition of a locally testable language. For every integer $k \geq 1$, let $=_k$ be the finite-index congruence defined on A^+ by setting $u =_k v$ if and only if $u = v$ or u and v are words of length $\geq k$, $p_{k-1}(u) = p_{k-1}(v)$, $s_{k-1}(u) = s_{k-1}(v)$ and $F_k(u) = F_k(v)$. We say that a language of A^+ is *k-testable* if it is a union of $=_k$-classes. A language is locally testable if it is k-testable for some k.

Now, for every $k \geq 1$, let $\sim_k$ be the finite-index equivalence on A^+ defined by $u \sim_k v$ if and only if $u = v$ or u and v are words of length $\geq k$, $p_{k-1}(u) = p_{k-1}(v)$ and $F_k(u) = F_k(v)$. We note that this equivalence is not a congruence in general. For instance, consider $A = \{a, b\}$, $u = aba$ and $v = abab$. One has $u \sim_2 v$, but $ua \not\sim_2 va$. Indeed a^2 is a factor of length 2 of ua but it is not a factor of va. We denote by $\equiv_k$ the congruence on A^+ generated by $\sim_k$. Observe that $\equiv_k$ is a finite-index congruence since it is coarser than $\sim_k$. The rest of this paper is devoted to showing that the languages that are a union of $\equiv_k$-classes for some k are exactly the $\mathcal{R}$-trivial and locally testable languages. This work is analogous to Selmi's work [15] on $\mathcal{J}$-trivial and locally testable languages.

We will use the notion of *locally testable* semigroup introduced by McNaughton [11] and Zalcstein [18], which we now describe. For a semigroup S we denote by S^+ the set of all finite sequences of elements of S.

Definition 5.1 *Let S be a finite semigroup and let $k \geq 1$. We say that S is* k-testable *if for each pair of elements $(x_1, \ldots, x_n), (y_1, \ldots, y_m)$ of S^+, with $n, m \geq k$, having the same prefix and suffix of length $k-1$ and the same set of factors of length k, one has $x_1 \cdots x_n = y_1 \cdots y_m$. A semigroup is* locally testable *if it is k-testable for some k.* ■

The set of all locally testable semigroups is denoted by $\mathbf{LT}$ and $\mathbf{LT_k}$ is the set of all k-testable semigroups. Zalcstein [18] proved that $\mathbf{LT_k}$ is a pseudovariety and corresponds to the class of k-testable languages. It follows from Brzozowski and Simon [9] and McNaughton [11], that $\mathbf{LT} = \bigcup_{k=1}^{\infty} \mathbf{LT_k}$ is the pseudovariety of locally idempotent and locally commutative semigroups, that is $\mathbf{LT} = \mathbf{LJ_1}$. Note that every k-testable semigroup is also m-testable for all $m \geq k$, so we have $\mathbf{LT_1} \subseteq \mathbf{LT_2} \subseteq \cdots \subseteq \mathbf{LT_k} \subseteq \cdots$. As a result, we have $\mathbf{R} \cap \mathbf{LJ_1} = \bigcup_{k=1}^{\infty} (\mathbf{R} \cap \mathbf{LT_k})$.

Let us now return to the congruence $\equiv_k$ and prove the following results.

Proposition 5.2 *The semigroup $A^+/\!\equiv_k$ lies in $\mathbf{R} \cap \mathbf{LT_k}$.*

Proof. The congruence $\equiv_k$ is coarser than the congruence $=_k$. So, the semigroup $A^+/\equiv_k$ is a homomorphic image of the semigroup $A^+/=_k$ which lies in $\mathbf{LT_k}$. Hence, since $\mathbf{LT_k}$ is a pseudovariety, $A^+/\!\equiv_k$ lies in $\mathbf{LT_k}$. To prove that $A^+/\!\equiv_k$ is $\mathcal{R}$-trivial we show that it satisfies the pseudoidentity $(xy)^\omega x = (xy)^\omega$ which defines $\mathbf{R}$.

Let $u, v \in A^+$. It follows from the definition of $\sim_k$ that $(uv)^n u \sim_k (uv)^n$ for all $n \geq k$. So, $(uv)^n u \equiv_k (uv)^n$ for all $n \geq k$, and hence $A^+/\!\equiv_k$ satisfies $(xy)^\omega x = (xy)^\omega$. ■

Proposition 5.3 *The semigroup $A^+/\!\equiv_k$ is the free object of $\mathbf{R} \cap \mathbf{LT_k}$ on A.*

Proof. Let $\pi : A^+ \to A^+/\!\equiv_k$ be the natural morphism. We need to prove that, for any semigroup S in $\mathbf{R} \cap \mathbf{LT_k}$ and any morphism $\eta : A^+ \to S$, there exists a morphism $\varphi : A^+/\!\equiv_k \to S$ such that $\varphi \circ \pi = \eta$. It suffices to prove that the morphism $\varphi : A^+/\!\equiv_k \to S$ given by $\varphi(\pi(u)) = \eta(u)$ for any $u \in A^+$ is well-defined, that is, to prove that $u \equiv_k v$ implies $\eta(u) = \eta(v)$. By transitivity and multiplicativity, this in turn reduces to showing that if $u \sim_k v$, then $\eta(u) = \eta(v)$.

If $|u| < k$ or $|v| < k$, then $u = v$ so that $\eta(u) = \eta(v)$ trivially. Let us now suppose that $|u|, |v| \geq k$. By definition of $\sim_k$ we have $p_{k-1}(u) = p_{k-1}(v)$ and $F_k(u) = F_k(v)$. Let s be the suffix of length $k-1$ of u. Then s occurs in v, so that $v = xsy$ for some $x, y \in A^*$. Put $w = uy$. We claim that $w =_k v$. Indeed $p_{k-1}(w) = p_{k-1}(u) = p_{k-1}(v)$. Next, $s_{k-1}(w) = s_{k-1}(uy) = s_{k-1}(sy) = s_{k-1}(v)$, since $|sy| \geq k-1$. Now, since $F_k(u) = F_k(v)$, each factor of length k of v is a factor of u and hence a factor of w. Conversely, let z be a factor of length k of w. Then, z is either a factor of u, or a factor of sy. In each case, it is also a factor of v, which proves the claim.

Let $u = u_1 \cdots u_l$, $y = y_1 \cdots y_m$ (supposing $y \neq 1$) and $v = v_1 \cdots v_n$, where $u_h, y_i, v_j \in A$ for all h, i and j. Since $uy =_k v$, the sequences of elements of S

$$(\eta(u_1), \ldots, \eta(u_l), \eta(y_1), \ldots, \eta(y_m)) \text{ and } (\eta(v_1), \ldots, \eta(v_n))$$

have the same prefix and suffix of length $k-1$ and the same set of factors of length k. But $S \in \mathbf{LT_k}$, so

$$\begin{aligned} \eta(u)\eta(y) &= \eta(u_1) \cdots \eta(u_l)\eta(y_1) \cdots \eta(y_m) \\ &= \eta(v_1) \cdots \eta(v_n) \\ &= \eta(v). \end{aligned}$$

A dual argument would show that $\eta(u) \,\mathcal{R}\, \eta(v)$. But S is $\mathcal{R}$-trivial, so that $\eta(u) = \eta(v)$. ■

Proposition 5.4 *A language $L \subseteq A^+$ is recognized by a semigroup in $\mathbf{R} \cap \mathbf{LT_k}$ if and only if it is a union of $\equiv_k$-classes.*

Proof. Let $\eta : A^+ \to S(L)$ be the syntactic morphism of L and let $\pi : A^+ \to A^+/{\equiv_k}$ be the natural morphism. Suppose first that $S(L) \in \mathbf{R} \cap \mathbf{LT_k}$. Then, by Proposition 5.3, there is a surjective morphism $\varphi : A^+/{\equiv_k} \to S(L)$ such that $\varphi \circ \pi = \eta$. So, L is a union of $\equiv_k$-classes.

Conversely, suppose that L is a union of $\equiv_k$-classes. Then, L is recognized by the natural morphism π. So, $S(L)$ divides $A^+/\equiv_k$, whence, by Proposition 5.2, $S(L) \in \mathbf{R} \cap \mathbf{LT_k}$. ∎

The announced result is now an immediate consequence of this last proposition and of the equality $\mathbf{R} \cap \mathbf{LJ_1} = \bigcup_{k=1}^{\infty}(\mathbf{R} \cap \mathbf{LT_k})$.

Theorem 5.5 *A language $L \subseteq A^+$ is recognized by a semigroup in $\mathbf{R} \cap \mathbf{LJ_1}$ if and only if it is a union of $\equiv_k$-classes for some k.* ∎

Acknowledgements

I would like to thank Pascal Weil, my supervisor, for suggesting the study which is the object of this paper and for his useful comments and suggestions. I am also very grateful to Assis Azevedo for the attention he paid to my work and for his many valuable pieces of advice.

References

[1] Almeida, J., *The algebra of implicit operations*, Algebra Universalis **26** (1989), 16-72.

[2] Almeida, J., *Implicit operations on finite $\mathcal{J}$-trivial semigroups and a conjecture of I. Simon*, J. Pure and Applied Algebra **69** (1990), 205-218.

[3] Almeida, J., "Finite Semigroups and Universal Algebra", World Scientific, Singapore, 1994.

[4] Almeida, J., and A. Azevedo, *On regular implicit operations*, Portugaliæ Mathematica **50** (1993), 35-61.

[5] Almeida, J., and P. Weil, *Relatively free profinite monoids: an introduction and examples*, in J. Fountain (Ed.), "Semigroups, Formal Languages and Groups", Kluwer (1995), 73-117.

[6] Almeida, J., and P. Weil, *Free profinite $\mathcal{R}$-trivial semigroups*, Int. J. Algebra and Computation, to appear.

[7] Azevedo, A., *Operations preserving homomorphisms on the class of finite semigroups* **DS**, in J. Almeida (Ed.), "Actas II Encontro de Algebristas Portugueses", Universidade do Porto (1987), 33-43.

[8] Azevedo, A., "Operações Implícitas sobre Pseudovariedades de Semigrupos. Aplicações", Doctoral dissertation, Universidade do Porto, 1989.

[9] Brzozowski, J., and I. Simon, *Characterization of locally testable events*, Discrete Mathematics **4** (1973), 243-271.

[10] Eilenberg, S., "Automata, Languages and Machines", vol. B, Academic Press, New York, 1976.

[11] McNaughton, R., *Algebraic decision procedures for local testability*, Math. Systems and Theory **8** (1974), 60-76.

[12] Pin, J.-E., "Variétés de Langages Formels", Masson, Paris, 1984. English translation: "Varieties of formal languages", Plenum, New York and North Oxford, London, 1986.

[13] Pin, J.-E., and H. Straubing, *Monoids of upper triangular matrices*, Colloquia Mathematica Societatis Janos Bolyai, Semigroups, Szeged **39** (1981), 259-272.

[14] Reiterman, J., *The Birkhoff theorem for finite algebras*, Algebra Universalis **14** (1982), 1-10.

[15] Selmi, C., *Over testable languages*,Theoretical Computer Science **161** (1996), 157-190.

[16] Simon, I., *Piecewise testable events*, in "Proc. 2^{nd} GI Conf.", Lecture Notes in Computer Science **33**, Springer, Berlin (1975), 214-222.

[17] Weil, P., *Implicit operations on pseudovarieties: an introduction*, in J. Rhodes (Ed.), "Monoids and semigroups with applications", World Scientific, Singapore, 1991.

[18] Zalcstein, Y., *Locally testable languages*, J. Computer and System Sciences **6** (1972), 151-167.

[19] Zeitoun, M., "Opérations Implicites et Variétés de Semigroups Finis", Doctoral dissertation, Université Paris VII, 1993.

Dep. Matemática
Universidade do Minho
Campus de Gualtar
4700-320 Braga
Portugal
email: jcosta@math.uminho.pt